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Maximal Cohen-Macaulay Modules over Gorenstein Rings and Bourbaki-Sequences

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Introduction

Motivated by recent work of Knörrer [17], Buchweitz, Greuel and Schreyer [9] who proved that a hypersurface singularity R over $\boldsymbol{C}$ is of finite Cohen-Macaulay representation type if and only if R is a simple hypersurface singularity, we were led to study, quite in general, maximal Cohen-Macaulay modules (*MCM*-modules) over Gorenstein rings. Of course, there is no deeper reason why one should restrict one's attention to Gorenstein rings. It seems, however, that representation theory over non-Gorenstein rings is fundamentally more complicated. For instance, the question of the finite Cohen-Macaulay representation type is not completely settled, though some beautiful techniques have been developed just for that purpose by M. Auslander and I. Reiten [20]. They also give two examples of non-Gorenstein rings of dim ≥ 3, which are of finite Cohen-Macaulay representation type. No other such examples are known. In [14] the first named author of the paper showed that $\boldsymbol{C}[\![x, y]\!]^G$ is of finite Cohen-Macaulay representation type, where $G \subseteq Gl(2; \boldsymbol{C})$ is finite. That these are the only 2-dimensional rings with this property was shown by Artin-Verdier [2], Auslander [3] and Esnault [13].

The great technical advantage of Gorenstein rings is that MCM-modules over Gorenstein rings are reflexive, and that the R-dual of an MCM-module is again an MCM-module.

For the rest of the paper let us always assume that (R, m) is a local Gorenstein ring. The most general question one may raise in this connection is to determine all isomorphism classes of indecomposable MCM-modules over R. Of course, this problem is posed far too generally, and should be considered only as a "Leitmotiv". The following problem seems to be more accessible: Determine all pairs of numbers (m, n) for which there exists an MCM-module M which has rank m and is minimally generated by n elements. We call (m, n) the data of M.

In [10] D. Eisenbud gives a very explicit description of the MCM-

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modules over a hypersurface domain. It follows from this description that if the hypersurface domain $R=A/(f)$ admits an MCM-module with data (m, n), then $f^m=\det \alpha$, where α is a square matrix of size n with entries in the maximal ideal of A. It is not clear in general which powers of f allow such a presentation as a determinant. However, as a consequence of our theorem (3.1) we can show at least the following result: If $f\in k[\![x, y, z]\!]$ is a homogeneous polynomial of degree e, and k is an infinite field, then for $t=1, 2, \cdots, e$ there exists a square matrix α of size $2t$ with coefficients in (x, y, z) such that $f^2=\det \alpha$.

In the first section of this paper we list some generalities about MCM-modules over Gorenstein domains that will be used later. In particular, we show in (1.3) that if M has data (m, n) then the following inequalities hold: $(e\cdot m-p)/(e-1)\leq n\leq e\cdot m$.

Here e is the multiplicity of R, and p the rank of a maximal free direct summand of M. We call an MCM-module with data (m, n) an Ulrich-module if $n=e\cdot m$. Ulrich asks in his paper [21] whether such a module always exists for any Cohen-Macaulay ring. In (1.4) we show that if R is a hypersurface ring of multiplicity 2, then any MCM-module is a direct sum of an Ulrich-module and a free module. More results about Ulrich-modules can be found in [6].

The main technique used in this paper to study MCM-modules are Bourbaki-sequences associated with a module. Recall from [4] that if M is a torsionfree module over a normal domain, then there exists an exact sequence $0\to F\to M\to I\to 0$, where F is free and I is an ideal of R. We call such a sequence a Bourbaki-sequence if M is an MCM-module, and I is a codim 2 CM-ideal or if $I=R$. If R is normal, then an MCM-module M can be inserted into Bourbaki-sequence if and only if it is orientable; this means that $(\Lambda^m M)^{**}\simeq R$, where $m=\operatorname{rank} M$. On the other hand, if I is any codim 2 CM-ideal then there is a Bourbaki-sequence ending with I: $0\to F\to M\to I\to 0$, such that $\operatorname{rk} F=$ Cohen-Macaulay type of R/I. Moreover, the module M in this exact sequence is orientable and unique up to stable isomorphism, see (1.8) and (1.9).

We say that the Bourbaki-sequence $0\to F\to M\to I\to 0$ is tight if the induced map $F\otimes k\to M\otimes k$ is injective. In that case the data (m, n) of M are determined by data of I: $n=\mu(I)+r(I)$, $m=r(I)+1$. Here $\mu(I)$ denotes the minimal number of generators of I, and $r(I)$ the Cohen-Macaulay type of R/I. If R is a hypersurface ring, tightness of the Bourbaki-sequence is easy to check, as we show in (1.10). In this way the question of the existence of an MCM-module with given data can be translated into a question on ideals with a given number of generators and given type. Since ideals are better understood, the latter question is usually easier to answer.

The main result of section two is Theorem (2.1), in which we clarify the precise connection between MCM-modules and codim 2 CM-ideals. There we show that if we are given two Bourbaki-sequences $0 \to F' \to M' \to I' \to 0$ and $F'' \to M'' \to I'' \to 0$, then the following conditions are equivalent:

(a) M' and M'' are stably isomorphic,

(b) I' and I'' belong to the same even linkage class.

M. Miller who proved a special case of this theorem in his dissertation pointed out to us that P. Rao has proved a similar theorem in [19] for vector bundles in $\boldsymbol{P}^n$ with $H^1(\boldsymbol{P}^n, \mathscr{E}(p))=0$ for all $p \in \boldsymbol{Z}$. Our result on MCM-modules is exactly in the same spirit as his theorem and its proof. As an immediate consequence of the above theorem one gets the following Rao-correspondence (2.3): If R is a normal Gorenstein domain, then there exists a bijection between stable isomorphism classes of orientable MCM-modules and even linkage classes of codim 2 CM-ideals.

If M is an MCM-module over R, we set $D(M)=\operatorname{syz}_1(M)^*$, where $\operatorname{syz}_1(M)$ denotes the first syzygy-module of M. Notice that $D(D(M))$ is stably isomorphic to M. It follows quite easily from the main theorem (2.1) of Section 2 that M_1 and $D(M_2)$ are stably isomorphic if and only if the corresponding ideals in the Bourbaki-sequences for M_1 and M_2 are oddly linked, see (2.5).

In the last section we apply this theory to study orientable MCM-modules of rank 2 over hypersurface rings. In (3.1) we show that any such module is evenly generated. For the proof we use that any codim 3 Gorenstein ideal is oddly generated, as known by [8] and [22]. This example demonstrates how the knowledge about ideals yields results on MCM-modules.

More delicate is the question of which even numbers actually occur as numbers of generators for a rank 2 MCM-module. One certainly has to restrict the dimension of the hypersurface domain to get a complete answer. In fact, W. Bruns has shown the inequality $2 \cdot \operatorname{rank} M+1 \geq \dim R$ for any non-trivial module M over an isolated hypersurface singularity R. We would like to thank W. Bruns who allowed us to include this result in our paper, see (3.4). However, if $\dim R=2$, then there are not so many restrictions for the existence of MCM-modules. For instance, we can show that if $R=k[\![x, y, z]\!]/(f)$, where f is homogeneous of degree e, then there exist orientable MCM-modules of rank 2 with $2t$ generators for $t=1, 2, \cdots, e$. For the proof of this result we use a beautiful theorem of D. Eisenbud on matrices of linear forms with no generalized zeros. It states that "quite general" square matrices of linear forms have a non-vanishing determinant. The precise statement can be found in the theorem preceding (3.3).

The rest of the paper is devoted to the study of orientable MCM-modules with data (2, 4) over a 2-dimensional hypersurface domain R whose associated graded ring $\mathrm{gr}_m(R)$ is factorial. We show that if R has multiplicity 2, then there exists up to isomorphism just one, while if the multiplicity is 3, exactly two such modules, see (3.5) and (3.6). For the proof of these results we use the Rao-correspondence in its full strength.

As an amazing corollary of these classifications we are able to show that if R is a 2-dimensional hypersurface ring of multiplicity ≤ 3, such that $\mathrm{gr}_m(R)$ is factorial then R itself is factorial. C. Huneke supplied us with an example of a hypersurface domain R of dim 4 for which $\mathrm{gr}_m(R)$ is factorial but R is not factorial, so that the above conclusion is not true in general, and we also learned from U. Storch that D. Günther in his dissertation "Divisorenklassengruppen und Picardsche Gruppen" Essen (1976) has essentially proved the same result with different methods. In this dissertation it is also shown that if (R, m) is a noetherian complete normal domain such that $\mathrm{gr}_m(R)$ is factorial and depth $\mathrm{gr}_m(R) \geq 3$, then R is factorial as well.

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§ 1. Some general observations about maximal Cohen-Macaulay modules and Bourbaki-sequences

Throughout this section we only consider finitely generated modules over a local noetherian Gorenstein domain (R, m, k) of dimension d with infinite residue class field k. As usual we set rank $M = \dim_{Q(R)} M \otimes_R Q(R)$ for an R-module M, where $Q(R)$ denotes the quotient field of R. For most of the statements in this section the assumption that R is a domain can be weakened. If one drops this condition, one has to redefine rank M and has to restrict oneself to modules that actually have a rank. Similarly an infinite residue class field is not needed for all the proofs. But in order to avoid pedantic technical assumptions all the time we prefer to stick on the above frame.

An R-module M is called a *maximal Cohen-Macaulay* module (*MCM*-module) if depth $M = d(=\dim R)$. The R-dual of M will be denoted by M^*. It is well known that M^* is again an MCM-module and that M is reflexive, see for instance [15], Satz 6.1. Moreover, the functor $M \to M^*$ is exact on the full subcategory of MCM-modules over R.

The R-modules M and N are called *stably isomorphic*, if there exist free R-modules F and G (of finite rank) such that $M \oplus F \simeq N \oplus G$. We write in this case $M \underset{\text{st}}{\simeq} N$.

Since R is local, it is clear that stably isomorphic R-modules of equal rank are actually isomorphic.

We define the *free rank* of an R-module M (*f-rank* M) as the rank of a maximal free direct summand of M.

