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Euler Characteristics and Swan Conductors

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The purpose of this note is to describe some recent work on Euler characteristics in degenerating families of curves, with particular emphasis on mixed characteristic degenerations. Let $S=\operatorname{Spec}(\Lambda)$ be the spectrum of a complete discrete valuation ring with algebraically closed residue field. Write s (resp. η, resp. $\bar{\eta}$) for the closed (resp. generic, resp. geometric generic) point of S. Let $f\colon X\to S$ be flat and proper with fibre dimension 1. We assume X is regular and the generic fibre $X_\eta\to\eta$ is smooth. For z either s or $\bar{\eta}$, let

$$\chi(X_z)=\sum(-1)^i \dim H^i_{et}(X_z, \boldsymbol{Q}_l),$$

the étale Euler characteristic of the corresponding fibre. We ask for a formula calculating $\chi(X_s)-\chi(X_{\bar{\eta}})$.

In characteristic zero ($\Lambda=\boldsymbol{C}[[t]]$) the result is understood (cf. [2] ex. 14.1.5); dt gives a section of the sheaf $\Omega^1_{X/\boldsymbol{C}}$ of Kähler differentials, to which one can associate a localized chern class $\boldsymbol{Z}(s_f)\in \mathrm{CH}_0(X_s)$. (I will follow the notation in Fulton's book op. cit. except that I prefer to denote the Chow group of dim. n cycles by CH_n.) One gets in this case for X a degenerating family of varieties of any dimension, $\deg \boldsymbol{Z}(s_f)=(-1)^{\dim X}(\chi(X_s)-\chi(X_{\bar{\eta}}))$. Note that one can (and we will) think of $(-1)^{\dim X}\boldsymbol{Z}(s_f)$ as a local contribution to the cycle-theoretic self-intersection of the diagonal.

$$(1)\qquad (\Delta_X\cdot\Delta_X)_s=(-1)^{\dim X}\boldsymbol{Z}(s_f)=\chi(X_s)-\chi(X_{\bar{\eta}}).$$

We will be most interested in the mixed characteristic and pure characteristic p analogues of this result. There are two problems with (1) in these cases. First, in mixed characteristic, our construction of $\boldsymbol{Z}(s_f)$ doesn't make sense. (What is $\Omega^1_{X/\boldsymbol{C}}$?) Second, even in the pure char. p case, the formula is wrong! It is clear, however, from the global *Grothendieck-Ogg-Shafarevich formula* [6] that the appropriate correction factor is the *Swan conductor* (cf. [8] as well as the discussion below)

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$$\mathrm{sw}(X/S)=\dim \mathrm{Hom}_{\mathrm{gal}}(\mathrm{sw}_S, H^*(X_{\bar\eta})).$$

(Notation like $\mathrm{Hom}(A, H^*(X_{\bar\eta}))$ or $\mathrm{tr}\,\sigma|H^*(X_{\bar\eta})$ are used as shorthand for the familiar alternating sums.) In the pure char. p case, the local formula for a family of curves reads

$$(2) \qquad (\Delta_X \cdot \Delta_X)_s = -\mathrm{sw}(X/S)+\chi(X_s)-\chi(X_{\bar\eta}).$$

In the remainder of this note I will outline a proof of (2) in the mixed characteristic case.

This work was in large measure inspired by the work of K. Kato and S. Saito, and I profited enormously from conversations with them both in Tokyo and in Sendai at the conference. L. Illusie showed me how to use the Lefschetz trace formula to prove (2) in equal characteristic p. Finally, the book of Fulton [2] contains a wealth of ideas and techniques, based on his work and that of MacPherson, which can profitably be applied to the study of cycles on arithmetic varieties.

The class $(\Delta_X \cdot \Delta_X)_s$

In the mixed characteristic case, there is no nice sheaf of absolute differentials, but we can work with $\Omega^1_{X/S}$. When X/S has fibre dimension $d-1$, we should think of $(\Delta_X \cdot \Delta_X)_s$ as a localization of $(-1)^d c_d(\Omega^1_{X/S})$. Let X be a noetherian scheme of finite type over a regular scheme, and let $Z \subset X$ be a closed subscheme of pure codimension r. Let E^* be a finite complex of coherent, locally-free $\mathcal{O}_X$-modules. If the homology of E^* is supported on Z, one can define (op. cit. chap. 18, and ex. 18.1.3) localized chern classes $c^X_{Z,i}(E^*)$ which induce cap product maps $\cap : \mathrm{CH}_*(X) \to \mathrm{CH}_*(Z)$, lowering dimension by i. In fact, these arguments give somewhat more. Suppose F is a coherent sheaf of finite homological dimension on X such that $F|_{X-Z}$ locally free of some rank m. Then localized chern classes $c^X_{Z,i}(F)$ can be defined for all $i \geq m+1$.

