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Stability of the Pluricanonical Maps of Threefolds

Masaki Hanamura

Introduction

Let X be a non-singular projective variety with the non-negative Kodaira dimension $\kappa(X)$ defined over the field of complex numbers.

Letting K_X to be a canonical divisor of X, we denote by $\Phi_{|nK_X|}$ the rational map associated with the linear system $|nK_X|$. Then there exists some positive integer n such that $\dim(\operatorname{Im}\Phi_{|nK_X|})=\kappa(X)$ and that the generic fiber of $\Phi_{|nK_X|}: X \dashrightarrow \operatorname{Im}\Phi_{|nK_X|}$ is geometrically irreducible. Such a fibration is called the stable canonical map or the Iitaka fibration.

In this paper we consider the following problem:

Let X be a non-singular projective variety with $\kappa(X)>0$. For which value of n, does $|nK_X|$ define the stable canonical map?

For surfaces, this problem was studied in detail by Bombieri [2], Kodaira [17] and Iitaka [6]. For non-singular threefolds of general type with K_X numerically effective, Wilson [27], Benveniste [1] and Matsuki [21] treated the problem. Recently Kollár [18] investigated threefolds with irregularity $\geqq 4$.

The purpose of the present paper is to generalize their results.

In Section 3, we study the birationality of the pluricanonical maps of threefolds of general type. Our result is stated as follows:

Theorem (3.4). *Let X be a non-singular threefold of general type which has a minimal model of index r. Then $\Phi_{|nK_X|}$ is a birational map for $n\geqq n_0$ where*

$$\begin{aligned} n_0&=9 && \text{if } r=1,\\ n_0&=13 && \text{if } r=2,\\ n_0&=4r+4 && \text{if } 3\leqq r\leqq 5,\\ n_0&=4r+3 && \text{if } r\geqq 6. \end{aligned}$$

For the definition of the minimal model and its index, see Section 1. Note that the index is independent of the choice of a minimal model.

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In Section 4, we consider threefolds X with $\kappa(X)=1$ or 2, and treat the stability of the pluricanonical maps. The result is roughly stated as follows: *there exists a positive integer n such that for a threefold X with* $\kappa(X)=1$ *or* 2, $\Phi_{|nK_X|}$ *is the stable canonical map unless*

(I) *the irregularity* $q(X)=0$, *or*

(II) $q(X)=1$ *or* 2 *and* X *is a certain type of quotient of* (*curve*)$\times$(*surface*). (See (4.6) for the precise statement).

In Section 5, we give a couple of general results on the pluricanonical systems of varieties of arbitrary dimension.

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§ 1. Preliminaries

Throughout this paper, we consider varieties over $\boldsymbol{C}$.

(1.1) Let X be a normal projective variety of dimension n. We denote by $\operatorname{Div} X$ the group of Weil divisors on X. An element $D \in (\operatorname{Div} X)\otimes\boldsymbol{Q}$ is said to be a ***Q**-divisor.* A $\boldsymbol{Q}$-divisor D is said to be a ***Q**-Cartier divisor* if mD is a Cartier divisor for some positive integer m. For a $\boldsymbol{Q}$-Cartier divisor D, we can define the intersection number $(D\cdot C)$ for any irreducible curve $C\subset X$, and the self-intersection number (D^n). We say that a $\boldsymbol{Q}$-Cartier divisor is *nef* (or *numerically effective*) if $(D\cdot C)\geqq 0$ for any curve $C\subset X$. A nef Cartier divisor D is called big if $(D^n)>0$. Further, D is said to be ample if mD is an ample Cartier divisor for some positive integer m. If every Weil divisor on X is a $\boldsymbol{Q}$-Cartier divisor, we say that X is ***Q**-factorial.*

There is a one-to-one correspondence between the isomorphism classes of reflexive sheaves of rank one on X and the linear equivalence classes of Weil divisors on X. For a Weil divisor D, the corresponding reflexive sheaf is denoted by $\mathcal{O}_X(D)$. K_X means a *canonical divisor* of X, that is, a Weil divisor satisfying $\mathcal{O}_X(K_X)=(\Omega_X^n)^{**}$, the right hand side denoting the double dual of Ω_X^n. We shall also write $\omega_X^{[s]}$ instead of $\mathcal{O}_X(sK_X)$ for a positive integer s.

Let $D=\Sigma_i a_i D_i$ be a $\boldsymbol{Q}$-divisor where $\{D_i\}$ are distinct prime divisors, and $a_i \in \boldsymbol{Q}$. We define the integral part of D, the round-up of D, and the fractional part of D by:

$$[D]:=\Sigma_i[a_i]D_i,\ [a_i]=\text{the integral part of } a_i,$$

$$\ulcorner D\urcorner:=-[-D],$$

$$\{D\}:=\ulcorner(D-[D])\urcorner.$$

(1.2) **Definition** (Reid [22]). Let X be a normal projective variety, and K_X a canonical divisor. We say that X has only *canonical singularities*, if K_X is a $\boldsymbol{Q}$-Cartier divisor and for a resolution of singularities $\mu: \tilde{X} \to X$, there are natural morphisms $\mu^*\omega_X^{[s]} \to \omega_{\tilde{X}}^{\otimes s}$ for any positive integer s. The *index* of X is defined to be the smallest positive integer r such that rK_X is a Cartier divisor.

(1.3) **Definition** (Reid [23]). Let X be a normal projective variety. X is said to be *minimal* or a *minimal model*, if X has only canonical singularities and K_X is nef. Moreover if K_X is ample, we say that X is a canonical variety.

(1.4) **Proposition** (birational invariance of the index). *If X and X' are birationally equivalent minimal models of general type, they have the same index.*

Proof. By Kawamata [13], for a positive integer m which is large enough and is a multiple of the index of X, the linear system $|mK_X|$ is base-point free and $\varphi := \Phi_{|mK_X|}$ is a birational morphism onto

$$X_{\mathrm{can}} = \mathrm{Proj} \oplus_{t \geq 0} H^0(X, \mathcal{O}_X(tK_X)),$$

which is the canonical model of X.

Let $r =$ (index of X). We know that

$$rK_X = \varphi^* L$$

for an ample divisor L on X_{can}. We see immediately that $L = rK_{X_{\mathrm{can}}}$. Thus the index s of X_{can} divides r. We can write

$$sK_X = \varphi^*(sK_{X_{\mathrm{can}}}) + \Delta$$

for an effective Weil divisor Δ. Multiplying both sides with r/s and comparing the result with the above, we get $\Delta = 0$. Hence sK_X is a Cartier divisor and we must have $r = s$.

(1.5) For the definition of the Kodaira dimension $\kappa(X)$ of an algebraic variety X, we refer the reader to Iitaka [5] or [7]. X is said to be of general type (resp. of fiber type) if $\kappa(X) = \dim X$ (resp. $0 < \kappa(X) < \dim X$).

(1.6) A surjective morphism $f: X \to Y$ between projective algebraic varieties is called a fiber space if the generic fiber X_η of f is geometrically irreducible, η denoting the generic point of Y.

