Advanced Studies in Pure Mathematics 10, 1987
Algebraic Geometry, Sendai, 1985
pp. 207–251

On p-Adic Vanishing Cycles
(Application of ideas of Fontaine-Messing)

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§ 0. Introduction

Let K be a complete discrete valuation field with perfect residue field k such that $\mathrm{char}(K)=0$ and $\mathrm{char}(k)=p>0$. Let X be a smooth scheme over the valuation ring O_K of K, and fix notations as

$$\begin{array}{lll} X_K=X\otimes_{O_K}K, & Y=X\otimes_{O_K}k, & \\ X_{\bar{K}}=X_K\otimes_K\bar{K}, & \bar{Y}=Y\otimes_k\bar{k}, & \bar{X}=X\otimes_{O_K}O_{\bar{K}}, \\ Y\xrightarrow{i}X\xleftarrow{j}X_K, & \bar{Y}\xrightarrow{\bar{i}}\bar{X}\xleftarrow{\bar{j}}X_{\bar{K}} & \end{array}$$

where $\bar{K}$ (resp. $\bar{k}$) denotes the algebraic closure of K (resp. k) and $O_{\bar{K}}$ denotes the integral closure of O_K in $\bar{K}$.

The sheaf of p-adic vanishing cycles $\bar{i}^*R^q\bar{j}_*(\boldsymbol{Z}/p^n\boldsymbol{Z})$ was studied in [3] [4] and used for the study of the etale cohomology group $H^*(X_{\bar{K}}, \boldsymbol{Z}/p^n\boldsymbol{Z})$. The purpose of Chapter I of this paper is to prove the following new result concerning $\bar{i}^*R^q\bar{j}_*(\boldsymbol{Z}/p^n\boldsymbol{Z})$. In Section 1, we define certain complexes $\mathcal{S}_n(r)_{\bar{X}}$ ($n\geqq 1, 0\leqq r<p$) on $\bar{Y}_{\mathrm{et}}$ which come from the crystalline cohomology theory, following the ideas of J.-M. Fontaine and W. Messing in their "syntomic cohomology theory" ([10]).

Theorem (Ch. I (4.3)). *Let $0\leqq r<p-1$. Then for any $n\geqq 1$, there is a canonical isomorphism*

$$\mathcal{H}^q(\mathcal{S}_n(r)_{\bar{X}})\cong\begin{cases}\bar{i}^*R^q\bar{j}_*(\boldsymbol{Z}/p^n\boldsymbol{Z}(r)) & \textit{if } q\leqq r\\ 0 & \textit{if } q>r.\end{cases}$$

Here $\boldsymbol{Z}/p^n\boldsymbol{Z}(r)$ denotes the Tate twist of $\boldsymbol{Z}/p^n\boldsymbol{Z}$. By the definition of $\mathcal{S}_n(r)_{\bar{X}}$ in Ch. I § 1, the result means that $\bar{i}^*R^q\bar{j}_*(\boldsymbol{Z}/p^n\boldsymbol{Z})$ can be described in terms of differential forms.

In Chapter II, we shall apply the above result to obtain the following theoerm on the etale cohomology groups $H^q(\bar{Y}, \bar{i}^*R^r\bar{j}_*(\boldsymbol{Z}/p^n\boldsymbol{Z}))$ in the case

Received December 2, 1985.

X is projective, assuming e_K $(=\mathrm{ord}_K(p))$ is one and p is sufficiently big. Recall that Y is called of Hodge-Witt if the W-modules $H^q(Y, W\Omega_Y^r)$ are of finite type for all q and r ([12] IV (4.6)).

Theorem (cf. Ch. II (4.3)(4.4)). *Assume X is projective over O_K, $e_K=1$ and $\dim(X_K)<p-1$. Then, the following three conditions are equivalent.*

(i) *Y is of Hodge-Witt.*

(ii) *The groups $H^q(\bar{Y}, \bar{i}^*R^r\bar{j}_*(Z/p^nZ))$ are finite for all q, r and n.*

(iii) *The spectral sequence*

$$E_2^{q,r}=H^q(\bar{Y}, \bar{i}^*R^r\bar{j}_*(Z/p^nZ))\Longrightarrow H^*(X_{\bar{K}}, Z/p^nZ)$$

degenerates for any n. If these equivalent conditions are satisfied, we have the following (1) *and* (2) *for any pair (q, r).*

(1) *There is a p-divisible group Γ over O_K without multiplicative part such that the Tate module $T_p(\Gamma)$ of Γ satisfies*

$$Q_p\otimes_{Z_p}T_p(\Gamma)(-r)\cong Q_p\otimes_{Z_p}\varprojlim_n H^q(\bar{Y}, \bar{i}^*R^r\bar{j}_*(Z/p^nZ))$$

as $Q_p[\mathrm{Gal}(\bar{K}/K)]$-modules.

(2) (Concerning the problem of Serre [13] 1.7.) *Assume $k=\bar{k}$. Let T be any simple subquotient of the* $\mathrm{Gal}(\bar{K}/K)$*-module $H^q(\bar{Y}, \bar{i}^*R^r\bar{j}_*(Z/pZ))$ and let $h=\dim_{F_p}(T)$. Then the action of* $\mathrm{Gal}(\bar{K}/K)$ *on T has the form*

$$\chi_h^{-(i_0+i_1p+\cdots+i_{h-1}p^{h-1})}$$

such that $i_m=r$ or $r-1$ for any m and $i_m=r$ for some m. Here χ_h denotes the "fundamental character of level h" ([9] 5.1; *cf.* Ch. II (3.6)).

This theorem generalizes, under the assumptions in this theorem, the corresponding results on $H^q(Y, \bar{i}^*R^r\bar{j}_*(Z/p^nZ))$ and $H^*(X_{\bar{K}}, Z/p^nZ)$ obtained in [3] [4] assuming Y is "ordinary" ([12] IV (4.12), [4] § 7), to the case Y is of Hodge-Witt. (For example, in the case Y is ordinary, the p-divisible group Γ is an etale p-divisible group. Cf. problems on the last page of [3]). In particular, it shows (by the above spectral sequence) that $H^*(X_{\bar{K}},Q_p)$ has a Hodge-Tate decomposition in the case $e_K=1$, $\dim(X_K)<p-1$ and Y is of Hodge-Witt.

This Hodge-Tate decomposition result was already proved with some deeper results by Fontaine and Messing [10] without the Hodge-Witt assumption.

This paper depends heavily on ideas of Fontaine and Messing [10]. Our method differs from theirs in that, to connect p-adic etale objects with differential objects, we use the local results of [3] [4] on p-adic vanishing cycles whereas p-adic vanishing cycles are not considered in [10]. This method enables us to obtain results on the cohomology of p-adic vanishing cycles as in the above theorem, and to understand the relation between the method of [3] [4] and that of [10].

I received recently the paper Faltings [18] which proves the Hodge-Tate decomposition conjecture completely generally. In the introduction of [18], it is explained that the main result of [18] furnishes a purely algebraic proof of the fact that the Hodge spectral sequence degenerates in characteristic zero. Our result Ch. II Prop. 2.5 (1) can be used for another purely algebraic proof of the degeneracy. I thank Professor W. Messing for a correspondence on this fact.

The study of $i^*R^qj_*(Z/p^nZ(r))$ (not only that of $\bar{i}^*R^q\bar{j}_*(Z/p^nZ)$ of this paper) will be important for the p-adic algebraic geometry.

I express my sincere gratitudes to Professors J.-M. Fontaine and W. Messing for explaining me their method in [10] and for helpful discussions. The reader will see in the content of this paper that I owe much to them. I am also obliged to Professor S. Bloch, from whom I learned much about the field considered in [3] [4].

Contents

Notations

In the following,

p denotes a fixed prime number
k denotes a perfect field of characteristic p,
W (resp. W_n) denotes the Witt ring $W(k)$ ($W_n(k)$),
σ denotes the frobenius $W \to W$.

For a W-module M, $M^{(\sigma)}$ denotes the W-module whose underlying abelian group is M but on which W acts by $W \times M \to M$; $(a, x) \mapsto \sigma(a)x$,

For a scheme T,

T_n denotes $T \otimes_Z Z/p^nZ$,

$D(T_{\text{et}})$ denotes the derived category of the category of all abelian group sheaves on the small etale site T_{et},

$\Omega_T^{\bullet}$ denotes the absolute de Rham complex $\Omega_{T/Z}^{\bullet}$ on T_{et}.

The notations for cohomology H^*, $R^q j_*$ etc. mean the etale cohomology unless the contrary is explicitly indicated.

Chapter I. Local study

§ 1. The definition of $\mathscr{S}_n(r)$

According to Fontaine and Messing, we call a morphism of schemes syntomic if it is flat and is complete intersection in the sense of [15] VIII §1.

Let X be a syntomic quasi-projective scheme over W. In this Section 1, we define objects $\mathscr{S}_n(r)_X$ ($n\geq 1$, $0\leq r<p$) of the derived category $D((X_1)_{\mathrm{et}})$ where $X_1=X\otimes \boldsymbol{Z}/p\boldsymbol{Z}$, by using the crystalline cohomology theory (cf. (1.6)).

Remark (1.1). J.-M. Fontaine and W. Messing defined a site X_{syn} called the syntomic site and a sheaf S_n^r on X_{syn} by using the crystalline cohomology theory. In terms of their theory, $\mathscr{S}_n(r)_X$ is isomorphic to $R\pi_*(S_n^r)$ where $\pi: X_{\mathrm{syn}}\to X_{\mathrm{et}}$ is the canonical morphism of sites, though we do not discuss this fact in this paper. We define $\mathscr{S}_n(r)_X$ directly in this section by following the ideas of Fontaine-Messing in their definition of S_n^r, and study $\mathscr{S}_n(r)_X$ in the subsequent sections by using its explicit definition. (We do not use the syntomic topology in this paper.)

Definition (1.2). Let T be a scheme over W. A morphism of scheme $f: T\to T$ is called a frobenius of T if $f\otimes \boldsymbol{Z}/p\boldsymbol{Z}: T_1\to T_1$ is the absolute frobenius induced by $\mathcal{O}_{T_1}\to\mathcal{O}_{T_1}$; $t\mapsto t^p$, and if the diagram

$$\begin{array}{ccc} T & \xrightarrow{f} & T \\ \downarrow & & \downarrow \\ \mathrm{Spec}(W) & \xrightarrow{\sigma} & \mathrm{Spec}(W) \end{array}$$

is commutative.

Take an immersion $i: X\xrightarrow{\subset} Z$ over W such that Z is a smooth scheme over W endowed with a frobenius $f: Z\to Z$. The existence of (Z, i, f) follows from the fact that the projective space has a frobenius. Let

$$D_n=D_{X_n}(Z_n)$$

be the divided power envelope with respect to the canonical divided power structure of $pW_n\subset W_n$ ([2] Section 3). Let J_{D_n} be the ideal of D_n defining X_n, and let $J_{D_n}^{[r]}$ ($r\geq 0$) be its r-th divided power. Since the underlying space of D_n coincides with that of X_1, we can regard sheaves on $(D_n)_{\mathrm{et}}$ as sheaves on $(X_1)_{\mathrm{et}}$. For $r \geqq 0$, we denote the well known complex of sheaves on $(X_1)_{\mathrm{et}}$

$$\underset{(\text{deg. } 0)}{J_{D_n}^{[r]}} \xrightarrow{d} \underset{(\text{deg. } 1)}{J_{D_n}^{[r-1]} \otimes_{\mathcal{O}_{Z_n}} \Omega_{Z_n}^1} \xrightarrow{d} J_{D_n}^{[r-2]} \otimes_{\mathcal{O}_{Z_n}} \Omega_{Z_n}^2 \longrightarrow \cdots$$

by $\boldsymbol{J}_{n,X,Z}^{[r]}$. (We denoted $i^{-1}(\Omega_{Z_n}^q)$ simply by $\Omega_{Z_n}^q$. For $i \leqq 0$, $J_{D_n}^{[i]} = \mathcal{O}_{D_n}$ by convention.) We denote $\boldsymbol{J}_{n,X,Z}^{[0]}$ by $\boldsymbol{E}_{n,X,Z}$.

Lemma (1.3). (1) *For* $0 \leqq r < p$, $f(J_{D_n}^{[r]}) \subset p^r \mathcal{O}_{D_n}$.

(2) *For any* r, m *and* n, *the sequence*

$$J_{D_{m+n}}^{[r]} \xrightarrow{p^m} J_{D_{m+n}}^{[r]} \xrightarrow{p^n} J_{D_{m+n}}^{[r]} \longrightarrow J_{D_n}^{[r]} \longrightarrow 0$$

is exact.

Corollary (1.4). (1) *For* $0 \leqq r < p$, $f(\boldsymbol{J}_{n,X,Z}^{[r]}) \subset p^r \boldsymbol{E}_{n,X,Z}$.

(2) *For any* r, m *and* n, *the sequence*

$$\boldsymbol{J}_{m+n,X,Z}^{[r]} \xrightarrow{p^m} \boldsymbol{J}_{m+n,X,Z}^{[r]} \xrightarrow{p^n} \boldsymbol{J}_{m+n,X,Z}^{[r]} \xrightarrow{\text{can.}} \boldsymbol{J}_{n,X,Z}^{[r]} \longrightarrow 0$$

is exact.

Indeed, (1.4) (1) follows from (1.3) (1) and from

$$f(\Omega_Z^i) \subset p^i \Omega_Z^i \quad (i \geqq 0).$$

Proof of (1.3). (I learned the following proof from Fontaine and Messing.) For (1), it suffices to show $f(x^{[r']}) \in p^r \mathcal{O}_{D_n}$ for $x \in J_{D_n}$ and for $r' \geqq r$. Write $f(x) = x^p + py$, $y \in \mathcal{O}_{D_n}$. We have

$$f(x^{[r']}) = f(x)^{[r']} = (x^p + py)^{[r']} = p^{[r']}((p-1)!\,x^{[p]} + y)^{r'},$$

but $p^{[r']} \in (p^r)$. For (2), the problem is etale local, so we may assume that X is defined by a regular sequence $(f_1, \cdots, f_s)$ in Z ([15] VIII, 1.2). Consider the cartesian diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\subset} & Z_n \\ \downarrow & & \downarrow \\ \mathrm{Spec}(W_n) & \xrightarrow{\subset} & \boldsymbol{A}_{W_n}^s = \mathrm{Spec}(W_n[t_1, \cdots, t_s]) \\ 0 \leftarrow\!\!\shortmid t_i. & & \end{array} \qquad \begin{array}{c} f_i \\ \uparrow \\ \vert \\ t_i. \end{array}$$

Since the right vertical arrow is flat on some open neighbourhood of X_n in Z_n ([16] Ch. 0, 15. 1.21), we can apply [2] 3.21 to obtain

$$D_{X_n}(Z_n) \cong Z_n \times_{A^s_{W_n}} D_{\mathrm{Spec}(W_n)}(A^s_{W_n}).$$

From this, the problem is reduced to the case $X=\mathrm{Spec}(W)$, $Z=A^s_W$ and $X \xrightarrow{\subset} Z$ is the immersion to the origin. In this case, $\mathcal{O}_{D_n}=W_n\langle t_1 \cdots, t_n\rangle$ ([2] 3.20 (5)) and $J^{[r]}_{D_n}$ is the free W_n-module with base $\{t_1^{[i_1]}\cdots t_s^{[i_s]};\ i_1+\cdots+i_s \geqq r\}$.

Corollary (1.5). *Let* $0 \leqq r < p$. *There is a unique homomorphism of complexes*

$$f_r : J^{[r]}_{n,X,Z} \longrightarrow E_{n,X,Z}$$

for which $f: J^{[r]}_{n+r,X,Z} \longrightarrow E_{n+r,X,Z}$ *factors as*

$$J^{[r]'}_{n+r,X,Z} \xrightarrow{\text{can.}} J^{[r]}_{n,X,Z} \xrightarrow{f_r} E_{n,X,Z} \xrightarrow{p^r} E_{n+r,X,Z}.$$

Definition (1.6). We define the complex $\mathcal{S}_n(r)_{X,Z}$ (this depends on f but we abbreviate it) to be the "mapping fiber" of

$$J^{[r]}_{n,X,Z} \xrightarrow{f_r-1} E_{n,X,Z}$$

(1 means the inclusion map), and we define $\mathcal{S}_n(r)_X$ to be the image of $\mathcal{S}_n(r)_{X,Z}$ in the derived category $D((X_1)_{\mathrm{et}})$.

Precisely, the degree i part of $\mathcal{S}_n(r)_{X,Z}$ is

$$(J^{[r-i]}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^i_{Z_n}) \oplus (\mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^{i-1}_{Z_n})$$

and the differential of $\mathcal{S}_n(r)_{X,Z}$ is given by

$$(x, y) \longmapsto (dx, (f_r-1)(x)-dy)$$
$$(x \in J^{[r-i]}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^i_{Z_n},\ y \in \mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^{i-1}_{Z_n}).$$

We see that $\mathcal{S}_n(r)$ is independent of the choice of (Z, f), as follows. For two pairs (Z, f) and (Z', f'), we take the third immersion $X \xrightarrow{\subset} Z \times_W Z'$ where $Z \times_W Z'$ is endowed with the frobenius $f \times f'$. Then, the canonical maps

$$\mathcal{S}_n(r)_{X,Z} \longrightarrow \mathcal{S}_n(r)_{X,Z\times_W Z'}, \qquad \mathcal{S}_n(r)_{X,Z'} \longrightarrow \mathcal{S}_n(r)_{X,Z\times_W Z'}$$

are quasi-isomorphisms as is seen from theory of crystalline cohomology ([2] Theorem 7.2).

Remark (1.7). In later sections, we shall be concerned with syntomic schemes over $O_{\bar{K}}$ (cf. § 0), which can not be syntomic over W (unless it is empty). So we slightly generalize the definition of $\mathscr{S}_n(r)$ to schemes of such type. Generally, let X be a scheme such that there are directed inverse systems $(X_\lambda)_{\lambda\in\Lambda}$ and $(Z_\lambda)_{\lambda\in\Lambda}$ of W-schemes, and a system of immersions $i_\lambda: X_\lambda\to Z_\lambda$ and a system of frobenius $f_\lambda: Z_\lambda\longrightarrow Z_\lambda$ ($\lambda\in\Lambda$) which commute with the transition maps such that

(1.7.1) Each X_λ is syntomic over W,

(1.7.2) Each Z_λ is smooth over W,

(1.7.3) The transition maps $X_{\lambda'}\to X_\lambda$ and $Z_{\lambda'}\to Z_\lambda$ ($\lambda'\geqq\lambda$) are affine,

(1.7.4) $X=\varprojlim X_\lambda$.

For $0\leqq r<p$, we define $\mathscr{S}_n(r)_X$ to be the image of $\varinjlim \mathscr{S}_n(r)_{X_\lambda,Z_\lambda}$ in $D((X_1)_{\mathrm{et}})$. Then, one sees that $\mathscr{S}_n(r)_X$ is independent of the choice of systems (X_λ), (Z_λ), (i_λ), (f_λ).

