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Representations of $GL(n)$ and Schur Algebras

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The fundamental representations of the algebraic group $GL(n)$ are the exterior powers $\Lambda^t(V)$ of the standard module V of dimension n and their formal characters are the elementary symmetric polynomials $e_t(x_1, \cdots, x_n)$ in n variables. The formal character of the Schur module $L_\lambda(V)$ for a partition $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_m)$ is a symmetric polynomial known classically as the Schur function $\{\tilde{\lambda}\}$ for the transpose partition $\tilde{\lambda}$. The symmetric polynomial $\{\tilde{\lambda}\}$ can be expressed in terms of the elementary ones as the determinant of the m by m matrix whose (i, j)-th entry is the elementary symmetric polynomial of degree $\lambda_i - i + j$. This expansion corresponds to the Giambelli expansion for a Schubert cycle on a Grassmannian in terms of the special Schubert cycles and it is actually just the transpose of the usual Jacobi-Trudi expansion for a Schur function. More will be said about the Jacobi-Trudi identity later.

There is a useful description of the above expansion for $\{\tilde{\lambda}\}$ in terms of the twisted action of the symmetric group on sequences. For any sequence $\gamma=(\gamma_1, \cdots, \gamma_m)$ of integers it is customary to let e_γ denote the monomial $e_{\gamma_1}\cdots e_{\gamma_m}$ of elementary symmetric polynomials. The expansion then can be written in the form

$$\{\tilde{\lambda}\}=\sum(-1)^w e_{w\cdot\gamma} \tag{1}$$

where the summation is over all permutations w of $\{1, \cdots, m\}$ and $w\cdot\lambda$ denotes the twisted action $w(\lambda+\zeta)-\zeta$ with $\zeta=(m-1, m-2, \cdots, 2, 1, 0)$. If for any sequence $\lambda=(\lambda_1, \cdots, \lambda_m)$ we let $\Lambda_\gamma(V)$ denote the tensor product representation $\Lambda^{\gamma_1}(V)\otimes\cdots\otimes\Lambda^{\gamma_m}(V)$ then the identity in (1) can be written as

$$[L_\lambda(V)]=\sum(-1)^w[\Lambda_{w\cdot\lambda}(V)] \tag{2}$$

in the formal character ring, or Grothendieck ring, of polynomial representations of $GL(n)$.

It was observed by A. Lascoux that the above identity should be realized as a resolution

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$$0 \to B_{\binom{m}{2}}(\lambda) \to \cdots \to B_1(\lambda) \to B_0(\lambda) \to L_\lambda(v) \to 0 \tag{3}$$

of $GL(n, k)$-modules where k is a field of characteristic zero and each $B_i(\lambda)$ is the direct sum $\sum \Lambda_{w\cdot\lambda}(V)$ over all permutations w of length i. Later, A. Nielsen conjectured a description for the boundary maps of such a complex using the Coxeter group structure on the symmetric group. In the last several years there has been growing interest in proving that the maps described by A. Nielsen are boundary operators and that the resulting complex is exact. One of the reasons for the interest in such a resolution is its formal resemblance to Bernstein-Gelfand-Gelfand resolutions involving infinite dimensional modules called Verma modules.

Let U denote the universal enveloping algebra of the Lie algebra $sl(n, C)$ of trace zero n by n matrices over C, and let h denote the Lie subalgebra of $sl(n, C)$ consisting of diagonal matrices of trace zero. For every sequence $\lambda = (\lambda_1, \cdots, \lambda_n)$ of complex numbers there is a linear function on h, which we shall also denote by λ, whose value on the diagonal matrix $(z_1, \cdots, z_n) \in h$ is $\lambda_1 z_1 + \cdots + \lambda_n z_n$. We let M_λ denote the Verma module of highest weight λ. The Verma module M_λ can be characterized as the universal highest weight U-module of highest weight λ and has a unique simple factor module V_λ, the irreducible U-module of highest weight λ. When $\lambda = (\lambda_1, \cdots, \lambda_n)$ is a nondecreasing sequence of nonnegative integers, i.e. a partition, then the simple U-module is finite dimensional and is in fact isomorphic as a U-module to the $GL(n, C)$-module $L_{\tilde{\lambda}}(V)$ associated to the transpose $\tilde{\lambda}$ of the partition λ. The Bernstein-Gelfand-Gelfand resolution of the finite dimensional module V_λ is an exact complex

$$0 \to M_{\binom{n}{2}}(\lambda) \to \cdots \to M_1(\lambda) \to M_0(\lambda) \to V_\lambda \to 0 \tag{4}$$

of U-modules where each $M_i(\lambda)$ is the direct sum $\sum M_{w\cdot\lambda}$ taken over all permutation w of length i [3].

