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Some Cohen-Macaulay Complexes arising in Group Theory

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In this brief survey we will review some main facts known about the ring-theoretic properties of three classes of simplicial complexes which naturally arise in finite group theory. Ring-theoretic concepts are applied to a finite simplicial complex Δ via its Stanley-Reisner ring $\underset{\sim}{k}[\Delta]$. In matters of definitions and notation we adhere to Stanley's book [20], to where the reader is referred for all unexplained terminology.

§ 1. Subgroup lattices

Let G be a finite group, which we assume not to be of prime order to avoid trivialities. The proper subgroups H of G (i.e., $H \neq G$, 1) are the vertices of a simplicial complex $\Delta(L_G)$ whose faces are the chains $H_0 \subset H_1 \subset \cdots \subset H_k$. Thus, $\Delta = \Delta(L_G)$ is a simplicial complex on which G acts by conjugation, so also the ring $\underset{\sim}{k}[\Delta]$ has an induced G-action.

The following characterization of Cohen-Macaulayness is obtained by combining results of Björner, Iwasawa and Stanley, see [3, § 3].

$\Delta(L_G)$ is Cohen-Macaulay if and only if G is supersolvable.

The smaller class of Gorenstein subgroup lattices and the larger class of Buchsbaum subgroup lattices have also been determined. The next result is due to Hibi [14].

$\Delta(L_G)$ is Gorenstein if and only if G is cyclic of prime-power or square-free order.

The following was shown by Björner and Smith [6].

$\Delta(L_G)$ is Buchsbaum if and only if G is either supersolvable or else a semidirect product of an elementary Abelian group N of order p^k, $k \geq 2$, and a cyclic group of order q acting irreducibly on N, for distinct primes p and q.

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It follows from this characterization that groups with a Buchsbaum subgroup lattice are always solvable. Also, the Buchsbaum rings $k[\Delta]$ coming from subgroup lattices either have depth equal to their Krull dimension (the supersolvable case) or else they have depth equal to one. The smallest non-supersolvable group with a Buchsbaum subgroup lattice is the alternating group A_4.

Suppose that $\Delta=\Delta(L_G)$. Let first G be supersolvable. Cohen-Macaulayness entails in particular that reduced homology of Δ vanishes in dimensions less than dim Δ. In fact, it was shown in [3] that such Δ has the homotopy type of a bouquet of (dim Δ)-dimensional spheres. A related result by Kratzer and Thévenaz [21, 25] states that if G is solvable then Δ has the homotopy type of a bouquet of n-dimensional spheres, where $n+2$ is the length of a chief series of G. Hence, for solvable G, unless Δ is contractible, one may conclude from Hochster's formula [20, Corollary 4.9] that depth $k[\Delta]\leq n+1$. It is not clear whether the depth of $k[\Delta]$ has some otherwise more tangible connection to the group-structure of G (as does its Krull dimension).

The p-subgroups form a subcomplex of $\Delta(L_G)$ which has been studied by Quillen [17], Thévenaz [22] and others. Quillen established Cohen-Macaulayness for the subcomplex of elementary Abelian p-subgroups in some cases.

§ 2. Coset complexes

Let G be a finite group and $\mathscr{H}=\{H_1, H_2, \cdots, H_d\}$ a system of subgroups. Define a simplicial complex $\Delta=\Delta(G, \mathscr{H})$ as follows. The vertices of Δ are all cosets gH_i, $g \in G$, $1\leq i\leq d$, and a collection of such cosets forms a face of Δ if the intersection is nonempty. Two simple properties of Δ are apparent: (i) all maximal faces of Δ have cardinality d (or, Δ is "pure $(d-1)$-dimensional"), and (ii) the vertex set V of Δ has a natural partitioning $V=\bigcup_{i=1}^{d} V_i$ such that $|F\cap V_i|=1$ for all maximal faces F and all $1\leq i\leq d$ (or, Δ is "balanced"). Furthermore, G acts on Δ by left multiplication and this action is type-preserving (i.e., $g(V_i)=V_i$ for all $g \in G$ and $1\leq i\leq d$) and transitive on maximal faces. Conversely, as observed by Lannér [16], if a simplicial complex Δ is pure $(d-1)$-dimensional and balanced and if the group G of type-preserving automorphisms of Δ acts transitively on its maximal faces, then Δ is of the form $\Delta(G, \mathscr{H})$.