By $\mu(M)$ we denote the minimal number of generators of M. If $\mu(M) = n$, then there exists an exact sequence $0 \to N \to R^n \to M \to 0$. The module N is uniquely determined by M up to isomorphism and is called the first *syzygy-module* ($\mathrm{syz}_1(M)$) of M. The i^{th} syzygy-module of M is defined to be $\mathrm{syz}_1(\mathrm{syz}_{i-1}(M))$. It is convenient to put $\mathrm{syz}_0(M) := M$.

The numbers $\beta_i(M) := \mu(\mathrm{syz}_i(M))$ are called the *Betti numbers* of M. We say that M *has data* (m, n) if $m = \mathrm{rank}\, M$, and $n = \mu(M)$. Let M be an MCM-module. We set $D(M) := \mathrm{syz}_1(M)^*$.

The assignment $M \rightsquigarrow D(M)$ satisfies the following rules:

(1.1) **Lemma.** *Suppose M is an MCM-module with data(m, n) and* f-rank p, *then*

(a) $D(M \oplus F) \simeq D(M)$, *where F is free*

(b) $M \simeq D(D(M)) \oplus R^p$

(c) $D(M)$ *has data* $(n-m, n-p)$ *and* f-rank 0.

Proof. (a) Clearly $D(M \oplus N) \simeq D(M) \oplus D(N)$, and $D(F) = 0$ if F is free. These two observations imply (a). (b) Using (a) we may assume that f-rank $M = 0$, and we have to show that $M \simeq D(D(M))$. Let

$$0 \longrightarrow \mathrm{syz}_1(M) \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

be a minimal presentation of M, then

$$(*) \qquad 0 \longrightarrow M^* \longrightarrow R^n \longrightarrow D(M) \longrightarrow 0$$

is exact.

Since f-rank $M = 0$, the presentation (*) of $D(M)$ is minimal as well, so that $M^* = \mathrm{syz}_1(D(M))$ and therefore $M \simeq M^{**} = \mathrm{syz}_1(D(M))^* = D(D(M))$. (c) It follows from (*) that $D(M)$ has data $(n-m, n)$ if f-rank $M = 0$. Using (a) the formula follows in general. Finally, f-rank $D(M) = 0$ follows from the next lemma.

(1.2) **Lemma.** *If M is an MCM-module then* f-rank $(\mathrm{syz}_i(M)) = 0$ *for all $i \geq 1$.*

Proof. It is enough to consider $i = 1$. Choose a minimal presentation $0 \to N \to F \to M \to 0$ of M, then $\mathrm{syz}_1(M) = N \subseteq mF$. Therefore $\varphi(N) \subseteq m$ for all $\varphi \in F^*$, and this together with the dual exact sequence

$$0 \longrightarrow M^* \longrightarrow F^* \longrightarrow N^* \longrightarrow 0$$

gives $\varphi(N) \subseteq m$ for all $\varphi \in N^*$, and consequently f-rank $N=0$. □

(1.3) **Corollary.** *Suppose R has multiplicity $e>1$, and that M is an MCM-module with data (m, n) and* f-rank p, *then*

$$\frac{e \cdot m - p}{e-1} \leq n \leq e \cdot m.$$

Proof. As R is assumed to have infinite residue class field there exists a system of parameters $\underline{x}$ for M such that the multiplicity of M is $e(M) = \text{length}\,(M/\underline{x}M)$. The associative law for multiplicities yields $e(M) = e \cdot \text{rank}\, M = e \cdot m$, and therefore

$$n = \mu(M) = \text{length}\,(M/mM) \leq \text{length}\,(M/\underline{x}M) = e \cdot m.$$

The lower bound can be deduced if we apply this upper estimate to $D(M)$ and use (1.1), (c). □

If an MCM-module $M \neq 0$ assumes the upper bound for the number of generators, i.e. $\mu(M) = e(R) \cdot \text{rank}\, M$, then we call M an *Ulrich-module.*

In his paper [21] Ulrich asks whether all local CM-rings admit such a module and remarks that rings of minimal multiplicity certainly do. Moreover, Ulrich informed the authors in a letter that he and Brennan have shown the existence of Ulrich-modules of rank e over any two-dimensional hypersurface ring of the form $k[\![X, Y, Z]\!]/(f)$, where degree $f=e$. In the last section of this paper we prove the existence of rank 2 Ulrich-modules over such a hypersurface ring. More results on Ulrich-modules can be found in a joint paper [6] by the first-named author of this paper together with Brennan and Ulrich.

As a consequence of the above considerations we now deduce a simple existence statement for Ulrich-modules.

(1.4) **Corollary.** *Suppose R has multiplicity* 2 (*in which case R is an abstract hypersurface ring*), *then any non-free MCM-module M can be decomposed $M \simeq U \oplus F$, where U is an Ulrich-module and F is free.*

In particular, any such ring admits an Ulrich-module.

Proof. Write $M \simeq U \oplus R^p$, where $p=$f-rank M. Then f-rank $U=0$, and (1.3) yields $\mu(U) = 2 \cdot \text{rank}\, U$.

Since R is not regular this ring admits a nontrivial MCM-module U with f-rank $U=0$. This module is then an Ulrich-module. □

Maximal MCM-modules over hypersurface rings are much better understood than MCM-modules over arbitrary Gorenstein domains. A basic reference on this subject is the paper [10] by D. Eisenbud.

We list a few elementary properties that will be used in the sequel.

(1.5) **Lemma.** *Let R be a hypersurface ring and M be an MCM-module over R, then $\mu(M)=\mu(M^*)$.*

Proof. By assumption $R=A/(f)$, where A is a regular local ring. If $n=\mu(M)$, then M admits a minimal free A-resolution $0\to A^n\to A^n\to M\to 0$. Dualizing this resolution shows that $n=\mu(\mathrm{Ext}^1(M, A))$.

On the other hand, the exact sequence $0\to A\xrightarrow{f} A\to R\to 0$ yields the exact sequence $0\to \mathrm{Hom}_A(M, R)\to \mathrm{Ext}_A^1(M, A)\xrightarrow{f}\cdots$. But f annihilates M, so $M^*=\mathrm{Hom}_R(M, R)=\mathrm{Hom}_A(M, R)\simeq \mathrm{Ext}_A^1(M, A)$, whence $\mu(M)=n=\mu(M^*)$. □

(1.6) **Corollary.** *Let M be an MCM-module over an hypersurface ring, then*

(a) f-rank $M=\beta_0(M)-\beta_1(M)$

(b) $\beta_i(M)=\beta_1(M)$ *for* $i\geq 1$.

Proof. (a) By (1.5) and (1.1) we have

$$\begin{aligned}\beta_1(M)&=\mu(\mathrm{syz}_1(M))=\mu(\mathrm{syz}_1(M)^*)=\mu(D(M))=\mu(M)-\text{f-rank } M\\&=\beta_0(M)-\text{f-rank } M.\end{aligned}$$

(b) follows from (a) and (1.2). □

(1.7) **Lemma** (D. Eisenbud [10]). *Let M be an MCM-module over an abstract hypersurface ring R. Put $p=$f-rank M, and write: $M\simeq U\oplus R^p$, then $\mathrm{syz}_2(M)\simeq U$. In particular, if $p=0$, then $\mathrm{syz}_2(M)\simeq M$ and M has periodic resolution of period 2.*

Proof. Since $\mathrm{syz}_2(M)\simeq \mathrm{syz}_2(U)$ we may assume that f-rank $M=0$, hence $\beta_1(M)=\beta_0(M)$ by (1.6). From a minimal A-resolution

$$0\longrightarrow A^n\xrightarrow{\alpha} A^n\longrightarrow M\longrightarrow 0$$

we obtain the exact sequence

$$0\longrightarrow \mathrm{Tor}_1^A(M, R)\xrightarrow{\beta} R^n\xrightarrow{\bar{\alpha}} R^n\longrightarrow M\longrightarrow 0.$$

Since $\beta_1(M)=\beta_0(M)=n$ we have $\mu(\mathrm{Im}\,\bar{\alpha})=n$, and we deduce $\mathrm{Im}\,\beta\subseteq mR^n$; therefore $\mathrm{Tor}_1^A(M, R)\simeq \mathrm{syz}_2(M)$. On the other hand, $0\to A\xrightarrow{f} A\to R\to 0$ yields the exact sequence $0\to \mathrm{Tor}_1^A(M, R)\to M\xrightarrow{f} M$, and $f\cdot M=0$ implies $\mathrm{Tor}_1^A(M, R)\simeq M$. □

One of our main tools for constructing MCM-modules will be the notion of Bourbaki-sequences which establishes a connection between modules and ideals.

Recall from [4], ch. VII, § 4, No. 9, Theorem 6 the following result:

Theorem. *If R is a noetherian integrally closed domain and M is a finitely generated torsion-free R-module, there exists a free submodule $F \subseteq M$ such that M/F is (as R-module) isomorphic to an ideal of R.*

In this paper, we will call an exact sequence of R-modules

$$0 \longrightarrow F \longrightarrow M \longrightarrow I \longrightarrow 0$$

a *Bourbaki-sequence* if F is free, M is an MCM-module and I is a codim 2 CM-ideal, or $I=R$ (notice that codim $I \leq 2$ if $I \neq R$, since M is an MCM-module). Here we use the convention that an ideal I is called a *CM-ideal* if R/I is a CM-ring. The number $r(I) := \dim_k \operatorname{Ext}^t(k, R/I)$ where $t = \dim R - \operatorname{codim} I$, is called the *(Cohen-Macaulay)-type* of I. For the existence of Bourbaki-sequences the following notion is of importance:

Suppose R is normal with divisor class group $\mathrm{Cl}(R)$. If J is a reflexive ideal in R we denote by $[J]$ the corresponding class in $\mathrm{Cl}(R)$. If M is an R-module of rank m, then $\det M := [(\Lambda^m M)^{**}]$ is called the *determinant* of M. M is called *orientable* if $\det M = 0$. Notice the following simple facts that will be used in the sequel:

(i) det is additive on exact sequence, i.e. if

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is exact, then

$$\det M_2 = \det M_1 + \det M_3$$

(ii) If M is a module of rank 1, then the following conditions are equivalent:

(a) M is orientable

(b) M is isomorphic to an ideal of codim ≥ 2, or $M \simeq R$.