In this case we define $\kappa(f) := \kappa(X_\eta)$. The variation var$(f)$ of a fiber space $f: X \to Y$ is defined to be the minimum of the dimensions of algebraic varieties Y' which are subject to the following condition: There exist a fiber space $f'\colon X'\to Y'$ between projective varieties, a projective variety $\bar{Y}$, a generically finite map $\tau\colon \bar{Y}\to Y$ and a surjective morphism $g\colon \bar{Y}\to Y'$ such that the main components of $X\times_Y \bar{Y}$ and $X'\times_{Y'}\bar{Y}$ are birational over $\bar{Y}$.

§ 2. Vanishing theorems and the plurigenus formula

We describe vanishing theorems which will be used in Sections 3 and 5.

(2.1) **Lemma.** *Let X be a normal projective variety with only canonical singularities. Let D be a $\boldsymbol{Q}$-Cartier Weil divisor such that $D-K_X$ is nef and big. Then $H^i(X, \mathcal{O}_X(D))=0$ for any $i>0$.*

For the proof, we refer the reader to Kawamata-Matsuki-Matsuda [16].

(2.2) **Lemma** (Kawamata [10], Viehweg [25]). *Let X be a non-singular projective variety, and D a $\boldsymbol{Q}$-divisor on X. Assume that the fractional part $\{D\}$ is a divisor with normal crossings and that $D-K_X$ is nef and big. Then $H^i(X, \mathcal{O}_X(K_X+\ulcorner D\urcorner))=0$ for $i>0$.*

(2.3) **Lemma.** *Let X be a minimal threefold of general type with index r. Then we have*

$$h^0(X, \mathcal{O}_X((mr+s)K_X))$$
$$=(1/12)\cdot(mr+s)(mr+s-1)(2mr+2s-1)(K_X^3)+am+c_s$$

for $0\leqq s<r$, $mr+s\geqq 2$, where a and c_s are constants.

Proof. Let $\mu\colon \tilde{X}\to X$ be a resolution of singularities such that the exceptional locus is a divisor with normal crossings and

$$K_{\tilde{X}}=\mu^*(K_X)+\Delta$$

with an effective divisor Δ, where $\mu(\Delta)$ consists of finite number of points.

By Lemma (2.2),

$$H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(K_{\tilde{X}}+\ulcorner (n-1)\mu^*K_X\urcorner))=0$$

for $n\geqq 2$ and $i>0$. Thus

$$\begin{aligned}
&h^0(X, \mathcal{O}_X(nK_X))\\
&\quad=h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(K_{\tilde{X}}+\ulcorner (n-1)\mu^*K_X\urcorner))\\
&\quad=\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}(K_{\tilde{X}}+\ulcorner (n-1)\mu^*K_X\urcorner))\\
&\quad=(1/6)(K_{\tilde{X}}+\ulcorner (n-1)\mu^*K_X\urcorner)^3-(1/4)(K_{\tilde{X}}+\ulcorner (n-1)\mu^*K_X\urcorner)^2\cdot K_{\tilde{X}}\\
&\qquad+(1/12)(K_{\tilde{X}}+\ulcorner (n-1)\mu^*K_X\urcorner)\cdot(K_{\tilde{X}}^2+c_2(\tilde{X}))+(1/24)\chi(\mathcal{O}_{\tilde{X}})
\end{aligned}$$

by the Riemann-Roch formula. Hence we get the desired expression with $a=(1/12)\cdot(c_2(\tilde{X})\cdot\mu^*(rK_X))$.

§ 3. Stability of the pluricanonical maps of threefolds of general type

In the rest of the paper, we shall denote simply by $P(n)$ the n-genus $P_n(X)$ of X.

(3.1) **Lemma** (Benveniste [1], Matsuki [21]). *Let S be a non-singular surface, R a nef and big divisor on S, and m a positive integer. Then the rational map $\Phi_{|K_S+mR|}$ is a birational map if the following conditions are satisfied:*

(1) *Given any two distinct points $x, y \in S$, let $\mu\colon \tilde{S}\to S$ be the blowing up of S at x and y. Then $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\mu^*(mR)-2L_x-2L_y))\neq 0$, where $L_x=\mu^{-1}(x)$ and $L_y=\mu^{-1}(y)$.*

(2) *$m\geqq 4$, or*

(2)′ *$m=3$ and $(R^2)\geqq 2$.*

(3.2) **Lemma.** *Let X be a minimal threefold of general type with index $r\geqq 2$.*

(i) *$P(n)\neq 0$ for any $n\geqq r+2$. $P(mr)\geqq 12$ for any $m\geqq 3$.*

(ii) *The linear system $|(mr+s)K_X|$ has dimension $\geqq 3$ and is not composed of a pencil, if the triple (r, s, m) satisfies one of the following conditions:*

Case $r=2$: $m\geqq 3$.

Case $r=3$: $m\geqq 2$.

Case $r=4, 5$: $0\leqq s\leqq 2$ and $m\geqq 2$; $s\geqq 3$ and $m\geqq 1$.

Case $r\geqq 6$: $0\leqq s\leqq 1$ and $m\geqq 2$; $s\geqq 2$ and $m\geqq 1$.

Proof. By Lemma (2.2), we can put

$$(1)\qquad P(mr+s)=(1/12)(mr+s)(mr+s-1)(2mr+2s-1)(K_X^3)+am+c_s,$$

where a and c_s $(0\leqq s<r)$ are constants. We consider the right hand side of (1) as a polynomial in m and denote it by $P_s(m)$. Let $Q_s(m)$ be the first term of $P_s(m)$, i.e.,

$$P_s(m)=Q_s(m)+am+c_s.$$

Obviously,

$$(2)_{s,m}\qquad P_s(m)\geqq 0 \qquad \text{for } m\geqq 1 \text{ or } m=0,\ s\geqq 2.$$

Assume $m\leqq -1$. By the Grothendieck duality, we have

$$\boldsymbol{R}\Gamma\boldsymbol{R}\mathcal{H}om\,(\mathcal{O}_X((mr+s)K_X),\ \omega_X^{\cdot})=\boldsymbol{R}\,\mathrm{Hom}\,(\boldsymbol{R}\Gamma(\mathcal{O}_X((mr+s)K_X),\ \boldsymbol{C}).$$

Since $\mathcal{O}_X((mr+s)K_X)$ is a Cohen-Macaulay sheaf (see [19]), we have

$$\boldsymbol{R}\operatorname{Hom}(\mathcal{O}_X((mr+s)K_X), \omega_X^{\cdot})=\mathcal{O}_X((1-mr-s)K_X)[3].$$