Remark (1.8). The hypothesis that X is quasi-projective in this section is not indispensable for the construction of $\mathscr{S}_n(r)$. Indeed, not assuming X quasi-projective, let $X'\to X$ be an affine open covering and take a W-immersion $X'\xrightarrow{\subset}Z$ such that Z is a smooth scheme over W endowed with a frobenius. Assume that X is separated (even this assumption is unnecessary if we use the technique in [2] 7.8, proof) and let

$$X^{(i)}=X'\times_X\cdots\times_X X',\qquad Z^{(i)}=Z\times_W\cdots\times_W Z\quad (i \text{ times}),$$
$$\pi_i: X^{(i)}\longrightarrow X\quad \text{the canonical morphism.}$$

Then, we define $\mathscr{S}_n(r)_{X,Z}$ ($0\leqq r<p$) to be the complex associated to the double complex

$$\underset{(\deg.\ 0)}{(\pi_1)^*(\mathscr{S}_n(r)_{X^{(1)},Z^{(1)}})}\longrightarrow\underset{(\deg.\ 1)}{(\pi_2)_*(\mathscr{S}_n(r)_{X^{(2)},Z^{(2)}})}\longrightarrow\cdots.$$

By the argument in (1.6) using the fact π_i are affine morphisms (for X is separated) and acyclic for quasi-coherent sheaves, we see that the image $\mathscr{S}_n(r)_X$ of $\mathscr{S}_n(r)_{X,Z}$ in the derived category is independent of the choices of X' and Z. In this paper, we restrict ourselves to the quasi-projective case for the simplicity of the description of the theory. This is sufficient for the study in Ch. I which is local. Indeed, the cohomology sheaves $\mathscr{H}^q(\mathscr{S}_n(r)_X)$ can be defined by considering only the quasi-projective case (for one can glue local definitions), and Ch. I concerns these cohomology sheaves. In Ch. II, if we use the above general definition for X not necessarily quasi-projective, we can generalize the results in Ch. II Section 2, Section 4 to smooth proper schemes X over W which need not be projective.

§ 2. The product structure

In this section, we define a product

$$\mathscr{S}_n(r)_{X,Z}\times\mathscr{S}_n(r')_{X,Z}\longrightarrow\mathscr{S}_n(r+r')_{X,Z}\quad (r, r'\geqq 0, r+r'<p).$$

(2.1) Generally, let T be a topos, and let $A=(A^q)_{q\in Z}$ be a complex of abelian groups in T. We call A a ring-complex if a global section 1 of A^0 and a homomorphism of complexes $A\otimes A\to A$; $x\otimes y\mapsto xy$ are given satisfying the usual axioms of the ring. By the definition of the tensor product of complexes, this means that $\oplus_{q\in Z}A^q$ is endowed with a ring structure with the unit 1 whose additive structure is the original one, such that $A^qA^{q'}\subset A^{q+q'}$ and

$$d(xy)=d(x)y+(-1)^q xd(y)\ (q, q'\in \boldsymbol{Z}, x\in A^q, y\in A^{q'}).$$

(Here $x\in A^q$ means that x is a local section of A^q.)

Now let A and B be ring-complexes in T, and let $g, h: A\to B$ be two homomorphisms of complexes preserving the ring-complex structures. Let S be the mapping fiber of $g-h: A\to B$. (So, $S^q=A^q\oplus B^{q-1}$, and $d: S^q\to S^{q+1}$ is $(x, y)\mapsto(dx, g(x)-h(x)-dy)$.) We define the product

$$S\otimes S\longrightarrow S\quad\text{by}\quad (x, y)(x', y')=(xx', (-1)^q g(x)y'+yh(x'))$$
$$((x, y)\in S^q=A^q\oplus B^{q-1},\quad (x', y')\in S^{q'}=A^{q'}\oplus B^{q'-1}).$$

We can prove easily

Lemma (2.2). *With the unit section $(1_A, 0)\in S^0$, the above product defines a ring-complex structure on S.*

Lemma (2.3). *Assume that A and B satisfy*

$$xy=(-1)^{qq'}yx$$

for any $q, q'\in \boldsymbol{Z}$ and for any homogeneous local sections x and y of degree q and q' respectively. Then we have

$$xy=(-1)^{qq'}yx$$

for any $q, q'\in \boldsymbol{Z}$ and for any local sections x (resp. y) of the cohomology sheaf $\mathscr{H}^q(S)$ (resp. $\mathscr{H}^{q'}(S)$).

(2.4) Now let X and Z be as in Section 1. We apply the above arguments to the case

$$A = \bigoplus_{0 \leq r < p} J^{[r]}_{n,X,Z}, \qquad B = \bigoplus_{0 \leq r < p} \boldsymbol{E}_{n,X,Z}$$
$$g = \bigoplus_{0 \leq r < p} f_r, \qquad h = \bigoplus_{0 \leq r < p} 1$$

(1 means the natural inclusion). Here the ring-complex structure of B is defined by

$$(x_0, \cdots, x_{p-1})(y_0, \cdots, y_{p-1}) = (z_0, \cdots, z_{p-1})$$

where
$$z_q = \sum_{i+j=q} x_i y_j$$

with the natural product $x_i y_j$ in $\boldsymbol{E}_{n,X,Z}$, and that of A is induced from this via the natural inclusion $A \xrightarrow{\subset} B$. We obtain thus a product

$$\mathscr{S}_n(r)_{X,Z} \otimes \mathscr{S}_n(r')_{X,Z} \longrightarrow \mathscr{S}_n(r+r')_{X,Z}$$

($r, r' \geqq 0$, $r + r' < p$). As is seen by the argument in (1.6), the induced product structure

$$\mathscr{S}_n(r)_X \overset{L}{\otimes} \mathscr{S}_n(r')_X \longrightarrow \mathscr{S}_n(r+r')_X$$

in the derived category is independent of the choice of Z.

§ 3. Symbols

Let X be a syntomic scheme over W. In this section, we define a "symbol map"

$$\underbrace{\mathcal{O}^\times_{X_{n+1}} \otimes \cdots \otimes \mathcal{O}^\times_{X_{n+1}}}_{r \text{ times}} \longrightarrow \mathscr{H}^r(\mathscr{S}_n(r)_X) \quad (r < p)$$

and prove that this map is surjective if $r < p-1$. (See Theorem (3.6)).

(3.1) In (3.1)–(3.5), we assume that X is quasi-projective. We first define a homomorphism

$$\mathcal{O}^\times_{X_{n+1}}[-1] \longrightarrow \mathscr{S}_n(1)_X$$

in $D((X_1)_{\text{et}})$. This will induce a canonical map

$$\mathcal{O}(X_{n+1})^\times = \Gamma(X_{n+1}, \mathcal{O}^\times_{X_{n+1}}) \longrightarrow H(X_1, \mathscr{S}_n(1)_X)$$

and by the product structure of $\mathscr{S}_n(r)_X$ in Section 2, a map

$$\underbrace{\mathcal{O}(X_{n+1})^\times \otimes \cdots \otimes \mathcal{O}(X_{n+1})^\times}_{r \text{ times}} \longrightarrow H^r(X_1, \mathscr{S}_n(r)_X) \quad (r < p)$$

defining (without the quasi-projective assumption) the above symbol map.

Take $i: X \overset{\subset}{\longrightarrow} Z$ and $f: Z \longrightarrow Z$ as in Section 1. Let

$$N = \mathrm{Ker}(i^{-1}(\mathcal{O}_{Z_{n+1}}^{\times}) \longrightarrow \mathcal{O}_{X_{n+1}}^{\times}),$$

and let C be the complex

$$\begin{array}{ccc} N & \longrightarrow & i^{-1}(\mathcal{O}_{Z_{n+1}}^{\times}) \\ (\text{deg. } 0) & & (\text{deg. } 1) \end{array}$$

on $(X_1)_{\mathrm{et}}$. Clearly, C is quasi-isomorphic to $\mathcal{O}_{X_{n+1}}^{\times}[-1]$. We define a homomorphism of complexes

$$s: C \longrightarrow \mathcal{S}_n(1)_{X,Z}$$

which induces the desired map $\mathcal{O}_{X_{n+1}}^{\times}[-1] \to \mathcal{S}_n(1)_X$ in $D((X_1)_{\mathrm{et}})$, as follows. Denote the map $i^{-1}(\mathcal{O}_{Z_{n+1}}) \longrightarrow \mathcal{O}_{D_{n+1}}$ as $a \mapsto \bar{a}$. Then, the degree zero part of s is

$$N \longrightarrow J_{D_n}^{[1]};\ a \longmapsto \log(\bar{a}).$$

The degree one part of s is

$$\begin{aligned} i^{-1}(\mathcal{O}_{Z_{n+1}}^{\times}) &\longrightarrow (\mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega_{Z_n}^1) \oplus \mathcal{O}_{D_n} \\ a &\longmapsto (a^{-1}da, p^{-1}\log(\overline{f(a)a^{-p}})). \end{aligned}$$

Here, since $\overline{f(a)a^{-p}} \in 1 + p\mathcal{O}_{D_{n+1}}$, we have $\log(\overline{f(a)a^{-p}}) \in p\mathcal{O}_{D_{n+1}}$. The notation $p^{-1}\log(\overline{f(a)a^{-p}})$ means the unique element of $\mathcal{O}_{D_n}$ whose image under $p: \mathcal{O}_{D_n} \to \mathcal{O}_{D_{n+1}}$ coincides with $\log(\overline{f(a)a^{-p}})$.

Proposition (3.2). *Assume $p \neq 2$. Let $h: \mathcal{O}(X_{n+1})^{\times} \to H^1(X_1, \mathcal{S}_n(1)_X)$ be the canonical map defined above, and let x and y be elements of $\mathcal{O}(X_{n+1})^{\times}$ such that $x+y=1$ or $x+y=0$. Then*

$$h(x)h(y) = 0 \qquad \text{in } H^2(X_1, \mathcal{S}_n(2)_X).$$

We give here the proof for the case $x+y=1$; the case $x+y=0$ is proved similarly and follows also from (2.3). We prove a slightly general

Lemma (3.3). *Assume $p \neq 2$. Let $x \in \mathcal{O}(X_{n+1})^{\times}$, $a \in \mathcal{O}(X_{n+1})$ and assume that $1 - a^{p^n}x$ is invertible. Then*

$$h(x)h(1-a^{p^n}x) = 0 \qquad \text{in } H^2(X_1, \mathcal{S}_n(2)_X).$$

Proof. Since the problem is local and the definition of h is functorial, we may assume that

$$X=\mathrm{Spec}(W[S, T, T^{-1}, (1-S^{p^n}T)^{-1}]),$$

and $x=T$, $a=S$. Let $Z=\mathrm{Spec}(W[S, T])$ and let $f\colon Z\to Z$ be the frobenius defined by $S\mapsto S^p$, $T\mapsto T^p$. Then, $D_{X_n}(Z_n)=X_n$, and the degree one part of the map s

$$\mathcal{O}^{\times}_{X_{n+1}}\longrightarrow \Omega^1_{X_n}\oplus\mathcal{O}_{X_n}$$

in (3.1) satisfies

$$\begin{aligned}
&T\longmapsto (T^{-1}dT, 0)\\
&1-S^{p^n}T\longmapsto (-(1-S^{p^n}T)^{-1}S^{p^n}dT,\\
&\qquad\qquad\qquad p^{-1}\log((1-S^{p^{n+1}}T^p)(1-S^{p^n}T)^{-p})).
\end{aligned}$$

From this, we see that $h(T)h(1-S^{p^n}T)(\in H^2(X_1, \mathcal{S}_n(2)_X))$ is the class of

$$(0, -p^{-1}\log((1-S^{p^{n+1}}T^p)(1-S^{p^n}T)^{-p})T^{-1}dT)\in\Gamma(X_n, \Omega^2_{X_n}\oplus\Omega^1_{X_n}).$$

This class is zero, for

$$p^{-1}\log((1-S^{p^{n+1}}T^p)(1-S^{p^n}T)^{-p})T^{-1}dT=du \quad \text{in } \Omega^1_{X_n},$$

where

$$u=\sum_{\substack{(i,p)=1\\ i\geqq 1}} i^{-2}(S^{p^n}T)^i\in\Gamma(X_n, \mathcal{O}_{X_n}).$$

(This formal power series $u\in W_n[[T, S]][T^{-1}]$ belongs to the subspace $\Gamma(X_n, \mathcal{O}_{X_n})$).

Remark (3.4). Proposition (3.2) suggests that there are chern class maps $K_q(X)\to H^{2r-q}(X_1, \mathcal{S}_n(r)_X)$, but the author has not yet studied this problem.

Remark (3.5). The complex $\mathcal{S}_n(r)_X$ is similar to the complex of Deligne-Beilinson

$$\underset{\text{(deg. 0)}}{\boldsymbol{Z}(r)}\longrightarrow\mathcal{O}_X\longrightarrow\Omega^1_{X/C}\longrightarrow\cdots\longrightarrow\Omega^{r-1}_{X/C}\longrightarrow 0$$

on a complex manifold ([1]). Indeed, if X is smooth over W and has a frobenius f, $\mathcal{S}_n(r)_X$ has the form

$$\begin{aligned}
&0\longrightarrow\underset{\text{(deg. 1)}}{\mathcal{O}_{X_n}}\xrightarrow{-d}\Omega^1_{X_n}\xrightarrow{-d}\cdots\xrightarrow{-d}\Omega^{r-2}_{X_n}\\
&\qquad\xrightarrow{(0, -d)}\Omega^r_{X_n}\oplus\Omega^{r-1}_{X_n}\xrightarrow{(f_r-1, -d)}\Omega^r_{X_n}\longrightarrow 0
\end{aligned}$$

in which the part $\Omega^r_{X_n}\xrightarrow{f_r-1}\Omega^r_{X_n}$ corresponds to $\boldsymbol{Z}(r)$ on a complex manifold.

Theorem (3.6). *Let X be a syntomic scheme over W.*

(1) *Let $0\leqq r<p$. Then the cohomology sheaves $\mathcal{H}^q(\mathcal{S}_n(r)_X)$ vanish for $q>r$.*

(2) *Assume $r<p-1$. Then the symbol map*

$$\mathcal{O}^\times_{X_{n+1}}\otimes\cdots\otimes\mathcal{O}^\times_{X_{n+1}}\longrightarrow\mathcal{H}^r(\mathcal{S}_n(r)_X)$$

is surjective.

Proof. We may assume $n=1$. Since the problem is local, we may assume that X is quasi-projective. Take Z as in Section 1. For the proof of (1), it suffices to show that the map

$$f_r-1:(\boldsymbol{J}^{[r]}_{1,X,Z})_q\longrightarrow(\boldsymbol{E}_{1,X,Z})_q$$

is an isomorphism if $q>r$ and is surjective if $q=r$. If $q>r$, $f_r=p^{q-r}f_q=0$ on $(\boldsymbol{J}^{[r]}_{1,X,Z})_q$ and hence f_r-1 is rewritten as $\mathcal{O}_{D_1}\otimes_{\mathcal{O}_{Z_1}}\Omega^q_{Z_1}\xrightarrow[\cong]{-1}\mathcal{O}_{D_1}\otimes_{\mathcal{O}_{Z_1}}\Omega^q_{Z_1}$. For $q=r$, the map is $f_r-1:\mathcal{O}_{D_1}\otimes_{\mathcal{O}_{Z_1}}\Omega^r_{Z_1}\longrightarrow\mathcal{O}_{D_1}\otimes_{\mathcal{O}_{Z_1}}\Omega^r_{Z_1}$ and is seen easily to be surjective.

(3.7) Now we prove (3.6) (2) in the case where X is smooth over W and has a frobenius. The triangle

$$\mathcal{S}_1(r)_{X,X}\longrightarrow\boldsymbol{J}^{[r]}_{1,X,X}\xrightarrow{f_r-1}\boldsymbol{E}_{1,X,X}$$

induces an exact sequence of cohomology sheaves

$$(3.7.1)\quad \mathcal{H}^{r-1}(\Omega^\cdot_{X_1})\xrightarrow{\delta}\mathcal{H}^r(\mathcal{S}_1(r)_X)\longrightarrow\Omega^r_{X_1,d=0}\xrightarrow{F-1}\mathcal{H}^r(\Omega^\cdot_{X_1}).$$

Here F is the inverse Cartier operator ([11] I, 2.1.4) and $\Omega^r_{X_1,d=0}$ denotes the kernel of $d:\Omega^r_{X_1}\to\Omega^{r+1}_{X_1}$. By [11] I, 2.4.2, the kernel of $F-1$ is generated locally by sections of the form $b_1^{-1}db_1\wedge\cdots\wedge b_r^{-1}db_r$ with $b_1,\cdots,b_r$ invertible. So, for X smooth with frobenius, (3.6) (2) follows from

Lemma (3.7.2). *Let X be a smooth scheme over W with frobenius. Let $0\leqq r<p$. Then;*

(1) *Let $n\geqq 1$. Then the composite*

$$\mathcal{O}^\times_{X_{n+1}}\otimes\cdots\otimes\mathcal{O}^\times_{X_{n+1}}\longrightarrow\mathcal{H}^r(\mathcal{S}_n(r)_X)\longrightarrow\mathcal{H}^r(\boldsymbol{J}^{[r]}_{n,X,X})=\Omega^r_{X_n,d=0}$$

is the map $b_1\otimes\cdots\otimes b_r\mapsto b_1^{-1}db_1\wedge\cdots\wedge b_r^{-1}db_r$.

(2) *Let a be a section of $\mathcal{O}_X$ and let $b_1,\cdots,b_{r-1}$ be local sections of $\mathcal{O}^\times_X$. Then the image of $(1+pa)\otimes b_1\otimes\cdots\otimes b_{r-1}$ in $\mathcal{H}^r(\mathcal{S}_1(r)_X)$ under the*

symbol map coincides with the image of $a^p b_1^{-1} db_1 \wedge \cdots \wedge b_r^{-1} db_{r-1}$ *under the boundary map* $\delta : \mathscr{H}^{r-1}(\Omega^{\cdot}_{X_1}) \to \mathscr{H}^r(\mathscr{S}_1(r)_X)$ (cf. (3.7.1)).

This lemma is immediate from the definition of the symbol map.

