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Abelian Surfaces with (1, 2)-Polarization

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§ 0.1 Introduction
§ 0.2 Polarizations

§ 1. The (1, 2)-polarization
1.1 Divisors defining the polarization
1.2 The pencil $|\mathscr{L}|$ on A
1.3 Doubly elliptic curves
1.4 The surface $\mathrm{Prym}(D/E)$

§ 2. The (2, 4)-polarization
2.1 The linear system $|\mathscr{L}^{\otimes 2}|$ and its symmetries
2.2 Quadrics Q_t through A of rank ≤ 4
2.3 Special quadrics Q_{EF} through A of rank ≤ 4
2.4 Quadratic equations for $A \subset \boldsymbol{P}_7$

§ 3. The parameters
3.1 Symplectic automorphisms of $T(2, 4)$
3.2 Degenerations
3.3 Properties of the expressions r_{jk}
3.4 Theta-Null values

§ 4. Kummer surfaces
4.1 The hypernet
4.2 The dual Kummer surface $K^{\vee} = \mathrm{Km}\, A^{\vee}$
4.3 The Kummer surface $K = \mathrm{Km}\, A$

§ 5. Haine's theorem
5.1 The pencil of odd sections
5.2 The complete intersection surface $A_C \subset \boldsymbol{P}_6$
5.3 The elliptic curve $E \subset \boldsymbol{P}_3$

§ 6. Comments

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§ 0.1. Introduction

The aim of this note is to study quadratic equations for abelian surfaces embedded in $\boldsymbol{P}_7$ by a linear system coming from a (2, 4)-polarization.

The motivation was the work of Adler-van Moerbeke [AM1, AM2] and Haine [H] on certain cases of geodesic flow on SO(4) leading to integrable Hamiltonian systems. The integration of these systems takes place on affine complete intersection surfaces in $\boldsymbol{C}^6$, which are affine parts of abelian surfaces. Of course, when extended to $\boldsymbol{P}_6$ these surfaces must be singular at infinity, because abelian surfaces are not simply-connected and therefore cannot be projective complete intersections. So it is quite interesting to understand what makes the affine surfaces complete intersections, and perhaps to look for a classification of all such examples, which at the moment is quite obscure.

The case to be treated here, case I from [AM2], was first understood geometrically by Mumford [M5]: The smooth complete model A of the surface lies in $\boldsymbol{P}_7$ the hyperplane sections providing it with a (2, 4)-polarization. A is then projected to $\boldsymbol{P}_6$ with singularities arising only along a hyperplane. Haine [H] has shown that A in $\boldsymbol{P}_7$ is described by six quadratic equations. Haine also identified A as Prym variety for a double cover $D \to E$, D of genus 3, E elliptic, which naturally arises from the geometry of the quadratic equations. He obtained this result by computing period matrices.

In this note we apply Mumford's technique of the Heisenberg group [EDAV]. Its symmetries almost completely determine the quadratic equations for $A \subset \boldsymbol{P}_7$. In fact, in the space of all quadratic polynomials there are three particular 4-dimensional representation spaces for this group, each consisting of two copies of a 2-dimensional representation. The 6-dimensional space of quadrics vanishing on A turns out to be a sum of three isotypical components, one in each of the three 4-dimensional spaces. The choice of the three isotypical components means the introduction of three projective parameters $\lambda_j : \mu_j$, the three moduli of A. These three parameters are analysed carefully and it is proven that they belong to an abelian surface if and only if for $j \neq k = 1, 2, 3$

$$\lambda_j/\mu_j \neq \pm \lambda_k/\mu_k \quad \text{or} \quad \pm \mu_k/\lambda_k.$$

The coarse moduli space of our abelian surfaces (with level structure) thus is identified with an open set in $\boldsymbol{P}_1 \times \boldsymbol{P}_1 \times \boldsymbol{P}_1$. Under the simultaneous embedding the 2-torsion points have nontrivial monodromy, so we do not obtain a fine moduli space in this way.