Coset complexes have been studied primarily for the case when G is a finite Coxeter group or finite group of Lie type of rank d and $\mathscr{H}$ is the system of maximal parabolic subgroups (in the Lie case, the maximal parabolic subgroups containing a fixed Borel subgroup), cf. Bourbaki [9], Lannér [16] and Tits [23]. Then $\Delta=\Delta(G, \mathscr{H})$ is the Coxeter complex and the Tits building, respectively. In the Coxeter case Δ is homeomorphic to a sphere, so $\underset{\sim}{k}[\Delta]$ is Gorenstein. In the Lie type case $\underset{\sim}{k}[\Delta]$ is Cohen-Macaulay, as can be deduced from the homology computation for buildings of Solomon and Tits [19] combined with the theorem of Reisner [18]. The following strengthening was given by Björner [4]. For ease of statement only the Chevalley group case will be formulated, for the general case see [4, Corollary 4.12].

If G is Chevalley group over $GF(q)$, then its building is $(q+1)$-Cohen-Macaulay.

The conclusion means that any collection of at most q vertices can be removed from Δ without losing Cohen-Macaulayness or reducing the dimension. The result is sharp in the sense that Δ is not $(q+2)$-Cohen-Macaulay. Baclawski [1, 2] has shown that this concept of higher Cohen-Macaulayness for complexes has good ring-theoretic consequences in terms of the Betti numbers $\dim_{k}\mathrm{Tor}_i^A(\underset{\sim}{k}[\Delta], \underset{\sim}{k})$ and the canonical module of Stanley-Reisner rings $\underset{\sim}{k}[\Delta]=A/I$. What precisely this says about the ring $\underset{\sim}{k}[\Delta]$ of a Tits building Δ over $GF(q)$ is discussed in [4, pp. 193–195]. For instance, the type of $\underset{\sim}{k}[\Delta]$ as a Cohen-Macaulay ring is $q^{l(w_0)}$, where w_0 is the Weyl group element of maximal length, and the canonical module of $\underset{\sim}{k}[\Delta]$ is isomorphic to the ideal generated by the apartments of the building. The Hilbert series of $\underset{\sim}{k}[\Delta]$ can also be computed, see Section 4.

P. Garst [13] has investigated the Cohen-Macaulayness of some coset complexes arising in symmetric groups and certain related complex reflection groups. E.g., let $G=S_n$, the symmetric group of all permutations of $\{1, 2, \cdots, n\}$, and let $\mathscr{H}_d=\{\mathrm{Stab}(\{i\})\mid i=1, 2, \cdots, d\}$. Then $\Delta_{n,d}=\Delta(S_n, \mathscr{H}_d)$ can be identified with the complex of all non-taking rook placements on a $d\times n$ chessboard. The following result is due to Garst.

$\Delta_{n,d}$ is Cohen-Macaulay if and only if $2d\leq n+1$.

As a corollary we deduce:

$\Delta_{n,d}$ is Buchsbaum if and only if $2d\leq n+2$.

The only complexes of this kind which are Gorenstein are $\Delta_{1,1}$, $\Delta_{2,1}$, and $\Delta_{3,2}$. The chessboard complex $\Delta_{4,3}$ is homeomorphic to the 2-dimensional torus.

Very little seems to be known about general necessary or sufficient conditions on systems $\mathscr{H}$ of subgroups of G such that the coset complex $\Delta(G, \mathscr{H})$ is Cohen-Macaulay. An interesting sufficient condition for simple-connectedness of coset complexes involving group presentation structure was found by Lannér [16]. A proof also appears in [13]. By adding a few ingredients one can from it deduce the following sufficient condition for Cohen-Macaulayness of 2-dimensional coset complexes.

Let G be a finite group and S a set of generators with 3 *given subsets* S_1, S_2, *and* S_3. *Assume* (i) *that* $S=S_i\cup S_j$, *for* $1\leq i<j\leq 3$, *and* (ii) *that for some presentation* $G=\langle S;\, R\rangle$ *each relation* $r\in R$ *involves only generators from some subset* S_i. *Then* $\Delta(G,\mathscr{H})$ *is Cohen-Macaulay, where* $\mathscr{H}=\{\langle S_i\rangle \mid i=1,2,3\}$.

It seems plausible that deeper connections may exist between group presentation structure and topological and ring-theoretic properties of coset complexes, that may lead to similar conditions for Cohen-Macaulayness of higher-dimensional coset complexes.

The following necessary condition was given by Garst [13]. For $\mathscr{H}=\{H_1, H_2, \cdots, H_d\}$ and $J\subseteq\{1,\cdots,d\}$, let $H_J=\bigcap_{i\in J}H_i$, $H_\phi=G$. Then $\Delta=\Delta(G,\mathscr{H})$ is Cohen-Macaulay only if the subgroup generated by H_I and H_J equals $H_{I\cap J}$, for all $I, J\subseteq\{1,\cdots,d\}$. This condition is, in fact, satisfied already if all links in Δ of dimension at least one are connected, which is a very weak consequence of Cohen-Macaulayness. E.g., the condition is satisfied by the chessboard complexes $\Delta_{n,d}$, $n\geq 2$, if and only if $d\leq n-1$.