Before discussing the connection between the complexes in (3) and (4), it will be useful to recall the involutory ring automorphism ω on the ring of symmetric functions in a countably infinite set $\{x_1, x_2, \cdots\}$ of variables. The involution ω takes the elementary symmetric function

$$e_t(x) = \sum_{i_1 < \cdots < i_t} x_{i_1} \cdots x_{i_t}$$

to the complete symmetric function

$$h_t(x) = \sum_{i_1 \leq \cdots \leq i_t} x_{i_1} \cdots x_{i_t}$$

and takes the Schur function $\{\lambda\}$ to $\{\tilde{\lambda}\}$. Applying ω to the expansion in

(2) we get the usual form

(5) $$\{\lambda\}=\sum(-1)^{w}h_{w\cdot\lambda}$$

of the Jacodi-Trudi identity. When restricted to a finite number of variables $x_1, \cdots, x_n$, the complete symmetric polynomial $h_t(x_1, \cdots, x_n)$ is the formal character of the symmetric power $S_t(V)$ of the standard $GL(n)$-module V. Over the complex numbers, tensoring an appropriate Bernstein-Gelfand-Gelfand resolution by an appropriate Schur algebra gives rise to a complex

(6) $$0\rightarrow C_{\binom{n}{2}}(\lambda)\rightarrow\cdots\rightarrow C_1(\lambda)\rightarrow C_0(\lambda)\rightarrow V_\lambda\rightarrow 0$$

where $C_i(\lambda)$ is the direct sum $\sum S_{w\cdot\lambda}(V)$ of tensor products of symmetric powers of V taken over all permutations w of length i. This complex realizes the Jacobi-Trudi identity in the same sense that the complex in (3) should realize the identity in (2). In fact the existence, structure, or exactness of these two types of complexes is equivalent over fields of characteristic zero in the sense that one can obtained from the other. Moreover, since the boundary maps of the resolutions in (4) are described explicitly, it is possible to trace them through to obtain a reasonable description of the boundary maps of the complexes in (6), and the description thus obtained is reminiscent of that given by A. Nielsen mentioned earlier. It still remains to prove that the complexes in (6) are exact but it seems hopeful that the connection with Bernstein-Gelfand-Gelfand resolutions will be useful for this purpose.(*)

We will now turn our attention to the case where we are no longer over a field of characteristic zero. Although the identity in (2) is valid over any field or ring, there can not be a resolution of the form as described in (3) over an arbitrary field, or over the ring $\boldsymbol{Z}$ of integers, for any partition λ whose second term λ_2 is greater than one. It follows easily from any of the various constructions of Schur modules that there is always a surjection from $\Lambda_\lambda(V)$ to $L_\lambda(V)$. In general, however, there are more relations necessary than those that are present in the term $B_1(\lambda)$ of (3). For simplicity let us consider the case where λ is a partition (λ_1, λ_2) of length 2 to illustrate this phenomenon. Over a field of characteristic zero the resolution in (3) takes the form of a well-known exact sequence

(7) $$0\rightarrow\Lambda_{(\lambda_1+1,\lambda_2-1)}(V)\rightarrow\Lambda_\lambda(V)\rightarrow L_\lambda(V)\rightarrow 0.$$

Over $\boldsymbol{Z}$ the relations have to be expanded to give the presentation

(8) $$\sum\Lambda_{(\lambda_1+t,\lambda_2-t)}(V)\rightarrow\Lambda_\lambda(V)\rightarrow L_\lambda(V)\rightarrow 0$$

where the direct sum is over all $t \in \{1, 2, \cdots, \lambda_2\}$. This phenomenon is essentially the consequence of the fact that the Kostant $\boldsymbol{Z}$-form $U_{\boldsymbol{Z}}$ of the universal enveloping algebra U of $sl(n, \boldsymbol{C})$ is the subring of U generated by the set