(3.8) Next we give the proof of (3.6) (2) for the general case. By (3.7), it suffices to prove the surjectivity of

$$i^{-1}(\mathscr{H}^r(\mathscr{S}_1(r)_Z)) \longrightarrow \mathscr{H}^r(\mathscr{S}_1(r)_X).$$

By $f_r = pf_{r+1} = 0$ on $\boldsymbol{J}^{[r+1]}_{1,X,Z}$ (here we used the condition $r+1 < p$), the map $f_r - 1 : \boldsymbol{J}^{[r]}_{1,X,Z} \to \boldsymbol{E}_{1,X,Z}$ induces an isomorphism of subcomplexes $\boldsymbol{J}^{[r+1]}_{1,X,Z} \xrightarrow[\cong]{} \boldsymbol{J}^{[r+1]}_{1,X,Z}$. Thus we have a distinguished triangle

$$\mathscr{S}_1(r)_{X,Z} \longrightarrow \boldsymbol{J}^{[r]}_{1,X,Z} / \boldsymbol{J}^{[r+1]}_{1,X,Z} \xrightarrow{f_r - 1} \boldsymbol{E}_{1,X,Z} / \boldsymbol{J}^{[r+1]}_{1,X,Z}.$$

From this, we see that, if I denotes the ideal of $\mathcal{O}_{Z_1}$ defining X_1, $\mathscr{H}^r(\mathscr{S}_1(r)_X)$ is isomorphic to the cohomology sheaf in degree r of a complex of the form

$$\cdots \longrightarrow \underset{(\text{deg. } r)}{\Omega^r_{Z_1}/I\Omega^r_{Z_1} \oplus \Omega^{r-1}_{Z_1}/I^2\Omega^{r-1}_{Z_1}} \xrightarrow{(f_r - 1,\ -d)} \underset{(\text{deg. } r+1)}{\Omega^r_{Z_1}/I\Omega^r_{Z_1}} \longrightarrow 0.$$

Let B be the kernel of this map $(f_r - 1, -d)$, and on the other hand, let $A = \mathrm{Ker}(\Omega^r_{Z_1} \oplus \Omega^{r-1}_{Z_1} \xrightarrow{(f_r - 1,\ -d)} \Omega^r_{Z_1})$. We have a commutative diagram with surjective horizontal arrows

$$\begin{array}{ccc} i^{-1}(A) & \twoheadrightarrow & i^{-1}(\mathscr{H}^s(\mathscr{S}_1(r)_Z)) \\ \downarrow & & \downarrow \\ B & \twoheadrightarrow & \mathscr{H}^r(\mathscr{S}_1(r)_X). \end{array}$$

So it suffices to prove that the natural map $i^{-1}(A) \to B$ is surjective, but this follows from the surjectivity of $f_r - 1 : I\Omega^r_{Z_1} \to I\Omega^r_{Z_1}$.

§ 4. *p*-adic vanishing cycles

Let K, X, Y etc. and

$$\bar{Y} \xrightarrow{\bar{i}} \bar{X} \xleftarrow{\bar{j}} X_{\bar{K}}$$

be as at the beginning of Section 0. As in [3] [4], let

$$\bar{M}^r_n = \bar{i}^* R^r \bar{j}_*(\boldsymbol{Z}/p^n\boldsymbol{Z}(r)) \quad (r \in \boldsymbol{Z},\ n \geqq 0)$$

where (r) denotes Tate's twist. The sheaf $\bar{M}^r_n$ was studied in [3] [4] by using "symbols". For any ring R, let

$$K_q^M(R)=(\underbrace{R^\times\otimes\cdots\otimes R^\times}_{q \text{ times}})/J$$

where J is the subgroup of the tensor product generated by all elements of the form $x_1\otimes\cdots\otimes x_q$ such that $x_i+x_j=1$ or $x_i+x_j=0$ for some $i\neq j$. An element $a_1\otimes\cdots\otimes a_q \bmod J$ of $K_q^M(R)$ is denoted by $\{a_1, \cdots, a_q\}$.

Definition (4.1). For a scheme T and $x \in T$, let $\mathcal{O}_{T,\bar{x}}$ be the strict henselization of $\mathcal{O}_{T,x}$. For $x \in \bar{Y}$, let

$$S_x^q=K_q^M\left(\mathcal{O}_{\bar{X},\bar{x}}\left[\frac{1}{p}\right]\right).$$

As in [4], we have a symbol map

$$S_x^q/p^nS_x^q \longrightarrow \bar{M}_{n,\bar{x}}^q$$

($\bar{M}_{n,\bar{x}}^q$ denotes the stalk of $\bar{M}_n^q$ at $\mathrm{Spec}(\mathcal{O}_{\bar{Y},\bar{x}})$). This map is surjective, and it is bijective if x is a generic point of $\bar{Y}$ (see [4] § 5, § 6). On the other hand, we have

Lemma (4.2). *Let $x \in \bar{Y}$. Then the symbol map*

$$K_q^M(\mathcal{O}_{\bar{X},\bar{x}}) \longrightarrow \mathcal{H}^q(\mathcal{S}_n(q)_{\bar{X}})_{\bar{x}} \quad (q<p)$$

(*cf.* (3.2)) *factors through the surjection*

$$K_q^M(\mathcal{O}_{\bar{X},\bar{x}}) \longrightarrow S_x^q/p^nS_x^q.$$

Here the surjectivity of the last map follows from the facts that $(\mathcal{O}_{\bar{X},\bar{x}}[1/p])^\times$ is generated by $(\mathcal{O}_{\bar{X},\bar{x}})^\times$ and $\bar{K}^\times$, and that $\bar{K}^\times/(\bar{K}^\times)^{p^n}=0$. By these facts, (4.2) follows from (3.3).

The purpose of this section is to prove the following

Theorem (4.3). *Let X be as above.*

(1) *For $0\leqq q<p-1$, there exists an isomprohism*

$$\bar{M}_n^q \cong \mathcal{H}^q(\mathcal{S}_n(q)_{\bar{X}})$$

compatible with symbol maps

$$S_x^q \twoheadrightarrow \bar{M}_{n,\bar{x}}^q \quad \textit{and} \quad S_x^q \twoheadrightarrow \mathcal{H}^q(\mathcal{S}_n(q)_{\bar{X}})_{\bar{x}} \quad (x \in \bar{Y}).$$

(2) *If $0\leqq q\leqq r<p-1$, we have a canonical isomorphism*

$$\mathcal{H}^q(\mathcal{S}_n(q)_{\bar{X}})(r-q)\xrightarrow[\cong]{}\mathcal{H}^q(\mathcal{S}_n(r)_{\bar{X}})$$

where $(r-q)$ *means the Tate twist.*

Note that by (3.6),

$$\mathcal{H}^q(\mathcal{S}_n(r)_{\bar{X}})=0 \quad \text{if } r<p \quad \text{and} \quad q>r.$$

In the following, we denote

$$B_n=H^0_{\mathrm{crys}}(\mathrm{Spec}\ (O_{\bar{K}}/p^n O_{\bar{K}})/W_n,\ \mathcal{O}_{\mathrm{crys}})$$

where $\mathcal{O}_{\mathrm{crys}}$ denotes the structural sheaf of the crystalline site,

$J_n=\mathrm{Ker}(B_n\longrightarrow O_{\bar{K}}/p^n O_{\bar{K}})$,

$J_n^{[r]}$ the r-th divided power of J_n (cf. [8] § 3).

Let $\mu_{p^n}(O_{\bar{K}})$ be the group of all p^n-th roots of unity in $O_{\bar{K}}$. Then we have a homomorphism

$$\mu_{p^n}(O_{\bar{K}})\longrightarrow J_n^{[1]};\ \zeta\longmapsto[\zeta]\underset{\mathrm{def}}{=}\log(\tilde{\zeta}^{p^n})$$

where $\tilde{\zeta}$ is any lifting of ζ to $\mathcal{O}_{\mathrm{crys}}$, and f_1 $([\zeta])=[\zeta]$. This induces a map (defined by [10] for the syntomic sheaf S_n^1)

$$\mu_{p^n}(O_{\bar{K}})\longrightarrow H^0(\bar{Y},\ \mathcal{S}_n(1)_{\bar{X}})\subset H^0_{\mathrm{crys}}(\bar{X}_n/W_n,\ \mathcal{O}_{\mathrm{crys}})$$

and hence a canonical homomorphism

$$\mathcal{S}_n(q)_{\bar{X}}(r-q)\longrightarrow\mathcal{S}_n(r)_{\bar{X}} \qquad (0\leqq q\leqq r<p,\ X \text{ quasi-projective}).$$

The isomorphism of (4.3) (2) is induced by this homomorphism. By (4.3) and by the remark after it, we have

Corollary (4.4). *Let* $0\leqq q<p-1$, *and assume that* X *is quasi-projective. Then there exists a distinguished triangle*

$$\mathcal{S}_n(q-1)_{\bar{X}}(1)\longrightarrow\mathcal{S}_n(q)_{\bar{X}}\longrightarrow\bar{M}_n^q[-q].$$

The rest of this section is devoted to the proof of (4.3). We give some remarks on the proof. The problem is etale local, so we may assume that X is isomorphic to the affine space $A^m_{O_K}$, and hence that X is obtained by base change from a smooth scheme over W. Thus we may assume $O_K=W$.

So in the following, we assume that X is a smooth quasi-projective scheme over $O_K=W$. From (4.7) on, we shall also assume that X has a frobenius f without loss of generality.

We begin with

Lemma (4.5). *Define the homomorphism* $\theta\colon O_{\bar{K}}/pO_{\bar{K}}\to B_1$ *by* $\theta(x)=\tilde{x}^p$ *where* $\tilde{x}$ *is any lifting of* x *to* B_1. *Fixing a* p*-th root* $(-p)^{1/p}$ *of* $-p$, *let* $\varepsilon=\theta((-p)^{1/p})$.

(1) *The homomorphism* θ *induces an isomorphism*

$$O_{\bar{K}}/pO_{\bar{K}}\xrightarrow[\cong]{}B_1/J_1^{[p]}.$$

(2) *If* $0\leqq r<p$, $J_1^{[r]}/J_1^{[p]}$ *is generated by* ε^r *as a* B_1*-module.*

(3) *If* $0\leqq r<p$, $f_r(\varepsilon^r)\equiv 1 \bmod J_1^{[p]}$.

(4) *For* $\zeta\in\mu_p(O_{\bar{K}})$, *we have*

$$[\zeta]\equiv-\sum_{i=1}^{p-1}\frac{1}{i}\theta((1-\zeta)^i) \bmod J_1^{[p]}.$$

Proof. (1) and (2) follow from [8] 3.4, and the proof of (4) is straightforward. Finally (3) follows from [9] Lemma 5.4, by the relation described in [8] Section 3 between B_n and the ring $W(R)$ of [9].

Lemma (4.6). *Let* $n\geqq 1$ *and* $0\leqq r<p$. *Then the complex*

$$\underset{\text{(deg. 0)}}{J_n^{[r]}\otimes_{W_n}\mathcal{O}_{X_n}}\xrightarrow{d}\underset{\text{(deg. 1)}}{J_n^{[r-1]}\otimes_{W_n}\Omega^1_{X_n}}\xrightarrow{d}\cdots\xrightarrow{d}\underset{\text{(deg. }q)}{J_n^{[r-q]}\otimes_{W_n}\Omega^q_{X_n}}\longrightarrow\cdots$$

represents $\boldsymbol{J}^{[r]}_{n,\bar{X}}$ *in* $D(\bar{Y}_{\text{et}})$. *Here* d *denotes* $1\otimes d$.

Proof. Let L be a finite extension of K, and let $\operatorname{Spec}(O_L)\to Z$ be an immersion where Z is a smooth W-scheme having a frobenius. Then, $X'=X\otimes_W O_L$ is embedded in $X\times_W Z$. We have

$$D_{X'}(X\times_W Z)\cong X\times_W D_{\operatorname{Spec}(O_L)}(Z)$$

by [2] 3.21. So, $\boldsymbol{J}^{[r]}_{n,X',X\times_W Z}$ is isomorphic to the tensor product of complexes $\boldsymbol{J}^{[r]}_{n,\operatorname{Spec}(O_L),Z}\otimes_{W_n}\Omega^{\bullet}_{X_n}$. By [8] Section 3 Theorem 1, the lemma follows by a limit argument.

From now on, we assume that X has a frobenius f.

Corollary (4.7). *The following complex represents* $\mathcal{S}_n(r)_{\bar{X}}$ *in* $D(\bar{Y}_{\text{et}})$:

$$\underset{\text{(deg. 0)}}{J_n^{[r]}\otimes_{W_n}\mathcal{O}_{X_n}}\xrightarrow{d^0}(J_n^{[r-1]}\otimes_{W_n}\Omega^1_{X_n})\oplus(B_n\otimes_{W_n}\mathcal{O}_{X_n})\xrightarrow{d^1}\cdots$$

$$\xrightarrow{d^{q-1}}(J_n^{[r-q]}\otimes_{W_n}\Omega^q_{X_n})\oplus(B_n\otimes_{W_n}\Omega^{q-1}_{X_n})\xrightarrow{d^q}\cdots.$$

Here each d^q is defined by

$$(x, y) \longmapsto (dx, (f_r - 1)(x) - dy).$$

For a scheme T over F_p, let F: $\Omega_T^q \to \Omega_T^q / d\Omega_T^{q-1}$ be the unique homomorphism which satisfies

$$F(ab_1^{-1}db_1 \wedge \cdots \wedge b_q^{-1}db_q) = a^p b_1^{-1}db_1 \wedge \cdots \wedge b_q^{-1}db_q \bmod d\Omega_T^{q-1}$$

for any local section a of $\mathcal{O}_T$ and for any local sections $b_1, \cdots, b_q$ of $\mathcal{O}_T^\times$ ([11] I. 2.1.4).

Lemma (4.8). *Let $0 \leqq q \leqq r < p-1$. Then, there exists an exact sequence*

$$0 \longrightarrow P \xrightarrow{\alpha} \mathcal{H}^q(\mathcal{S}_1(r)_{\bar{X}}) \xrightarrow{\beta} Q \longrightarrow 0$$

where P and Q are the following sheaves. Let c be the class of $(-p)^{(r-q+1)/p}$ in $O_{\bar{K}}/pO_{\bar{K}}$. Then, P is the cokernel of

$$\Omega_{\bar{X}_1}^{q-1} \longrightarrow \Omega_{\bar{X}_1}^{q-1}/d(\Omega_{\bar{X}_1}^{q-2}); \qquad w \longrightarrow F(w) - cw$$

($\bar{X}_1$ denotes $\bar{X} \otimes Z/pZ$). On the other hand, let

$$S = Y \otimes_k (O_{\bar{K}}/p^{(p-1-r+q)/(p-1)}O_{\bar{K}}).$$

Then, Q is the kernel of $F-1$: $\Omega_S^q \to \Omega_S^q / d\Omega_S^{q-1}$.

Proof. By (4.6), the triangle $\mathcal{S}_1(r)_{\bar{X}} \to J_{1,\bar{X}}^{[r]} \to E_{1,\bar{X}}^{[r]}$ induces an exact sequence

$$0 \longrightarrow P' \longrightarrow \mathcal{H}^q(\mathcal{S}_1(r)_{\bar{X}}) \longrightarrow Q' \longrightarrow 0$$

where

$$P' = \mathrm{Coker}\,(J_1^{[r-q+1]} \otimes_k \Omega_{Y,d=0}^{q-1} \xrightarrow{f_r - 1} B_1 \otimes_k \mathcal{H}^{q-1}(\Omega_Y^\cdot))$$

$$Q' = \mathrm{Ker}\,(J_1^{[r-q]} \otimes_k \Omega_{Y,d=0}^q / J_1^{[r-q+1]} \otimes_k d\Omega_Y^{q-1} \xrightarrow{f_r - 1} B_1 \otimes_k \mathcal{H}^{q-1}(\Omega_Y^\cdot)).$$

We show that there are canonical isomorphisms $P \cong P'$ and $Q \cong Q'$.

First, by $f_r = pf_{r+1} = 0$ on $J_1^{[r-q+2]} \otimes_k \Omega_{Y,d=0}^{q-1}$, we see that

$$P' \cong \mathrm{Coker}\,(J_1^{[r-q+1]}/J_1^{[r-q+2]} \otimes_k \Omega_{Y,d=0}^{q-1} \xrightarrow{f_r - 1} B_1 / J_1^{[r-q+2]} \otimes_k \mathcal{H}^{q-1}(\Omega_Y^\cdot)).$$

Thus P' is a quotient of $B_1/J_1^{[p]} \otimes_k \mathcal{H}^{q-1}(\Omega_Y^\cdot)$, whereas P is a quotient of $\Omega_{\bar{X}_1}^{q-1}$. By using (4.5), it is seen easily that the isomorphism

$$\Omega^{q-1}_{X_1}\cong O_{\bar K}/pO_{\bar K}\otimes_k\Omega^{q-1}_Y\xrightarrow[\cong]{\theta\otimes F}B_1/J_1^{[p]}\otimes_k\mathscr{H}^{q-1}(\Omega^\bullet_Y)$$

induces $P\cong P'$ by passing to the quotients.

Next we prove $Q\cong Q'$. Since $f_r=0$ on $J_1^{[r-q+1]}\otimes_k\Omega^q_{Y,d=0}$, we see

$$Q'\cong\operatorname{Ker}(J_1^{[r-q]}/J_1^{[r-q+1]}\otimes_k\Omega^q_{Y,d=0}\xrightarrow{f_r-1}B_1/J_1^{[r-q+1]}\otimes_k\mathscr{H}^q(\Omega^\bullet_Y)).$$

Let

$$M=J_1^{[r-q]}/J_1^{[r-q+1]}\otimes_k\Omega^q_{Y,d=0},\qquad N=B_1/J_1^{[r-q+1]}\otimes_k(\Omega^q_Y/d\Omega^{q-1}_Y).$$

For any integer $i\geqq 0$, let $a_i=(-p)^{(r-q)(p-1)^{-1}(1-p^{-i})}$, and let M^i (resp. N^i) be the B_1-submodule of M (resp. N) generated by $\theta(a_{i+1})\otimes\Omega^q_{Y,d=0}$ (resp. $\theta(a_i)\otimes(\Omega^q_Y/d\Omega^{q-1}_Y)$). Let M^∞ (resp. N^∞) be the B_1-submodule of M (resp. N) generated by $[\zeta]^{r-q}\otimes\Omega^q_{Y,d=0}$ (resp. $[\zeta]^{r-q}\otimes\Omega^q_Y/d\Omega^{q-1}_Y$) where ζ is a primitive p-th root of unity. Then by (4.5), we have

$$M=M^0\supset M^1\supset M^2\supset\cdots,\qquad M^\infty=\bigcap_{i\geqq 0}M^i,$$
$$N=N^0\supset N^1\supset N^2\supset\cdots,\qquad N^\infty=\bigcap_{i\geqq 0}N^i.$$

We can show easily $(f_r-1)(M^i)\subset N^i$ for $0\leqq i\leqq\infty$.

We prove

$$Q'\cong\operatorname{Ker}(f_r-1\colon M\longrightarrow N)=\operatorname{Ker}(f_r-1\colon M^\infty\longrightarrow N^\infty)\cong Q.$$

Indeed, for $0\leqq i<\infty$, $f_r-1=f_r\colon M^i/M^{i+1}\to N^i/N^{i+1}$ and this map is injective as is easily seen from (4.5). Consequently, $f_r-1\colon M/M^\infty\to N/N^\infty$ is injective and hence $\operatorname{Ker}(f_r-1\colon M\to N)=\operatorname{Ker}(f_r-1\colon M^\infty\to N^\infty)$. On the other hand, by (4.5) we have a commutative diagram

$$\begin{tikzcd}
{[\zeta]^{r-q}\theta(x)\otimes w} & M \arrow[r, "f_r-1"] & N & {[\zeta]^{r-q}\theta(x)\otimes w} \\
x^p\otimes w \arrow[u, mapsto] & R\otimes_k\Omega^q_Y \arrow[u, "\wr"'] \arrow[r] & R\otimes_k\Omega^q_Y/d\Omega^{q-1}_Y \arrow[u, "\wr"'] & x^p\otimes w \arrow[u, mapsto] \\
 & x\otimes w \arrow[r, mapsto] & x^p\otimes F(w)-x\otimes w &
\end{tikzcd}$$

where $R=O_{\bar K}/p^{(p-1-r+q)/(p-1)}O_{\bar K}$. This proves $\operatorname{Ker}(f_r-1\colon M^\infty\to N^\infty)\cong Q$.