Coset complexes for other groups, particularly for some finite simple groups, have been studied as examples of "diagram geometries", cf. Buekenhout [10], Tits [24], and [26, 27]. However, we are not aware of any results in this setting which have bearing on the Stanley-Reisner ring.

§ 3. Bruhat order

Let (W, S) be a finite Coxeter group. For the standard definitions and terminology see Bourbaki [9]. A partial ordering of W is defined by the transitive closure of the relations $w<wt$, if $l(w)<l(wt)$ where $w\in W$ and $t\in\{usu^{-1}\mid u\in W,\, s\in S\}$. This ordering was first studied as the inclusion ordering of cells in the Bruhat decomposition of flag manifolds, and is often called the Bruhat ordering. It gives rise to a multitude of Cohen-Macaulay complexes, induced on certain classes of subsets. The following results are quoted from Björner and Wachs [7, 8].

For $E\subseteq W$, let $W/E=\{w\in W \mid l(we)=l(w)+l(e)$ for all $e\in E\}$. One can prove that for every subset E there exists a unique element w_E (not necessarily a member of E) such that $W/E=W/\{w_E\}$. Furthermore, W/E has a unique maximal element $w_0w_E^{-1}$, and all maximal chains in the Bruhat ordering of W/E have cardinality $l(w_0w_E^{-1})+1$. Now, let $\Delta(W/E)$ be the simplicial complex of all Bruhat chains $w_1<w_2<\cdots<w_k$ in W/E.

$\Delta(W/E)$ is Cohen-Macaulay for all $E \subseteq W$.

When $W = S_n$, the symmetric group, some classes of familiar combinatorial objects can be shown to be of the form W/E. For instance, all standard Young tableaux of fixed shape read row by row to form permutations have this property.

For $\mathscr{J} \subseteq S$ one defines the descent class $\mathscr{D}_{\mathscr{J}} = \{w \in W \mid l(ws) < l(w)$ if and only if $s \in \mathscr{J}\}$. More generally, if $I \subseteq K \subseteq S$ let $\mathscr{D}_I^K = \bigcup \mathscr{D}_{\mathscr{J}}$, union over all $I \subseteq \mathscr{J} \subseteq K$. Such a class $\mathscr{D}_I^K$ has a unique minimal and a unique maximal element, and every maximal chain in the Bruhat ordering of $\mathscr{D}_I^K$ has cardinality equal to the difference of their lengths plus one. Again, let $\Delta(\mathscr{D}_I^K)$ be the complex of all Bruhat chains in $\mathscr{D}_I^K$.

$\Delta(\mathscr{D}_I^K)$ is Cohen-Macaulay for all $I \subseteq K \subseteq S$.

For $\mathscr{J} \subseteq S$ one sees that $W/\mathscr{J} = \mathscr{D}_{\phi}^{S-\mathscr{J}}$ is the set of minimal length left coset representatives modulo the parabolic subgroup $W_{\mathscr{J}}$. The Cohen-Macaulayness of the Stanley-Reisner ring $\underset{\sim}{k}[\Delta]$ for such $\Delta = \Delta(W/\mathscr{J})$, has in some cases been used to deduce the Cohen-Macaulayness of homogeneous coordinate rings for generalized Schubert varieties, see DeConcini et al [11, 12]. In [15] Huneke and Lakshmibai prove the Cohen-Macaulayness of some closely related complexes based on Bruhat order and put it to similar ring-theoretic use.

For some further properties of the complexes $\Delta(\mathscr{D}_I^K)$ of a topological nature and other additional remarks, see [5].

§ 4. Final remarks

While this survey has been limited mainly to cataloguing some of the ring-theoretic properties known about complexes arising in finite groups, it should be noted that much more precise information is available. In practically all cases Cohen-Macaulayness and Gorensteinness have been established using a constructive combinatorial procedure known as shellability. Shellability produces an explicit characteristic-free basis for $\underset{\sim}{k}[\Delta]$ as a free module over a certain natural system of parameters. See Section 1 of [4] for a detailed account. From the combinatorial description of such a basis it is in some cases possible to deduce the form of the Hilbert series of $\underset{\sim}{k}[\Delta]$ as a standard graded algebra. For instance, let Δ be the building of a finite Chevalley group G over $GF(q)$ with Weyl group (W, S). Then (cf. [4]):

$$\mathrm{Hilb}\,(\underset{\sim}{k}[\Delta], t) = \frac{\sum_{J \subseteq S} t^{|J|} \sum_{w \in \mathscr{D}_{\mathscr{J}}} q^{l(w)}}{(1-t)^{|S|}}.$$

The length generating functions $\sum q^{l(w)}$ for descent classes $\mathscr{D}_{\mathscr{J}}$ in W are computable by known formulas in all cases. E.g., if $G=PSL(4, GF(q))$, so $W=S_4$, then

$$\mathrm{Hilb}\,(\underset{\sim}{k}[\Delta], t)=(1-t)^{-3}(1+(3q+4q^2+3q^3+q^4)t + (q^2+3q^3+4q^4+3q^5)t^2+t^3).$$

Putting $q=1$ into these formulas one gets the Hilbert series for the ring $\underset{\sim}{k}[\Delta]$ of the Coxeter complex Δ of group (W, S).