Haine's isomorphism $A = \text{Prym}(D/E)$ is reduced to standard geometric constructions: In the space $\boldsymbol{P}_5$ parametrizing quadrics through A there appears the Kummer variety $K^\vee$ of the dual surface $A^\vee$ parametrizing (in general) all quadrics of rank ≤ 4. Haine's curve E lies on $K^\vee$, and D parametrizes the pencils of $\boldsymbol{P}_3$'s contained in these quadrics. In the usual way (Reid, Tyurin, Beauville) associating with $p \in D$ the class in $\mathrm{Pic}(A)=A^\vee$ of the pencil on A cut out by the $\boldsymbol{P}_3$'s, one obtains a geometric map $D \hookrightarrow A^\vee$. That this map induces an isomorphism of $\mathrm{Prym}(D/E)$ with A (and not $A^\vee$) is a fairly general property of genus-3 curves D contained in abelian surfaces. This is shown in Section 1, where it is also observed that these curves D are exactly the curves admitting a 2:1 map onto an elliptic curve. (This is very obvious, but I do not know a reference.)

This paper is quite lengthy, the reasoning is sometimes very explicit and never very deep. But I hope that it can be used as reference for abelian surfaces with (1, 2)-polarization. Almost all the theory of abelian varieties deals with principal polarizations. From a geometric point of view, however, those varieties deserve the most attention, which have the simplest geometric realisations, i.e., lowest degree of the embedding projective space, preferably quadratic equations. I do not know, whether from this point of view principally polarized surfaces are simpler than those considered here.

Convention. The base field (denoted by $\boldsymbol{k}$) of course has to be algebraically closed. Usually it suffices for my purpose that its characteristic does not divide some obvious integers. But at a few points I use transcendental arguments. So it is safer to put $\boldsymbol{k}=\boldsymbol{C}$.

§ 0.2. Polarizations

This section serves two purposes: to recall some basic facts on abelian varieties from [EDAV] and at the same time to introduce some notation.

By an *abelian variety* I do not mean a group (with a distinguished origin) but a homogeneous space A, on which its *translation group* $T(A)$ acts. The action of $t \in T(A)$ is denoted by $x \mapsto tx$. For $d \in \boldsymbol{N}$, by $T_d(A)$ we denote the *d-torsion subgroup* of $T(A)$. The *dual variety* of A is $A^\vee := \mathrm{Pic}^0(A)$, the group of line bundles on A, algebraically equivalent to zero.

Now let $\mathscr{L}$ be an *ample line bundle* on A. Associated with $\mathscr{L}$ we have the following objects:

1. A finite group $T(\mathscr{L})=\{t \in T(A): t^*\mathscr{L} \simeq \mathscr{L}\}$. The isogeny $T(A) \to A^\vee$, $t \mapsto \mathscr{L}^{-1} \otimes t^*\mathscr{L}$, with kernel $T(A)$ is called the *polarization* defined by $\mathscr{L}$.

2. A group $G(\mathscr{L})$ of bundle automorphisms of $\mathscr{L}$, which consists of liftings $\bar{t}$ of translations $t \in T(\mathscr{L})$, see [EDAV, p. 289]. It fits into an exact sequence

$$1 \longrightarrow \boldsymbol{k}^* \longrightarrow G(\mathscr{L}) \longrightarrow T(\mathscr{L}) \longrightarrow 0.$$

3. A skew-symmetric bilinear form $e^{\mathscr{L}}$ on $T(\mathscr{L})$ with values in $\boldsymbol{k}^*$. It is defined as follows: For $t_1, t_2 \in T(\mathscr{L})$ let $\bar{t}_1, \bar{t}_2 \in G(\mathscr{L})$ be liftings. Then $e^{\mathscr{L}}(t_1, t_2) := \bar{t}_1 \bar{t}_2 \bar{t}_1^{-1} \bar{t}_2^{-1}$.