Hence we get

$$H^{3-i}(X, \mathcal{O}_X(1-mr-s))\cong H^i(X, \mathcal{O}_X((mr+s)K_X))',$$

where the symbol $'$ denotes the dual vector space. Thus

$$\begin{aligned}-\chi(\mathcal{O}_X((mr+s)K_X))&=\chi(\mathcal{O}_X((1-mr-s)K_X))\\ &=h^0(\mathcal{O}_X((1-mr-s)K_X))\geqq 0\end{aligned}$$

by Lemma (2.1). Therefore

$$(2)_{s,m} \qquad -P_s(m)\geqq 0 \qquad \text{for } m\leqq -1.$$

We shall estimate the numbers a and c_s on the basis of $(2)_{s,m}$ ($m\geqq 1$ or $m=0$, $s\geqq 2$ or $m\leqq -1$). For any r and s,

$$(2)_{s,1} \qquad Q_s(1)+a+c_s\geqq 0$$

and

$$(2)_{s,-1} \qquad -Q_s(-1)+a-c_s\geqq 0$$

induce

$$(3)_s \qquad a\geqq(1/2)\{Q_s(-1)-Q_s(1)\}-(1/12)\{2r^2+(6s^2-6s+1)\}(rK_X^3).$$

The last term decreases in s; thus

$$(3)_0 \qquad a\geqq -(1/12)(2r^2+1)(rK_X^3)$$

is the strongest bound among $(3)_s$. Moreover if $r\geqq 3$, we have for $s\geqq 2$,

$$(2)_{s,0} \qquad Q_s(0)+c_s\geqq 0.$$

By $(2)_{s,-1}$ and $(2)_0$,

$$(4)_s \qquad \begin{aligned}a&\geqq -Q_s(0)+Q_s(-1)\\ &=(1/12)\{-2r^2+(6s-3)r-(6s^2-6s+1)\}(rK_X^3).\end{aligned}$$

Among them, the most strict estimate is

$$(4)_{(r+1)/2} \qquad a\geqq(1/12)\{-(1/2)r^2+1/2\}(rK_X^3) \qquad \text{if } r \text{ is odd},$$

$$(4)_{(r/2)} \qquad a\geqq(1/12)\{-(1/2)r^2-1\}(rK_X^3) \qquad \text{if } r \text{ is even}.$$

Next we assume that $|nK_X|$ is composed of a pencil. Letting $f\colon X'\to X$ to be a resolution of singularities of X and the base locus of $\Phi_{|nK_X|}$, we get a diagram

$$\begin{array}{ccc} X' & \xrightarrow{\quad g \quad} & W_n' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{\Phi_{|nK_X|}} & W_n \subset \boldsymbol{P}^{P(n)-1}. \end{array}$$

Here $W_n=\operatorname{Im}\Phi_{|nK_X|}$ is a curve and $X'\xrightarrow{g}W_n'\xrightarrow{h}W_n$ is the Stein factorization of $\Phi_{|nK_X|}\circ f\colon X'\to W_n$. Put $a=\deg W_n$, $b=\deg h$ and let F be the fixed part of $|nK_{X'}|$. Then we have

$$nK_{X'}\approx abS+F,$$

where S is a general fiber of g and the symbol $\approx$ denotes numerical equivalence. Multiplying this with $rf^*(K_X)^2$, which is a nef 1-cycle, we get

$$rn(K_X^3)\geqq ab(rS\cdot f^*(K_X^2))\geqq a.$$

But we know that $a\geqq P(n)-1$ since W_n is a non-degenerate curve in $\boldsymbol{P}^{P(n)-1}$. Thus

$$(5)\qquad 1\geqq P(n)-n(rK_X^3)$$

must be satisfied.

Now we divide the lemma into cases according to the value s. We first estimate the plurigenus formula (1) from below by a polynomial, using the restrictions on the pair (a, c_s) obtained before. (i) follows directly from the estimate. Combining this estimate with (5), we get the assertion (ii).

Case 1: $r\geqq 3$, $s\geqq 2$. If r is odd, by $(2)_{s,0}$ and $(4)_{(r+1)/2}$, we have

$$\begin{aligned} P(mr+s)&\geqq Q_s(m)+m\cdot(-1/12)\cdot(1/2\cdot r^2+1/2)\cdot(rK_X^3)-Q_s(0)\\ &\geqq(1/12)[(mr+s)(mr+s-1)(2mr+2s-1)\\ &\qquad +m\{-(1/2)r^3+(1/2)r\}-s(s-1)(2s-1)](K_X^3).\end{aligned}$$

From this inequality we deduce

(i) $P(mr+s)>0$ for $m\geqq 1$.

(ii) $P(mr+s)-(mr+s)(rK_X^3)>1$ and $P(mr+s)\geqq 4$ if $m\geqq 2$ or $m=1$, $s\geqq 3$, $r\geqq 5$ or $m=1$, $s=2$, $r\geqq 7$.

If r is even, by $(2)_{s,0}$ and $(4)_{r/2}$, we obtain

$$P(mr+s)\geqq(1/12)[2r^2m^3+(6s-3)rm^2+(6s^2-6s-(1/2)r^2)m](rK_X^3).$$

Hence

(i) $P(mr+s)>0$ for $m\geqq 1$.

(ii) $P(mr+s)-(mr+s)(rK_X^3)>1$ and $P(mr+s)\geqq 4$ if $m\geqq 3$ or $m=2$, $s\geqq 3$, $r\geqq 4$ or $m=2$, $s=2$, $r\geqq 6$.

Case 2: $s=1$. By $(2)_{1,1}$ and $(2)_{1,-1}$, we get

$$P(mr+1)\geqq(1/12)(m^2-1)r(2rm+3)(rK_X^3).$$

Hence

(i) $P(mr+1)>0$ for $m\geqq 2$.

(ii) $P(mr+1)-(mr+1)(rK_X^3)>1$ and $P(mr+1)\geqq 4$ if $m\geqq 3$ or $m=2$, $r\geqq 3$.

Case 3: $s=0$. By $(2)_{0,1}$ and $(2)_{0,-1}$, we have

$$P(mr)\geqq(1/12)(m^2-1)r(2rm-3)(rK_X^3).$$

Thus

(i) $P(mr)>0$ for $m\geqq 2$. $P(mr)\geqq 12$ for $m\geqq 3$.

(ii) $P(mr)-mr(rK_X^3)>1$ and $P(mr)\geqq 4$ if $m\geqq 3$ or $m=2$, $r\geqq 4$.

For $r=3$, using $(4)_2$ and $(2)_{0,1}$, we get

$$P(3m)-3m(3K_X^3)>1 \qquad \text{for } m=2.$$

(3.3) **Proposition.** *Let X be a minimal threefold of index $r\geqq 2$. Then $\Phi_{|nK_X|}$ is a birational map if s, a, k, n are integers satisfying the following conditions*:

(1) $0\leqq s<r$,

(2) $|(ar+s)K_X|$ *is not composed of a pencil,*

(3) $P(n-ar-s)\neq 0$,

(4) $P(n-kr-s-1)\neq 0$,

(5) $k-a\geqq 3$,

(6) $P(ar+s)\geqq 4$,

(7) $P((k-a)r)\geqq 9$.