(1.8) **Proposition.** (a) *Given a* codim 2 *CM-ideal I, then there exists a Bourbaki-sequence $0 \to F \to M \to I \to 0$ for I. Moreover, if R is normal, any such M is orientable.*

(b) *Suppose R is normal. If M is an orientable MCM-module, then there exists a Bourbaki-sequence $0 \to F \to M \to I \to 0$.*

Proof. (a) Let $\xi = \xi_1, \cdots, \xi_r$ be a system of generators of $\operatorname{Ext}^1_R(I, R)$, and let $0 \to R^r \to M \to I \to 0$ be the extension of I by R^r corresponding to

$\xi \in \mathrm{Ext}^1_R(I, R^r)$. We claim that M is an MCM-module. In fact, dualizing the above extension we obtain an exact sequence

$$M^* \longrightarrow (R^r)^* \xrightarrow{\delta} \mathrm{Ext}^1(I, R) \longrightarrow \mathrm{Ext}^1\,(M, R) \longrightarrow 0$$

and isomorphisms

$$\mathrm{Ext}^i(M, R) \simeq \mathrm{Ext}^i(I, R) \qquad \text{for } i \geq 2.$$

By the choice of the extension the map δ is surjective, and therefore $\mathrm{Ext}^1(M, R)=0$. Since I is a CM-ideal of codim 2 we have $\mathrm{Ext}^i(I, R)=0$ for $i \geq 2$. It follows that $\mathrm{Ext}^i(M, R)=0$ for $i \geq 1$. This implies that M is an MCM-module.

If R is normal, then the Bourbaki-sequence yields $\det M=[I]=0$, so that M is orientable.

(b) According to the theorem of Bourbaki there exists an exact sequence $0 \to F \to M \to I \to 0$, where I is an ideal of R. Assuming that $I \neq R$ and M is orientable we have $[I]=\det M=0$. Hence we may assume that codim $I \geq 2$. On the other hand since $\mathrm{Ext}^{i+1}(R/I, R) \simeq \mathrm{Ext}^i(I, R) \simeq \mathrm{Ext}^i(M, R)=0$ for $i \geq 2$, it follows that codim $I \leq 2$. Moreover, I is a CM-ideal since, as we have seen, $\mathrm{Ext}^i(R/I, R)=0$ for $i \neq 2$. □

We now give a complement to (1.8), (a).

(1.9) **Proposition.** *Given a* codim 2 *CM-ideal I of type r, there exists a Bourbaki-sequence*

$$(*) \qquad 0 \longrightarrow R^r \xrightarrow{\iota} M \longrightarrow I \longrightarrow 0.$$

Any other Bourbaki-sequence for I is isomorphic to some trivial extension of (*):

$$0 \longrightarrow R^r \oplus G \xrightarrow{(\iota,\, \mathrm{id})} M \oplus G \longrightarrow I \longrightarrow 0.$$

In particular, the Bourbaki-sequence (*) *and the module M are up to isomorphism uniquely determined by I.*

We call (*) the *natural Bourbaki-sequence* for I and denote M by $M(I)$.

Proof. $\mathrm{Ext}^1(I, R) \simeq \mathrm{Ext}^2(R/I, R) \simeq \omega_{R/I}$, the canonical module of R/I. Therefore $r=\mu(\mathrm{Ext}^1(I, R))$, and if $\xi=\xi_1, \cdots, \xi_r$ is a minimal system of generators for $\mathrm{Ext}^1(I, R)$, then the extension associated with ξ yields the Bourbaki-sequence (*). The remaining assertion follows from the fact that if

$$0 \longrightarrow F' \longrightarrow M' \longrightarrow I \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow F'' \longrightarrow M'' \longrightarrow I \longrightarrow 0$$

are any two Bourbaki-sequences for I, and if rank $F'=\text{rank } F''$, then there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \longrightarrow & M' & \longrightarrow & I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & F'' & \longrightarrow & M'' & \longrightarrow & I & \longrightarrow & 0 \end{array}$$

where the vertical maps are isomorphisms. This follows easily from the general properties of extensions ([5], § 7, No. 5, Theórème 1). □

The proposition shows that any codim 2 CM-ideal I of type r determines via a Bourbaki-sequence —up to stable isomorphism— a unique MCM-module $M(I)$ of rank $r+1$.

In Section 2, we will study the converse question: Given a MCM-module M, characterize all ideals I that can be inserted into a Bourbaki-sequence $0\to F\to M\to I\to 0$. We conclude this section by discussing how the data of M in the Bourbaki-sequence $0\to F\to M\to I\to 0$ are determined by invariants of I. Obviously we have rank $M=\text{rank } F+1$, and $\mu(M)\leq$ rank $F+\mu(I)$. Equality holds if and only if $F\otimes k\to M\otimes k$ is injective. In this case the Bourbaki-sequence is said to be *tight*. Notice, that if the natural Bourbaki-sequence for I is tight, then $M(I)$ has data $(r(I)+1, r(I)+\mu(I))$. In the case of hypersurface rings tightness is easy to control.

(1.10) **Lemma.** *Suppose R is a hypersurface ring and I is* codim 2 *CM-ideal of R. The following conditions are equivalent*:

(a) *The natural Bourbaki-sequence for I is tight*

(b) *Any Bourbaki-sequence for I is tight*

(c) $\beta_2(I)=\beta_1(I)$

(d) f-rank $\text{syz}_1(I)=0$.

Proof. In view of (1.9), (a) and (b) are equivalent.

(b)⇔(c): Let $0\to F\to M\to I\to 0$ be a Bourbaki-sequence for I. We obtain the exact sequence

$$0\longrightarrow \text{Tor}_1(k, M)\longrightarrow \text{Tor}_1(k, I)\longrightarrow F\otimes k\longrightarrow M\otimes k$$

and the isomorphism

$$\text{Tor}_2(k, M)\simeq \text{Tor}_2(k, I).$$

It follows that the above Bourbaki-sequence is tight if and only if $\beta_1(M)=\beta_1(I)$. However, by (1.6), (b) we have $\beta_1(M)=\beta_2(M)$, and since $\beta_2(M)=\beta_2(I)$, the assertion follows.

(c)⇔(d) follows from (1.6), (a) since $\text{syz}_1(I)$ is an MCM-module. □

For a hypersurface ring the module $M(I)$ attached to a codim 2 CM-ideal I can be interpreted in the following way:

(1.11) **Lemma.** *Suppose R is a hypersurface ring and I is a* codim 2 *CM-ideal, then $M(I) \simeq \mathrm{syz}_2(I) \oplus R^p$, where $p = r(I) - \beta_1(I) + \beta_0(I)$.*

Proof. If $0 \to R^r \to M(I) \to I \to 0$ is a natural Bourbaki-sequence for I it follows that $\mathrm{syz}_2(M(I)) \simeq \mathrm{syz}_2(I)$.

By (1.7) we have $\mathrm{syz}_2(M(I)) \oplus R^p \simeq M(I)$, where $p = \text{f-rank}\, M(I)$. Using (1.6), (a) and the long exact sequence

$$0 \longrightarrow \mathrm{Tor}_1(k, M(I)) \longrightarrow \mathrm{Tor}_1(k, I) \longrightarrow R^r \otimes k \longrightarrow M(I) \otimes k \longrightarrow I \otimes k \longrightarrow 0$$

we find that

$$\text{f-rank}\, M(I) = \beta_0(M) - \beta_1(M) = r(I) - \beta_1(I) + \beta_0(I). \qquad \square$$

§ 2. Bourbaki-sequences and linkage

Recall from [18] that two ideals I, J of R are said to be (directly) *linked* with respect to a regular sequence $\underline{x} = x_1, \cdots, x_g$ in $I \cap J$ (notation: $I \underset{(\underline{x})}{\sim} J$) if $(\underline{x})\colon I = J$ and $(\underline{x})\colon J = I$. If I is a CM-ideal of codimension g, $\underline{x} = x_1, \cdots, x_g$ is a regular sequence in I and we let $J = (\underline{x})\colon I$, then J is a CM-ideal of codimension g as well and $I = (\underline{x})\colon J$, i.e. I and J are linked.

I and J are said to be *evenly linked*, or J is said to belong to the *even linkage* class of I, abbreviated as $I \underset{\mathrm{e}}{\sim} J$, if there exists a sequence of ideals $I = I_0, I_1, \cdots, I_n = J$ such that n is even and $I_i \sim I_{i+1}$ directly for $i = 0, \cdots, n-1$.

The next result resembles the main theorem of [19].

(2.1) **Theorem.** *Let R, as before, be a local Gorenstein domain with infinite residue class field k. Let $0 \to F' \to M' \to I' \to 0$ and $0 \to F'' \to M'' \to I'' \to 0$ be any two Bourbaki-sequence (i.e., F', F'' are free, M', M'' are MCM-modules, and I', I'' are CM-ideals of codimension* 2). *Then the following statements are equivalent*:

(i) *M' and M'' are stably isomorphic*

(ii) *I' and I'' are evenly linked.*

The proof will reveal that, if the equivalent conditions of the theorem hold, I' and I'' can be linked in at most $4 \cdot \max\{\mathrm{rk}\, F', \mathrm{rk}\, F''\}$ double-steps. In case the given Bourbaki-sequences are tight this estimate may be improved to $2 \cdot \max\{\mathrm{rk}\, F', \mathrm{rk}\, F''\}$.

The proof of (ii)$\Rightarrow$(i), although being fairly easy, will be postponed until some notations have been fixed for the proof of (i)$\Rightarrow$(ii).

Recall the following construction from [18], Proposition 2.6: Let I be a CM-ideal of height 2 and $\underline{x}=(x_1, x_2)$ be a regular sequence in I. Assume $0\to M_2\to M_1\to I\to 0$ is an exact sequence, where M_1 (and hence M_2 also) are MCM-modules. Lift the natural embedding $(\underline{x})\hookrightarrow I$ to a commutative diagram

$$(*)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & M_2 & \xrightarrow{\iota} & M_1 & \longrightarrow & I & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \alpha_2} & & \uparrow{\scriptstyle \alpha_1} & & \cup & & \\ 0 & \longrightarrow & K_2(\underline{x}; R) & \xrightarrow{\partial_2} & K_1(\underline{x}; R) & \xrightarrow{\partial_1} & (\underline{x}) & \longrightarrow & 0 \end{array}$$

where the lower-side row is the Koszul-complex $(K_{\bullet}(\underline{x}; R), \partial)$ on the sequence $\underline{x}$. Then the linked ideal $J=(\underline{x})\colon I$ appears as the cokernel of the mapping cone associated to the dual of (*):

$$0\longrightarrow M_1^* \xrightarrow{(\iota^*, \alpha_1^*)} M_2^*\oplus K_1^*(\underline{x}; R) \xrightarrow{\rho} J \longrightarrow 0,$$

where ρ is induced by $\begin{pmatrix} \alpha_2^* \\ -\partial_2^* \end{pmatrix}\colon M_2^*\oplus K_1^*(\underline{x}; R)\to K_2^*(\underline{x}; R)\simeq R$.