4. A representation of $G(\mathscr{L})$ on the vector space $H^0(\mathscr{L})$.

Now fix a $T_2(A)$-orbit on A (consisting of $2^{2 \dim A}$ points) and denote by ι the involution of A having exactly these points as fixed points. The bundle $\mathscr{L}$ is called *symmetric*, if $\iota^* \mathscr{L} \simeq \mathscr{L}$. In this case we can consider the *extended translation group* $ET(\mathscr{L}) = T(\mathscr{L}) \rtimes \langle \iota \rangle$, the *extended automorphism group* $EG(\mathscr{L})$, which again is a central $\boldsymbol{k}^*$-extension of $ET(\mathscr{L})$, and the representation of $EG(\mathscr{L})$ on $H^0(\mathscr{L})$.

All these objects were identified by Mumford [EDAV] as follows:

1. $T(\mathscr{L})$ is of the form $K \oplus \hat{K}$, where $\hat{K}$ denotes the dual group $\mathrm{Hom}_{\boldsymbol{Z}}(K, \boldsymbol{k}^*)$ of K [EDAV, Theorem 1, p. 293]. Let $d_1, \cdots, d_k \in \boldsymbol{N}$ be the elementary divisors of K, i.e., $K \simeq \hat{K} \simeq \boldsymbol{Z}_{d_1} \oplus \cdots \oplus \boldsymbol{Z}_{d_k}$ and $d_i \mid d_{i+1}$ for $i = 1, \cdots, k-1$. Then

$$\delta = \underbrace{1, \cdots, 1}_{\dim(A)-k}, d_1, \cdots, d_k$$

is called the *type* of $\mathscr{L}$ (or of the polarization).

2. $G(\mathscr{L})$ is isomorphic to the *Heisenberg group* $G(\delta)$ defined as follows: As a set, $G(\delta) = \boldsymbol{k}^* \times K \times \hat{K}$. On this set the group law is

$$(\alpha, t, l)(\alpha', t', l') = (\alpha \cdot \alpha' \cdot l'(t), t + t', l + l')$$

[EDAV, Corollary of Theorem 1].

3. It follows now that

$$e^{\mathscr{L}}((t_1, l_1), (t_2, l_2)) = l_2(t_1)/l_1(t_2).$$

In particular $e^{\mathscr{L}}((t_1, 0), (0, l_2)) = l_2(t_1)$ is the natural pairing $K \times \hat{K} \to \boldsymbol{k}^*$.

4. The representation of $G(\mathscr{L})$ on $H^0(\mathscr{L})$ is isomorphic to the *Schroedinger representation* of $G(\delta)$ defined as follows: Let $V(\delta)$ be the vector space of all $\boldsymbol{k}$-valued functions on $K = \boldsymbol{Z}_{d_1} \oplus \cdots \oplus \boldsymbol{Z}_{d_k}$. On $V(\delta)$ the group $G(\delta)$ acts by

$$((\alpha, t, l)f)(x) = \alpha \cdot l(x) \cdot f(t + x), \qquad f \in V(\delta)$$

[EDAV, Proposition 3 on p. 295 and Theorem 2 on p. 297].

Notice that the type δ of $\mathscr{L}$ determines the self-intersection

$$\mathscr{L}^{\dim A}=n!\, d_1\cdot\cdots\cdot d_k,$$

hence by Riemann-Roch we have

$$h^0(\mathscr{L})=\frac{1}{n!}\mathscr{L}^{\dim A}=d_1\cdot\cdots\cdot d_k,$$

and this is just the dimension of $V(\delta)$.

If $\mathscr{L}$ is symmetric, ι can be lifted to an involution $\tilde{\iota}$ on $\mathscr{L}$. The bundle $\mathscr{L}$ is called *totally symmetric*, if $\tilde{\iota}$ acts in the same way over all fixed points of ι, i.e., if we may assume $\tilde{\iota}\,|\,\mathscr{L}_a=+1$ for all $a\in A$ in the distinguished $T_2(A)$-orbit. If, e.g., $\mathscr{L}\simeq\mathscr{M}^{\otimes 2}$ with symmetric $\mathscr{M}$, then $\mathscr{L}$ clearly is totally symmetric. If $\mathscr{L}$ is totally symmetric, the isomorphism $G(\mathscr{L})\simeq G(\delta)$ can be chosen in such a way that for all $g=(\alpha, t, l)$