In the following, we use the notation

$$e(T)=\sum_{i=0}^{p-1}\frac{T^i}{i!}\in\mathbf{Z}_p[T].$$

Lemma (4.9). (1) $e(T_1+T_2)\equiv e(T_1)e(T_2)\bmod (T_1,T_2)^p$.

(2) $e(-T)\equiv\prod_{i=1}^{p-1}(1-T^i)^{\mu(i)/i} \bmod T^p\boldsymbol{Z}_p[[T]]$ (μ: *Moebius function*).
(3) $e(T^p)e(T)^{-p}\equiv e(-pT) \bmod pT^p\boldsymbol{Z}_p[T]$.

Proof. The congruence (1) is classical. For (2), see [5] III Section 1. We can prove easily the congruence (3) modulo (p) and also modulo (T^p), but $(p)\cap(T^p)=(pT^p)$.

In the following, by fixing a primitive p^n-th root of unity ζ, we call the composite map

$$S_{\bar{x}}^q/p^nS_{\bar{x}}^q\xrightarrow{(4.2)}\mathscr{H}^q(\mathscr{S}_n(q)_{\bar{X}})_{\bar{x}}\xrightarrow{t\mapsto[\zeta]^{r-q}t}\mathscr{H}^q(\mathscr{S}_n(r)_{\bar{X}})_{\bar{x}}$$

$(0\leqq q\leqq r<p)$ the symbol map.

Lemma (4.10). *Let* $0\leqq q\leqq r<p-1$, *and let the exact sequence*

$$0\longrightarrow P\xrightarrow{\alpha}\mathscr{H}^q(\mathscr{S}_1(r)_{\bar{X}})\xrightarrow{\beta}Q\longrightarrow 0$$

be as in (4.8). *Let* $x\in\bar{Y}$.

(1) *The stalk* $\alpha_{\bar{x}}$ *factors as*

$$P_{\bar{x}}\longrightarrow S_{\bar{x}}^q/pS_{\bar{x}}^q\xrightarrow{\text{symbol}}\mathscr{H}^q(\mathscr{S}_1(r)_{\bar{X}})_{\bar{x}}$$

where the first arrow is induced by

$$(4.10.1)\quad \Omega_{\bar{X}_1,\bar{x}}^{q-1}\longrightarrow S_{\bar{x}}^q/pS_{\bar{x}}^q$$

$$ab_1^{-1}db_1\wedge\cdots\wedge b_{q-1}^{-1}db_{q-1}\longmapsto\{e(-\tilde{a}(\zeta-1)^{p-1-r+q}),\ \tilde{b}_1,\ \cdots,\ \tilde{b}_{q-1}\}$$

(ζ *is the fixed primitive p-th root of unity, and* $\tilde{a}$ *and* $\tilde{b}_i$ *denote liftings to* $\mathcal{O}_{\bar{X}}$).

(2) *The composite*

$$S_{\bar{x}}^q/pS_{\bar{x}}^q\xrightarrow{\text{symbol}}\mathscr{H}^q(\mathscr{S}_1(r)_{\bar{X}})_{\bar{x}}\xrightarrow{\beta_{\bar{x}}}Q_{\bar{x}}$$

is the unique map which sends $\{a_1,\cdots,a_q\}$ $(a_1,\cdots,a_q\in(\mathcal{O}_{\bar{X},\bar{x}})^\times)$ *to* $a_1^{-1}da_1\wedge\cdots\wedge a_q^{-1}da_q$.

The proof of (2) is easy and we omit it.

Proof of (1). We first show that the map (4.10.1) is well defined and annihilates $d\Omega_{\bar{X},\bar{x}}^{q-2}$. For any local ring A, the kernel of

$$A\otimes\underbrace{A^\times\otimes\cdots\otimes A^\times}_{q\text{ times}}\longrightarrow\!\!\!\!\rightarrow\Omega^q/d\Omega^{q-1};\ a\otimes b_1\otimes\cdots\otimes b_q\mapsto ab_1^{-1}db_1\wedge\cdots\wedge b_q^{-1}db_q$$

is generated by element of the form $a_0\otimes a_1\otimes\cdots\otimes a_q$ (each $a_i\in A^\times$) such that $a_i=a_j$ for some $i\neq j$ ([4] (4.2)). So, it is sufficient to show

$$\{e(-a(\zeta-1)^{p-1-r+q}),\, a\}=0 \quad \text{in } S_x^2/pS_x^2 \quad (a \in \mathcal{O}_{X,\bar{x}}),$$

but this follows easily from (4.9) (2).

Next, from (4.8) (3), it is deduced that the map (4.10.1) annihilates elements of the form

$$a^p b_1^{-1}db_1\wedge\cdots\wedge b_{q-1}^{-1}db_{q-1}-cab_1^{-1}db_1\wedge\cdots\wedge b_{q-1}^{-1}db_{q-1}$$

($a \in \mathcal{O}_{X_1,\bar{x}}$, $b_1, \cdots, b_q \in \mathcal{O}_{X_1,\bar{x}})^\times$) where c is the class of $(-p)^{(r-q+1)/p}$ (cf. (4.9)). Thus, (4.10.1) induces $P_{\bar{x}}\to S_x^q/pS_x^q$.

Finally we prove that the diagram

$$\begin{tikzcd} P_{\bar{x}} \arrow[rr, "\alpha_{\bar{x}}"] \arrow[dr, "\text{by (4.10.1)}"'] & & \mathscr{H}^q(\mathscr{S}_1(r)_X)_{\bar{x}} \\ & S_x^q/pS_x^q \arrow[ur, "\text{symbol}"'] & \end{tikzcd}$$

is commutative. (The proof is rather long.) By (4.7), $\mathscr{H}^q(\mathscr{S}_1(r)_X)$ is the cohomology sheaf of the complex

$$J_1^{[r-q+1]}\otimes_k \Omega_Y^{q-1}\xrightarrow{(d,\, f_r-1)}(J_1^{[r-q]}\otimes_k \Omega_{Y,d=0}^q)\oplus(B_1\otimes_k \Omega_Y^{q-1}/d\Omega_Y^{q-2})$$
$$\xrightarrow{(f_r-1,\, -d)}B_1\otimes_k \Omega_Y^q.$$

Let $a \in \mathcal{O}_{X,\bar{x}}$ and let $b_1, \cdots, b_{q-1}$ be elements of $(\mathcal{O}_{X,\bar{x}})^\times$ such that $f(b_i)\equiv b_i^p \bmod p^2$ $(1\leqq i\leqq q-1)$. By changing f if necessary, we may assume that $P_{\bar{x}}$ is generated by the classes of $ab_1^{-1}db_1\wedge\cdots\wedge b_{q-1}^{-1}db_{q-1}$ where $a, b_1, \cdots, b_{q-1}$ are as taken here. Let $\tilde{a} \in B_2\otimes_{W_2}\mathcal{O}_{X_2,\bar{x}}$ be a lifting of $a \bmod p^2$, let $t \in B_2$ be a lifting of $(\zeta-1)^{p-1-r+q} \in O_{\bar{K}}/p^2O_{\bar{K}}$ such that $f(t)=t^p$ (for example, we can take as t the p^2-th power of any lifting of $(\zeta-1)^{(p-1-r+q)/p^2} \in O_{\bar{K}}/p^2O_{\bar{K}}$ to B_2), and let

$$u=e(t\tilde{a}) \in (B_2\otimes_{W_2}\mathcal{O}_{X_2,\bar{x}})^\times.$$

By the definition of the product structure (Section 2), the image of $ab_1^{-1}db_1\wedge\cdots\wedge b_{q-1}^{-1}db_{q-1}$ in $\mathscr{H}^q(\mathscr{S}_1(r)_X)_{\bar{x}}$ under

$$P_{\bar{x}}\xrightarrow{(4.10.1)}S_x^q/pS_x^q\xrightarrow{\text{symbol}}\mathscr{H}^q(\mathscr{S}_1(r)_X)_{\bar{x}}$$

coincides with the class of

$$(v,\, w) \in J_1^{[r-q]}\otimes_{\bar{k}} \Omega_{Y,\bar{x},d=0}^q\oplus B_1\otimes_{\bar{k}} \Omega_{Y,\bar{x}}^{q-1}/d\Omega_{Y,\bar{x}}^{q-2}$$

where

$$v=-[\zeta]^{r-q}u^{-1}du\wedge b_1^{-1}db_1\wedge\cdots\wedge b_{q-1}^{-1}db_{q-1},$$
$$w=-[\zeta]^{r-q}(p^{-1}\log(f(u)u^{-p}))b_1^{-1}db_1\wedge\cdots\wedge b_{q-1}^{-1}db_{q-1}.$$

Note that on the other hand, $\alpha_{\bar{x}}(ab_1^{-1}db_1\wedge\cdots\wedge b_{q-1}^{-1}db_{q-1})\in\mathscr{H}^q(\mathscr{S}_1(r)_{\bar{X}})_{\bar{x}}$ coincides with the class of $(0,\theta(a)b_1^{-1}db_1\wedge\cdots\wedge b_{q-1}^{-1}db_{q-1})$. We have easily $v\in J_1^{[r-q+1]}\otimes_k\Omega^q_{\bar{Y},\bar{x},d=0}$ and hence

$$dw=(f_r-1)(v)=(pf_{r+1}-1)(v)=-v.$$

On the other hand, by (4.9) (3) and $f(t)=t^p$, we obtain

$$w\equiv[\zeta]^{r-q}t\tilde{a}b_1^{-1}db_1\wedge\cdots\wedge b_{q-1}^{-1}db_{q-1}\ \bmod J_1^{[p]}\otimes_{\bar{k}}\Omega^{q-1}_{\bar{Y},\bar{x}}/d\Omega^{q-2}_{\bar{Y},\bar{x}}.$$

From this we have

$$w\in J_1^{[r-q+1]}\otimes_{\bar{k}}\Omega^{q-1}_{\bar{Y},\bar{x}}/d\Omega^{q-2}_{\bar{Y},\bar{x}},$$
$$f_r(w)=\theta(a)b_1^{-1}db_1\wedge\cdots\wedge b_{q-1}^{-1}db_{q-1}\qquad\text{in } B_1/J_1^{[p]}\otimes_{\bar{k}}\mathscr{H}^{q-1}(\Omega^{\bullet}_{\bar{Y}})_{\bar{x}}.$$

Since the class of $(dw,(f_r-1)(w))$ in $\mathscr{H}^q(\mathscr{S}_1(r)_{\bar{X}})_{\bar{x}}$ is zero, the class of (v,w) coincides with that of

$$(v,w)+(dw,(f_r-1)(w))=(0,\theta(a)b_1^{-1}db_1\wedge\cdots\wedge b_{q-1}^{-1}db_{q-1}).$$

This completes the proof of (4.10).

Lemma (4.11). *Let $0\leqq q\leqq r<p-1$ and let $x\in\bar{Y}$.*

(1) *The symbol map $S_x^q/p^nS_x^q\to\mathscr{H}^q(\mathscr{S}_n(r)_{\bar{X}})_{\bar{x}}$ is surjective for any $x\in\bar{Y}$, and is bijective if x is a generic point of $\bar{Y}$.*

(2) *In the case $n=1$, more precisely, let U be the subgroup of S_x^q/pS_x^q generated by elements of the form*

$$\{1+a,b_1,\cdots,b_{q-1}\}\quad\bmod pS_x^q;$$
$$a\in\operatorname{Ker}(\mathcal{O}_{\bar{X},\bar{x}}\to\mathcal{O}_{\bar{Y},\bar{x}}),\qquad b_1,\cdots,b_{q-1}\in\left(\mathcal{O}_{\bar{X},\bar{x}}\left[\frac{1}{p}\right]\right)^{\times}.$$

Then the restriction of the symbol map $S_x^q/pS_x^q\to\mathscr{H}^q(\mathscr{S}_1(r)_{\bar{X}})_{\bar{x}}$ to U is injective, and we have a commutative diagram

$$\begin{array}{cccc}
\{\tilde{a}_1,\cdots,\tilde{a}_q\} & (S_x^q/p)/U & \xrightarrow{\text{symbol}} & \mathscr{H}^q(\mathscr{S}_1(r)_{\bar{X}})_{\bar{x}}/\operatorname{Im}(U)\\
\uparrow & \wr\| & & \wr\|\\
\{a_1,\cdots,a_q\} & K_q^M(\mathcal{O}_{\bar{Y},\bar{x}})/p & \xrightarrow{d\log} & \nu(q)_{\bar{Y},\bar{x}}
\end{array}$$

whose vertical arrows are bijective. Here $\nu(q)_{\bar{Y},\bar{x}}$ denotes the kernel of

$$F-1\colon \Omega^q_{Y,\bar{x}} \longrightarrow \Omega^q_{Y,\bar{x}}/d\Omega^{q-1}_{Y,\bar{x}},$$

and $d\log$ *denotes the map*

$$\{a_1, \cdots, a_q\} \longmapsto a_1^{-1}da_1 \wedge \cdots \wedge a_q^{-1}da_q.$$

Here, in the diagram in (2), the existence of the left vertical isomorphism is easily seen. Note that the map $d\log$ in this diagram is surjective by [11] I, 2.4.2, and it is bijective if x is a generic point of $\bar{Y}$ by [4] Section 2.

Remark (4.12). O. Gabber (unpublished) proved that $d\log$ is bijective for any x (he proved $K_q^M(A)/p \cong \nu(q)_A$ for quite general A including $\mathcal{O}_{\bar{Y},\bar{x}}$ as above). If we use his result, we have $S_{\bar{x}}^q/p^n S_{\bar{x}}^q \cong \mathcal{H}^q(\mathcal{S}_n(r)_{\bar{X}})_{\bar{x}}$ improving (4.11) which will simplify the arguments in the rest of this section.

Proof of (4.11). By the diagram

$$\begin{tikzcd}
 & S_{\bar{x}}^q/p^{n-1} \arrow[r, "p"] \arrow[d] & S_{\bar{x}}^q/p^n \arrow[r] \arrow[d] & S_{\bar{x}}^q/p \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \mathcal{H}^q(\mathcal{S}_{n-1}(r)_{\bar{X}})_{\bar{x}} \arrow[r] & \mathcal{H}^q(\mathcal{S}_n(r)_{\bar{X}})_{\bar{x}} \arrow[r] & \mathcal{H}^q(\mathcal{S}_1(r)_{\bar{X}})_{\bar{x}} \arrow[r] & \cdots,
\end{tikzcd}$$

(1) is reduced to the case $n=1$ and hence to (2).

For a rational number $s>0$, let U_s be the subgroup of $S_{\bar{x}}^q/pS_{\bar{x}}^q$ generated by elements of the form

$$\{1+p^s a, b_1, \cdots, b_{q-1}\} \mod pS_{\bar{x}}^q;$$
$$a \in \mathcal{O}_{\bar{X},\bar{x}}, \qquad b_1, \cdots, b_{q-1} \in \left(\mathcal{O}_{\bar{X},\bar{x}}\left[\frac{1}{p}\right]\right)^\times.$$

Then, (4.10) shows that for $s=(p-1-r+q)/(p-1)$, we have the following commutative diagram of exact sequences in which the left vertical arrow is bijective.

$$\begin{tikzcd}
0 \arrow[r] & U_s \arrow[r] \arrow[d, equal, "\wr"'] & S_{\bar{x}}^q/p \arrow[r] \arrow[d] & (S_{\bar{x}}^q/p)/U_s \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_{\bar{x}} \arrow[r] & \mathcal{H}^q(\mathcal{S}_1(r)_{\bar{X}})_{\bar{x}} \arrow[r] & Q_{\bar{x}} \arrow[r] & 0
\end{tikzcd}$$

Furthermore, the right vertical arrow is surjective by [11] I, 2.4.2. So it remains to prove that $U/U_s \longrightarrow Q_{\bar{x}}$ $(s=(p-1-r+q)/(p-1))$ is injective and that the composite map $\mathcal{H}^q(\mathcal{S}_1(r)_{\bar{X}})_{\bar{x}} \rightarrow Q_{\bar{x}} \rightarrow \nu(q)_{Y,\bar{x}}$ induces an isomorphism $\mathcal{H}^q(\mathcal{S}_1(r)_{\bar{X}})_{\bar{x}}/\mathrm{Im}\,(U) \cong \nu(q)_{Y,\bar{x}}$. The former fact is reduced to the following (4.13). For the latter fact, it suffices to prove that the map

$U \to \mathrm{Ker}(Q_{\bar{x}} \to \nu(q)_{\bar{Y},\bar{x}})$ is surjective, and this is reduced to the following (4.14) (note that $O_{\bar{K}}/p^s O_{\bar{K}}$ is an inductive limit of artinian local rings).

Lemma (4.13). *Let $x \in \bar{Y}$. For a rational number s such that $0<s \leq 1$, let $A_s = \mathcal{O}_{\bar{X},\bar{x}}/p^s\mathcal{O}_{\bar{X},\bar{x}}$. Let U and U_s be as above. Then, the restriction of the map*

$$(S_{\bar{x}}^q/p)/U_s \longrightarrow \Omega_{A_s}^q; \qquad \{a_1, \cdots, a_q\} \longmapsto a_1^{-1}da_1 \wedge \cdots \wedge a_q^{-1}da_q$$

$(a_1, \cdots, a_q \in (\mathcal{O}_{\bar{X},\bar{x}})^\times)$ to U/U_s is injective.

Proof. It is sufficient to prove that for $0<s\leq 1/2$, the induced map $U_s/U_{2s} \to \Omega_{A_{2s}}^q$ is injective. Let $\Omega_{A_s}^{q-1} \to U_s/U_{2s}$ be the surjective homomorphism

$$ab_1^{-1}db_1 \wedge \cdots \wedge b_{q-1}^{-1}db_{q-1} \longmapsto \{1+p^s\tilde{a}, \tilde{b}_1, \cdots, \tilde{b}_{q-1}\}.$$

Then, this map annihilates $\Omega_{A_s,d=0}^{q-1}$. Indeed, $\Omega_{A_s,d=0}^{q-1}$ is generated by $d\Omega_{A_s}^{q-2}$ and elements of the form $a^p b_1^{-1}db_1 \wedge \cdots \wedge b_{q-1}^{-1}db_{q-1}$, but

$$1+p^s a^p = (1+p^{s/p}a)^p \qquad \text{in } A_{2s}.$$

Now the composite map

$$\Omega_{A_s}^{q-1}/\Omega_{A_s,d=0}^{q-1} \twoheadrightarrow U_s/U_{2s} \longrightarrow p^s\Omega_{A_{2s}}^q \xrightarrow[\cong]{p^{-s}} \Omega_{A_s}^q$$

is just $w \mapsto dw$, and hence the second arrow is injective.