Consequently, if a codim 2 CM-ideal I is inserted into a Bourbaki-sequence $0\to F\to M\to I\to 0$ and $J=(\underline{x})\colon I$, where $\underline{x}=x_1, x_2$ is a regular sequence in I, then $\mathrm{syz}_1(J)$ is stably isomorphic to M^*. The following statements are trivial:

(i) If (A, m, k) is a noetherian local ring (with infinite residue class field k) and we are given a non-zero-divisor $x\in m$ and an arbitrary element $y\in m$, there exists a finite set $X\subset k$ such that, for all units ε of R such that $\varepsilon+m\notin X$, the element $x+\varepsilon\cdot y$ is a non-zero-divisor of R.

(ii) Similarly, if M is a finitely generated A-module, $x\in M\backslash mM$, and $y\in M$ is arbitrary, there exists a finite set $X\subset k$ (in fact, $\sharp X\leq 1$), such that for all units ε of R with $\varepsilon+m\notin X$ we have $x+\varepsilon\cdot y\notin mM$.
The quintessential tool in our proof of (2.1) will be

(2.2) **Lemma.** *Suppose*

$$(*)\qquad 0\longrightarrow F\xrightarrow{\iota} M\xrightarrow{\pi} I\longrightarrow 0$$

is a Bourbaki-sequence, and choose a basis $\{e_1, \cdots, e_r\}$ for F. Take 2 elements $f_1, f_2\in M$, $x_i:=\pi(f_i)$ $(i=1, 2)$, such that $\underline{x}=x_1, x_2$ is a regular sequence in I. Pick any $t\in\{1, \cdots, r\}$. Then there exists a finite subset X of k such that for all units ε of R with residue class $\varepsilon+m\notin X$ the submodule $L:=\langle e_1, \cdots, e_{t-1}, e_t+\varepsilon\cdot f_1, e_{t+1}, \cdots, e_r\rangle$ of M is free of rank r, *and there exists a Bourbaki-sequence*

$$(\substack{*\\*})\quad 0\longrightarrow L\longrightarrow M\longrightarrow J\longrightarrow 0$$

where J is evenly linked to I in one double-step $I\underset{(x)}{\sim}K\underset{(y)}{\sim}J$. In case the Bourbaki-sequence (*) *was tight, i.e. $mM\cap F=mF$, X may be arranged such that $(\substack{*\\*})$ is tight as well for all $\varepsilon+m\notin X$.*

Proof. Choose a homomorphism α from the Koszul-complex

$$K_{\bullet}(\underline{x};R)=0\longrightarrow K_2(\underline{x};R)\xrightarrow{\partial_2}K_1(\underline{x};R)\xrightarrow{\partial_1}(\underline{x})\longrightarrow 0$$

to the Bourbaki-sequence (*) which lifts the natural embedding $(\underline{x})\hookrightarrow I$, such that $\alpha_1(u_i)=f_i$ $(i=1,2)$, where $\{u_1,u_2\}$ is a basis for $K_1(\underline{x};R)$ with $\partial_1(u_i)=x_i$. According to Peskine-Szpiro, the linked ideal $K=(\underline{x}):I$ has a presentation

$$0\longrightarrow M^*\xrightarrow{(\iota^*,\alpha_1^*)}F^*\oplus K_1^*(\underline{x};R)\xrightarrow{\rho}K\longrightarrow 0$$

where ρ is being induced by $\begin{pmatrix}\alpha_2^*\\-\partial_2^*\end{pmatrix}$: $F^*\oplus K_1^*(\underline{x};R)\to K_2^*(\underline{x};R)\simeq R$. We have $\partial_2(u_1\wedge u_2)=x_2u_1-x_1u_2$; moreover, there exist (uniquely determined) $\lambda_1,\cdots,\lambda_r\in R$ with $x_2f_1-x_1f_2=\sum_{i=1}^r\lambda_ie_i$. Hence $\rho(e_i^*)=\lambda_i$ $(i=1,\cdots,r)$, $\rho(u_1^*)=-x_2$, $\rho(u_2^*)=x_1$. We know that there exists a finite subset $X\subset k$ such that for all units ε of R with $\varepsilon+m\notin X$ the sequence x_1, $\varepsilon\cdot\lambda_t+x_2$ is regular. In case (*) was tight we know that $\{e_1,\cdots,e_r\}$ makes up part of a minimal system of generators for M, and we may further assume that $\{e_1,\cdots,e_{t-1},e_t+\varepsilon\cdot f_1,e_{t+1},\cdots,e_r\}$ is part of a minimal system of generators as well, for $\varepsilon+m\notin X$. Now take any unit $\varepsilon\in R$ with $\varepsilon+m\notin X$, and put $y_1:=x_1$, $y_2:=\varepsilon\cdot\lambda_t+x_2$, to the effect that $\underline{y}=y_1,y_2$ is a regular sequence on R, contained in K. Let $\{v_1,v_2\}$ be a free basis on which the Koszul-complex

$$K_{\bullet}(\underline{y};R)=0\longrightarrow K_2(\underline{y};R)\xrightarrow{\partial_2}K_1(\underline{y};R)\xrightarrow{\partial_1}(\underline{y})\longrightarrow 0$$

is being built, $\partial_1(v_i)=y_i$ $(i=1,2)$, and consider a morphism β of comparison

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^* & \xrightarrow{(\iota^*,\alpha_1^*)} & F^*\oplus K_1^*(\underline{x};R) & \xrightarrow{\rho} & K & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle\beta_2} & & \uparrow{\scriptstyle\beta_1} & & \cup & & \\
0 & \longrightarrow & K_2(\underline{y};R) & \xrightarrow{\partial_2} & K_1(\underline{y};R) & \xrightarrow{\partial_1} & (\underline{y}) & \longrightarrow & 0
\end{array}$$

where $\beta_1(v_1)=u_2^*$ and $\beta_1(v_2)=\varepsilon\cdot e_t^*-u_1^*$.

We use Peskine-Szpiro's argument again to deduce an exact sequence

$$0 \longrightarrow F^{**} \oplus K_1^{**}(\underline{x};R) \xrightarrow{\left(\begin{pmatrix} \iota^{**} \\ \alpha_1^{**} \end{pmatrix}, \beta_1^*\right)} M^{**} \oplus K_1^*(\underline{y};R) \longrightarrow J \longrightarrow 0,$$

where $J=(\underline{y}):K$. Upon identification of the double duals $F^{**} \simeq F$, $K_1^{**}(\underline{x};R) \simeq K_1(\underline{x};R)$, $M^{**} \simeq M$, this sequences translates into

$$0 \longrightarrow F \oplus K_1(\underline{x};R) \xrightarrow{\varphi} M \oplus K_1^*(\underline{y};R) \longrightarrow J \longrightarrow 0$$

where φ is induced by $\left(\begin{pmatrix} \iota^{**} \\ \alpha_1^{**} \end{pmatrix}, \beta_1^*\right)$,

i.e. $\qquad \varphi(e_t)=e_t+\varepsilon \cdot v_2^*, \quad \varphi(e_i)=e_i \ (i \neq t), \quad \varphi(u_1)=f_1-v_2^*,$
$\varphi(u_2)=f_2+v_1^*.$

In particular, the last 2 of these equations show that the composition

$$\psi := p_2 \circ \varphi \colon F \oplus K_1(\underline{x};R) \xrightarrow{\varphi} M \oplus K_1^*(\underline{y};R) \xrightarrow{p_2} K_1^*(\underline{y};R)$$

is surjective. As for its kernel, obviously

$$\tilde{L} := \operatorname{Ker} \psi = \langle e_1, \cdots, e_{t-1}, e_t+\varepsilon \cdot u_1, e_{t+1}, \cdots, e_r \rangle,$$

and

$$L := (p_1 \circ \varphi)(\tilde{L}) = \langle e_1, \cdots, e_{t-1}, e_t+\varepsilon \cdot f_1, e_{t+1}, \cdots, e_r \rangle \subseteq M.$$

Apply the snake-lemma to the diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & L \arrow[d] & M \arrow[d] & J \arrow[d] & \\
0 \arrow[r] & F \oplus K_1(\underline{x};R) \arrow[r, "\varphi"] \arrow[d, "\varphi"] & M \oplus K_1^*(y;R) \arrow[r] \arrow[d, "p_2"] & J \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_1^*(\underline{y};R) \arrow[r, "\mathrm{id}"] \arrow[d] & K_1^*(y;R) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

to find $0 \to L \to M \to J \to 0$ exact. Finally, $\operatorname{rk} F=r$, so $\operatorname{rk} M=r+1$ and $\operatorname{rk} J=1$ yields $\operatorname{rk} L=r$. As $\mu(L) \le r$, L must be free. Since $I_{(\tilde{x})} K_{(\tilde{y})} J$, J is a CM-ideal of codimension 2. In case the original Bourbaki-sequence for I was tight the basis of L can be arranged, as we have seen, to make up part of a minimal system of generators for M, so that the Bourbaki-sequence for J is tight again. $\square$

We are now in the position to prove the theorem.