Proof. Combining (2) and (3), we obtain the following diagram:

$$\begin{tikzcd}
X' \arrow[d, "f"'] \arrow[dr, "g"] & \\
X \arrow[r, dashed, "\Phi_{|(ar+s)K_X|}"] \arrow[d, dashed, "\Phi_{|nK_X|}"'] & W_{ar+s} \\
W_n \arrow[ur, dashed, "h"'] &
\end{tikzcd}$$

Here f is a proper birational morphism resolving singularities of X and the base locus of $|(ar+s)K_X|$, $g:=\Phi_{|(ar+s)K_X|}\circ f$, and h is the projection induced from an inclusion $|(ar+s)K_X|\subset|nK_X|$. We denote by $|M|$ the movable part of $|(ar+s)K_{X'}|$, which is an irreducible linear system by Bertini's theorem. Obviously, $\Phi_{|nK_X|}\circ f=\Phi_{|nK_{X'}|}$. By (3), we have inclusions

$$\begin{aligned}|nK_{X'}|&\supset|(n-kr-s)K_{X'}+(k-a)f^*(rK_X)+S|\\&\supset|K_{X'}+(k-a)f^*(rK_X)+S|\end{aligned}$$

for a general member $S\in|M|$. Note that S is a non-singular irreducible surface. Thus $\Phi_{|nK_{X'}|}$ is birational if so is $\Phi_{|K_{X'}+(k-a)f^*(rK_X)+S|}$.

The exact sequence

$$\begin{aligned}0&\longrightarrow\mathcal{O}_{X'}(K_{X'}+(k-a)f^*(rK_X))\longrightarrow\mathcal{O}_{X'}(K_{X'}+(k-a)f^*(rK_X)+S)\\&\longrightarrow\mathcal{O}_S(K_S+(k-a)f^*(rK_X)|_S)\longrightarrow0\end{aligned}$$

induces the long exact sequence of cohomologies:

$$\begin{aligned}0&\longrightarrow H^0(X',\mathcal{O}_{X'}(K_{X'}+(k-a)f^*(rK_X)))\\&\longrightarrow H^0(X',\mathcal{O}_{X'}(K_{X'}+(k-a)f^*(rK_X)+S))\\&\longrightarrow H^0(S,\mathcal{O}_S(K_S+(k-a)f^*(rK_X)|_S))\\&\longrightarrow H^1(\mathcal{O}_{X'}(K_{X'}+(k-a)f^*(rK_X))),\end{aligned}$$

where $H^1(X',\mathcal{O}_{X'}(K_{X'}+(k-a)f^*(rK_X)))=0$ by Lemma (2.1), since $f^*(rK_X)$ is nef and big. Therefore, letting $R=f^*(rK_X)$, we have $\Phi_{|K_{X'}+(k-a)f^*(rK_X)+S|}|_S=\Phi_{|K_S+(k-a)R|}$. Hence, $\Phi_{|K_{X'}+(k-a)f^*(rK_X)+S|}$ is birational if and only if so is $\Phi_{|K_S+(k-a)R|}$. We shall show that $\Phi_{|K_S+(k-a)R|}$ is a birational map by making use of Lemma (3.1).

First note that $(R^2)=(f^*(rK_X)^2\cdot S)=r(rf^*(K_X^2)\cdot S)\geqq r\geqq2$, and $k-a\geqq3$. Given two points x, y of S, we denote by $\mu\colon\tilde{X}'\to X'$ the blow-up of X' at x and y, which induces a birational morphism $\mu_S\colon\tilde{S}\to S$. We put $E_x=\mu^{-1}(x)$, $E_y=\mu^{-1}(y)$ and $L_x=E_x\cap\tilde{S}$, $L_y=E_y\cap\tilde{S}$. The linear system $|(k-a)\mu^*f^*(rK_X)-2E_x-2E_y|$ is not empty, since its dimension $\geqq P((k-a)r)-9\geqq0$ by (7).

Claim. $\tilde{S}\not\subset F$, where F is defined to be the fixed part of

$$|(k-a)\mu^*f^*(rK_X)-2E_x-2E_y|.$$

Proof. Otherwise $h^0(\tilde{X}',\mathcal{O}_{\tilde{X}'}(\tilde{S}))=1$. On the other hand, $\tilde{S}=\mu^*(S)-E_x-E_y$. Thus

$$h^0(\tilde{X}',\mathcal{O}_{\tilde{X}'}(\mu^*(S)-E_x-E_y))\geqq P(ar+s)-2\geqq2,$$

by (6), a contradiction.

By the above claim we deduce

$$H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}((k-a)R-2L_x-2L_y))\neq 0.$$

Now we have checked all the conditions needed. Applying Lemma (3.1), we complete the proof of Proposition (3.3).

(3.4) **Theorem.** *Let X be a non-singular threefold of general type which has a minimal model of index r. Then $\Phi_{|nK_X|}$ is a birational map for $n\geqq n_0$, where*

$$\begin{aligned} &n_0=9 && \textit{if } r=1,\\ &n_0=13 && \textit{if } r=2,\\ &n_0=4r+4 && \textit{if } 3\leqq r\leqq 5,\\ &n_0=4r+3 && \textit{if } r\geqq 6. \end{aligned}$$

Proof. To prove the theorem we may assume that X is itself a minimal model.

When $r=1$, under the notation of Lemma (2.3), $(c_2(\tilde{X})\cdot\Delta)=0$ (see Reid [22]), hence we have the same plurigenus formula as that for a non-singular minimal threefold. Thus we can prove that $\dim\Phi_{|nK_X|}\geqq 2$ for $n\geqq 4$ by the argument of Benveniste [1] or Matsuki [21, Theorem 7], and thus conclude that $\Phi_{|nK_X|}$ is birational for $n\geqq 9$ by the argument of [21, Theorem 8, Case 1].

We shall next assume $r\geqq 2$ and shall apply Proposition (3.3). Let (r, s, m) satisfy the condition (ii) in Lemma (3.2). Put $a=m$, $k=a+3$, $n=kr+s+1$. We check the conditions (1) through (7) in Proposition (3.3). (1) and (5) are trivial. (2) and (6) are satisfied by Lemma (3.2), (ii), while (7) by Lemma (3.2) (i). (4) is clear, since $n-kr-s-1=0$. The rest of the proof is quite easy.

(3.5) **Remark.** Recently the author learned that, for a minimal threefold X,

(1) we can find a terminal minimal model which is $\boldsymbol{Q}$-factorial (Kawamata);

(2) letting $\mu\colon Y\to X$ be a resolution of singularities, the 1-cycle $\mu_*(3c_2(Y)-c_1(Y)^2)$ is pseudo-effective (Miyaoka).

Using these results, we can prove that, in case $r=1$, we can take $n_0=7$.

(3.6) In our theorem, the number n_0 depends on r. It is natural to pose:

Problem 1. Is it possible to choose an n_0 such that for any non-singular threefold of general type X, $\Phi_{|n_0K_X|}$ is a birational map?

We feel that such an absolute number n_0 might not exist. By Kollár [18], we know that the above problem is equivalent to the following:

Problem 2. Does there exist a positive integer n_1 such that $P_{n_1}(X) \geqq 2$ for any non-singular threefold of general type?

Even the answer to the following question is unknown.

Problem 3. For an arbitrarily large m, does there exist a non-singular threefold of general type X with $P_m(X)=0$?

It is certain by Lemma (3.2) that if such examples exist and they have minimal models, the indices are unbounded.