Lemma (4.14). *Let R be an artinian local ring over $\mathbf{F}_p$ and let A be a local ring which is essentially smooth over R. Then, for any ideal I of R such that $R \neq I$, the sequence*

$$(1+IA)\otimes(A^\times)^{\otimes(q-1)} \longrightarrow I\Omega_{A/R}^q \xrightarrow{F-1} \Omega_{A/R}^q/d\Omega_{A/R}^{q-1}$$

is exact, where the first map is given by $a_1\otimes\cdots\otimes a_q \mapsto a_1^{-1}da_1 \wedge \cdots \wedge a_q^{-1}da_q$.

Proof. We may assume $I^2=0$. Then, $F\colon I\Omega_{A/R}^q \to \Omega_{A/R}^q/d\Omega_{A/R}^{q-1}$ is the zero map, and $I\Omega_{A/R}^q \cap d\Omega_{A/R}^{q-1} = I\cdot d\Omega_{A/R}^{q-1}$ as $d\Omega_{A/R}^{q-1}$ is an R-direct summand of $\Omega_{A/R}^q$. But for $t \in I$ and $a \in A$,

$$\begin{aligned} &t\cdot d(ab_1^{-1}db_1 \wedge \cdots \wedge b_{q-1}^{-1}db_{q-1}) \\ &\quad = (1+ta)^{-1}d(1+ta) \wedge b_1^{-1}db_1 \wedge \cdots \wedge b_{q-1}^{-1}db_{q-1}. \end{aligned}$$

Now we can prove Theorem (4.3). Let S be the set of generic points of $\bar{Y}$, and let $\tau\colon \coprod_{\nu\in S}\mathrm{Spec}(\kappa(\nu)) \to \bar{Y}$ be the canonical morphism. By [4]

Section 5, the symbol map $S^q_x/p^n S^q_x \to \bar{M}^q_{n,\bar{x}}$ is bijective if $x \in S$. So, by (4.11) (1), we have an isomorphism

$$\tau^*(\mathscr{H}^q(\mathscr{S}_n(r)_{\bar{X}})) \cong \tau^*(\bar{M}^q_n)$$

(we neglect the Tate twist by fixing a primitive p^n-th root of unity). This defines a homomorphism $\mathscr{H}^q(\mathscr{S}_n(r)_{\bar{X}}) \to \tau_* \tau^*(\bar{M}^q_n)$. By [4] Section 6, the canonical map $\bar{M}^q_n \to \tau_* \tau^*(\bar{M}^q_n)$ is injective, and the diagram

$$\begin{tikzcd} S^q_x \arrow[r, two heads] \arrow[d] & \mathscr{H}^q(\mathscr{S}_n(r)_{\bar{X}})_{\bar{x}} \arrow[d] \\ \bar{M}^q_{n,\bar{x}} \arrow[r] & \tau_* \tau^*(\bar{M}^q_n)_{\bar{x}} \end{tikzcd} \qquad (x \in \bar{Y})$$

shows that the image of $\mathscr{H}^q(\mathscr{S}_n(r)_{\bar{X}})$ in $\tau_* \tau^*(\bar{M}^q_n)$ is contained in the image of $\bar{M}^q_n$. Thus we obtain a homomorphism

$$\mathscr{H}^q(\mathscr{S}_n(r)_{\bar{X}}) \longrightarrow \bar{M}^q_n$$

which is compatible with the symbol maps. We prove that this is an isomorphism. For this, we may assume $n=1$. By [4] Section 4 and Section 6, the composite $U \to \mathscr{H}^q(\mathscr{S}_1(r)_{\bar{X}})_{\bar{x}} \to \bar{M}^q_{1,\bar{x}}$ is injective ($x \in \bar{Y}$ and U is as in (4.11)). So the commutative diagram

$$\begin{tikzcd} \mathscr{H}^q(\mathscr{S}_1(r)_{\bar{X}})_{\bar{x}}/\mathrm{Im}\,(U) \arrow[dr, "\cong"] \arrow[d] & \\ \bar{M}^q_{1,\bar{x}}/\mathrm{Im}\,(U) \arrow[r, "\cong", "\text{[4] Section 6}"'] & \nu(q)_{Y,\bar{x}} \end{tikzcd}$$

proves $\mathscr{H}^q(\mathscr{S}_1(r)_{\bar{X}})_{\bar{x}} \xrightarrow[\cong]{} \bar{M}^q_{1,\bar{x}}$.

Chapter II Application to Schemes with Hodge-Witt Reduction

§ 1. Varieties of Hodge-Witt

As in [12] IV (4.6), a smooth proper variety Y over k is called of Hodge-Witt if the W-modules $H^q(Y, W\Omega^r_Y)$ are of finite type over W. Then, Y is of Hodge-Witt if and only if the spectral sequence

$$E_1^{r,q} = H^q(Y, W\Omega^r_Y) \Longrightarrow H^*_{\mathrm{crys}}(Y/W)$$

degenerates.

In this section (resp. the next section), for a smooth proper scheme Y over k (resp. a smooth projective scheme X over W of dimension $\leq p$), we give a sufficient and necessary condition for Y (resp. for $X\otimes_W k$) to be of Hodge-Witt, in terms of F-gauges (resp. filtered Dieudonné modules) associated to Y (resp. to X). (See Theorem (1.16) and Theorem (2.15).)

The theory of F-gauges described below was developed by Ekedahl and Fontaine. All results of this section are essentially contained in Ekedahl [17], but since some definitions and results (e.g. the part concerning $GH^m(Y/k)$) seem not explicitly given in [17], we give the self-contained introduction to their theory and a complete proof of (1.16).

(1.1) An F-gauge over k is a graded W-module $D=\oplus_{j\in Z} D_i$ endowed with W-linear maps $\tilde{F}: D\to D$ and $\tilde{V}: D\to D$ such that $\tilde{F}$ is of degree 1 and $\tilde{V}$ is of degree -1 satisfying $\tilde{F}\cdot\tilde{V}=\tilde{V}\cdot\tilde{F}=p$, and with a σ-linear isomorphism $\varphi: D_\infty \xrightarrow{\cong} D_{-\infty}$ where

$$D_\infty=\varinjlim (D_i \xrightarrow{\tilde{F}} D_{i+1} \xrightarrow{\tilde{F}} \cdots)$$
$$D_{-\infty}=\varinjlim (D_i \xrightarrow{\tilde{V}} D_{i-1} \xrightarrow{\tilde{V}} \cdots).$$

F-gauges over k form an abelian category which we denote by $\boldsymbol{FG}_k$.

(1.2) Let Y be a smooth proper variety over k. For $m\in \boldsymbol{Z}$ and $n\geqq 1$, we define the F-gauge $GH^m(Y/W_n)$ as follows. For $i\in \boldsymbol{Z}$, let $W_n\Omega_Y^{\cdot}(i)$ be the complex on Y

$$(W_n\mathcal{O}_Y)^{(\sigma)} \xrightarrow{d} \cdots \xrightarrow{d} (W_n\Omega_Y^{i-1})^{(\sigma)} \xrightarrow{d} ((W_n\Omega_Y^i)')^{(\sigma)} \xrightarrow{d\circ C} W_n\Omega_Y^{i+1}$$
$$\xrightarrow{d} W_n\Omega_Y^{i+2} \xrightarrow{d} \cdots$$

where $(W_n\Omega_Y^i)'$ is the image of F: $W\Omega_Y^i\to W_n\Omega_Y^i$ and C: $(W_n\Omega_Y^i)'\to W_n\Omega_Y^i$ is the unique map for which the diagram

$$\begin{tikzcd} W\Omega_Y^i \arrow[r, "\text{proj.}"] \arrow[d, "F"'] & W_n\Omega_Y^i \arrow[d, "``p\text{''}"] \\ (W_n\Omega_Y^i)' \arrow[ur, "C"] \arrow[r, "V"'] & W_{n+1}\Omega_Y^i \end{tikzcd}$$

is commutative. (The existence of C follows from the injectivity of "p".) The F-gauge $D=GH^m(Y/W_n)$ is defined as follows:

$$D_i=\boldsymbol{H}^m(Y,\ W_n\Omega_Y^{\cdot}(i)) \qquad \text{for } i\in \boldsymbol{Z},$$

$\tilde{F}$: $D_i\to D_{i+1}$ (resp. $\tilde{V}$: $D_i\to D_{i-1}$) is the map induced by the following homomorphism of complexes

$$\begin{tikzcd}
(W_n\mathcal{O}_Y)^{(\sigma)} \arrow[r, "d"] \arrow[d, "1"] & \cdots \arrow[r, "d"] & ((W_n\Omega_Y^i)')^{(\sigma)} \arrow[r, "d\circ C"] \arrow[d, "1"] & W_n\Omega_Y^{i+1} \arrow[r, "d"] \arrow[d, "pF"] & W_n\Omega_Y^{i+2} \arrow[r, "d"] \arrow[d, "p"] & W_n\Omega_Y^{i+3} \arrow[r] \arrow[d, "p"] & \cdots \\
(W_n\mathcal{O}_Y)^{(\sigma)} \arrow[r, "d"] & \cdots \arrow[r, "d"] & (W_n\Omega_Y^i)^{(\sigma)} \arrow[r, "d"] & ((W_n\Omega_Y^{i+1})')^{(\sigma)} \arrow[r, "d\circ C"] & W_n\Omega_Y^{i+2} \arrow[r, "d"] & W_n\Omega_Y^{i+3} \arrow[r] & \cdots
\end{tikzcd}$$

(resp.

$$\begin{tikzcd}
(W_n\mathcal{O}_Y)^{(\sigma)} \arrow[r, "d"] \arrow[d, "p"] & \cdots \arrow[r, "d"] & (W_n\Omega_Y^{i-1})^{(\sigma)} \arrow[r, "d"] \arrow[d, "p"] & ((W_n\Omega_Y^i)')^{(\sigma)} \arrow[r, "d\circ C"] \arrow[d, "C"] & W_n\Omega_Y^{i+1} \arrow[r, "d"] \arrow[d, "1"] & W_n\Omega_Y^{i+2} \arrow[r, "d"] \arrow[d, "1"] & \cdots \\
(W_n\mathcal{O}_Y)^{(\sigma)} \arrow[r, "d"] & \cdots \arrow[r, "d"] & ((W_n\Omega_Y^{i-1})')^{(\sigma)} \arrow[r, "d\circ C"] & W_n\Omega_Y^i \arrow[r, "d"] & W_n\Omega_Y^{i+1} \arrow[r] & W_n\Omega_Y^{i+2} \arrow[r, "d"] & \cdots)
\end{tikzcd}$$

and $\varphi\colon D_\infty \to D_{-\infty}$ is induced by the identity map $(W_n\Omega_Y^\cdot)^{(\sigma)} \to W_n\Omega_Y^\cdot$.

In the case $n=1$, we denote $GH^m(Y/W_1)$ by $GH^m(Y/k)$. We denote by $GH^m(Y/W)$ the F-gauge D such that $D_i = \varprojlim_n \boldsymbol{H}^m(Y, W_n\Omega_Y^\cdot(i))$ with $\tilde{F}$, $\tilde{V}$ and φ obtained as the inverse limits of those of $GH^m(Y/W_n)$. We remark that the procomplex "$\varprojlim$" $W_n\Omega_X^\cdot(i)$ is isomorphic to $W\Omega^\cdot(i, 1)$ of Nygaard ([12] III § 3).

Lemma (1.3). *The map $W_n\Omega_Y^\cdot(i) \xrightarrow{p} W_{n+1}\Omega_Y^\cdot(i)$ is injective and the canonical map $W_{n+1}\Omega_Y^\cdot(i)/pW_{n+1}\Omega_Y^\cdot(i) \to \Omega_Y^\cdot(i)$ is a quasi-isomorphism. So, in the derived category, we have a distinguished triangle*

$$W_n\Omega_Y^\cdot(i) \xrightarrow{p} W_{n+1}\Omega_Y^\cdot(i) \longrightarrow \Omega_Y^\cdot(i).$$

This is proved just as the case of $W_n\Omega_Y^\cdot \xrightarrow{p} W_{n+1}\Omega_Y^\cdot \longrightarrow \Omega_Y^\cdot$ given in [11] I (3.13).

Corollary (1.4). (1) *There are long exact sequences*

$$\cdots \longrightarrow GH^m(Y/W_n) \longrightarrow GH^m(Y/W_{n+1}) \longrightarrow GH^m(Y/k)$$
$$\longrightarrow GH^{m+1}(Y/W_n) \longrightarrow GH^{m+1}(Y/W_{n+1}) \longrightarrow GH^{m+1}(Y/k) \longrightarrow \cdots,$$
$$\cdots \longrightarrow GH^m(Y/W) \xrightarrow{p} GH^m(Y/W) \longrightarrow GH^m(Y/k)$$
$$\longrightarrow GH^{m+1}(Y/W) \xrightarrow{p} GH^{m+1}(Y/W) \longrightarrow GH^{m+1}(Y/k) \longrightarrow \cdots.$$

(2) *Each graded component $GH^m(Y/W)_i$ ($i \in \boldsymbol{Z}$) is a finitely generated W-module.*

Here (2) follows from the second exact sequence in (1).

Now we give the definition of an F-gauge of Hodge-Witt. We say that an interval I in $\boldsymbol{R}$ is integral if any endpoint of I is an integer. (The intervals $[r, \infty)$ and (r, ∞) for $r \in \boldsymbol{Z}$ are integral as they have unique endpoint r.)

Definition (1.5). Let I be a closed integral interval. As in [17] II 2.1

(ii), we say that an F-gauge D is of level I if $\tilde{F}\colon D_i \to D_{i+1}$ is an isomorphism for any $i \geqq \sup(I)$ and $\tilde{V}\colon D_i \to D_{i-1}$ is an isomorphism for any $i \leqq \inf(I)$.

Definition (1.6). We say that an F-gauge D is of Hodge-Witt if the following two conditions are satisfied.

(1.6.1) Each D_i ($i \in \boldsymbol{Z}$) is a W-module of finite type.

(1.6.2) As an F-gauge, D is isomorphic to the direct sum of a finite family of F-gauges each of whose members is of level $[i, i+1]$ for some $i \in \boldsymbol{Z}$ (i may depend on each member).

We denote by $\boldsymbol{FG}_{k,HW}$ the full subcategory of $\boldsymbol{FG}_k$ consisting of all F-gauges of Hodge-Witt. This subcategory has the following nice properties.

Proposition (1.7). (1) *If D and D' are F-gauges of Hodge-Witt and $h\colon D \to D'$ is a homomorphism of F-gauges, the F-gauges* Ker(h) *and* Coker(h) *are F-gauges of Hodge-Witt.*

(2) *Let $0 \to D' \to D \to D'' \to 0$ be an exact sequence of F-gauges and assume that D' and D'' are of Hodge-Witt. Then D is of Hodge-Witt.*

Corollary (1.8). *The category $\boldsymbol{FG}_{k,HW}$ is abelian.*

For the proof of (1.7), we give some preliminaries.

Definition (1.9). A Dieudonné module over k is a W-module L having a σ-linear homomorphism $F\colon L \to L$ and a σ^{-1}-linear homomorphism $V\colon L \to L$ such that $F \circ V = V \circ F = p$.

To a Dieudonné module L, we associate an F-gauge D as follows:

$$D_i = L \quad \text{if } i \leqq 0, \quad D_i = L^{(\sigma)} \quad \text{if } i \geqq 1,$$

$\tilde{F}\colon D_i \to D_{i+1}$ is $p\colon L \to L$ if $i<0$ (resp. is F; $L \to L^{(\sigma)}$ if $i=0$, resp. is the identity map $L^{(\sigma)} \to L^{(\sigma)}$ if $i>0$),

$\tilde{V}\colon D_{i+1} \longrightarrow D_i$ is the identity map $L \longrightarrow L$ if $i<0$

(resp. is $V\colon L^{(\sigma)} \to L$ if $i=0$, resp. is $p\colon L^{(\sigma)} \to L^{(\sigma)}$ if $i>0$),

$\varphi\colon D_{\infty} \longrightarrow D_{-\infty}$ is the identity map $L^{(\sigma)} \longrightarrow L$.

We see easily

Lemma (1.10). *The above correspondence induces an equivalence between the category of Dieudonné modules over k and the category of F-gauges over k of level* $[0, 1]$.

By the theory of Dieudonné modules (cf. [5] Ch. III for example), we have

Corollary (1.11). *Let D be an F-gauge of Hodge-Witt. Then, D has a direct decomposition in* $\boldsymbol{FG}_k$

$$(1.11.1) \qquad D=(\bigoplus_{i\in Z} D^{[i]})\oplus(\bigoplus_{i\in Z} D^{(i,i+1)})$$

such that for each $i \in \boldsymbol{Z}$, $D^{[i]}$ is an F-gauge of level $[i, i]$ and the translation $D^{(i,i+1)}[i]$ is the F-gauge corresponding to a Dieudonné module L (under the correspondence (1.10)) such that $F: L\to L$ and $V: L\to L$ are topologically nilpotent for the p-adic topology. (The translation $D[m]$ of an E-gauge D is defined in the evident way.)

Definition (1.12). Let D be an F-gauge of Hodge-Witt, and let I be an integral interval which need not be closed. We say that D is of level I if D has a direct decomposition as (1.11.1) such that $D^{[i]}=0$ unless $i \in I$ and $D^{(i,i+1)}=0$ unless $(i, i+1)\subset I$.

By (1.11), for a closed integral interval, the definition (1.12) coincides with (1.5).

Lemma (1.13). *Let I be an integral interval and J a closed integral interval such that $I\cap J=\phi$. Let D be an F-gauge of Hodge-Witt of level I such that each D_i is of finite length over W. On the other hand, let D' be an F-gauge of level J. Then*

$$\mathrm{Hom}(D, D')=(0), \qquad \mathrm{Hom}(D', D)=(0),$$
$$\mathrm{Ext}^1(D, D')=(0), \qquad \mathrm{Ext}^1(D', D)=(0).$$

Here Hom *and* Ext^1 *are taken with respect to the abelian category* $\boldsymbol{FG}_k$.

Proof. By translation and by the theory of Dieudonné modules, it suffices to consider the following two cases.

(1) $I=[0, 0]$.

(2) $I=(0, 1)$ and the Dieudonné module corresponding to D satisfies $F=V=0$.

In these cases, the proof of Hom$=(0)$ is straightforward, and so we consider the proof of $\mathrm{Ext}^1=(0)$. We give here the proof of $\mathrm{Ext}^1(D, D')=(0)$ assuming $I=(0, 1)$, $J=[1, \infty)$ and assuming that the Dieudonné module corresponding to D satisfies $F=V=0$. The proofs for the other cases are similar and left to the reader.