(9) $$\{\theta_{ij}^t/t! \mid i \neq j,\ t \geq 1\}$$

of divided powers of the elementary matrices θ_{ij} in $sl(n, \boldsymbol{C})$. Over a field of characteristic p it is sufficient to restrict the integers t in (8) or (9) to powers of p. With these observations, the problem becomes that of attempting to realize the identities in (2) as resolutions where one is allowed to expand the terms as in (3) by adding more terms of tensor products of exterior powers of V. There is a recursive procedure described in [1], [2] for constructing such exact finite complexes $\Lambda(\lambda)$ which resolve the Schur modules $L_\lambda(V)$. In the case $\lambda = (\lambda_1, \lambda_2)$ discussed above, the resolution $\Lambda(\lambda)$ is relatively simple to describe. The components $\Lambda_i(\lambda)$ in dimension i is the direct sum

$$\sum \binom{t-1}{i-1} \Lambda_{(\lambda_1+t, \lambda_2-t)}(V)$$

where the sum is over $t \in \{1, 2, \cdots, \lambda_2\}$ and the summand corresponding to each t appears with multiplicity equal to the binomial coefficient $\binom{t-1}{i-1}$.

Analogous comments can be made about the identity in (5) which can be realized as an exact complex $D(\lambda)$ which resolves the Weyl module $K_\lambda(V)$. The terms of the resolution $D(\lambda)$ consist of direct sums of tensor products of divided powers of the module V. The divided power module $D_t(V)$ is isomorphic to the submodule of $V^{\otimes t}$ consisting of symmetric tensors of degree t and thus has the complete symmetric polynomial $h_t(x_1, \cdots, x_n)$ as its formal character. The modules $D_t(V)$ and $S_t(V)$ are distinct integral forms with the same character, and the appearance of $D_t(V)$ is forced by the consideration of right resolutions. Such distinctions do not arise over fields of characteristic zero because of the complete reducibility of finite dimensional $GL(n)$-modules.

Before continuing with resolutions, it will be beneficial to discuss briefly the significance of Schur modules, Weyl modules, and Schur algebras in the characteristic-free representation theory of $GL(n)$. If V is the standard $GL(n, R)$-module of dimension n over a commutative ring R then the Schur algebra $A(n, t, R)$ is the endomorphism algebra of the t-fold tensor power $V^{\otimes t}$ as a module over the symmetric group on t letters, and it is the universal algebra for the homogeneous polynomial represen-

*) This can now be shown to be true.

tations of degree t of the algebraic group scheme $GL(n, R)$. When k is a field of characteristic zero, the Schur algebra $A(n, t, k)$ is semisimple and the modules $V_\lambda \cong L_{\tilde{\lambda}}(V) \cong K_\lambda(V)$ where $\lambda=(\lambda_1, \cdots, \lambda_n)$ is a partition of sum t form a complete set of nonisomorphic simple left modules for $A(t, n, k)$. Although no part of the above statement holds over arbitrary fields or rings, the Schur and Weyl modules play a central role in the representation theory of Schur algebras. Various constructions of these modules can be found in the literature (see [8]). The Schur module $L_\lambda(V)$ can be characterized over any field as the global sections of the line bundle on the flag variety of $GL(n)$ formed by taking the tensor product bundle of the collection of line bundles consisting of the λ_i-th exterior power bundle of the universal λ_i-plane bundle for each term λ_i of the partition $\lambda=(\lambda_1, \cdots, \lambda_n)$. The Weyl module $K_\lambda(V)$ can be characterized over any field as the universal highest weight module for $GL(n)$ of highest weight λ. Over the ring $\boldsymbol{Z}$, the modules $K_\lambda(V)$ and $L_{\tilde{\lambda}}(V)$ can be characterized as the minimal and maximal integral forms, respectively, of the irreducible $GL(n, Q)$-module V_λ in the sense of [15]. For example, when $\lambda=(t, 0, \cdots, 0)$, $K_\lambda(V)$ is $D_t(V)$ and $L_{\tilde{\lambda}}(V)$ is $S_t(V)$.

Given any left $GL(n, R)$-module M, the linear dual $M^* = \mathrm{Hom}_R(M, R)$ is naturally a right $GL(n, R)$-module and thus can be made into a left $GL(n, R)$-module, denoted M^0, by using the anti-automorphism of $GL(n, R)$ which takes every matrix to its transpose. For example, $(D_t(V))^0$ is isomorphic to $S_t(V)$. More generally, Schur modules and Weyl modules are dual in the sense that $(K_\lambda(V))^0$ is isomorphic to $L_{\tilde{\lambda}}(V)$.