Proof of (2.1), (i)$\Rightarrow$(ii). We may, after trivial extension of one of the Bourbaki-sequences, if necessary, assume that in fact $M'=M''=:M$, i.e. we are given Bourbaki-sequences

$$0\longrightarrow F'\longrightarrow M\xrightarrow{\pi'}I'\longrightarrow 0 \quad\text{and}\quad 0\longrightarrow F''\longrightarrow M\xrightarrow{\pi''}I''\longrightarrow 0.$$

Let $r:=rkF'=rkF''$. We first claim that there exists an ideal $\tilde{I}$ in the even linkage class of I' that allows a *tight* Bourbaki-sequence

$$0\longrightarrow\tilde{F}\longrightarrow M\longrightarrow\tilde{I}\longrightarrow 0$$

Proof of this claim: Put $t:=1+\mathrm{rk}_k(F'+mM/mM)$, so that $1\leq t\leqq r+1$. Induct on t and assume $t\geq 1$, $F'=\bigoplus_{i=1}^r Re_i'$, and $\{e_1',\cdots,e_{t-1}'\}$ is part of a minimal system of generators for M. By Nakayama it is possible to choose $f_1\in M\backslash(mM+F')$. Let $x_1:=\pi'(f_1)$ and choose $f_2\in M$ such that $\underline{x}=x_1, x_2$ is a regular sequence in R, where $x_2=\pi'(f_2)$. According to (2.2) there exists $\tilde{I}$ evenly linked to I' having a Bourbaki-sequence $0\rightarrow L\rightarrow M\rightarrow\tilde{I}\rightarrow 0$, where $L=\langle e_1',\cdots,e_{t-1}',e_t'+\varepsilon\cdot f_1,e_{t+1}',\cdots,e_r'\rangle$, for some suitable unit $\varepsilon\in R$ such that $e_1',\cdots,e_{t-1}',e_t',e_t'+\varepsilon\cdot f_1$ have linearly independent residue classes in M/mM. Replacing I' by $\tilde{I}$, the claim now follows by induction.

Applying this same procedure to the second Bourbaki-sequence, we may assume that both $0\rightarrow F'\rightarrow M\rightarrow I'\rightarrow 0$ and $0\rightarrow F''\rightarrow M\rightarrow I''\rightarrow 0$ are in fact tight Bourbaki-sequences. Introduce $t:=1+\sup\{\mathrm{rk}\,G\,|\,G\subseteq F'\cap F''$ is a free direct summand of both F' and $F''\}$ and use decreasing induction on t. If $t>r=\mathrm{rk}\,F'=\mathrm{rk}\,F''$, then $F'=F''$, therefore $I'\simeq I''$ as R-modules. But it is well known that isomorphic ideals of grade ≥ 2 are necessarily equal.

So assume the theorem is proved for t and consider $t-1$. To fix some notation, let $F'=\bigoplus_{i=1}^r R\cdot e_i'$, $F''=\bigoplus_{i=1}^r R\cdot e_i''$ and $e_i'=e_i''$ for $1\leq i<t$. Our assumptions imply $e_t'\notin F$, for $G:=\bigoplus_{i=1}^{t-1}\mathrm{Re}_i'$ is a maximal common direct summand of both F' and F'', and in case $e_t'\in F''$ we could, via $e_t'\notin G+mF'\Rightarrow e_t'\notin G+mM$ (by tightness)$\Rightarrow e_t'\notin G+mF''$, deduce that $G\oplus R\cdot e_t'$ would be a free direct summand of rank t of both F' and F'', contradicting the definition of t.

So $e_t'\notin F''$, and likewise $e_t''\notin F'$. Let $f_1':=e_t''$, $f_1'':=e_t'$. Because of codim $I', I''=2$, $f_1'\notin\mathrm{Ker}\,\pi'$, $f_1''\notin\mathrm{Ker}\,\pi''$, and R being a CM-domain, there exist $f_2', f_2''\in M$ in such a way that, with $x_i':=\pi'(f_i')$ and $x_i'':=\pi''(f_i'')$ $(i=1,2)$, $\underline{x}'=x_1', x_2'$ and $\underline{x}''=x_1'', x_2''$ are regular sequences on R (in I' and

I'', respectively). Now use (2.2) to find finite subsets X', X'' of k so that for all units ε', ε'' of R with $\varepsilon'+m \notin X'$, $\varepsilon''+m \notin X''$, and

$$L'=\langle e'_1, \cdots, e'_{t-1}, e'_t+\varepsilon'\cdot e''_t, e'_{t+1}, \cdots, e'_r\rangle \subseteq M$$
$$L''=\langle e_1, \cdots, e'_{t-1}, e''_t+\varepsilon''\cdot e'_{t'}, e''_{t+1}, \cdots, e''_r\rangle \subseteq M,$$

there are *tight* Bourbaki-sequences $0 \to L' \to M \to J' \to 0$ and $0 \to L'' \to M \to J'' \to 0$ where J' and J'' are, in one double-step, linked to I' and I'', respectively. In particular, it is possible to find a unit ε of R such that $\varepsilon+m \notin X'$ and $\varepsilon^{-1}+m \notin X''$. Upon this choice of ε, and $\varepsilon' := \varepsilon$, $\varepsilon'' := \varepsilon^{-1}$, the parameter of induction for L' and L'' exceeds t, hence, by the induction hypothesis, J' and J'', hence I' and I'' as well, belong to the same even linkage class.

This finishes the proof of (2.1) (i)$\Rightarrow$(ii).

Proof of (2.1) (ii)$\Rightarrow$(i). Since "stably isomorphic" defines an equivalence relation, we may assume $I' \underset{(\underline{x})}{\sim} K \underset{(\underline{y})}{\sim} I''$ for some ideal K and regular sequences $\underline{x}$, $\underline{y}$. As was remarked before (2.2), the given Bourbaki-sequences $0 \to F' \to M' \to I' \to 0$ and $0 \to F'' \to M'' \to I'' \to 0$ yield free presentations of K:

$$0 \longrightarrow M'^* \longrightarrow F'^* \oplus K_1^*(\underline{x}; R) \longrightarrow K \longrightarrow 0 \quad \text{and}$$
$$0 \longrightarrow M''^* \longrightarrow F''^* \oplus K_1^*(\underline{y}; R) \longrightarrow K \longrightarrow 0.$$

Apply Shanuel's lemma to conclude

$$M'^* \oplus F''^* \oplus K_1^*(\underline{y}; R) \simeq M''^* \oplus F'^* \oplus K_1^*(\underline{x}; R),$$

i.e. M'^* and M''^* are stably isomorphic, whence $M'^{**} \simeq M'$ and $M''^{**} \simeq M''$ are stably isomorphic. $\square$

As an immediate consequence of (2.1) we find

(2.3) **Corollary.** *Suppose R is a normal Gorenstein domain. There exists a bijection between even linkage classes of codimension 2 CM-ideals I of R and stable isomorphism classes of orientable MCM-modules M over R.*

(2.4) **Corollary.** *Let I' and I'' be two CM-ideals of codimension 2 in R. Then I' and I'' are evenly linked if and only if* $\mathrm{syz}_1(I')$ *and* $\mathrm{syz}_1(I'')$ *are stably isomorphic.*

Proof. Perform (arbitrary) direct links $I' \sim J'$ and $I'' \sim J''$. Take the natural Bourbaki-sequences $0 \to F' \to M' \to J' \to 0$ and $0 \to F'' \to M'' \to J'' \to 0$. Then: I' is evenly linked to $I'' \Leftrightarrow J'$ is evenly linked to $J'' \Leftrightarrow M'$

is stably isomorphic to $M'' \Leftrightarrow M'^*$ is stably isomorphic to $M''^* \Leftrightarrow \mathrm{syz}_1(I')$ is stably isomorphic to $syz_1(I'')$, by the remark following the description of the Peskine-Szpiro-construction. □

(2.5) **Corollary.** *Suppose* $0 \to F' \to M' \to I' \to 0$ *and* $0 \to F'' \to M'' \to I'' \to 0$ *are Bourbaki-sequences. Then* I' *and* I'' *belong to the same odd linkage class if and only if* M' *and* $D(M'')$ *are stably isomorphic.*

Proof. Perform one (arbitrary) direct link $J \sim I'$, so that $\mathrm{syz}_1(J) \underset{\mathrm{st}}{\simeq} M'^*$. Then I' and I'' are oddly linked if and only if J and I'' are evenly linked. The result now follows from (2.4). □

To close this section with, we will consider the (even linkage class of the) ideal that corresponds to the direct sum of two orientable MCM-modules. Recall that, for any ideals I' and I'', grade $I' \cdot I'' =$ grade $I' \cap I''$ $= \min\{\text{grade } I', \text{grade } I''\}$.

(2.6) **Remark.** Suppose

$$0 \longrightarrow F' \longrightarrow M' \xrightarrow{\pi'} I' \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow F'' \longrightarrow M'' \xrightarrow{\pi''} I'' \longrightarrow 0$$

are two Bourbaki-sequences. Choose any regular sequence x', x'' such that $x' \in I'$ and $x'' \in I''$. Then there exists a Bourbaki-sequence

$$0 \longrightarrow F' \oplus F'' \oplus R \longrightarrow M' \oplus M'' \longrightarrow x''I' + x'I'' \longrightarrow 0.$$

Proof. The epimorphism $I' \oplus I'' \to x''I' + x'I''$ has kernel $R \cdot (-x', x'')$

$$(y', y'') \longrightarrow x''y' + x'y''$$

The given Bourbaki-sequences combine to

$$0 \longrightarrow F' \oplus F'' \longrightarrow M' \oplus M'' \xrightarrow{(\pi', \pi'')} I' \oplus I'' \longrightarrow 0.$$

Choose any $e \in M' \oplus M''$ such that $(\pi', \pi'')(e) = (-x', x'')$. Then, clearly, the sum $F' \oplus F'' + \mathrm{Re} \subset M' \oplus M''$ is direct, and

$$M' \oplus M'' / F' \oplus F'' \oplus \mathrm{Re} \simeq I' \oplus I'' / R \cdot (-x', x'') \simeq x''I' + x'I''. \qquad \square$$

In particular, we find that, had we taken any different regular sequence $y' \in I'$, $y'' \in I''$, the ideals $x''I' + x'I''$ and $y''I' + y'I''$ are evenly linked.

§ 3. Maximal Cohen-Macaulay rings of rank 2 over hypersurface rings

For the rest of the paper we will assume that R is a hypersurface domain; i.e. $R = A/f$, where A is a regular local ring, and f is a prime element in A.

From the homological point of view hypersurface rings are the most simple among the Gorenstein rings. It is therefore natural to begin the study of MCM-modules over this class of rings. As a first step in this direction we will try to understand the rank 2 orientable MCM-modules over a hypersurface domain.