Consider an exact sequence of F-gauges

$$0\longrightarrow D'\longrightarrow E\xrightarrow{g} D\longrightarrow 0.$$

Our task is to define a section $D\to E$. The commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D_1' & \longrightarrow & E_1 & \xrightarrow{g_1} & D_1 & \longrightarrow & 0 \\ & & {\scriptstyle \tilde{V}}\downarrow \wr\!\! \| & & {\scriptstyle \tilde{V}}\downarrow & & {\scriptstyle 0}\downarrow & & \\ 0 & \longrightarrow & D_0' & \longrightarrow & E_0 & \xrightarrow[g_0]{} & D_0 & \longrightarrow & 0 \end{array}$$

shows that g_1 induces an isomorphism

$$\mathrm{Ker}(\tilde{V}\colon E_1 \longrightarrow E_0) \xrightarrow{\cong} D_1.$$

Let $h_1\colon D_1 \to E_1$ be the inverse of this isomorphism. For $i \geqq 1$, let $h_i\colon D_i \to E_i$ be the composite

$$D_i \underset{\cong}{\xleftarrow{\tilde{F}^{i-1}}} D_1 \xrightarrow{h_1} E_1 \xrightarrow{\tilde{F}^{i-1}} E_i.$$

Let $h_\infty\colon D_\infty \to E_\infty$ be the inductive limit of $(h_i)_{i\geqq 1}$ and let $h_{-\infty}\colon D_{-\infty} \to E_{-\infty}$ be the unique homomorphism such that $\varphi_E \circ h_\infty = h_{-\infty} \circ \varphi_D$. For $i \leqq 0$, we define $h_i\colon D_i \to E_i$ to be the composite

$$D_i \xrightarrow{\text{by } \tilde{V}\text{'s}} D_{-\infty} \xrightarrow{h_{-\infty}} E_{-\infty} \underset{\cong}{\xleftarrow{\text{by } \tilde{V}\text{'s}}} E_i.$$

Then, it is easily seen that $(h_i)_{i \in \boldsymbol{Z}}$ is a section of $g\colon E \to D$.

Corollary (1.14). *Let I and J be integral intervals such that $I \cap J = \phi$ and let D (resp. D') be an F-gauge of Hodge-Witt of level I (resp. J). Then,*

$$\mathrm{Hom}(D, D') = (0), \qquad \mathrm{Ext}^1(D, D') = (0).$$

Proof. This follows from (1.13) applied to D/p^nD and D'/p^nD' $(n \geqq 1)$ by the inverse limit argument.

Corollary (1.15). *For an F-gauge D of Hodge-Witt the direct decomposition* (1.11.1) *is unique.*

Now, Proposition (1.7) is a consequence of (1.14).

Theorem (1.16). *Let Y be a smooth proper variety over k. Then the following four conditions are equivalent.*

(i) *Y is of Hodge-Witt.*
(ii) *The F-gauges $GH^m(Y/k)$ are of Hodge-Witt for all m.*
(iii) *The F-gauges $GH^m(Y/W_n)$ are of Hodge-Witt for all m and n.*
(iv) *The F-gauges $GH^m(Y/W)$ are of Hodge-Witt for all m.*

Proof. For $q, r \in \boldsymbol{Z}$, $H^q(Y, W\Omega_Y^r)$ (resp. $H^q(Y, W\Omega_Y^r/p)$ where $W\Omega_Y^r/p = W\Omega_Y^r/pW\Omega_Y^r$) is a Dieudonné module with the usual operators F and V.

By writting the corresponding F-gauge (1.10) by D, we define $A_Y^{q,r}$ (resp. $B_Y^{q,r}$) to be the translation $D[-r]$. By the definitions of $GH^m(Y/W)$ and $GH^m(Y/k)$, and by (1.3), we have spectral sequences

$$(1.17.1) \qquad E_1^{r,q}=A_Y^{q,r} \Longrightarrow GH^*(Y/W)$$

$$(1.17.2) \qquad E_1^{r,q}=B_Y^{q,r} \Longrightarrow GH^*(Y/k).$$

Assume now that Y is of Hodge-Witt. From the fact that the slope spectral sequence of Y degenerates, it is easily deduced that the spectral sequence (1.17.1) degenerates. Consequently, in $\boldsymbol{FG}_k$, $GH^m(Y/W)$ has a filtration whose graded quotients are isomorphic to $A_Y^{q,m-q}$ $(0\leqq q\leqq m)$. By (1.7) (2), this shows that $GH^m(Y/W)$ is of Hodge-Witt. Next, the equivalence (ii)$\Leftrightarrow$(iii) and the implication (iv)$\Rightarrow$(ii) follow from (1.4) and (1.7).

Finally we prove (ii)$\Rightarrow$(i). To prove that $H^q(Y, W\Omega_Y^r)$ are of finite type for all q and r (i.e. that Y is of Hodge-Witt), it is sufficient to show that $H^q(Y, W\Omega_Y^r/p)$ are finite dimensional over k for all q and r. Let $W\Omega_Y^{\geqq r}(i)$ be the (degree $\geqq r$)-part of $W\Omega_Y^{\cdot}(i)$ (so, the degree j part of $W\Omega_Y^{\geqq r}(i)$ is that of $W\Omega_Y^{\cdot}(i)$ if $j\geqq r$, and is zero if $j<r$). Let $B_Y^{q,\geqq r}$ be the F-gauge whose i-th graded component is $\boldsymbol{H}^q(Y, W\Omega_Y^{\geqq r}(i)/p)$ and whose $\tilde{F}$, $\tilde{V}$ and φ are defined just as in the definition of $GH^q(Y/W)$. Then, $B_Y^{q,\geqq r}$ is of level $[r, \infty)$, and we have a long exact sequence

$$\cdots \longrightarrow B_Y^{q,\geqq r+1} \longrightarrow B_Y^{q,\geqq r} \longrightarrow B_Y^{q,r} \longrightarrow B_Y^{q+1,\geqq r+1} \longrightarrow \cdots.$$

We prove the following statements by induction on r.

(S_r) *$B_Y^{q,\geqq r}$ are of Hodge-Witt for all q.*

(S'_r) *$B_Y^{q,r}$ are of Hodge-Witt for all q.*

Note that (S'_r) is sufficient for the proof of the finiteness of $H^q(Y, W\Omega_Y^r/p)$. By assumption, (S_0) is true. We prove that (S_r) implies (S_{r+1}) and (S'_r). Let $B_Y^{q,\geqq r}=P^q\oplus Q^q$ be the direct decomposition such that P^q (resp. Q^q) is of level $[r+1, \infty)$ (resp. $[r, r+1)$).

Claim (1.17.3). The image of $B_Y^{q,\geqq r+1}\to B_Y^{q,\geqq r}$ coincides with P^q.

Indeed, the composite $B_Y^{q,\geqq r+1}\to B_Y^{q,\geqq r}\xrightarrow{\text{proj.}}Q^q$ is zero by (1.13), and hence it is sufficient for the proof of (1.17.3) to show the composite $P^q\to B_Y^{q,\geqq r}\to B_Y^{q,r}$ is zero. The graded k-module $\oplus_{i\in \boldsymbol{Z}}(B_Y^{q,r})_i/V^n(B_Y^{q,r})_i$ is a quotient F-gauge of $B_Y^{q,r}$, and is of Hodge-Witt of level $[r, r+1)$. Since each $(B_Y^{q,r})_i/V^n(B_Y^{q,r})_i$ is finite dimensional over k and $(B_Y^{q,r})_i=\varprojlim_n (B_Y^{q,r})_i/V^n(B_Y^{q,r})_i$, the vanishing of $P^q\to B_Y^{q,r}$ follows from (1.13).

Now, by (1.17.3) we have an exact sequence of F-gauges

$$(1.17.4) \qquad 0\longrightarrow Q^q\longrightarrow B_Y^{q,r}\longrightarrow B_Y^{q+1,\geqq r+1}\longrightarrow P^{q+1}\longrightarrow 0.$$

Consider the following diagram of exact sequences induced by the maps $\tilde{V}$ of the F-gauges in (1.17.4)

(1.17.5)

$$\begin{tikzcd}
0 \arrow[r] & (Q^q)_{r+1} \arrow[r] \arrow[d] & H^q(Y, W\Omega_Y^r/p) \arrow[r] \arrow[d, "V"] & (B_Y^{q+1,\geqq r+1})_{r+1} \arrow[r] \arrow[d, "\wr\|"] & (P^{q+1})_{r+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (Q^q)_r \arrow[r] & H^q(Y, W\Omega_Y^r/p) \arrow[r] & (B_Y^{q+1,\geqq r+1})_r \arrow[r] & (P^{q+1})_r \arrow[r] & 0.
\end{tikzcd}$$

Since $(P^{q+1})_i$ (resp. $(Q^q)_i$) has the same finite dimension for any i (this is a property of an F-gauge of Hodge-Witt annihilated by p), one sees from (1.17.5)

$$\dim_k \mathrm{Coker}\,(V) = \dim_k \mathrm{Ker}\,(V), \text{ where } V \text{ is as in (1.17.5).}$$

Since $H^q(Y, W\Omega_Y^r/p)$ is a module of finite type over the non-commutative ring k[[V]], this implies that $H^q(Y, W\Omega_Y^r/p)$ is finite dimensional over k. Hence $B_Y^{q,r}$ is of Hodge-Witt, and by (1.7) and (1.17.4), we see that $B_Y^{q,\geqq r+1}$ is also of Hodge-Witt.

§ 2. Filtered Dieudonné module of Hodge-Witt

(2.1) As in Fontaine-Laffaille [9], by a filtered Dieudonné module over W, we mean a W-module M endowed with the following structures.

(2.1.1) A decreasing filtration $(M^i)_{i\in Z}$ by W-submodules such that $M^i=0$ for $i\gg 0$ and $M^i=M$ for $i\ll 0$.

(2.1.2) A family of σ-linear homomorphisms $(\varphi^i\colon M^i\to M)_{i\in Z}$ such that for each i, the restriction of φ^i to M^{i+1} coincides with $p\varphi^{i+1}$.

As in Winterberger [14], we denote by $\boldsymbol{MF}_{W,tf}$ the category of filtered Dieudonné modules over W satisfying the following conditions.

(2.1.3) For each i, M^i is of finite type over W.

(2.1.4) For each i, M^i is a direct summand of M (as a W-submodule).

(2.1.5) $$M=\sum_{i\in Z}\varphi^i(M^i).$$

Then, $\boldsymbol{MF}_{W,tf}$ is an abelian category by [14] 1.4.1.

The contents of this section is as follows. First, we shall show that if X is a projective smooth scheme over W, and if $m<p$ or $\dim(X)\leqq p$, then the de Rham cohomology $H_{DR}^m(X)$ has a structure of an object of $\boldsymbol{MF}_{W,tf}$. (This fact was proved in Fontaine [8] assuming that $H_{DR}^*(X)$ is

torsion free.) Next we shall define a full subcategory $MF_{W,HW}$ of $MF_{W,tf}$ consisting of objects of Hodge-Witt, and we prove that if $\dim(X)\leqq p$, $H^m_{DR}(X)$ are of Hodge-Witt for all m if and only if $X\otimes_W k$ is of Hodge-Witt (See (2.15)).

(2.2) We begin with the study of the relation between filtered Dieudonné modules and F-gauges. Let $MF'_{W,tf}$ be the following category. An object D of $MF'_{W,tf}$ is an F-gauge over k endowed with a W-submodule D^i of D_i for each $i\in \boldsymbol{Z}$ satisfying the following conditions (2.2.1)–(2.2.4).

(2.2.1) Each D_i is of finite type over W.

(2.2.2) There are integers m, n such that the F-gauge D is of level in $[m, n]$ and such that $D^i=D_i$ if $i\leqq m$ and $D^i=0$ if $i\geqq n$.

(2.2.3) $\tilde{V}(D^i)\subset D^{i-1}$ for any i.

(2.2.4) The sequence

$$0\longrightarrow D^i\xrightarrow{\tilde{V}}D_{i-1}\xrightarrow{\tilde{F}}D_i/D^i\longrightarrow 0$$

is exact.

Note that the condition (2.2.4) is equivalent to the exactness of

$$(2.2.5)\qquad 0\longrightarrow D^i\xrightarrow{(\tilde{V},p)}D_{i-1}\oplus D^i\xrightarrow{(\tilde{F},-1)}D_i\longrightarrow 0.$$

The morphism in $MF'_{W,tf}$ is defined in the evident way.

Lemma (2.3). *The two categories $MF_{W,tf}$ and $MF'_{W,tf}$ are equivalent.*

The definition of the equivalence is the following. For an object M of $MF_{W,tf}$, we define the corresponding object D of $MF'_{W,tf}$ as follows. Let $D^i=M^i$ for all i. For $i\ll 0$, let $D_i=M$, $\tilde{F}$: $D_i\rightarrow D_{i+1}$ the multiplication by p, and $\tilde{V}$: $D_i\rightarrow D_{i-1}$ the identity map. We proceed by induction on i. Assume that D_{i-1} and the inclusion $D^{i-1}\hookrightarrow D_{i-1}$ are already defined. Then, D_i is defined as the push out

$$\begin{tikzcd} M^i \arrow[r, hook] \arrow[d, "p"'] & M^{i-1}=D^{i-1} \arrow[r, hook] & D_{i-1} \arrow[d, dotted] \\ M^i=D^i \arrow[rr, dotted] & & D_i, \end{tikzcd}$$

and the map $\tilde{F}$: $D_{i-1}\rightarrow D_i$ and the inclusion $D^i\hookrightarrow D_i$ are defined as the dotted arrows in this diagram. The map $\tilde{V}$: $D_i\rightarrow D_{i-1}$ is defined to be the unique homomorphism such that $\tilde{V}\circ\tilde{F}=p$ and such that the composite maps $M^i \to D_i \xrightarrow{V} D_{i-1}$ and $M^i \xrightarrow{\subset} M^{i-1} \to D_{i-1}$ coincide. Finally, the map φ: $D_\infty \to D_{-\infty}$ is defined to be the unique homomorphism such that the composite

$$M^i \xrightarrow{\subset} D_i \xrightarrow{\tilde{F}'s} D_\infty \xrightarrow{\varphi} D_{-\infty} = M$$

coincides with φ^i for any i. By [14] 1.6, φ is an isomorphism. It is clear that D satisfies the conditions (2.2.1)–(2.2.4).

Conversely, for an object D of $\boldsymbol{MF}'_{W,tf}$, we define the corresponding filtered Dieudonne modulé M as follows.

Let $M = D_{-\infty}$, M^i the image of the composite $D^i \hookrightarrow D_i \xrightarrow{V's} D_{-\infty}$, and let φ^i; $M^i \to M$ be the composite

$$M^i \underset{\cong}{\xleftarrow{\tilde{V}'s}} D^i \xrightarrow{\tilde{F}'s} D_\infty \xrightarrow{\varphi} D_{-\infty}.$$

To see that M is an object of $\boldsymbol{MF}_{W,tf}$, the only non-trivial thing is that M^i is a direct summand of M. To show this, it suffices to prove that the maps $M^i/p^n M^i \to M^{i-1}/p^n M^{i-1}$ are injective for all $n \geqq 1$ ([14] 1.5.3), and so it suffices to show the injectivity of $\tilde{V}$: $D^i/p^n D^i \to D_{i-1}/p^n D_{i-1}$. Let T_i be the kernel of this map. Then, by (2.2.5), we have an exact sequence

$$0 \longrightarrow T_i \longrightarrow D^i/p^n D^i \longrightarrow D_{i-1}/p^n D_{i-1} \oplus D^i/p^n D^i \longrightarrow D_i/p^n D_i \longrightarrow 0$$

from which we obtain

$$l(D_i/p^n D_i) = l(D_{i-1}/p^n D_{i-1}) + l(T_i).$$

Here l means the length over W. However, for $i \gg 0$ and $j \ll 0$,

$$l(D_i/p^n D_i) = l(D_\infty/p^n D_\infty) = l(D_{-\infty}/p^n D_{-\infty}) = l(D_j/p^n D_j).$$

Thus we have $T_i = 0$ for all i.

(2.4) Let X be a projective smooth scheme over W. We show that if $m < p$ or if $\dim(X) \leqq p$, the de Rham cohomology groups $H^m_{DR}(X) = \boldsymbol{H}^m(X, \Omega^\cdot_{X/W})$ and $H^m_{DR}(X_n) = \boldsymbol{H}^m(X_n, \Omega^\cdot_{X_n/W_n})$ $(n \geqq 1)$ have structures of objects of $\boldsymbol{MF}_{W,tf}$. Let $n \geqq 1$. Let $M = H^m_{DR}(X_n)$ and let $(M^i)_{i \in \boldsymbol{Z}}$ be the Hodge-filtration on M, i.e. the filtration induced by the spectral sequence

$$E_1^{r,q} = H^q(X_n, \Omega^r_{X_n/W_n}) \Longrightarrow H^*_{DR}(X_n). \tag{2.4.1}$$

Assume $m < p$ or $\dim(X) \leqq p$. Then, as in (2.5) (1) below, the map $\boldsymbol{H}^m(Y, \Omega^{\geqq i}_{X_n/W_n}) \to M^i$ is bijective. The homomorphism φ^i: $M^i \to M$ $(i < p)$ is defined to be

$$M^i \xleftarrow{\cong} H^m(Y, \Omega^{\geqq i}_{X_n/W_n}) \cong H^m(Y, J^{[i]}_{n,X,Z}) \xrightarrow{f_i} H^m(Y, E_{n,X,Z})$$

($Y = X \otimes_W k$) with the notations in Ch. I Section 1. (Note $M^p = 0$).

Proposition (2.5). *Let X be as above and assume that $m < p$ or* $\dim(X) \leqq p$. *Let $n \geqq 1$, Then*;

(1) *The map $H^m(Y, \Omega^{\geqq i}_{X_n/W_n}) \to H^m_{DR}(X_n)$ is injective. Consequently, the Hodge spectral sequence*

$$E_1^{r,q} = H^q(X_n, \Omega^r_{X_n/W_n}) \Longrightarrow H^*_{DR}(X_n)$$
$$(\textit{resp. } E_1^{r,q} = H^q(X, \Omega^r_{X/W}) \Longrightarrow H^*_{DR}(X))$$

degenerates if $\dim(X) \leqq p$.

(2) *With the filtration and with the homomorphisms φ^i defined above (in the case of $H^m_{DR}(X)$, we take the inverse limit of the above definition), $H^m_{DR}(X_n)$ ($n \geqq 1$) and $H^m_{DR}(X)$ become objects of $\boldsymbol{MF}_{W,tf}$.*

This (2.5) (2) is proved in [8] Section 1 under the assumption that $H^*_{DR}(X)$ is torsion free.

(2.6) Let Z be as in Ch. I Section 1, and fix $m \geqq 0$ and $n \geqq 1$. To prove (2.5), we give some definitions. Let $E^{(0)}_{n,X,Z} = E_{n,X,Z}$ and define the complex $E^{(r)}_{n,X,M}$ ($r \geqq 1$) by induction on r, to be the push out

$$\begin{tikzcd} J^{[r]}_{n,X,Z} \arrow[r, hook] \arrow[d, "p"'] & J^{[r-1]}_{n,X,Z} \arrow[r] & E^{(r-1)}_{n,X,Z} \arrow[d, dotted] \\ J^{[r]}_{n,X,Z} \arrow[rr, dotted] & & E^{(r)}_{n,X,Z} \end{tikzcd}$$

(cf. (2.3)). We define an object D of $\boldsymbol{MF}'_{W,tf}$ as follows. For $0 \leqq i < p$, let

$$D_i = H^m(Y, E^{(i)}_{n,X,Z}), \qquad D^i = H^m(Y, J^{[i]}_{n,X,Z})$$

For $i \leqq 0$, let $D_i = D^i = H^m_{DR}(X_n)$. For $i \geqq p$, let $D_i = H^m(E^{(p-1)}_{n,X,Z})$. The maps $\tilde{F}$, $\tilde{V}$ and φ of D are defined following the method in the definitions in (2.3) except that for $i \geqq p$, the map $\tilde{F}$: $D_{i-1} \to D_i$ (resp. $\tilde{V}$: $D_i \to D_{i-1}$) is defined to be the identity map (resp. to be the multiplication by p) and that φ is defined to be the map induced by f_{p-1}: $E^{(p-1)}_{n,X,Z} \to E_{n,X,Z}$.