In the representation theory of $GL(n, R)$ it is very useful to consider Weyl and Schur modules for skew diagrams λ/μ where λ, μ are partitions. These modules can be characterized by their relationships to the ordinary Weyl and Schur modules, and they also satisfy the duality $(K_{\lambda/\mu}(V))^0 \cong L_{\tilde{\lambda}/\tilde{\mu}}(V)$. Any ordinary partition λ can be identified with skew diagram $\lambda/(0)$ where (0) is the zero partition, so the ordinary Weyl and Schur modules are contained in this large family. A uniform construction for all of these modules is given in [1]. The formal character of the skew Weyl module $K_{\lambda/\mu}(V)$ is the skew Schur function $\{\lambda/\mu\}$ and there is a determinantal expansion for $\{\lambda/\mu\}$ which can be written in the form

$$\{\lambda/\mu\} = \det\,(h_{\lambda_i-\mu_j+j-i}) \tag{10}$$

Analogously, the formal character of the skew Schur module $L_{\lambda/\mu}(V)$ is the skew Schur function $\{\tilde{\lambda}/\tilde{\mu}\}$.

For all skew diagrams $\alpha=\lambda/\mu$, there is a recursive procedure given in [1], [2] for constructing finite resolutions $D(\alpha)$

$$\cdots \to D_1(\alpha) \to D_0(\alpha) \to K_\alpha(V) \to 0 \tag{11}$$

of $GL(n, R)$-modules realizing the identities in (10) over an arbitrary ring R. Each $D_i(\alpha)$ is a direct sum of tensor products of divided powers of the standard module V. We recall that the divided power module $D_t(V)$, the homogeneous component in degree t of the divided power algebra $D(V)$ of the module V, is isomorphic to the submodule of symmetric tensors in the t-fold tensor power of V. If $\lambda=(\lambda_1, \cdots, \lambda_n)$ is any sequence of nonnegative integers of sum t then the tensor product

$$D_\lambda(V)=D_{\lambda_1}(V)\otimes\cdots\otimes D_{\lambda_n}(V)$$

can be characterized as the universal weight module for the Schur algebra $A(n, t, R)$ of weight λ and its formal character is the monomial h_λ. Consequently, for any $\alpha=(\lambda_1, \cdots, \lambda_n)/(\mu_1, \cdots, \mu_n)$ the complex $D(\alpha)$ in (11) is a finite projective resolution of the Weyl module $K_\alpha(V)$ over the Schur algebra $A(n, t, R)$, where t is the degree of $\{\alpha\}$, and $D(\alpha)$ can be used to compute the extension groups of $K_\alpha(V)$ by any module whose weight structure is known.

The fundamental problem in the representation theory of a finite dimensional algebra is the determination of its Cartan matrix C whose entries enumerate the composition multiplicities of the principal indecomposable submodules of A. If A is a Schur algebra over a field of positive characteristic then it follows from a theorem of Brauer on modular reduction of integral forms of semisimple algebras that $C=D^t\cdot D$ where D is the matrix whose entries enumerate the composition multiplicities of the Weyl modules of the Schur algebra. Actually a stronger statement can be made about this reciprocity in the case of Schur algebras: every principal indecomposable projective A-module has a filtration whose factors are Weyl modules and D^t is the matrix whose entries enumerate the multiplicities of the Weyl module factors.

The Grothendieck group of the Schur algebra $A=A(n, t, R)$ is the homogeneous component in degree t of the ring of symmetric polynomials in n variables, at least in the case where R is a principal ideal domain. The involutory ring automorphism ω on symmetric functions preserves the classical inner product which corresponds to the intertwining numbers of pairs of A-modules for $n\geq t$. Since the algebra A is semisimple over a field of characteristic zero, it follows in this case that ω can be realized as an exact involutory functor on the category of finite dimensional A-modules whenever $n\geq t$. The situation over arbitrary fields or rings is quite a bit more complicated. Although one can define a functor Ω realizing, it is neither exact, involutory, nor unique. However, it is true that any such Ω must take a tensor product $D_\lambda(V)$ of divided powers to the corresponding tensor product $\Lambda_\lambda(V)$ of exterior powers. Such a functor Ω is used in [2] to construct, for all skew diagrams α, finite resolutions $\Lambda(\alpha)$

(12) $$\cdots \to \Lambda_1(\alpha) \to \Lambda_0(\alpha) \to L_\alpha(V) \to 0$$

of $GL(n, R)$-modules over an arbitrary ring R where each $\Lambda_i(\alpha)$ is a sum of tensor products of exterior powers of V. Moreover, using the fact that $\Lambda(\alpha)$ can be obtained from $D(\alpha)$ by an application of the functor Ω and the homological properties of the resolutions $D(\alpha)$, it is shown in [2] that there are natural isomorphisms