§ 4. Stability of the pluricanonical maps of threefolds of fiber type

(4.1) In this section we let X to be a non-singular projective threefold with $\kappa(X)=1$ or 2. We use the following notation:

Assume that $\Phi_{|lK_X|}$ is the stable canonical map, where l is a positive integer. By taking suitable non-singular birational models X' and Y of X and $\operatorname{Im} \Phi_{|lK_X|}$, respectively, we get a morphism $X' \to Y$. Replacing X' by X, we have a fiber space $f: X \to Y$ which is birationally equivalent to the stable canonical map.

Let $\alpha: X \to A$ be the Albanese map of X, $\alpha(X)$ the image of α, and $X \xrightarrow{\beta} Z \xrightarrow{\gamma} \alpha(X)$ the Stein factorization of $\alpha: X \to \alpha(X)$. Put $P(n)=\dim H^0(X, \omega_X^{\otimes n})$ as before.

(4.2) **Proposition.** *Let X be a non-singular projective threefold with $\kappa(X)=1$ or 2. Suppose $P(n) \geqq 2$ for some positive integer n.*

(1) *If $\kappa(X)=1$, there exists a positive integer m_1, depending only on n, such that $\Phi_{|12mK_X|}$ is the stable canonical map for $m \geqq m_1$.*

(2) *If $\kappa(X)=2$, there exists a positive integer m_2, depending only on n, such that $\Phi_{|mK_X|}$ is the stable canonical map for $m \geqq m_2$.*

Remark. As the proof below shows, the following suffice:

$$m_1 = 12 \cdot (24n^2 + 26n + 1);$$
$$m_2 = 2n^2 + 28n + 13.$$

Proof. We use an argument similar to that in Kollár [18, Theorem 4.6].

Assume that $\kappa(X)=2$ (resp. 1). Take a subpencil Λ of the linear system $|nK_X|$. By blowing up the base locus of Λ, we get a birational morphism $X' \to X$ and a morphism $g: X' \to \boldsymbol{P}^1$. Replacing X' by X, we may assume that we have a morphism $g\colon X\to \boldsymbol{P}^1$ and an inclusion of invertible sheaves $g^*\mathcal{O}_{\boldsymbol{P}^1}(1)\to\omega_X^{\otimes n}$.

We denote by S a general fiber of g, which is a non-singular surface, not necessarily connected. By the easy addition formula of the Kodaira dimension (see Iitaka [7]) applied to the Stein factorization of g, we have $\kappa(X)=2$ (resp. $1)\leqq\kappa(S_0)+1$, where S_0 is a connected component of S. Hence $\kappa(S_0)\geqq 1$ (resp. 0).

Let k be a positive integer such that

$$\begin{aligned}&k \text{ is divisible by } 12 \quad \text{if } \kappa(S_0)=0,\\ &k\geqq 14 \quad \text{if } \kappa(S_0)=1,\\ &k\geqq 5 \quad \text{if } \kappa(S_0)=2.\end{aligned}$$

Claim. A general fiber of the $(2nk+n+k)$-ple canonical map of X is irreducible.

Proof. $g_*(\omega_{X/\boldsymbol{P}^1}^{\otimes k})$ is a semipositive vector bundle on $\boldsymbol{P}^1$ by Kawamata [12]. Thus it can be expressed in the form $\oplus_{i=1}^{r}\mathcal{O}_{\boldsymbol{P}^1}(a_i)$, where $r=\operatorname{rank} g_*(\omega_{X/\boldsymbol{P}^1}^{\otimes k})$ and a_i are non-negative integers. Hence we obtain the inclusions of the linear systems:

$$|(2nk+n+k)K_X|\supset|kK_X+(2k+1)g^*H|\supset|g^*H|,$$

where we denote by H the hyperplane section of $\boldsymbol{P}^1$. Thus if we denote by W the image of the rational map $\Phi_{|(2nk+n+k)K_X|}$, there exists a rational map $h\colon W\cdots\to\boldsymbol{P}^1$ and a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, dotted, "\Phi_{|(2nk+n+k)K_X|}"'] & \boldsymbol{P}^1 \\ W \arrow[ur, dotted, "h"'] & \end{tikzcd}$$

From the above diagram we see easily that the claim is equivalent to saying that the restriction of $\Phi_{|(2nk+n+k)K_X|}$ to S is a fiber space. Thus it suffices to prove that the restriction of $\Phi_{|kK_X+(2k+1)g^*H|}$ to S is a fiber space with the image of dimension $\kappa(S_0)$.

Since $g_*\mathcal{O}_X(kK_X+(2k+1)g^*H)=g_*(\omega_{X/\boldsymbol{P}^1}^{\otimes k})\otimes H$ is a vector bundle generated by global sections, the restriction map

$$H^0(X,\mathcal{O}_X(kK_X+(2k+1)g^*H))\longrightarrow H^0(S,\mathcal{O}_S(kK_S))$$

is surjective for a general S. This means that the restriction of $\Phi_{|kK_X+(2k+1)g^*H|}$ to S coincides with $\Phi_{|kK_S|}$. By Bombieri [2] and Katsura-Ueno [8], we know that the k-ple canonical map of S is stable.

We go back to the proof of the proposition. Assume that $\kappa(X)=2$ (the case $\kappa(X)=1$ is similar). We infer that $\kappa(S)=1$, since $\dim W=\kappa(S)+1=2$. For each i with $0\leqq i<n$, let $t_i=\ulcorner(14-i)/n\urcorner$. Since $t_in+i\geqq 14$, the $\{(2n+1)\cdot(t_in+i)+n\}$-ple canonical map is stable. By $t_in\leqq 13-i+n$, we have $(2n+1)(t_in+i)+n\leqq 2n^2+28n+13$. Any integer $m\geqq 2n^2+28n+13$ can be written as $m=(2n+1)(t_in+i)+n+$(some positive multiple of n), where $i\equiv m$ modulo n. Hence we complete the proof of Proposition (4.2).

(4.3) Recall a conjecture by Iitaka and Viehweg: Let $f\colon V\to W$ be a fiber space between non-singular projective varieties with $\dim V=n$ and $\dim V=m$. Then the following "*addition formula*" should hold:

$$C^+_{n,m}\colon \kappa(V)\geqq\kappa(V_w)+\operatorname{Max}(\kappa(W),\operatorname{var}(f))\qquad \text{if } \kappa(W)\geqq 0.$$

This "addition formula" is known to be true in some cases among which are:

$C^+_{n,n-1}$ by Viehweg [25];

$C^+_{n,n-2}$ by Kawamata [14].

(4.4) If we apply to $f\colon X\to Y$ the addition formulas $C^+_{3,1}$ and $C^+_{3,2}$, we obtain

$$(C^+)_f\colon \kappa(X)\geqq\operatorname{Max}(\kappa(Y),\operatorname{var}(f))\qquad \text{if } \kappa(Y)\geqq 0.$$

If the right hand side is positive, we can estimate $P(n)$ from below:

(4.5) **Proposition.** *We have some positive integer n_0 which satisfies the following*:

Let X be a non-singular threefold with $\kappa(X)=1$ or 2. Suppose that $\kappa(Y)\geqq 0$, and $\operatorname{Max}(\kappa(Y),\operatorname{var}(f))>0$. *Then $P(n_0)\geqq 2$.*

Remark. It suffices to put $n_0=2\cdot 24^2$.