Our main result is the following:

(3.1) **Theorem.** (a) *If M is an orientable MCM-module of* rank 2, *then M is evenly generated.*

(b) *Suppose $R=k[\![x_1, x_2, x_3]\!]/f$, where k is an infinite field and f is a homogeneous polynomial of degree e. For all numbers $t=1, \cdots, e$ there exists an orientable MCM-module of rank* 2 *with* $2t$ *generators.*

(3.2) **Remarks.** (a) Since for any rank 2 MCM-module M one has $\mu(M)\leq 2\cdot e(R)=2e$, statement (3.1), (a), (b) shows that all possible numbers of generators for M actually occur. On the other hand if dim $R\geq 3$, there need not to exist any rank 2 MCM-module as we shall see later in (3.3) and (3.4).

(b) In the proof of (3.1), (b) we will not use our general assumption that f be a prime element; hence f could be reducible as well.

(c) D. Eisenbud shows in [10] that the existence of an MCM-module with data (m, n) over a hypersurface domain A/f is equivalent to a matrix factorization $fE_n=B\cdot C$ with det $B=f^m$, where E_n is the unit matrix of size n, and B and C are square matrices with entries in the maximal ideal of A.

In particular, (3.1), (b) implies that for any $t=1, \cdots, e$, there exists a square matrix B of size $2t$ with coefficients in the maximal ideal of $k[\![x_1, x_2, x_3]\!]$ such that $f^2=\det B$.

Suppose $R=k[\![x_1, x_2, x_3]\!]/f$ is factorial (in which case all MCM-modules over R are orientable), then (3.1), (a) implies that if $f^2=\det B$ for some square matrix B of size ≥ 2 with entries in the maximal ideal of $k[\![x_1, x_2, x_3]\!]$, then B is of even size.

Notice that f itself can be written as determinant of a square matrix of size ≥ 2 with coefficients in the maximal ideal of $k[\![x_1, x_2, x_3]\!]$ if and only if R is not factorial. This has been observed by Andreotti and Salmon [1], see also [11].

Proof of (3.1), (a): Let M be any orientable MCM-module of rank 2 and choose a tight Bourbaki-sequence $0\to R\to M\to I\to 0$. It follows from (1.9) that I is a Gorenstein ideal of codim 2. Let J be the inverse image of I in A, so that $I=J/f$ and $A/J\simeq R/I$. It follows that J is a Gorenstein ideal of codim 3 in A.

According to J. Watanabe [22], or the structure theorem of Buchsbaum and Eisenbud [8], $\mu(J)$ is odd. We claim that $f \in \mathfrak{n}J$. Once this has been shown we find that $\mu(M)=\mu(I)+1=\mu(J)+1$ is even.

In order to prove our claim, suppose to the contrary that $f \in J \backslash \mathfrak{n}J$, and let $0 \to F_3 \xrightarrow{\alpha} F_2 \to F_1 \to J \to 0$ be a minimal free A-resolution of J. Since J is Gorenstein, $F_3 \simeq A$ and J coincides with the first-order determinantal ideal $I_1(\alpha)$, hence $f \notin \mathfrak{n}I_1(\alpha)$.

From the above resolution we obtain the following minimal A-resolution of I

$$0 \longrightarrow F_3 \xrightarrow{\alpha} F_2 \longrightarrow F_1/gA \longrightarrow I \longrightarrow 0,$$

where g is a generator of F_1 which is mapped to f by $F_1 \to J$. Using the Bourbaki-sequence we obtain a commutative diagram

$$(*) \qquad \begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A \arrow[r, "\beta"] \arrow[d, "f"'] & F \arrow[d] & & \\
 & A \arrow[r, "\iota"'] \arrow[d] & F \arrow[d] & & \\
0 \arrow[r] & R \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & I \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where $0 \to F \to F \to M$ is a minimal free A-resolution of M. ι must be split injective, as the Bourbaki-sequence is tight. Taking the mapping cone and cancelling split morphisms we derive a minimal A-free resolution of I.

$$0 \longrightarrow A \xrightarrow{\beta} F \longrightarrow \bar{F} \longrightarrow I \longrightarrow 0,$$

where $\bar{F}=F/\iota(A)$. Comparing with the other resolution of I we find that $I_1(\alpha)=I_1(\beta)$, so that $f \notin \mathfrak{n}I_1(\beta)$. On the other hand diagram (*) yields $f \in \mathfrak{n}I_1(\beta)$, a contradiction.

Proof of (3.1), (b): Let t be an integer with $1 < t \leq e$. We claim that there exists a Gorenstein ideal J of codim 3 in $A=k[\![x_1, x_2, x_3]\!]$ with $2t-1$ generators such that $f \in \mathfrak{n}J$.

Having shown the existence of such an ideal J we let $I=J/f$. Then I is a codim 2 Gorenstein ideal in R. Let $0 \to R \to M \to I \to 0$ be the natural

Bourbaki-sequence for I. Since $\mu(I)=\mu(J)=2t-1$ is an odd number it follows from (a) that this Bourbaki-sequence is tight, whence $\mu(M)=2t$.

Let $P=k[x_1, x_2, x_3]$, then A is the (x_1, x_2, x_3)-adic completion $\hat{P}$ of P. We actually construct a homogeneous Gorenstein ideal L in P such that $J=LA$ has the above properties.

Given t with $1<t\leq e$, we put $s=2t-4$ and let φ: $P_s \to k$ be a non-trivial k-linear map. It is well-known and easy to check that any such φ defines a Gorenstein ideal $L(\varphi)$ whose homogeneous components are given by $L(\varphi)_i=\{a \in P_i \mid \varphi(aP_{s-i})=0\}$, and for which the degree of the socle of $R(\varphi):=P/L(\varphi)$ is s.

Suppose we can choose φ such that $L(\varphi)_{t-2}=0$, then all generators of $L(\varphi)$ have degree $\geq t-1$. Let us compute the number of generators of $L(\varphi)$ in degree $t-1$: Since the Hilbert-function of $R(\varphi)$ is symmetric it follows that

$$\begin{aligned}\dim_k L(\varphi)_{t-1} &= \dim_k P_{t-1} - \dim_k R(\varphi)_{t-1} = \dim_k P_{t-1} - \dim_k R(\varphi)_{s-t+1} \\ &= \dim_k P_{t-1} - \dim_k P_{t-3} = \binom{t+1}{2} - \binom{t-1}{2} = 2t-1.\end{aligned}$$

We claim that $L(\varphi)$ has no generators in degree higher than $t-1$, so that $\mu(L(\varphi))=2t-1$. In fact, by the structure theorem of Buchsbaum-Eisenbud [8] $R(\varphi)$ admits a homogeneous resolution

$$0 \longrightarrow P(-2t+1) \longrightarrow F_2 \longrightarrow F_1 \longrightarrow P \longrightarrow R(\varphi) \longrightarrow 0$$

with an algebra structure, which induces a perfect homogeneous pairing $F_1 \times F_2 \to P(-2t+1)$. Hence $F_1=\bigoplus_i P(-n_{1i})$, $F_2=\bigoplus_i P(-n_{2i})$ where $n_{1i}+n_{2i}=2t-1$ for all i. Since, clearly $n_{2i}\geq \min_j n_{1j}+1=t$, we conclude that $n_{1i}\leq(2t-1)-t=t-1$, as asserted.

Therefore we have seen that if for some φ: $P_s\to k$ the ideal $L(\varphi)$ satisfies $L(\varphi)_{t-2}=0$, then $L(\varphi)$ is a Gorenstein ideal with $2t-1$ generators all in degree $t-1$. If, in addition, we can choose φ such that $f \in L(\varphi)$, then actually $f \in \mathfrak{n}J$, since $e>t-1$.

As a result of our discussion so far we have to show the following: Given an integer $e\geq 2$, an integer $1<t\leq e$, and a homogeneous polynomial of degree e. Then, for $s=2t-4$, there exists a linear map φ: $P_s\to k$ such that $L(\varphi)_{t-2}=0$, and $f \in L(\varphi)$.

By the definition of $L(\varphi)$ it is clear that $L(\varphi)_{t-2}=0$ if and only if the induced bilinear form

$$\begin{aligned}\tilde{\varphi}\colon P_{t-2}\times P_{t-2} &\longrightarrow k \\ (a, b) &\longrightarrow \varphi(a\cdot b)\end{aligned}$$

is non-degenerate.

We choose a basis $\{a_1, \cdots, a_n\}$ of P_{t-2}, and a basis $\{b_1, \cdots, b_m\}$ of P_s. Let $\{b_i^*\}$ denote the dual basis in P_s^*, which establishes an isomorphism $P_s^* \simeq k^m$, then $\varphi = \sum_{l=1}^m x_l b_l^*$ for some $x_l \in k$. If we write $a_i a_j = \sum_{l=1}^m y_{ij}^l b_l$ for $i, j = 1, \cdots, n$, then

$$\tilde{\varphi}(a_i, a_j) = \sum_{l=1}^m y_{ij}^l x_l.$$

It follows that the matrix $(\tilde{\varphi}(a_i, a_j))_{i,j=1,\dots,n}$ is a matrix of linear forms in the x_l. Replacing the x_l by indeterminates Z_l we obtain a matrix α of linear forms in the indeterminates $Z_1, \cdots, Z_m$. Let $X = V(\det \alpha)$ be the set of zeros of $\det \alpha$ in P_s^*, then $\tilde{\varphi}$ is non-degenerate if and only if $\varphi \notin X$. The other condition that f be an element of $L(\varphi)$ is equivalent to $fP_{s-e} \subseteq \operatorname{Ker} \varphi$. The set of all $\varphi \in P_s^*$ with $fP_{s-e} \subseteq \operatorname{Ker} \varphi$ is a linear subspace H of P_s^*.

Hence we have to pick $\varphi \in H \backslash X$; this is possible if $H \not\subseteq X$. To see that, in fact, $H \not\subseteq X$ we shall need the following theorem of D. Eisenbud: Let k be a field and let $\alpha = (l_{ij})$ be a square matrix of linear forms $l_{ij} \in k[Z_1, \cdots, Z_m]$. α is said *to have no generalized zeros* if no component of any non-trivial k-linear combination of the rows (resp. columns) of α has a zero.

Examples. (a) The generic matrix $\alpha = (X_{ij})$ of indeterminates is a matrix of no generalized zeros.

(b) Let $R = \bigoplus_{i \geq 0} R_i$ be a graded noetherian domain where $R_0 = k$ is a field.