(13) $$\mathrm{Ext}^i_A(K_\alpha, K_\beta) \cong \mathrm{Ext}^i_A(L_\alpha, L_\beta)$$

over any Schur algebra $A(n, t, R)$, for any ring R, where $n \geq t$ and the standard module V has been suppressed. Combined with the properties of the dualizing functor $(-)^0$ the above can be written as a reciprocity

(14) $$\mathrm{Ext}^i_A(K_\alpha, K_\beta) \cong \mathrm{Ext}^i_A(K_{\tilde{\beta}}, K_{\tilde{\alpha}})$$

which is a strong generalization of the useful classical reciprocity

(15) $$\langle \{\alpha\}, \{\beta\} \rangle = \langle \{\tilde{\beta}\}, \{\tilde{\alpha}\} \rangle$$

of intertwining numbers of skew Schur functions.

The projective resolutions $D(\alpha)$ have been useful in obtaining other information on extensions of Weyl modules. Some of these computations have led to a conjecture on $\mathrm{Ext}^1_A(K_\lambda, K_\mu)$ when the partition μ can be obtained from the partition λ by adding t to the i-th term λ_i and subtracting t from the j-th term λ_j, $i<j$. The conjecture states that over $\boldsymbol{Z}$ the group $\mathrm{Ext}^1_A(K_\lambda, K_\mu)$ is cyclic of order equal to the greatest common divisor of the bionomial coefficients

(16) $$\binom{\lambda_i - \lambda_j + j - i + h}{h}$$

where $h \in \{1, \cdots, t\}$. Such a formula would explain various results obtained in recent years about nonzero homomorphisms from K_λ to K_μ over fields of positive characteristic, the connection being as follows. Since $\mathrm{Hom}_A(K_\lambda, K_\mu)$ is zero over $\boldsymbol{Q}$, the group $\mathrm{Hom}_A(K_\lambda, K_\mu)$ over a field of characteristic p is determined by the p-torsion part of $\mathrm{Ext}^1_A(K_\lambda, K_\mu)$ over $\boldsymbol{Z}$.

References

[1] K. Akin and D. A. Buchsbaum, Characteristic-free representation theory of the general linear group, to appear in Adv. in Math.

[2] ——, Characteristic-free representation theory of the general linear group II: Homological considerations, in preparation.

[3] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, Differential operators on the base affine space and a study of g-modules, "Proceedings of the Summer School on Group Representations," Bolyai Janos Math. Soc. (Budapest, 1971), Akademiai Kiado, Budapest, 1975.

[4] R. W. Carter and G. Lusztig, On the modular representations of the general linear and symmetric groups, Math Z., **136** (1975).

[5] E. Cline, B. Parshall and L. L. Scott, Cohomology, hyperalgebras, and representations, J. Algebra. **63** (1980).

[6] S. Donkin, Finite resolutions of modules for reductive algebraic groups, University of London preprint, 1984.

[7] R. M. Fossum, Decompositions revisited, in "Seminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin," Lecture Notes in Mathematics, No. **924**, Springer-Verlag, Berlin, 1982.

[8] J. A. Green, "Polynomial Representations of GL_n," Lecture Notes in Methematics, No. **830**, Springer Verlag, Berlin, 1980.

[9] W. J. Haboush, Central differential operators on split semi-simple groups over fields of positive characteristic, in "Seminaire d' Algèbre Paul Dubreil et Marie-Paule Malliavin," Lecture Notes in Mathematics, No. **795**, Springer-Verlag, Berlin, 1980.

[10] J. E. Humphreys and J. C. Jantzen, Blocks and indecomposable modules for semisimple algebraic groups, J. Algebra, **54** (1978).

[11] A. Lascoux, "Polynomes symétriques, foncteurs de Schur, et Grassmanniennes," Thèse, Université Paris VII, 1977.

[12] J. Lepowsky, Lectures on classical invariant theory, Brandeis University Preprint, 1971.

[13] H. A. Nielsen, Tensor functors of complexes, Aarhus University Preprint Series, No. **15**, 1978.

[14] J. Sullivan, Simply connected groups, the hyperalgebra, and Verma's conjecture, Amer. J. Math., **100** (1978).

[15] D.-N. Verma, The rôle of affine Weyl groups in the representation theory of algebraic Chevalley groups and their Lie algebras, in "Lie groups and their representations," Ed. I.M. Gelfand, Halsted, New York, 1975.

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