$$\tilde{\iota}g\tilde{\iota}=(\alpha, -t, -l)$$

[EDAV, Remark 2, p. 318]. So $EG(\mathscr{L})\simeq EG(\delta)$, where $EG(\delta)=G(\delta)\times\boldsymbol{Z}_2$ as a set, with the group structure defined by the formula above. And the representation of $EG(\mathscr{L})$ on $H^0(\mathscr{L})$ is equivalent to the representation of $EG(\delta)$ on $V(\delta)$ obtained from the Schroedinger representation, by letting $\tilde{\iota}$ act as

$$(\tilde{\iota}f)(x)=f(-x),\quad f\in V(\delta)$$

see [EDAV, Inverse formula, p.331].

§ 1. Polarizations of type (1, 2)

1.1. Divisors defining the polarization

Let A be an *abelian surface.* Several times we shall use the following simple:

(1.1) **Lemma.** *If $C, D\subset A$ are curves with $C\cdot D=0$, then $C=\sum E_i$, $D=\sum F_j$, where the $E_i, F_j\subset A$ are elliptic curves, all algebraically equivalent to each other.*

Proof. It suffices to consider the case that C and D are irreducible. After applying a translation to D we may assume $C\cap D\neq\phi$. This implies $C=D$, because $C\cdot D>0$ otherwise. Now $C^2=0$ implies that C has arithmetical genus 1, so C is elliptic, because A does not contain rational curves. □

Next let $\mathscr{L}$ be some *ample line bundle* on A with self-intersection $\mathscr{L}\cdot\mathscr{L}=4$. The polarization defined by $\mathscr{L}$ necessarily is of type (1, 2). Since $h^0(\mathscr{L})=2$, there are effective divisors defining $\mathscr{L}$. The following is a complete list of curves D with $D^2=4$, that can lie on an abelian surface:

(1.2) *Table of curves D* with $D^2=4$:
a) *smooth connected of genus* 3,
b) *irreducible of geometric genus* 2, *with one double point,*
c) $E+F$ *with* E, F *elliptic and* $E\cdot F=2$,
d) $E+F_1+F_2$ *with* E, F_1, F_2 *elliptic and* $E\cdot F_k=1$, $F_1\cdot F_2=0$.

Proof (That this list is complete). If D is smooth, then D is connected by (1.1) and $g(D)=3$ by the adjunction formula. If D is irreducible but not smooth, then its normalisation $\tilde{D}$ cannot be rational or elliptic, so $g(\tilde{D})=2$ and D has exactly one double point.

If $D=E+F$ is reducible, then $E\cdot F>0$ by (1.1). From

$$\frac{1}{2}E^2+\frac{1}{2}F^2+E\cdot F=2$$

we find that
—either $E\cdot F=2$, $E^2=F^2=0$. So E, F are irreducible elliptic (case c) or one of them is reducible (case d)
—or $E\cdot F=1$ and say $E^2=1$, $F^2=0$. Then F is irreducible elliptic and under $A\to A/F$ the curve E is projected onto the elliptic curve A/F with degree 1. Then necessarily E splits into two elliptic curves, one meeting F and one disjoint with F. This leads to type d. □

Notice that in case c) the surface A admits an isogeny onto a product of elliptic curves and that the polarization of A is the pull-back of the principal product polarization.

In case d) the surface A itself is a product $E\times F$ with $\mathscr{L}=\mathcal{O}_A(E+2F)$ and the polarization is a *product polarization.* This case leads to so many exceptions that from now on we take:

(1.3) **Assumption.** *The line bundle* $\mathscr{L}$ *is not isomorphic to some* $\mathcal{O}_A(E+2F)$ *of type* d.