Pick some $i \geq 1$, and choose a k-basis $a_1, \cdots, a_n$ for R_i and a k-basis $b_1, \cdots, b_m$ for R_{2i}. Let $a_i a_j = \sum_{l=1}^m y_{ij}^l b_l$, $y_{ij}^l \in k$, for $i, j = 1, \cdots, n$, and let $\alpha = (l_{ij})$, where $l_{ij} = \sum_{l=1}^m y_{ij}^l Z_l$. We claim that α is a matrix with no generalized zeros. In fact, suppose there exists a non-trivial k-linear combination $l = \sum_{i=1}^n \lambda_i l_i$ of the row vectors $l_i = (l_{i1}, \cdots, l_{in})$ such that the j-th component of l is zero, then $\sum_{i=1}^n \lambda_i l_{ij} = 0$, and therefore $\sum_{i=1}^n \lambda_i y_{ij}^l = 0$ for $l = 1, \cdots, m$. On the other hand, let $a = \sum_{i=1}^n \lambda_i a_i$, then $a \neq 0$, and since R is a domain, we have $a \cdot a_j \neq 0$ as well. Since

$$a \cdot a_j = \sum_{l=1}^m \left(\sum_{i=1}^n \lambda_i y_{ij}^l \right) b_1$$

it follows that $\sum_{i=1}^n \lambda_i y_{ij}^l$ cannot be zero for all l, a contradiction.

We quote

Theorem (Eisenbud [12]). *Suppose* $\alpha = (l_{ij})_{i,j=1,\dots,n}$ *is a matrix of linear forms with no generalized zeros. Then* $\det \alpha \not\equiv 0 \mod (l_1, \cdots, l_{n-1})$ *for any linear forms* $l_1, \cdots, l_{n-1}$.

To conclude the proof of (3.1), (b) we may apply this theorem to our matrix $\mathfrak{a}$, which determines the locus X of degeneracy of the bilinear form $\tilde{\varphi}$ associated with $\varphi \in P_s^*$, since by example (b) $\mathfrak{a}$ has no generalized zeros. The size of our matrix is $\dim_k P_{t-2}$, while the codimension of the linear subspace H is $\dim_k P_{s-e}<$size of $\mathfrak{a}$, since $s-e\leq(2t-4)-t<t-2$. Hence if $l_1, \cdots, l_r$ define H, then $r<$size of $\mathfrak{a}$, and therefore $\det \mathfrak{a}\not\equiv 0 \mod (l_1, \cdots, l_r)$. But since k is infinite, this implies $H\not\subseteq X$. □

The following example shows that in dimension higher than 2 there need not exist any nontrivial orientable rank 2 MCM-modules.

(3.3) **Example.** Suppose the ring $R=\boldsymbol{R}[\![x_1, x_2, x_3, x_4]\!]/f$, $f=x_1^2+x_2^2+x_3^2+x_4^2$ admits an orientable nontrivial rank 2 MCM-module. Then it follows from the proof of (3.1) that there exists a codim 3 ideal J with $\mu(J)=3$ and $x_1^2+x_2^2+x_3^2+x_4^2 \in \mathfrak{n}J$. Hence if $J=(a_1, a_2, a_3)$, then $f=\sum_{i=1}^3 b_i a_i$ with $b_i \in \mathfrak{n}$. Since f is homogeneous of degree 2, the above equation implies that $f \in (l_1, l_2, l_3)$, where l_i denotes the component of degree 1 of a_i. In particular f would have a zero in $\boldsymbol{R}^4$, a contradiction.

If we modify this example slightly and consider instead

$$S=\boldsymbol{R}[\![x_1, x_2, x_3, x_4]\!]/(x_1^2+x_2^2+x_3^2-x_4^2), \quad \text{then } f=x_1^2+x_2^2+x_3^2-x_4^2 \in \mathfrak{n}J,$$

where $J=(x_1, x_2, x_3-x_4)$. Hence by the proof of (3.1) S admits an orientable module with data (2. 4).

These considerations show that the existence of rank 2 MCM-modules for hypersurface rings of dimension 3 depend on the particular chosen ring. If, however, R is a hypersurface ring with isolated singularity of dimension at least 6, then R never admits any non-trivial rank 2 MCM-modules. This follows from a far more general result that was communicated to the authors by W. Bruns during the conference on commutative algebra in May 85 at Oberwolfach.

(3.4) **Theorem** (Bruns). *Suppose M is a non-trivial MCM-module over a hypersurface ring R with isolated singularity, then*

$$2\cdot \operatorname{rank} M+1\geq \dim R.$$

Proof. For any noetherian domain R and any finitely generated non-trivial R-module M it is shown in [7], Corollary 2 that the codimension of the non-free locus of M is bounded by the number $\operatorname{rank} M+\operatorname{rank} \operatorname{syz}_2(M)+1$. In our situation, M is free on the punctured spectrum of R, and $syz_2(M)\simeq M$ if f-rank $M=0$, as we may assume. The assertion follows. □

We conclude this section by studying orientable MCM-modules with data (2, 4) for 2-dimensional hypersurface domains. It follows from (3.1) that such modules are the "smallest" orientable non-trivial MCM-modules.

(3.5) **Corollary.** *Any 2-dimensional hypersurface domain admits an orientable MCM-module with data* (2, 4). *If, in addition, R has multiplicity* 2 *and* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is factorial, then any orientable MCM-module with data* (2, 4) *is isomorphic to* $M(\mathfrak{m})$.

Proof. Since $\mu(\mathfrak{m})=3$ and $r(\mathfrak{m})=1$ it follows that $M(\mathfrak{m})$ has data $(2, n)$, where $n=3$ or 4. By (3.1), (a) n is even, so $n=4$.

Now, we assume that R has multiplicity 2 and that $\mathrm{gr}_{\mathfrak{m}}(R)$ is factorial. Let M be an orientable MCM-module with data (2, 4), and let $0 \to R \to M \to I \to 0$ be a tight Bourbaki-sequence. We show that $M \simeq M(\mathfrak{m})$ by proving that $\mathfrak{m}=I$.

In the proof of (3.1) (a) we have seen that $I=J/f$ with $f \in \mathfrak{n}J$. In particular we have $\mu(J)=3$. Let $J=(a_1, a_2, a_3)$ and write $f=\sum_{i=1}^{3} b_i a_i$ with $b_i \in \mathfrak{n}$. By assumption, the initial form f^* of f has degree 2. Hence, if we set $\bar{c}=c+\mathfrak{n}^2$ for an element $c \in \mathfrak{n}$, we get the equation $f^*=\sum_{i=1}^{r} \bar{b}_i \bar{a}_i$ in $\mathrm{gr}_{\mathfrak{n}}(A)$.

Suppose $\bar{a}_1, \bar{a}_2, \bar{a}_3$ don't generate the irrelevant maximal ideal of $\mathrm{gr}_{\mathfrak{n}}(A)$; then f^* is a linear combination of at most two linear forms and hence can be written as a determinant. This, however, by remark (3.2) (c) implies that $\mathrm{gr}_{\mathfrak{m}}(R)$ is not factorial, contrary to our assumption.

Therefore we conclude that $\mathfrak{n}=(a_1, a_2, a_3)+\mathfrak{n}^2$, thus $J=\mathfrak{n}$ and $I=\mathfrak{m}$, by Nakayama.

We will now consider the (2, 4)-case in multiplicity 3.

(3.6) **Proposition.** *Suppose R is a* 2*-dimensional hypersurface domain of multiplicity* 3 *such that* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is factorial. Then* $M(\mathfrak{m})$ *and* $D(M(\mathfrak{m}))$ *are non-isomorphic orientable MCM-modules with data* (2, 4), *and any orientable MCM-module with data* (2, 4) *is isomorphic to either* $M(\mathfrak{m})$ *or* $D(M(\mathfrak{m}))$.

Proof. We have already seen in (3.5) that $M(\mathfrak{m})$ has data (2, 4). Since f-rank $M(\mathfrak{m})=0$ it follows from (1.1) that $D(M(\mathfrak{m}))$ has data (2, 4) as well.

Now let M be any orientable MCM-module with data (2, 4) and choose a tight Bourbaki-sequence $0 \to R \to M \to I \to 0$. As in the proof of (3.5), we have $I=J/f$, where $f \in \mathfrak{n}J$, and J is generated by a regular sequence a_1, a_2, a_3. In order to show that M is isomorphic to either $M(\mathfrak{m})$ or $D(M(\mathfrak{m}))$ we show that I is in the linkage class of $\mathfrak{m}$ by proving that there is a sequence of links (*) $J=J_0 \sim J_1 \sim \cdots \sim J_n=\mathfrak{n}$ with the following properties:

1) Each J_i is generated by a regular sequence (of length 3).

2) For all i, the linking regular sequence $\underline{x}$ with $J_{i+1}=(\underline{x}): J_i$ is of the form $\underline{x}= f, a, b$, where a, b is part of a minimal system of generators of J_i as well as of J_{j+1}. Modulo f this chain of links establishes the linkage between I and $\mathfrak{m}$.

The existence of such a sequence of links will follow quite easily from the next

Claim. Let L be generated by the regular sequence c_1, c_2, c_3 and assume that $f \in \mathfrak{n}L$. Write $f=\sum_{i=1}^{3} b_i c_i$, $b_i \in \mathfrak{n}$, and set $K=(f, c_1, c_2): L$, then

(a) K is generated by the regular sequence c_1, c_2, b_3

(b) $f \in \mathfrak{n}K$

(c) $\operatorname{edim} A/K=\begin{cases}\operatorname{edim} A/L-1, & \text{if } \operatorname{edim} A/(c_1, c_2)=\operatorname{edim} A/L \\ \operatorname{edim} A/L+1, & \text{otherwise.}\end{cases}$

Before we prove the claim let us construct the sequence (*). We proceed by induction on edim A/J. If edim $A/J=0$, then $J=\mathfrak{m}$, and nothing is to be proved. If edim $A/J>0$, we can find a regular sequence c_1, c_2, c_3 generating J such that edim $A/J=$edim $A/(c_1, c_2)$. We set $J_1=(f, c_1, c_2): J$. By (a) c_1, c_2 is part of a minimal system of generators of J_1, by (b) $f \in \mathfrak{n}J_1$, and by (c) edim $A/J_1=$edim $A/J-1$. Hence the assertion **follows** from the induction hypothesis applied to J_1.