References

[1] K. Baclawski, Cohen-Macaulay connectivity and geometric lattices, European J. Combin., **3** (1982), 293–305.
[2] ——, Canonical modules of partially ordered sets, J. Algebra, **83** (1983), 1–5.
[3] A. Björner, Shellable and Cohen-Macaulay partially ordered sets, Trans. Amer. Math. Soc., **260** (1980), 159–183.
[4] ——, Some combinatorial and algebraic properties of Coxeter complexes and Tits buildings, Adv. in Math., **52** (1984), 173–212.
[5] ——, Orderings of Coxeter Groups, in "Combinatorics and Algebra" (C. Greene, Ed.), Contemporary Math., Vol. 34, Amer. Math. Soc., Providence, R.I., 1984, pp. 175–195.
[6] A. Björner and S. Smith, Finite groups whose subgroup lattice is Buchsbaum, to appear.
[7] A. Björner and M. Wachs, Bruhat order of Coxeter groups and shellability, Adv. in Math., **43** (1982), 87–100.
[8] A. Björner and M. Wachs, Generalized quotients in Coxeter groups, preprint, 1986.
[9] N. Bourbaki, "Groupes et algèbres de Lie," Chap. 4–6, Hermann, Paris, 1968.
[10] F. Buekenhout, Diagrams for geometries and groups, J. Combin. Theory, Ser. A, **27** (1979), 121–151.
[11] C. DeConcini, D. Eisenbud and C. Procesi, "Hodge algebras," Astérisque, Vol. 91, Soc. Math. France, Paris, 1982.
[12] C. DeConcini and V. Lakshmibai, Arithmetic Cohen-Macaulayness and arithmetic normality for Schubert varieties, Amer. J. Math., **103** (1981), 835–850.
[13] P F. Garst, Cohen-Macaulay complexes and group actions, Thesis, Univ. of Wisconsin, Madison, 1979.
[14] T. Hibi, For which finite groups G is the lattice $L(G)$ of subgroups Gorenstein?, Nagoya Math. J., **105** (1987).
[15] C. Huneke and V. Lakshmibai, A characterization of Kempf varieties by means of standard monomials and its geometric consequences, J. Algebra, **94** (1985), 52–105.
[16] F. Lannér, On complexes with transitive groups of automorphisms, Medd. Lunds Univ. Mat. Sem., **11** (1950), 1–71.
[17] D. Quillen, Homotopy properties of the poset of non-trivial p-subgroups of a group, Adv. in Math., **28** (1978), 101–128.
[18] G. Reisner, Cohen-Macaulay quotients of polynomial rings, Adv. in Math., **21** (1976) 30–49.
[19] L. Solomon, The Steinberg character of a finite group with a BN-pair, in "Theory of Finite Groups" (R. Brauer and C.H. Sah, Eds.), Benjamin,

New York, 1969, pp. 213–221.

[20] R. Stanley, "Combinatorics and Commutative Algebra," Birkhäuser, Boston, 1983.

[21] J. Thévenaz, The top homology of the lattice of subgroups of a soluble group, Discrete Math., **55** (1985), 291–303.

[22] ——, Generalizations of Sylow and Brown theorems, preprint, 1985.

[23] J. Tits, Buildings of spherical type and finite BN-pairs, Lecture Notes in Math., No. **386**, Springer, Berlin, 1974.

[24] ——, Buildings and Buekenhout geometries, in "Finite Simple Groups, II" (M.J. Collins, Ed.), Academic Press, London, 1980, pp. 309–320.

[25] C. Kratzer and J. Thévenaz, Type d'homotopie des treillis et treillis des sous-groupes d'un groupe fini, Comment. Math. Helv., **60** (1985), 85–106.

[26] Finite Groups—Coming of Age, J. McKay (Ed.), Contemporary Math., Vol. 45, Amer. Math. Soc., Providence, R.I., 1985.

[27] Buildings and the Geometry of Diagrams, L.A. Rosati (Ed.), Lecture Notes in Math., Vol. 1181, Springer, Berlin, 1986.

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