1.2. The pencil $|\mathscr{L}|$ on A

Let $\mathscr{L}$ be as in 1.1. By Riemann-Roch $h^0(\mathscr{L})=(\mathscr{L}^2)/2=2$, so $|\mathscr{L}|$ is a pencil. Table (1.2) and Assumption (1.3) imply:

(1.4) *The pencil* $|\mathscr{L}|$ *has no fixed component. In particular* (by Bertini's theorem), *its general member is irreducible.*

The translation group $T(\mathscr{L})\simeq \boldsymbol{Z}_2\times \boldsymbol{Z}_2$ has order 4. The base of the pencil $|\mathscr{L}|$ is invariant under $T(\mathscr{L})$ and consists of at most $\mathscr{L}^2=4$ points. This proves:

(1.5) *The base locus of* $|\mathscr{L}|$ *consists of four distinct points* $e_1, \cdots, e_4$ *forming some* $T(\mathscr{L})$*-orbit. No curve* $D \in |\mathscr{L}|$ *is singular at one of these base points* (because then $\mathscr{L}\cdot D \geq 5$), *hence the general* $D \in |\mathscr{L}|$ *is smooth* (Bertini).

Notice, however, that $|\mathscr{L}|$ always contains singular curves too: The surface obtained from A by blowing up $e_1, \cdots, e_4$ has Euler number 4. If all $D \in |\mathscr{L}|$ were smooth, by [BPV, III Proposition 11.3] this Euler number would equal $e(\boldsymbol{P}_1)\cdot e(D)=-8$, a contradiction.

If $D=E+F \in |\mathscr{L}|$ is a curve of type c), not all the four points $e_1, \cdots, e_4$ can lie on E: In this case $tE=E$, $tF\neq F$ for some $t \in T(\mathscr{L})$, and

$$\mathcal{O}_A(tF-F)=\mathcal{O}_A(E+tF-(E+F))=\mathscr{L}\otimes\mathscr{L}^{-1}=\mathcal{O}_A$$

a contradiction. So the distribution of $e_1, \cdots, e_4$ on $E+F$ is as in Figure 1.

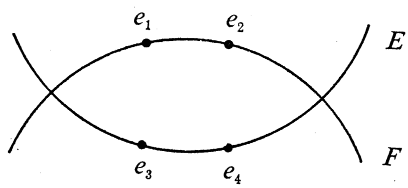


Figure 1

Now let ι: $A\rightarrow A$ be the *involution* with fixed points $e_1, \cdots, e_4$ and 12 other points $e_5, \cdots, e_{16}$ forming *a* $T_2(A)$-orbit altogether.

(1.6) **Proposition.** $\mathscr{L}$ *is symmetric, i.e.,* $\iota^*\mathscr{L}\simeq\mathscr{L}$. *In fact* $\iota D=D$ *for all* $D \in |\mathscr{L}|$.

Proof. It is known that ιD is algebraically equivalent with D. (This holds for any divisor D on any abelian variety.) In particular, $\iota D\cdot D=4$. But at the base points $e_1, \cdots, e_4$ the involution ι does not change the tangent directions. So $\iota D=D$ for all $D \in |\mathscr{L}|$, because otherwise $\iota D\cdot D \geq 5$, a contradiction. □

The involution ι can be lifted to $\mathscr{L}$ as an involution $\tilde{\iota}$ in two ways differing in sign. For each section $s \in H^0(\mathscr{L})$ we therefore have $\tilde{\iota}s=\pm s$. We distinguish one lifting $\tilde{\iota}$ by the property that it should act as $+1$ on the fibre of $\mathscr{L}$ over e_5 (not a base point of $|\mathscr{L}|$). Then clearly $\tilde{\iota}s=s$ for all $s \in H^0(\mathscr{L})$:

(1.7) *The involution ι can be lifted to $\mathscr{L}$ such that on $H^0(\mathscr{L})$ it acts trivially.* (The action of $EH(\mathscr{L})$ on $H^0(\mathscr{L})$ therefore is isomorphic to the Schroedinger representation of $EH(1, 2)$.)

If $s \in H^0(\mathscr{L})$ vanishes at some point e_i, $5 \le i \le 16$, then by [EDAV, Proposition 2, p. 307], it vanishes there to an even order. This shows: Any $D \in |\mathscr{L}|$ passing through some e_i, $5 \le i \le 16$, is singular there.