Proof of the claim. Reducing modulo c_1, c_2, assertion (a) follows immediately, while (b) is a consequence of (a) and the equation $f=\sum_{i=1}^{3} b_i c_i$.

We will use the following notations in the proof of (c): For an element s of a local ring $(S, \mathfrak{m})$ we set $v(s):=\sup\{n \mid s \in \mathfrak{m}^n\}$. We further put $\bar{A}=A/(c_1, c_2)$, and denote by $\bar{a}$ the residue class modulo (c_1, c_2) of an element $a \in A$. It is clear that $v(\bar{a})\geq v(a)$ for any $a \in A$.

We will show in a moment that $v(\bar{b}_3\bar{c}_3)=3$, but let us first derive (c) from this fact.

If edim $A/(c_1, c_2)=$edim A/L, then $v(\bar{c}_3)\geq 2$. Since $v(\bar{b}_3)+v(\bar{c}_3)=3$, it follows that $v(\bar{b}_3)=1$. Hence (a) implies that edim $A/K=$edim $A/(c_1, c_2)$ -1. This proves the first part of (c).

If edim $A/(c_1, c_2)\neq$edim A/L, then $v(\bar{c}_3)=1$, and edim $A/L=$edim $A/(c_1, c_2)-1$. But then $v(\bar{b}_3)=2$, and therefore edim $A/K=$edim $A/(c_1, c_2)=$ edim $A/L+1$.

To see that, in fact, $v(\bar{b}_3\bar{c}_3)=3$, observe first that $\bar{f}=\bar{b}_3\bar{c}_3$, so that $v(\bar{b}_3\bar{c}_3)=v(\bar{f})\geq v(f)=3$. Assume $v(\bar{b}_3\bar{c}_3)>3$, then there exists $g \in R$ with $v(g)\geq 4$ such that

(i) $-b_3c_3+g=r_1c_1+r_2c_2$ with some $r_i \in R$.

Using the equation

(ii) $f = b_1c_1 + b_2c_2 + b_3c_3$ we get

$$f - g = (b_1 - r_1)c_1 + (b_2 - r_2)c_2$$

If r_1 is a unit we use (i) and (ii) to obtain

$$f - r_1^{-1}gb_1 = (b_2 - r_1^{-1}r_2b_1)c_2 + (b_3 - r_1^{-1}b_3b_1)c_3.$$

If r_2 is a unit, then

$$f - r_2^{-1}gb_2 = (b_1 - r_2^{-1}r_1b_2)c_1 + (b_3 - r_2^{-1}b_3b_2)c_3$$

and therefore $\tilde{f} = \tilde{b}_ic_i + \tilde{b}_jc_j$ for some $i, j \in \{1, 2, 3\}$, $i \neq j$, $\tilde{b}_i, \tilde{b}_j \in n$ and $f^* = \tilde{f}^*$.

To get the contradiction to the assumption $v(\bar{b}_3\bar{c}_3) > 3$, we will show that $f^* \in nC$ for some ideal C generated by at most 2 elements. By Remark (3.2), (c) this is impossible since $\mathrm{gr}_m(R) = \mathrm{gr}_n(A)/(f^*)$ is factorial.

We will distinguish three cases. Note that $\min\{v(\tilde{b}_ic_i), v(\tilde{b}_jc_j)\} \leq v(\tilde{f}) = 3$.

Case 1. $v(\tilde{b}_ic_i) \geq 3$, then $v(\tilde{b}_jc_j) \geq 3$. If $v(\tilde{b}_ic_i) > 3$, then $f^* = \tilde{b}_j^*c_j^*$, a contradiction. Similarly, $v(\tilde{b}_jc_j) > 3$ is impossible. So we may assume that $v(\tilde{b}_ic_i) = v(\tilde{b}_jc_j) = 3$; but then $f^* = \tilde{b}_i^*c_i^* + \tilde{b}_j^*c_j^*$, again a contradiction.

We now may assume that $v(\tilde{b}_ic_i) = 2$. Here two cases are possible.

Case 2. $\dim_k(c_i, c_j) + n^2/n^2 = 2$, then $A/(c_i, c_j)$ is regular and therefore $\mathrm{Ker}\,(\mathrm{gr}_n(A) \to \mathrm{gr}_{\bar{n}}(\bar{A})) = (c_i^*, c_j^*)$. Since f^* is always contained in this kernel we get a contradiction.

Case 3. $\dim_k(c_i, c_j) + n^2/n^2 = 1$, then $c_j = \varepsilon c_i + c$, where ε is a unit and $c \in n^2$. Hence we can write $\tilde{f} = (\tilde{b}_i + \varepsilon\tilde{b}_j)c_i + \tilde{b}_jc$. Since $v(\tilde{b}_jc) \geq 3$ and $v(\tilde{f}) = 3$, it follows that $v((\tilde{b}_i + \varepsilon\tilde{b}_j)c_i) \geq 3$, and we are in case 1.

This completes the proof of the claim.

It remains to prove that $M(m) \not\simeq D(M(m))$. Suppose, to the contrary, that these two modules are isomorphic and let $0 \to R \to D(M(m)) \to I \to 0$ be a tight Bourbaki-sequence for $D(M(m))$. By (2.1), m and I belong to the same even linkage class. Again, we write $I = J/f$ with $f \in nJ$ and show more precisely that there is an even number of linkages

$$J = J_0 \sim J_1 \sim \cdots \sim J_{2n} = n$$

with the properties (1) and (2) of the linkage sequence (*).

It then follows from the claim (c) that $|\mathrm{edim}\, A/J_{i+1} - \mathrm{edim}\, A/J_i| = 1$ for all i, so that $\mathrm{edim}\, A/J$ is even. On the other hand by (2.5) $D(M(m)) \simeq$

$M(J_{2n-1})$. Since edim A/J_{2n-1} is odd, this is a contradiction. It is clear that a sequence of links $J=J_0\sim J_1\sim\cdots\sim J_{2n}=\mathfrak{n}$ satisfies the conditions 1) and 2) if and only if the corresponding sequence of links modulo f, the sequence

$$I=I_0\sim I_1\sim I_2\sim\cdots\sim I_{2n}=\mathfrak{m}$$

has the property that for all $i=0, 1, \cdots, 2n-1$, the linking regular sequence x_1, x_2 with $I_{i+1}=(x_1, x_2): I_i$ is part of a minimal system of generators of I_i. This sequence is then automatically part of a minimal system of generators of I_{i+1} as well, as can be seen using claim (a).

Identifying $M(\mathfrak{m})$ and $D(M(\mathfrak{m}))$ and calling this module M, we have two tight Bourbaki-sequences.

$$0\longrightarrow R\xrightarrow{\iota_1}M\xrightarrow{\varepsilon_1}\mathfrak{m}\longrightarrow 0$$

$$0\longrightarrow R\xrightarrow{\iota_2}M\xrightarrow{\varepsilon_2}I\longrightarrow 0$$

Since, by our general assumptions, k is infinite, we can pick an element $m\in M$ such that

(i) $m\otimes 1\neq 0$ in $M\otimes_R k$

(ii) M/Rm is torsionfree, so that we obtain a tight Bourbaki-sequence

$$0\longrightarrow Rm\longrightarrow M\xrightarrow{\varepsilon}\tilde{I}\longrightarrow 0$$

(iii) $\varepsilon_1(m)\notin\mathfrak{m}^2$, $\varepsilon(\iota_1(1))\notin\mathfrak{m}\tilde{I}$
$\varepsilon_2(m)\notin\mathfrak{m}I$, $\varepsilon(\iota_2(1))\notin\mathfrak{m}\tilde{I}$

By the proof of (2.1) there are two double links $\mathfrak{m}\underset{\underline{x}}{\sim}\mathfrak{m}_1\underset{\underline{y}}{\sim}\mathfrak{m}_2$ and $\tilde{I}\underset{\underline{w}}{\sim}\tilde{I}_1\underset{\underline{z}}{\sim}\tilde{I}_2$ with $\mathfrak{m}_2=\tilde{I}_2$. Considering how $\underline{x}$ and $\underline{y}$ (resp. $\underline{w}$ and $\underline{z}$) are chosen in the proof of (2.1), we see that under the assumption (iii) we may assume that the linking regular sequences are minimal in the corresponding ideals. In a similar way I and $\tilde{I}$ are linked, and this finishes the proof of (3.6). □

In (3.5) and (3.6) we assumed that $\mathrm{gr}_{\mathfrak{m}}(R)$ be factorial. Of course, if R is factorial, $\mathrm{gr}_{\mathfrak{m}}(R)$ need not to be even normal, let alone factorial. However, we don't know in general whether $\mathrm{gr}_{\mathfrak{m}}(R)$ factorial implies R itself to be factorial if R is a 2-dimensional hypersurface ring. C. Huneke gave us an example of a hypersurface ring of $\dim>2$ for which this fails.

(3.7) **Corollary.** *Let R be a 2-dimensional hypersurface ring of multiplicity ≤ 3 whose associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is factorial, then R is factorial.*

Proof. Assume that R is not factorial and that I is a non-trivial divisorial ideal. Since R is 2-dimensional, I is an MCM-module over R, and we have $\mu(I)\leq \text{rank } I\cdot e(R)=e(R)$. If $e(R)=2$, then $\mu(I)=2$ and I is an Ulrich-module. It is shown in [6] that if M is an Ulrich-module, then $\text{gr}_m(M)$ is again an MCM-module (of the same rank). Hence $\text{gr}_m(I)$ defines a non-trivial element in the divisor class group of $\text{gr}_m(R)$, a contradiction. If $e(R)=3$, then $\mu(I)=2$ or $\mu(I)=3$. If $\mu(I)=3$, then I is an Ulrich-module, and we get a contradiction. If $\mu(I)=2$, then we have an exact sequence

$$0\longrightarrow I^{-1}\longrightarrow R^2\longrightarrow I\longrightarrow 0, \quad \text{and} \quad \mu(I^{-1})=2.$$

It follows that $I\oplus I^{-1}$ is an orientable MCM-module with data (2, 4). We conclude from the above exact sequence that $D(I\oplus I^{-1})=(I\oplus I^{-1})^*\simeq I\oplus I^{-1}$. This contradicts (3.6).

We remark that if any 2-dimensional normal local ring R would admit a rank 1 Ulrich-module, then $\text{gr}_m(R)$ factorial would imply R factorial for this class of rings.

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