To see that D is indeed an object of $\boldsymbol{MF}'_{W,tf}$ in the case $m < p$ or $\dim(X) \leqq p$, it suffices to prove the following (2.6.1) and (2.6.2).

Lemma (2.6.1). *Let $0 \leqq r < p$. Then, the map f_r: $H^q(E^{(r)}_{n,X,Z}) \to H^q(E_{n,X,Z})$ ($H^q = H^q(Y,\ )$) is an isomorphism for $q \leqq r$. If* $\dim(X) - 1 \leqq r$, *it is an isomorphism for any q.*

Lemma (2.6.2). *Let $r<p$. Then the sequence*

$$0\longrightarrow H^m(J_n^{[r]})\xrightarrow{\alpha}H^m(E_n^{(r-1)})\oplus H^m(J_n^{[r]})\xrightarrow{\beta}H^m(E_n^{(r)})\longrightarrow 0$$

(cf. (2.2.5)) is exact for $m<p$. If $\dim(X)\leqq p$, *it is exact for any m.*

We first prove that f_r: $H^q(E_{n,X,Z}^{(r)})\to H^q(E_{n,X,Z})$ is bijective if $q<r$ and is injective if $q=r$, and that it is bijective for any q if $\dim(X)-1\leqq r$. For this, by Lemma (2.6.3) below, it is sufficient to prove that the map between cohomology sheaves f_r: $\mathscr{H}^q(E_{n,X,Z}^{(r)})\to\mathscr{H}^q(E_{n,X,Z})$ is bijective if $q<r$ and is injective if $q=r$, and that it is bijective for any q if $\dim(X)-1\leqq r$. This is a local problem and hence we may assume that X has a frobenius (and need not be projective). We may also assume $n=1$. Then, f_r: $E_{1,X,X}^{(r)}\to E_{1,X,X}$ is described as

$$\begin{array}{ccccccccccccc}
E_{1,X,X}^{(r)}= & (\mathscr{O}_Y & \xrightarrow{0} & \Omega_Y^1 & \xrightarrow{0} & \cdots & \xrightarrow{0} & \Omega_Y^r & \xrightarrow{d} & \Omega_Y^{r+1} & \xrightarrow{d} & \Omega_Y^{r+2} & \longrightarrow\cdots) \\
f_r\big\downarrow & f_0\big\downarrow & & f_1\big\downarrow & & & & f_r\big\downarrow & & 0\big\downarrow & & 0\big\downarrow & \\
E_{1,X,X}= & (\mathscr{O}_Y & \xrightarrow{d} & \Omega_Y^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega_Y^r & \xrightarrow{d} & \Omega_Y^{r+1} & \xrightarrow{d} & \Omega_Y^{r+2} & \longrightarrow\cdots).
\end{array}$$

So we are reduced to the fact that f_i induces an isomorphism $\Omega_Y^i\xrightarrow{\cong}\mathscr{H}^i(\Omega_Y^\cdot)$ ([2] Theorem 8.5).

In the above argument, we used the following

Lemma (2.6.3). *Let $E=(E_r^{i,j})$ and $E'=((E')_s^{i,j})$ $(s\geqq 2)$ be spectral sequences and let $E\to E'$ be a morphism of spectral sequences. Fix $r\in\boldsymbol{Z}$, $s\geqq 2$ and assume that $E_s^{i,j}\to(E')_s^{i,j}$ $(i,j\in\boldsymbol{Z})$ is bijective if $i+j<r$ and is injective if $i+j=r$. Then $E_{s+1}^{i,j}\to(E')_{s+1}^{i,j}$ is bijective if $i+j<r$ and is injective if $i+j=r$.*

We next prove (2.6.2) by induction on m. Let C_r be the cokernel of the map β in (2.6.2). By induction on m, we may assume that α is injective and hence we have

$$l(H^m(E_n^{(r)}))=l(H^m(E_n^{(r-1)}))+l(C_r).$$

However, as we have seen above, the map f_{p-1}: $H^m(E_n^{(p-1)})\to H^m(E_n)$ is injective. Hence

$$l(H^m(E_n))\geqq l(H^m(E_n^{(p-1)}))=l(H^m(E_n))+\sum_{i=1}^{p-1}l(C_i).$$

This shows $C_i=0$ for $1\leqq i<p$.

Finally, if $0\leqq q<p$, the above argument shows $l(H^q(E_n^{(q)}))=l(H^q(E_n))$, and hence the injection f_q: $H^q(E_n^{(q)})\to H^q(E_n)$ is an isomorphism.

This completes the proofs of (2.6.1) and (2.6.2).

Now, Proposition (2.5) is a consequence of the existence of the object D of $MF'_{W,tf}$ defined above, and of (2.3).

Remark (2.6.4) (added after I received the preprint [18] and a letter of Prof. Messing). Proposition (2.5) (1) furnishes an algebraic proof of the degeneration of the Hodge spectral sequence

$$E_1^{r,q}=H^q(V,\Omega^r_{V/F})\longrightarrow H^*_{DR}(V/F)$$

for a projective smooth scheme V over a field F of characteristic zero.*) The following is a modification of an argument by Fontaine and Messing in a letter of Prof. Messing to the author. Take a subring A of F which is of finite type over $\boldsymbol{Z}$, a projective smooth scheme $\tilde{V}$ over A such that $V=\tilde{V}\otimes_A F$, and a prime number p such that p is a prime element in A and $\dim(V)<p$. It suffices to show that the Hodge spectral sequence for the scheme $\tilde{V}\otimes_A A_{(p)}$ over $A_{(p)}$ degenerates. Let k' be the residue field of $A_{(p)}$ and let $k=(k')^{p^{-\infty}}$. Then there is a faithfully flat embedding $A_{(p)}\to W(k)$, and hence we are reduced to (2.5) (1).

Definition (2.7). Let X be as in (2.6), and assume $m<p$ or $\dim(X)\leqq p$. We denote by $H^m_{DR,n}(X)$ the W-module $H^m_{DR}(X_n)$ regarded as an object of $MF_{W,tf}$ as above.

We next define and study filtered Dieudonné modules of Hodge-Witt. We give some preliminary definitions and lemmas.

Definition (2.8). Let M be an object of $MF_{W,tf}$ and let I be a closed integral interval. We say that M is of level I if $M^i=M$ for $i\leqq\inf(I)$ and $M^{i+1}=0$ for $i\geqq\sup(I)$.

Definition (2.9). Let G: $MF_{W,tf}\to FG_K$ be the composite functor $MF_{W,tf}\xrightarrow[(2.3)]{\sim}MF'_{W,tf}\to FG_k$ where the second arrow is the "forgetful functor".

The following (2.10) is proved easily.

Lemma (2.10). *Let M be an object of $MF_{W,tf}$ and let I be a closed integral interval. Then, M is of level I in the sense of* (2.8) *if and only if $G(M)$ is of level I in the sense of* (1.5).

Lemma (2.11). *Let M be an object of $MF_{W,tf}$ which is of finite length over W. Let $m\in\boldsymbol{Z}$, and let S be a subobject of $G(M)$ such that S is of level*

*) Added in proof: Better treatments of this problem have been given by Fontaine-Messing and Deligne-Illusie.

$(-\infty, m]$ *and such that* $G(M)/S$ *is of level* $[m, \infty)$. *Then there exists a unique subobject* L *of* M *in* $MF_{W,tf}$ *such that* $S=G(L)$.

Proof. The uniqueness follows from the fact that G is a faithful exact functor between abelian categories.

Let D be the object of $MF'_{W,tf}$ corresponding to M. So, $D_i=G(M)_i$ and $D^i=M^i$. Let $S^i=S_i\cap M^i$ where the intersection is taken in D_i. For the proof of the existence of L, it suffices to show that $(S_i, S^i)_{i\in Z}$ is a subobject of M in $MF'_{W,tf}$. For this, it is sufficient to prove that the sequence

$$(2.11.1)\qquad 0\longrightarrow S^i\xrightarrow{\tilde{V}} S_{i-1}\xrightarrow{\tilde{F}} S_i/S^i\longrightarrow 0$$

is exact for any $i\in Z$. First, let $i\leqq m$. By the surjectivity of $\tilde{F}$; $D_{i-1}\to D_i/D^i$ (cf. (2.2.4)), and by the bijectivity of $\tilde{V}$: $D_i/S_i\to D_{i-1}/S_{i-1}$ which follows from the condition on the level of $G(M)/S$, we see that the multiplication by $p=\tilde{F}\circ\tilde{V}$ on $D_i/(D^i+S_i)$ is surjective. This proves $D_i=D^i+S_i$. Thus

$$(2.11.2)\qquad D^i/S^i\cong D_i/S_i\overset{\tilde{V}}{\cong} D_{i-1}/S_{i-1}.$$

Now the exactness of (2.11.1) follows easily from (2.11.2) and (2.2.4). Next, let $i\geqq m+1$. By the injectivity of $\tilde{V}$: $S^i\to S_{i-1}$, and by the bijectivity of $\tilde{F}$: $S_{i-1}\xrightarrow{\cong} S_i$ which follows from the condition on the level of S, we see that the multiplication by p on S^i is injective. Hence $S^i=0$ and this proves the exactness of (2.11.1).

Definition (2.12). Let M be an object of $MF_{W,tf}$. We say that M is of Hodge-Witt if there is a finite sequence of subobjects $(M_{(i)})_{0\leqq i\leqq r}$ in $MF_{W,tf}$ such that

$$0=M_{(0)}\subset M_{(1)}\subset\cdots\subset M_{(r)}=M$$

and such that each $M_{(i)}/M_{(i-1)}$ is of level $[m_i, m_i+1]$ for some integer m_i.

We denote by $MF_{W,HW}$ the full subcategory of $MF_{W,tf}$ consisting of objects of Hodge-Witt.

Proposition (2.13). *Let* M *be an object of* $MF_{W,tf}$. *Then,* M *is of Hodge-Witt if and only if the F-gauge* $G(M)$ *is of Hodge-Witt.*

Proof. The implication

$$M\in MF_{W,HW}\Longrightarrow G(M)\in FG_{k,HW}$$

follows from (1.7) (2) and (2.10). The converse follows from

Lemma (2.14). *Let M be an object of $\boldsymbol{MF}_{W,tf}$ such that $G(M)$ is of Hodge-Witt, let $m \in \boldsymbol{Z}$, and let $G(M)=S\oplus T$ be the direct decomposition in $\boldsymbol{FG}_{k,HW}$ such that S is of level $(-\infty, m]$ and T is of level (m, ∞). Then, there exists a unique subobject L of M in $\boldsymbol{MF}_{W,tf}$ such that $S=G(L)$.*

Proof. By (2.11), let L_n be the subobject of M/p^nM such that $G(L_n)=S/p^nS$ in $G(M/p^nM)$. Then, $L=\varprojlim_n L_n$ is the unique subobject of M such that $G(L)=S$.

Theorem (2.15). *Let X be a smooth projective scheme over W, and let $Y=X\otimes_W k$.*

(1) *Assume $m<p$ or $\dim(X)\leq p$. Then, $H^m_{DR,1}(X)$ (see (2.7)) is of Hodge-Witt if and only if $GH^m(Y/k)$ is of Hodge-Witt.*

(2) *If $\dim(X)\leq p$, the following three conditions are equivalent.*

(i) *Y is of Hodge-Witt.*

(ii) *$H^m_{DR}(X)$ are of Hodge-Witt as objects of $\boldsymbol{MF}_{W,tf}$ for all m.*

(iii) *$H^m_{DR,n}(X)$ are of Hodge-Witt for all m and n.*

By (2.13), (2.15) (1) is a consequence of

Lemma (2.16). *Let X be as above and assume $m<p$ or $\dim(X)\leq p$. Then*

$$G(H^m_{DR,1}(X))\cong GH^m(Y/k).$$

Proof. Let $r<p$ and let $\boldsymbol{E}^{(r)}_{1,X,Z}$ be as in the proof of (2.6). By the definition of $\boldsymbol{E}^{(r)}_{1,X,Z}$, we have

$$\text{(2.16.1)} \qquad \boldsymbol{E}^{(r)}_{1,X,Z}=\bigoplus_{0\leq i<r}(\boldsymbol{J}^{[i]}_{1,X,Z}/\boldsymbol{J}^{[i+1]}_{1,X,Z})\oplus\boldsymbol{J}^{[r]}_{1,X,Z}.$$

By using (2.16.1) we obtain a quasi-isomorphism

$$h_r\colon \boldsymbol{E}^{(r)}_{1,X,Z}\longrightarrow\Omega^{\cdot}_Y(r) \qquad (0\leq r<p)$$

as follows. For $0\leq i<r$, let $\boldsymbol{J}^{[i]}_{1,X,Z}/\boldsymbol{J}^{[i+1]}_{1,X,Z}\rightarrow\Omega^{\cdot}_Y(r)$ be the map of complexes whose component of degree t is

$$J^{[i-t]}_{D_1}/J^{[i-t+1]}_{D_1}\otimes_{\mathcal{O}_{Z_1}}\Omega^t_{Z_1}\xrightarrow{f_i}\mathcal{O}_{D_1}\otimes_{\mathcal{O}_{Z_1}}\Omega^t_{Z_1}\xrightarrow{\text{nat.}}\Omega^t_Y$$

if $t\leq i$, and is zero if $t>i$. Let $\boldsymbol{J}^{[r]}_{1,X,Z}\rightarrow\Omega^{\cdot}_Y(r)$ be the map of complexes whose component of degree t is

$$J^{[r-t]}_{D_1}\otimes_{\mathcal{O}_{Z_1}}\Omega^t_{Z_1}\xrightarrow{f_r}\mathcal{O}_{D_1}\otimes_{\mathcal{O}_{Z_1}}\Omega^t_{Z_1}\xrightarrow{\text{nat.}}\Omega^t_Y$$

if $t\leq r$, and is the natural map $\mathcal{O}_{D_1}\otimes_{\mathcal{O}_{Z_1}}\Omega^t_{Z_1}\rightarrow\Omega^t_Y$ if $t>r$. The map h_r is defined as the sum of these maps with respect to the identification (2.16.1).

To see that h_r is a quasi-isomorphism, we may assume that X has a frobenius (and need not be projective). Then, the map h_r is described as

$$\begin{array}{ccccccccccc}
E_{1,X,X}^{(r)}= & (\mathcal{O}_Y & \xrightarrow{0} & \Omega_Y^1 & \xrightarrow{0} & \cdots & \xrightarrow{0} & \Omega_Y^r & \xrightarrow{d} & \Omega_Y^{r+1} & \longrightarrow\cdots) \\
\downarrow h_r & \downarrow f_0 & & \downarrow f_1 & & & & \downarrow f_r & & \downarrow 1 & \\
\Omega_Y^{\cdot}(r)= & (\mathcal{O}_Y & \xrightarrow{d} & \Omega_Y^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega_{Y,d=0}^r & \xrightarrow{d\circ C} & \Omega_Y^{r+1} & \longrightarrow\cdots).
\end{array}$$

This shows that h_r ($r \in \boldsymbol{Z}$) are quasi-isomorphisms and induce $G(H_{DR,1}^m(X)) \cong GH^m(Y/k)$.

Now we prove (2.15) (2). By (1.16) and by (2.15) (1), the condition (i) is equivalent to the fact that $H_{DR,1}^m(X)$ are of Hodge-Witt for all m. So we have (ii)$\Rightarrow$(i) and (i)$\Rightarrow$(iii). The implication (iii)$\Rightarrow$(ii) follows by the inverse limit argument.

§ 3. Theory of Fontaine-Laffaille

Let K be the field of fractions of W and $\bar{K}$ the algebraic closure of K. In this section, we apply the theory of Fontaine-Laffaille [9] concerning the functorial correspondence between filtered Dieudonné modules and p-adic representations of Gal$(\bar{K}/K)$, to filtered Dieudonné modules of Hodge-Witt. The definition of their correspondence was modified by Fontaine and Messing using the crystalline cohomology of $O_{\bar{K}}/p^n O_{\bar{K}}$ (see [8] § 3), and we use this latter definition in the following. Let B_n, $J_n^{[i]}$, etc. be as in Ch. I Section 4.

Definition (3.1) ([9] [14]). (1) Let $\boldsymbol{MF}_{W,lf}$ be the full subcategory of $\boldsymbol{MF}_{W,tf}$ consisting of all objects whose underlying W-modules are of finite length. Let $\boldsymbol{MF}_{W,tf}^p$ (resp. $\boldsymbol{MF}_{W,lf}^p$) be the full subcategory of $\boldsymbol{MF}_{W,tf}^p$ (resp. $\boldsymbol{MF}_{W,lf}^p$) consisting of objects of level $[0, p-1]$.

(2) For an object M of $\boldsymbol{MF}_{W,lf}^p$ and for $0 \leq r < p$, let $\psi_r(M)$ (resp. $\Lambda_r(M)$) be the kernel (resp. cokernel) of the homomorphism

$$f_r - 1 \colon \mathrm{Fil}^r(B_n \otimes_{W_n} M) \longrightarrow B_n \otimes_{W_n} M$$

where n is any integer ≥ 1 such that $p^n M = 0$,

$$\mathrm{Fil}^r(B_n \otimes_{W_n} M) = \sum_{i \geq 0} J_n^{[i]} \otimes_{W_n} M^{r-i} \subset B_n \otimes_{W_n} M,$$

and f_r is the unique map which coincides with $f_i \otimes \varphi^{r-i}$ on each $J_n^{[i]} \otimes_{W_n} M^{r-i}$ $(0 \leq i < p)$. (The existence of f_r follows from the fact that $J_n^{[i]}/J_n^{[i+1]}$ are free W_n-modules. It is clear that $\psi_r(M)$ and $\Lambda_r(M)$ are independent of the choice of n.)