We can apply this to $\Omega^1_{X/S}$ where $\dim X = m+1$. (Exercise: show X regular implies $\Omega^1_{X/S}$ has homological dimension ≤ 1.) Define

$$(\Delta_X \cdot \Delta_X)_s = (-1)^{\dim X} c^X_{X_s,\, \dim X}(\Omega^1_{X/S}) \cap [X] \in \mathrm{CH}_0(X_s) \to \mathbf{Z}.$$

We will abuse notation and write $(\Delta_X \cdot \Delta_X)_s$ also for the degree of this cycle, an integer.

One non-classical property of this intersection number is its dependence on the choice of base. For example, suppose one has $X \to T \to S$ with T/S a finite, totally ramified extension of discrete valuation rings. Then

$$(\Delta_X \cdot \Delta_X)_s - (\Delta_X \cdot \Delta_X)_t = (-1)^{\dim X - 1} \mathrm{ord}_S(d_{T/S}) \cdot \chi(X_{\bar\eta}),$$

where $d_{T/S}$ is the discriminant (so $\operatorname{ord}_S(d_{T/S}) = \mathrm{sw}_{T/S}(1) + \deg(T/S) - 1$).

The Swan conductor

Let $T \to S$ be a finite, totally ramified extension of discrete valuation rings, with $T = \operatorname{Spec}(\Lambda')$. Assume the extension of quotient fields K'/K is galois with group G, and let $\pi \in \Lambda'$ be a uniformizing parameter. Define $j\colon G - \{e\} \to \boldsymbol{Z}$ by

$$j(\sigma) = \operatorname{ord}_{\Lambda'}(\sigma(\pi) - \pi).$$

This is independent of the choice of π. The Swan conductor is the central function defined on G by

$$\mathrm{sw}_{T/S}(\sigma) = 1 - j(\sigma),\ \sigma \neq e;\ \ \mathrm{sw}_{T/S}(e) = -\sum_{\sigma \neq e} \mathrm{sw}_{T/S}(\sigma).$$

Note $\mathrm{sw}_{T/S}$ is supported on the wild ramification subgroup $I^w \subset G$. It is a non-trivial theorem that $\mathrm{sw}_{T/S}$ is the character of a representation.

In our geometric situation, the action of the wild ramification subgroup of $\operatorname{Gal}(\bar{K}/K)$ on $H^*(X_{\bar{\eta}})$ factors through a finite quotient [1]. Thus, for T sufficiently "large"

$$\mathrm{sw}(X/S) = |G|^{-1}\{\textstyle\sum(1 - j(\sigma)) \cdot \operatorname{tr} \sigma | H^*(X_{\bar{\eta}}) + \mathrm{sw}_{T/S}(e) \cdot H^*(X_{\bar{\eta}})\}$$

is well defined.

The plan

We assume henceforth that X/S has fibre dimension 1. Let T/S be sufficiently large so $X \times_S T$ has a *stable* regular model V; i.e. the fibre V_t is *reduced* with normal crossings [9]. One knows that the galois representation of $\bar{K}/K'$ on $H^*(X_{\bar{\eta}})$ is tame. The rational map $V \to X$ is not necessarily everywhere defined, but after a succession of blowings up of closed points, we get a scheme $W \to V$ fitting into a diagram

$$(3)\qquad \begin{tikzcd} & W \arrow[r] \arrow[dl] & X \arrow[dd] \\ V \arrow[dr] & & \\ & T \arrow[r] & S. \end{tikzcd}$$

Unfortunately, it is not true in general that the special fibre W_t is reduced, but we can suppose $W_{t,\,\mathrm{red}}$ is a union of smooth components with normal crossings.

Step 1. Formula (2) holds for V/T.

This is easy. There is no Swan term, and $(\Delta_V \cdot \Delta_V)_t$ is the number of double points in the special fibre. A standard geometric argument shows this is equal to $\chi(V_t)-\chi(V_{\bar{\eta}})$.

Step 2. Formula (2) holds for W/T.

This amounts to showing if $Y=\mathrm{BL}(x/X)$ for $x \in X$, then $(\Delta_Y \cdot \Delta_Y)_s = (\Delta_X \cdot \Delta_X)_s + 1$.

A digression

Before continuing with step 3, we need to discuss Fulton intersection theory in this context. Consider a cartesian diagram

$$\begin{tikzcd} A \cap \Delta \arrow[r] \arrow[d, "g"] & A \arrow[d] \\ \Delta_X \arrow[r] & X \underset{S}{\times} X \end{tikzcd}$$

with A proper over $X \times_S X$ and $\dim A=2$. Assume $A \cap \Delta$ is set-theoretically supported over the closed point $s \in S$. Fulton defines *Segre classes*

$$s_*(A \cap \Delta/A) \in \mathrm{CH}^*(A \cap \Delta).$$

Since $\Omega^1_{X/S}$ has finite homological dimension, we can mimic his approach and define

$$(\Delta_X \cdot A) = \{\textstyle\sum (-1)^i g^* c_i(\Omega^1_{X/S}) \cdot s_*(A \cap \Delta/A)\}_{\dim 0} \in \mathrm{CH}_0(A \cap \Delta).$$

(As before, we use the same notation for the degree of this cycle.) Using properties of Segre classes (op. cit. Lemma 4.2) one checks that if A has irreducible components A_i appearing with multiplicity m_i, then $(\Delta_X \cdot A) = \sum m_i (\Delta_X \cdot A_i)$.