Proof. (1) $\kappa(X)=1$.

(i) *Case* $\operatorname{var}(f)>0$. A general fiber of f is a non-singular surface with the Kodaira dimension 0. Therefore, in the argument of Kawamata [14, Section 6 and Step 1 of Section 7], we can take $m_0=12$, $i=13$, $m=24$, and $k=m^2=24^2$, and we have $\kappa(f_*(\omega_{X/Y}^{\otimes k}),Y)=1$. Hence $d:=\deg f_*(\omega_{X/Y}^{\otimes k})>0$.

Using the Riemann-Roch formula, we deduce

$$\begin{aligned} h^0(Y,(f_*(\omega_{X/Y}^{\otimes k}))^{\otimes l}\otimes\omega_Y^{\otimes kl}) &\geqq \chi(Y,(f_*(\omega_{X/Y}^{\otimes k}))^{\otimes l}\otimes\omega_Y^{\otimes kl})\\ &=(1-g(Y))+dl+kl(2g(Y)-2)\geqq dl\geqq 2\end{aligned}$$

for any integer $l \geqq 2$, $g(Y)$ being the genus of Y. Since there is a natural inclusion $(f_*(\omega_X^{\otimes k}))^{\otimes l} \to f_*(\omega_X^{\otimes kl})$, we get $P(kl) \geqq 2$ for $l \geqq 2$.

(ii) *Case* $\kappa(Y)>0$. In this case, $g(Y)>0$ and $\deg f_*(\omega_{X/Y}^{\otimes 12}) \geqq 0$. Thus

$$h^0(Y, f_*(\omega_X^{\otimes 12})) \geqq \chi(Y, f_*(\omega_X^{\otimes 12}))$$
$$= 23(g(Y)-1) + \deg f_*(\omega_{X/Y}^{\otimes 12}) \geqq 23.$$

Hence $P(12) \geqq 2$.

(2) $\kappa(X)=2$.

We may assume, by changing models, that there exists a Zariski open subset Y_0 of Y over which f is smooth and $D := Y \setminus Y_0$ is a divisor with normal crossings on Y.

(i) *Case* var $(f)>0$. Let $\{D_i\}$ be the irreducible components of D. The canonical bundle formula in Kawamata [12] says that

$$(f_*(\omega_{X/Y}))^{\otimes 12} = \mathcal{O}_Y(\Sigma(12a_i \cdot D_i)) \otimes J^* \mathcal{O}_{P^1},$$

where a_i are non-negative rational numbers determined by the types of the singularities over general points of D_i and J is the period map associated with f. We thus have inclusions

$$f_*(\omega_X^{\otimes 12}) \supset (f_*(\omega_X))^{\otimes 12} \supset J^* \mathcal{O}_{P^1}$$

since $\omega_Y^{\otimes 12}$ has a global section. Hence $P(12) \geqq h^0(\boldsymbol{P}^1, \mathcal{O}_{P^1}(1)) = 2$.

(ii) *Case* $\kappa(Y)=1$. In this case, by the same canonical bundle formula as above, we have $f_*(\omega_{X/Y})^{\otimes 12} \supset \mathcal{O}_Y$. Hence

$$f_*(\omega_X^{\otimes 24}) \supset (f_*(\omega_X))^{\otimes 24} \supset \omega_Y^{\otimes 24}$$

and thus $P(24) \geqq 2$ by Katsura-Ueno [8].

(iii) *Case* $\kappa(Y)=2$. Similar to the case (ii) above. The inclusions

$$f_*(\omega_X^{\otimes 12}) \supset (f_*(\omega_X))^{\otimes 12} \supset \omega_Y^{\otimes 12}$$

and $P_{12}(Y) \geqq 2$ imply that $P(12) \geqq 2$.

(4.6) **Theorem.** *There exist positive integers n_1 and n_2 such that the following holds:*

For a non-singular projective threefold X with $\kappa(X)=1$ or 2, $\Phi_{|nK_X|}$ is the stable canonical map for

$$n = 12n',\ n' \geqq n_1 \qquad \text{if } \kappa(X)=1;$$
$$n \geqq n_2 \qquad \text{if } \kappa(X)=2,$$

provided that X belongs to one of the families S_i ($i=0, 1, \cdots, 4$) below (see (4.1) for the notation):

$S_0:=\{X \mid \kappa(Y)\geqq 0$, *and* $\operatorname{Max}(\kappa(Y), \operatorname{var}(f))>0.\}$;
$S_1:=\{X \mid q(X)>0$, *and* α *is not surjective.*$\}$;
$S_2:=\{X \mid q(X)=3$, *and* α *is surjective.*$\}$;
$S_3:=\{X \mid q(X)=1$ *or* 2, α *is surjective,* $\kappa(\beta)=0$, *and* $\operatorname{Max}(\operatorname{var}(\beta), \kappa(Z))>0.\}$;
$S_4:=\{X \mid q(X)=1$, α *is surjective, and* $\operatorname{var}(\beta)>0\}$.

Those X which belong to neither of the families above are classified as follows; note that we have either (I) $q(X)=0$, *or* (II) $q(X)=1$ *or* 2 *and* X *is a quotient by a finite group of* (*curve*)$\times$(*surface*); *more precisely,* $\operatorname{var}(f)=0$ *or* $\operatorname{var}(\beta)=0$ *in each case.*

(I) *Case* $q(X)=0$.
(0.a) $\kappa(Y)=-\infty$;
(0.b) $\kappa(Y)=\operatorname{var}(f)=0$.

(II) *Case* $q(X)=1$ or 2, *and* α *is surjective.*
Subcase 1. $\kappa(Y)=\operatorname{var}(f)=0$ *and* $\kappa(\beta)=\operatorname{var}(\beta)=\kappa(Z)=0$.
(1.a) $q(X)=\kappa(X)=1$;
(1.b) $q(X)=2$, *and* $\kappa(X)=1$;
(1.c) $q(X)=\kappa(X)=2$.
Subcase 2. $\kappa(Y)=\operatorname{var}(f)=0$, *and* $\kappa(\beta)>0$.
(2.a) $q(X)=1$, *and* $\kappa(X)=2$;
(2.b) $q(X)=2$, *and* $\kappa(X)=2$, *and* $\operatorname{var}(\beta)=0$.
Subcase 4. $\kappa(Y)=-\infty$, *and* $\kappa(\beta)>0$.
(4.a) $q(X)=\kappa(X)=1$, *and* $\operatorname{var}(\beta)=0$;
(4.b) $q(X)=2$, $\kappa(X)=1$, *and* $\operatorname{var}(\beta)=0$;
(4.c) $q(X)=1$, $\kappa(X)=2$, *and* $\operatorname{var}(\beta)=0$;
(4.d) $q(X)=\kappa(X)=2$, *and* $\operatorname{var}(f)=0$.