Since the functors $M \mapsto M^r$ are exact on the category $\boldsymbol{MF}_{W,lf}$ (this is a consequence of (2.3)) and since $J_n^{[i]}/J_n^{[i+1]}$ are free W_n-modules, we see that the functors $M\mapsto \mathrm{Fil}^r(B_n\otimes_{W_n} M)$ (n is any integer such that $p^nM=0$) are exact on $\boldsymbol{MF}^p_{W,lf}$. Consequently,

Lemma (3.2). *If* $0\to D'\to D\to D''\to 0$ *is an exact sequence in* $\boldsymbol{MF}^p_{W,lf}$, *we have an exact sequence*

$$0\longrightarrow\psi_r(D')\longrightarrow\psi_r(D)\longrightarrow\psi_r(D'')\longrightarrow\Lambda_r(D')\longrightarrow\Lambda_r(D)$$
$$\longrightarrow\Lambda_r(D'')\longrightarrow 0.$$

(3.3) We recall the classification of simple objects of $\boldsymbol{MF}_{W,lf}$ in the case $k=\bar{k}$ established in [9]. For an integer $h\geqq 1$ and for a function i: $\boldsymbol{Z}/h\boldsymbol{Z}\to\boldsymbol{Z}$, denote by $E(h, i)$ the following object M of $\boldsymbol{MF}_{W,lf}$. As a W-module, M is the h-dimensional k-vector space with base $(e_m)_{m\in\boldsymbol{Z}/h\boldsymbol{Z}}$. For $r\in\boldsymbol{Z}$, M^r is the subspace generated by e_m's such that $i_m\geqq r$. The map φ^r: $M^r\to M$ is defined by

$$\varphi^r(e_m)=\begin{cases} e_{m+1} & \text{if } i_m=r \\ 0 & \text{if } i_m>r. \end{cases}$$

If the period of i is just h (i.e. if the map i does not factor through $\boldsymbol{Z}/h\boldsymbol{Z}\to\boldsymbol{Z}/h'\boldsymbol{Z}$ for any divisor h' of h different from h), $E(h, i)$ is a simple object of $\boldsymbol{MF}_{W,lf}$. If $k=\bar{k}$, any simple object of $\boldsymbol{MF}_{W,lf}$ is isomorphic to such $E(h, i)$ for a unique (h, i).

The following proposition is a consequence of [9] Theorem 5.3, and so we omit the proof.

Proposition (3.4). *Let* $h\geqq 1$, *i a function* $\boldsymbol{Z}/h\boldsymbol{Z}\to\boldsymbol{Z}$ *such that* $0\leqq i_m<p$ *for any* $m\in\boldsymbol{Z}/h\boldsymbol{Z}$, *and let* $0\leqq r<p-1$.

(1) *If* $r<\sup_m(i_m)$, *then* $\psi_r(E(h, i))=0$.

(2) *If* $r\geqq\sup_m(i_m)$, $\psi_r(E(h, i))$ *is an* $\boldsymbol{F}_p$*-vector space of dimension h. There is a one-to-one correspondence between the set of all* (p^h-1)*-th roots of* $-p$ *and the set of all non-zero elements of* $\psi_r(E(h, i))$ *defined by*

$$\alpha\longmapsto\sum_{m\in\boldsymbol{Z}/h} f_{r-i_m}(\theta(\alpha^{\sum_{s=1}^{h} p^{s-1}(r-i_{m-s})}))\otimes e_{m+1}\in\psi_r(E(h, i))$$

($\alpha^{p^h-1}=-p$, *see Ch.* I *Section* 4 *for the definition of* θ: $O_{\bar{K}}\to B_1$.)

Definition (3.5). Let $\boldsymbol{Rep}_{K,tf}$ (resp. $\boldsymbol{Rep}_{K,lf}$) be the category of all $\boldsymbol{Z}_p$-modules of finite type (resp. of finite length) endowed with a continuous action of $\mathrm{Gal}(\bar{K}/K)$.

By (3.4), $D\mapsto\psi_r(D)$ ($0\leqq r<p-1$) are functors $\boldsymbol{MF}^p_{W,lf}\to\boldsymbol{Rep}_{K,lf}$.

Definition (3.6). (1) For an integer $h\geqq 1$, let χ_h: $\mathrm{Gal}(\bar{K}/K_{\mathrm{nr}})\to(\boldsymbol{F}_{p^h})^\times$

be the unique homomorphism (K_{nr} denotes the maximal unramified extension of K) such that $\chi_h(g)$ is the image of $g(\alpha)\alpha^{-1}$ in $\bar{k}$ for any $g \in \mathrm{Gal}(\bar{K}/K_{\mathrm{nr}})$ and for any (p^h-1)-th root α of any prime element of K.

(2) For $h \geqq 1$ and for a function i: $\boldsymbol{Z}/h\boldsymbol{Z} \to \boldsymbol{Z}$, let $S(h, i)$ be the following object of $\boldsymbol{Rep}_{K_{\mathrm{nr}}, tf}$; $S(h, i) = \boldsymbol{F}_{p^h}$, and an element g of $\mathrm{Gal}(\bar{K}/K_{\mathrm{nr}})$ acts on $S(h, i)$ as the multiplication by

$$\chi_h(g)^{i_0 + p i_{-1} + \cdots + p^{h-1} i_{1-h}} \in (\boldsymbol{F}_{p^h})^{\times}.$$

Then, a simple object of $\boldsymbol{Rep}_{K_{\mathrm{nr}}, tf}$ is isomorphic to some $S(h, i)$. Furthermore,

$$\psi_r(E(h, i))(-r) \cong S(h, i) \qquad \text{if } \sup_m (i_m) \leqq r < p-1$$

by (3.4) (2) ([9] Theorem 5.3).

Corollary (3.7). *Let M be an object of $\boldsymbol{MF}^p_{W, tf}$ and let $0 \leqq r < p-1$.*

(1) *The canonical map $\psi_{r-1}(M)(1) \to \psi_r(M)$ induced by $\mu_{p^n}(O_{\bar{K}}) \to B_n$ (Ch.* I *Section* 4) *is injective.*

(2) *Assume $k = \bar{k}$, and let T be a simple subquotient of $\psi_r(M)(-r)/\psi_r(M)(1-r)$. Then T is isomorphic to some $S(h, i)$ such that $\inf_m (i_m) \geqq 0$ and $\sup_m (i_m) = r$. If M is of Hodge-Witt, T is isomorphic to $S(h, i)$ such that $i_m = r$ or $r-1$ for any m and such that $i_m = r$ for some m.*

Proposition (3.8). *Let $h \geqq 1$, i a function $\boldsymbol{Z}/h\boldsymbol{Z} \longrightarrow \boldsymbol{Z}$ such that $0 \leqq i_m < p$ for any $m \in \boldsymbol{Z}/h\boldsymbol{Z}$, and let $0 \leqq r < p-1$. Then;*

(1) *If $r \geqq \sup_m (i_m)$ or $r \leqq \inf_m (i_m)$, then $\Lambda_r(E(h, i)) = 0$.*

(2) *If $\inf_m (i_m) < r < \sup_m (i_m)$, then $\Lambda_r(E(h, i))$ is an infinite group.*

Proof. Let $E = E(h, i)$. Assume first $r \geqq \sup(i_m)$. Since $f_r(\mathrm{Fil}^{r+1}(B_1 \otimes_k E)) = 0$, the image of $\mathrm{Fil}^{r+1}(B_1 \otimes_k E)$ in $\Lambda_r(E)$ is zero. So it suffices to prove the surjectivity of

$$f_r - 1 \colon gr^r(B_1 \otimes_k E) \longrightarrow (B_1 \otimes_k E)/\mathrm{Fil}^{r+1}(B_1 \otimes_k E).$$

Here, the groups on the both sides are $B_1/J_1^{[p]}$-modules. Since $B_1/J_1^{[p]} \cong O_{\bar{K}}/pO_{\bar{K}}$ (Ch. I (4.5)), we are reduced to (3.9) below. (To see the surjectivity of f_r, use $f_r(a \otimes e_m) = f_{r-i_m}(a) \otimes e_{m+1}$ ($a \in J_1^{[r-i_m]}$) and Ch. I (4.5) (3).)

Next assume $r \leqq \inf(i_m)$. Then, $\mathrm{Fil}^r(B_1 \otimes_k E) = B_1 \otimes_k E$. So, it is easily seen that for each m, the image of $B_1 \otimes e_m$ in $\Lambda_r(E)$ is contained in the image of $B_1 \otimes e_{m+1}$. We may assume $i_m > r$ for some m (otherwise, $i_m = r$ for any m and this case is very easy). For this m, $f_r(B_1 \otimes e_m) = 0$ and hence the image of $B_1 \otimes e_m$ in $\Lambda_r(E)$ is zero. This proves $\Lambda_r(E) = 0$.

Finally assume $\inf(i_m) < r < \sup(i_m)$. Then we find $a, b \in \boldsymbol{Z}$ such that

$a<b$ and $i_a>r>i_b$, and such that $i_m=r$ if $a<m<b$. The composite map

$$\mathrm{Fil}^r(B_1\otimes_k E)\xrightarrow{f_r-1}B_1\otimes_k E\xrightarrow{s}\!\!\!\rightarrow B_1/J_1,$$

where s is the surjection

$$\sum_{m\in Z/h} x_m\otimes e_m\longmapsto \sum_{a<m\leqq b}(x_m)^{p^{b-m}},$$

is the zero map. This shows that $\Lambda_r(E)$ is infinite.

Lemma (3.9). *Let $R=O_{\bar K}/pO_{\bar K}$, let M and N be R-modules of finite type, and let $f\colon M\to N$ be a surjective additive map such that $f(ax)=a^pf(x)$ ($\forall a\in R$, $\forall x\in M$). Let $h\colon M\to N$ be an R-homomorphism. Then, $f+h\colon M\to N$ is surjective.*

Proof. We may assume $M=N=R^n$ and f is the map $(x_i)_{1\leqq i\leqq n}\mapsto(x_i^p)_{1\leqq i\leqq n}$. Let $\tilde h\colon O_{\bar K}^n\to O_{\bar K}^n$ be an $O_{\bar K}$-linear map which lifts h. Thus the morphism of schemes $A^n_{O_{\bar K}}\to A^n_{O_{\bar K}}\colon x=(x_i)_i\mapsto(x_i^p)_i+\tilde h(x)$ is finite and faithfully flat, and hence it induces a surjection $O_{\bar K}\to O_{\bar K}$ between the sets of $O_{\bar K}$-rational points. This proves (3.9).

Corollary (3.10). *Let M be an object of $\boldsymbol{MF}^p_{W,tf}$.*

(1) *Let $0\leqq r<p-1$. If $\Lambda_r(M)$ is not zero, it is an infinite group.*

(2) *If M is of Hodge-Witt, $\Lambda_r(M)=0$ for any $0\leqq r<p-1$. If M is not of Hodge-Witt, there is an integer r such that $0\leqq r<p-1$ and such that $\Lambda_r(M)$ is an infinite group.*

Corollary (3.11). *Let $0\leqq r<p-1$. Then, the functor ψ_r is exact on the full subcategory of $\boldsymbol{MF}^p_{W,tf}$ consisting of objects of Hodge-Witt.*

For an object M of $\boldsymbol{MF}^p_{W,tf}$ of Hodge-Witt and for $0\leqq r<p-1$, let

$$\psi_r(M)=\varprojlim_n \psi_r(M/p^nM),$$

which is an object of $\boldsymbol{Rep}_{K,tf}$.

Proposition (3.12). *Let M be an object of $\boldsymbol{MF}^p_{W,tf}$ of Hodge-Witt and let $0\leqq r<p-1$. Then, there is a p-divisible group Γ over W without multiplicative part such that the Tate module $T_p(\Gamma)$ satisfies*

$$\boldsymbol{Q}_p\otimes_{Z_p}T_p(\Gamma)\cong\boldsymbol{Q}_p\otimes_{Z_p}\psi_r(M)/(\psi_{r-1}(M)(1))$$

as $\boldsymbol{Q}_p[\mathrm{Gal}(\bar K/K)]$-modules.

Proof. By (2.14) and (3.4), we may assume that the F-gauge corresponding to M is of level $(r-1, r]$. Let $M[r]$ be the translation (defined by $(M[r])^i = M^{i+r}$ etc.) of M and let N be the object $\mathcal{H}om(M[r], W)$ of $\boldsymbol{MF}_{W,tf}$ (cf. [14] 1.7). We may assume $p \neq 2$. Then, N corresponds to a p-divisible group Γ over W via the contravariant functor ILM of [9] Section 9. Since the F-gauge $G(N)$ is of level $[0, 1)$, Γ has no multiplicative part. By [9] 9.12 and [8] 3.8, we have

$$\boldsymbol{Q}_p \otimes \psi_r(M) \cong \mathrm{Hom}_{MF_K}(N, \boldsymbol{Q}_p \otimes \varprojlim_n B_n) \cong \boldsymbol{Q}_p \otimes T_p(\Gamma)$$

where Hom_{MF_K} means homomorphisms of "filtered modules" in the sense of [6].

§ 4. Cohomology of the sheaf of p-adic vanishing cycles

In this section, X denotes a projective smooth scheme over W. Let K be the field of fractions of W, and $\bar{K}$ the algebraic closure of K. Let $Y = X \otimes_W k$, $\bar{Y} = Y \otimes_k \bar{k}$, $\cdots$ etc. be as in Ch. I Section 4.

Lemma (4.1). *Assume that $m < p$ or* $\dim(X_K) < p$, *and let $r < p$. Then we have an exact sequence*

$$0 \longrightarrow \Lambda_r(H^{m-1}_{DR,n}(X)) \longrightarrow H^m(\bar{Y}, \mathcal{S}_n(r)_{\bar{X}}) \longrightarrow \psi_r(H^m_{DR,n}(X)) \longrightarrow 0.$$

The author learned this type of result from Fontaine and Messing (who considered the syntomic cohomology $H^m(\bar{X}_{\mathrm{syn}}, S^r_n)$).

Proof. By Ch. I (4.6), there is a spectral sequence

$$E_1^{t,u} = J_n^{[r-t]} \otimes H^u(X_n, \Omega^t_{X_n}) \Longrightarrow H^*(\bar{Y}, \boldsymbol{J}^{[r]}_{n,\bar{X}})$$

which degenerates at $E_s^{m-t,t}$ ($s \geqq 1$, $t \geqq 0$) by (2.5). From this, we see that the canonical maps

$$J_n^{[i]} \otimes H^m(Y, \boldsymbol{J}^{[r-i]}_{n,X}) \longrightarrow H^m(\bar{Y}, \boldsymbol{J}^{[r]}_{n,\bar{X}}). \qquad (0 \leqq i \leqq r)$$

induce an isomorphism

$$\mathrm{Fil}^r(B_n \otimes H^m_{DR,n}(X)) \xrightarrow{\cong} H^m(\bar{Y}, \boldsymbol{J}^{[r]}_{n,\bar{X}}).$$

So (4.1) follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{Fil}^r(B_n \otimes H^m_{DR,n}(X)) & \xrightarrow{f_r - 1} & B_n \otimes H^m_{DR,n}(X) \\ \wr\Vert & & \wr\Vert \\ H^m(\bar{Y}, \boldsymbol{J}^{[r]}_{n,\bar{X}}) & \xrightarrow{f_r - 1} & H^m(\bar{Y}, \boldsymbol{E}_{n,\bar{X}}). \end{array}$$

By (4.1) and (3.10), we obtain

Lemma (4.2). *Assume that* $m<p$ *or* $\dim(X_K)<p$. *Then the following two conditions are equivalent.*

(4.2.1) *The F-gauge* $GH^{m-1}(Y/k)$ *is of Hodge-Witt.*

(4.2.2) *For any* r *such that* $0\leqq r<p-1$, $H^m(\bar{Y}, \mathcal{S}_1(r)_{\bar{X}})$ *are finite groups.*

Furthermore, if $GH^{m-1}(Y/k)$ *is of Hodge-Witt, we have*

$$H^m(\bar{Y}, \mathcal{S}_n(r)_{\bar{X}})\cong\psi_r(H^m_{DR,n}(X)).$$

Theorem (4.3). *Assume* $m<p-1$ *or* $\dim(X_K)<p$, *and assume that* $GH^m(Y/k)$ *and* $GH^{m-1}(Y/k)$ *are of Hodge-Witt. Let* $r<p-1$ *and* $n\geqq 1$. *Then;*

(1) *The group* $H^{m-r}(\bar{Y}, \bar{M}^r_n)$ *is finite and there is an exact sequence*

$$0\longrightarrow\psi_{r-1}(H^m_{DR,n}(X))(1)\longrightarrow\psi_r(H^m_{DR,n}(X))\longrightarrow H^{m-r}(\bar{Y}, \bar{M}^r_n)\longrightarrow 0.$$

(2) *Let* T *be a simple subquotient of the* $\mathrm{Gal}(\bar{K}/K_{\mathrm{nr}})$*-module* $H^{m-r}(\bar{Y}, \bar{M}^r_n)(-r)$. *Then* T *is isomorphic to* $S(h, i)$ (cf. (3.6)) *for some* $h\geqq 1$ *and* i: $\boldsymbol{Z}/h\boldsymbol{Z}\rightarrow\boldsymbol{Z}$ *such that* $i_m=r$ *or* $r-1$ *for any* m, *and such that* $i_m=r$ *for some* m.

(3) *There is a p-divisible group* Γ *over* W *without multiplicative part such that the Tate module* $T_p(\Gamma)$ *of* Γ *satisfies*

$$\boldsymbol{Q}_p\otimes T_p(\Gamma)\cong\boldsymbol{Q}_p\otimes\varprojlim_n H^{m-r}(\bar{Y}, \bar{M}^r_n)$$

as $\boldsymbol{Q}_p[\mathrm{Gal}(\bar{K}/K)]$*-modules. In particular, for any simple subquotient* T *of the* $\boldsymbol{Q}_p[\mathrm{Gal}(\bar{K}/K)]$*-module* $\boldsymbol{Q}_p\otimes\varprojlim_n H^{m-r}(\bar{Y}, \bar{M}^r_n)$, T *has a Hodge-Tate decomposition of the form*

$$\boldsymbol{C}_p\otimes T\cong\boldsymbol{C}_p^i\oplus\boldsymbol{C}_p(1)^j \qquad \textit{with } i>0.$$

Indeed, (1) follows from (4.2) and Ch. I (4.4). Next (2) (resp. (3)) follows from (1) and (3.7) (resp. (3.12)).

Theorem (4.4). *Assume* $\dim(X_K)<p-1$. *Then the following conditions are equivalent.*

(i) *Y is of Hodge-Witt.*

(ii) $H^q(\bar{Y}, \bar{M}^r_1)$ *are finite for any* q *and* r.

(iii) $H^q(\bar{Y}, \bar{M}^r_n)$ *are finite for any* q, r *and* n.

(iv) *The spectral sequence*

$$E_2^{q,r}=H^q(\bar{Y}, \bar{M}^r_1)(-r)\Longrightarrow H^*(X_{\bar{K}}, \boldsymbol{Z}/p\boldsymbol{Z})$$

degenerates.

(v) *The spectral sequence*

$$E_2^{q,r}=H^q(\bar{Y}, \bar{M}_n^r)(-r) \Longrightarrow H^*(X_{\bar{K}}, Z/p^nZ)$$

degenerates for any n.

Indeed, the implication (i)$\Rightarrow$(v) follows from (4.3) (2). Note $\bar{M}_n^r=0$ if $r \geqq p-1$, by the assumption $\dim(X_K)<p-1$. The implications (v)$\Rightarrow$(iii)$\Rightarrow$(ii) and (v)$\Rightarrow$(iv)$\Rightarrow$(ii) are easy by using the finiteness of $H^*(X_{\bar{K}}, Z/p^nZ)$. Finally the implication (ii)$\Rightarrow$(i) follows from (4.2) and Ch. I (4.4).

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Added in proof: The theory of Fontaine and Messing is now given in their paper "p-adic periods and p-adic etale cohomology" (preprint, 1986).