We will be particularly interested in two cases:

(a) $A=\Gamma_\sigma$, where σ is an automorphism of X lifting a *non-trivial* automorphism σ of S.

(b) $A=E \times E'$ where $E, E' \subset X_s$ are components of the special fibre.

In case (b), $(\Delta \cdot E \times E') = (E \cdot E')_X$ is the usual intersection number. In case (a), define $\mathrm{fix}(\sigma) = \Gamma_\sigma \cap \Delta_X$, and write $\mathrm{fix}(\sigma) = D \cup R$ where $D \subset X$ is a Cartier divisor and R is the 0-dimensional residual scheme. (If x and y are local coordinates at a point, then $\mathrm{fix}(\sigma)$: $\sigma(x)-x=\sigma(y)-y=0$. D is defined locally by F, the g.c.d. of these equations, and R: $(\sigma(x)-x)/F = (\sigma(y)-y)/F=0$.) We have (op. cit. 9.2.2)

$$(\Delta_X \cdot \Gamma_\sigma) = -g^*(\omega_{X/S}) \cdot D - D^2 + [R]. \tag{4}$$

The plan (cont.)

Step 3. With notation as in (3), let σ be an automorphism of V lifting a non-trivial automorphism σ of T over S. Then

$$(5) \qquad (\Delta_V \cdot \Gamma_\sigma) = (j(\sigma)-1) \cdot \mathrm{tr}\, \sigma | H^*(V_{\bar{\eta}}) + \mathrm{tr}\, \sigma | H^*(V_t).$$

The main idea in proving this is to note that if $Y \subset V_t$ is a smooth curve appearing in fix(σ) with multiplicity i, $0 < i < j(\sigma)$, then by means of the action of $\sigma - 1$ on $\mathcal{O}_{(i+1)Y}$, one can define a non-zero derivation $D_{\sigma,Y}$: $\mathcal{O}_Y \to \mathcal{O}_Y(-iY)$. Studying the zeroes of $D_{\sigma,Y}$ gives the key relation between $\chi(Y)$, $(Y. Y)$, and $[R]$ necessary to identify the right hand sides of (4) and (5).

Step 4. The same as step 3, with V replaced by W.

Step 5. The scheme $W \times_X W$ has pure dimension 2 and is a local complete intersection. As a cycle on $W \times_S W$, $[W \times_X W] = \sum_{\sigma \in G} \Gamma_\sigma + R$, where R is a sum with suitable multiplicities of components $E \times E'$ with E, $E' \subset W$ collapsing to the same point of X. Then

$$|G|^{-1} \cdot (\Delta_W \cdot R) = \chi(X_s) - \chi(W_t)^G,$$

where $\chi(W_t)^G$ denotes the alternating sum of the dimensions of the G-fixed part of the cohomology of W_t.

This is proved by first replacing W with the normal scheme W/G. The question is thus replaced by a birational one, which is shown to depend only on the graph of exceptional curves for W/G over X. This means the question can be studied with X replaced by $\boldsymbol{P}^2_{\boldsymbol{C}}$, where standard topological techniques suffice. The essential simplification which makes this possible is the fact that the (locally complicated) normal scheme W/G admits a regular and birational map to the (locally simple) regular scheme X.

Step 6. Putting it all together—the projection formula.

We refer again to diagram (3). It is natural to think

$$[W \underset{X}{\times} W] = (h \times h)^* \Delta_X.$$

Since $(h \times h)_* \Delta_W = |G| \cdot \Delta_X$, one might expect a projection formula

$$|G| \cdot (\Delta_X \cdot \Delta_X)_s = (\Delta_W \cdot [W \underset{X}{\times} W])_s = (\Delta_W \cdot \Delta_W)_s + \sum_{\sigma \neq e} (\Delta_W \cdot \Gamma_\sigma) + (\Delta_W \cdot R).$$

Let us check that such a formula is what we need. Substituting from the previous steps, we get

$$\begin{aligned}(\Delta_X \cdot \Delta_X)_s = |G|^{-1} \{ & \chi(W_t) - \chi(X_{\bar{\eta}}) - (\mathrm{sw}_{T/S}(1) + |G| - 1)\chi(X_{\bar{\eta}}) \\ & + \sum_{\sigma \neq e} (j(\sigma)-1)\, \mathrm{tr}\, \sigma | H^*(X_{\bar{\eta}}) + \sum_{\sigma \neq e} \mathrm{tr}\, \sigma | H^*(W_t)\end{aligned}$$

$$+|G|\chi(X_s)-|G|\chi(W_t)^G\}$$
$$=-\mathrm{sw}(X/S)-\chi(X_{\bar{\eta}})+\chi(X_s).$$

The proof of the projection formula is by a careful analysis of the structure of the normal cone of Δ_W in $W\times_X W$. The details of this analysis will appear in the proceedings of the 1985 AMS summer institute in algebraic geometry, to be published by the AMS.

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