Remark. (i) Let k_i be the following integers:

$$k_0=2\cdot 24^2,\quad k_1=6,\quad k_2=4,\quad k_3=2\cdot 24^2,\quad k_4=2\cdot 28^2.$$

Then $P_{k_i}(X)\geqq 2$ for any $X\in S_i$. We thus can calculate the values of n_1 and n_2 by making use of Remark to Proposition (4.2):

$$n_1=12\cdot(24\cdot 2^2\cdot 24^4+26\cdot 2\cdot 24^2+1);$$
$$n_2=2\cdot 2^2\cdot 28^4+28\cdot 2\cdot 28^2+13.$$

(ii) There is no Subcase 3 in (II) for a reason having to do with the way the proof is organized.

Proof. Let us say that a family S of threefolds with the Kodaira dimension 1 or 2 satisfies the condition ($*$) if there exists a positive integer

n such that $P(n) \geqq 2$ for all $X \in S$.

By Proposition (4.5), the family S_0 satisfies the condition $(*)$. We next claim that S_1 satisfies $(*)$.

Indeed, note that for any $X \in S_1$, $\alpha(X)$ is not a translation of an abelian subvariety of A. Thus $\kappa(\alpha(X))>0$ and for some abelian subvariety $B \subset A$, we have the following Cartesian diagram:

$$\begin{tikzcd} \alpha(X) \arrow[r, hook] \arrow[d, "p"'] & A \arrow[d, "\pi"] \\ Z \arrow[r, hook] & A/B, \end{tikzcd}$$

where π denotes the natural projection and $p: \alpha(X) \to Z$ is an analytic fiber bundle with fiber B over a subvariety $Z \subset A/B$ of general type (see Ueno [24]). By assumption, $\dim Z=1$ or 2. The same argument as that in Kollár [18, Theorem 6.1] proves that $P(6) \geqq 2$. (See the Case 2 and replace $\omega_{X/C}^{\otimes 12}$ by $\omega_{X/C}^{\otimes 2}$ in the Case 3.)

The family S_2 satisfies $(*)$. This fact is also contained in Kollár [18, Proposition 4.3], where he proved that $P(4) \geqq 2$ for all $X \in S_2$.

By the same argument as in Proposition (4.5), we can show that S_3 satisfies $(*)$. S_4 also satisfies $(*)$ by the same argument as in Proposition (4.5), (1) (i).

It remains to classify those X which belong to neither of the families S_i. They satisfy:

(1) $\kappa(Y)=-\infty$ or $\kappa(Y)=\operatorname{var}(f)=0$.

(2) $q(X) \leqq 2$, and α is surjective.

(3) If $q(X)=1$ or 2, then $\kappa(\beta)>0$ or $\kappa(\beta)=\operatorname{var}(\beta)=\kappa(Z)=0$.

(4) If $q(X)=1$ and α is surjective, then $\operatorname{var}(\beta)=0$.

In the case $q(X)=0$, we have (0.a) or (0.b); we thus assume $q(X)=1$ or 2 in the following.

1. *Case where* $\kappa(Y)=\operatorname{var}(f)=0$ *and, moreover,* $\kappa(\beta)=\operatorname{var}(\beta)=\kappa(Z)=0$. Since a general fiber of β is contracted to a point by f, there exists a rational map $h: Z \dashrightarrow Y$ which makes the following diagram commutative:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\beta"'] & Y \\ Z \arrow[ur, dotted, "h"'] & \end{tikzcd}$$

Therefore, $q(X) \geqq \kappa(X)$, hence the possible cases are:

(1.a) $q(X)=\kappa(X)=1$;

(1.b) $q(X)=2$ and $\kappa(X)=1$;

(1.c) $q(X)=\kappa(X)=2$. |

2. *Case where* $\kappa(Y)=\mathrm{var}\,(f)=0$ *and, moreover,* $\kappa(\beta)>0$. We shall prove $\kappa(X)=2$ in this case. Assume $\kappa(X)=1$. Since Y is an elliptic curve, we have a morphism $g\colon A\to Y$ and a commutative diagram:

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[dl, "\beta"'] \arrow[d, "\alpha"] & Y \\ Z \arrow[r, "\gamma"'] & A \arrow[ur, "g"'] \end{tikzcd}$$

If $q(X)=1$, then g must be an isomorphism, $A\cong Z$ and $\alpha=\beta$. This contradicts $\kappa(\beta)>0$.

If $q(X)=2$, take a general point $y\in Y$ and consider the fibers X_y and Z_y of $\gamma\circ g$, respectively. Denote by $\beta_y\colon X_y\to Z_y$ the induced fiber space. By the addition formula $C^+_{2,1}$ in (4.4), we have

$$\kappa(X_y)=0\geqq\kappa(\beta)+\kappa(Z_y).$$

But $\kappa(\beta)>0$ and $\kappa(Z_y)\geqq 0$ by our hypothesis, which is in contradiction to the above inequality. Thus we have either

(2.a) $q(X)=1$, $\kappa(X)=2$, and $\mathrm{var}\,(\beta)=0$, or

(2.b) $q(X)=2$ and $\kappa(X)=2$.

3. *Case where* $\kappa(Y)=-\infty$ *and, moreover,* $\kappa(\beta)=\mathrm{var}\,(\beta)=\kappa(Z)=0$. As in the case 1, we have a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\beta"'] & Y \\ Z \arrow[ur, dotted, "h"'] & \end{tikzcd}$$

If $\dim Z=\dim Y$, then h is a birational map. This contradicts $\kappa(Y)=-\infty$ and $\kappa(Z)=0$. Thus we have $\dim Z=2$ and $\dim Y=1$, which we now show to be a contradiction.

Indeed, a general fiber Z_y of h is an irreducible curve. Applying to $\beta_y\colon X_y\to Z_y$ the addition formula $C^+_{2,1}$, we have $\kappa(X_y)=0\geqq\kappa(X_z)+\kappa(Z_y)$, where X_z is a general fiber of β. Hence $\kappa(Z_y)=0$. By Ueno [24], Z_y is a translate of an abelian subvariety of Z. Since there exist only countably many abelian subvarieties of Z, we can take an abelian subvariety $E\subset Z$ of dimension one in such a way that the set $\{y\in Y\,|\,\text{the fiber } Z_y \text{ over } y \text{ are translations of } E.\}$ is a dense subset of Y. Therefore, h factors through the quotient map $q\colon Z\to Z/E$:

$$\begin{tikzcd} Z \arrow[r, dotted, "h"] \arrow[d, "q"'] & \boldsymbol{P}^1 \\ Z/E \arrow[ur] & \end{tikzcd}$$

Since g is a fiber space, we necessarily have $Z/E \cong \boldsymbol{P}^1$, which is a contradiction.

4. *Case where* $\kappa(Y) = -\infty$ *and moreover* $\kappa(\beta) > 0$. Assume $q(X)=2$ and $\kappa(X)=1$. Then we must have $\kappa(\beta)=1$ and $\mathrm{var}(\beta)=0$ by $C_{3,1}^+$ in (4.3).

Next we consider the case $q(X)=\kappa(X)=2$. A general fiber X_y of f cannot be contracted by β. Thus $\alpha(X_y)$ is an elliptic curve. Hence we have $\mathrm{var}(f)=0$.

Thus the following four cases are possible and we conclude the proof of the Theorem (4.6):

(4.a) $q(X)=\kappa(X)=1$ and $\mathrm{var}(\beta)=0$;
(4.b) $q(X)=2$, $\kappa(X)=1$ and $\mathrm{var}(\beta)=0$;
(4.c) $q(X)=1$, $\kappa(X)=2$ and $\mathrm{var}(\beta)=0$;
(4.d) $q(X)=\kappa(X)=2$ and $\mathrm{var}(f)=0$.

(4.7) **Remarks.** (1) We have examples of varieties for each type (n, α) ($n=0, 1, 2, 4$; $\alpha=a, b, c, d$) listed in the statement, though we shall not describe them in this paper.

(2) The author does not know whether there exists a positive integer n_1 (resp. n_2) such that $\Phi_{|n_1K_X|}$ (resp. $\Phi_{|n_2K_X|}$) is the stable canonical map for any threefold X with $\kappa(X)=1$ (resp. $\kappa(X)=2$).

(3) Another possible approach to the problem of stability is to establish an "exact" plurigenus formula (cf. an asymptotic one in (5.2)).

(4.8) **Example.** Here is a sample question to show that the remaining cases in (4.6) are not so easy to deal with: *for an arbitrarily large integer* n, *is there a non-singular projective surface* S *of general type which admits an action of a cyclic group* G *such that* $H^0(S, \mathcal{O}_S(nK_S))^G=0$? We show below that, if such an S exists, the answer to the question (2) is negative.

Take an elliptic curve E. Let $a \in E$ be a point of order $m=\#G$, and let g be a generator of G. Define an action of G on E as $g \cdot x = x+a$ for any $x \in E$. Consider the diagonal action of G on $S \times E$. Letting $X=(S \times E)/G$, we obtain the following diagram:

$$\begin{tikzcd} X \arrow[d, "f"'] & S \times E \arrow[l, "\pi"'] \arrow[d, "p_2"] \\ E/G & E \arrow[l, "\tau"'] \end{tikzcd}$$

Here τ and π are the quotient maps, p_2 the second projection, and f the induced morphism. Since π is étale, we have $\kappa(X)=\kappa(S \times E)=2$. For the integer n in question we have

$$H^0(X, \mathcal{O}_X(nK_X)) = H^0(S\times E, \mathcal{O}_{S\times E}(nK_{S\times E}))^G$$
$$= H^0(S, \mathcal{O}_S(nK_S))^G = 0$$

by hypothesis.

§ 5. Further results on pluricanonical systems on varieties of arbitrary dimension

We now turn to the case of an arbitrary dimension. It is rather easy to derive the following general theorem, although the evaluation is far from the best possible one.

(5.1) **Theorem.** *Let d be a fixed positive integer. There exists a linear function $n_0(r)$ of r such that for any minimal d-fold X of general type with index r, the n-ple canonical map $\Phi_{|nK_X|}$ is birational for $n \geqq n_0(r)$.*

Proof. We put $P(n) = h^0(X, \omega_X^{[n]})$ as before. By the Riemann-Roch formula and Lemma (2.1), we can write

$$P(mr) = (rK_X)^d \cdot P_0(m),$$

with

$$P_0(m) = m^d/d! + a_1 m^{d-1} + \cdots + a_d,$$

where a_i $(1 \leqq i \leqq d)$ are rational numbers. We shall prove that there exists a number m_0 such that for some m with $2 \leqq m \leqq m_0$,

$$(*) \qquad P(mr) \geqq (rK_X)^d \cdot m^{d-1} + d$$

is satisfied. Then by Lieberman-Mumford [20, Lemma 2.1], $\Phi_{|mrK_X|}$ is a generically finite map. By Wilson [27], we conclude that $\Phi_{|(1+mr)K_X|}$ is a birational map.

We may assume that $(*)$ is not satisfied for $2 \leqq m \leqq d+1$. Then

$$P_0(m) < m^{d-1} + d \qquad \text{for } 2 \leqq m \leqq d+1,$$

or explicitly,

$$0 \leqq 2^{d-1} \cdot a_1 + \cdots + a_d + 2^d/d! < 2^{d-1} + d,$$
$$\cdots\cdots\cdots\cdots$$
$$0 \leqq (d+1)^{d-1} \cdot a_1 + \cdots + a_d + (d+1)^d/d! < (d+1)^{d-1} + d.$$

Thus the d-ple $(a_1, \cdots, a_d)$ must be contained in a bounded set in $\boldsymbol{R}^d$. In particular, there exists a positive number M such that $a_i \geqq -M$ for all i. Hence, if we take a number m_0 sufficiently large, then we have

$$P_0(m)-(m^{d-1}+d)\geqq(m^d/d!-M\cdot m^{d-1}-\cdots-M)-(m^{d-1}+d)\geqq 0$$

for any $m\geqq m_0$, which shows our claim.

(5.2) **Proposition** (asymptotic plurigenus formula). *Let X be a good minimal model (see Kawamata* [15] *for the definition) with index r, and the Kodaira dimension $\kappa=\kappa(X)$. Then there exists a positive integer m which is a multiple of r, a positive rational number a, and polynomials $P_s(t)\in \boldsymbol{Q}[t]$ $(0\leqq s<m)$ such that*

$$P_s(t)=0, \quad \textit{or}$$
$$P_s(t)=a\cdot t^{\kappa}+(\textit{lower terms})$$

for each s $(0\leqq s<m)$ and that

$$P(km+s)=P_s(k) \qquad \textit{for any positive integer } k.$$

Proof. Let m be a sufficiently large multiple of r such that the invertible sheaf $\mathcal{O}_X(mK_X)$ is generated by global sections. Taking a general member $D\in|mK_X|$, we construct the cyclic covering

$$\pi\colon X'=\mathrm{Spec}\,\oplus_{i=0}^{m-1}\mathcal{O}_X(-iK_X)\longrightarrow X$$

in a natural manner. Kawamata [15] proved that X' has only rational Gorenstein singularities.

Let $\varphi=\Phi_{|mK_X|}\colon X\to Y=\mathrm{Im}\,\Phi_{|mK_X|}$ be the associated projective morphism. There exists an ample line bundle L on Y such that $mK_X=\varphi^*L$. Let $\varphi'=\varphi\circ\pi\colon X'\to Y$ be the composition.

For any positive integer k, we have $H^i(Y,\varphi'_*\mathcal{O}_{X'}(K_{X'})\otimes L^{\otimes k})=0$ for $i>0$ by the vanishing theorem of Kollár [18]. Hence

$$H^i(Y,\varphi_*\mathcal{O}_X(sK_X)\otimes L^{\otimes k})=0$$

for $i>0$, $k>0$, and s $(0\leqq s<m)$. Thus

$$H^0(X,\mathcal{O}_X((km+s)K_X))=H^0(Y,\varphi_*\mathcal{O}_Y(sK_X)\otimes L^{\otimes k})$$
$$=\chi(Y,\varphi_*\mathcal{O}_Y(sK_X)\otimes L^{\otimes k}).$$

Hence we are done by the Riemann-Roch formula.

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Department of Mathematics
Faculty of Science
Osaka University
Osaka 560
Japan