If $D \in |\mathscr{L}|$ is smooth, ι has exactly four fixed points on it. By the genus formula the quotient D/ι is elliptic. The genus 3-curve is thus *doubly elliptic*: it admits a 2:1 map onto an elliptic curve. This map has four branch points (namely $e_1, \cdots, e_4$). The covering involution is called an *elliptic involution.*

1.3. Doubly elliptic curves

The aim of this section is to prove:

(1.8) **Proposition.** *For a smooth genus-3 curve D the following properties are equivalent:*

i) *D admits an elliptic involution i_E.*

ii) *D admits an embedding into an abelian surface A.*

For the proof we need the elementary:

(1.9) **Lemma.** *If a smooth genus-3 curve admits an elliptic involution i_E and a hyperelliptic involution i_p, then $i_E(x) \neq i_p(x)$ for all $x \in D$.*

Proof. The product of the two double covers defines a birational map of D onto a curve $D' \subset E \times \boldsymbol{P}_1$ algebraically equivalent with $2(E \times pt) + 2(pt \times \boldsymbol{P}_1)$. From the adjunction formula one computes $\deg \omega_{D'} = 4$, hence $D \to D'$ is biregular. But any $x \in D$ with $i_E(x) = i_p(x)$ would cause a singularity on D'. □

Proof of Proposition (1.8). In view of Section 1.2 we must prove the direction i)⇒ii) only. So fix a double covering π: $D \to E$ onto an elliptic curve E. Let $e_0 \in D$ be some branch point for π and embed $D \to J :=$ Pic$^0(D)$ via $p \to \mathcal{O}_D(p-e_0)$. Further, use $x_0 = \pi e_0$ to identify $E \to$ Pic$^0(E)$ via $x \to \mathcal{O}_E(x-x_0)$. Then we have a natural injection π^*: $E \to J$. We put $A = J/\pi^* E$ and consider the map

$$\varphi: D \hookrightarrow J \xrightarrow{\bmod \pi^* E} A.$$

We claim φ is an embedding. To show that it suffices to prove:

a) φ is injective,

b) $\varphi(D)\cdot\varphi(D)=4$.

Indeed, b) implies that $\varphi(D)$ has arithmetic genus 3 and by a) this is possible only if $\varphi\,|\,D$ is biregular.

Proof of a). If $p, q \in D$ are points with $\varphi(p)=\varphi(q)$, then $\mathcal{O}_D(p-q) =\pi^*\mathcal{O}_E(x-x_0)$ for some $x \in E$. There is some $y \in E$ such that $\mathcal{O}_E(x-x_0) =\mathcal{O}_E(\pi p-y)$. Then putting $\pi^*y=r+i_E r$ we find

$$\mathcal{O}_D(p-q)=\mathcal{O}_D(p+i_E p-r-i_E(r)), \quad \text{i.e.}$$
$$\mathcal{O}_D(r+i_E r)=\mathcal{O}_D(q+i_E p).$$

By Lemma (1.9) this implies $q=i_E(i_E p)=p$.

Proof of b). By definition A has a distinguished origin 0_A and a group structure. Denote by ι the involution $a\to -a$. Then we have the equivariance $\iota\varphi(p)=\varphi(i_E p)$ for all $p \in D$, because

$$\mathcal{O}_D(p-p_0)\otimes\mathcal{O}_D(i_E p-p_0)=\mathcal{O}_D(p+i_E p-2p_0) \in \pi^*E.$$

We compute $\varphi(D)\cdot\varphi(D)$ as the number of points in $\varphi(D)\cap t(\varphi(D))$ for a general translation $t \in A$. The map $\varphi(D)\times\varphi(D)\to A$ sending p, q to $\varphi(p) -\varphi(q)$ is surjective. So we may take $t=\varphi(p)-\varphi(q)$ with $p, q \in D$, $q\neq\iota p$, $\mathcal{O}_D(p+\iota q)\neq\omega\otimes\mathcal{O}_D(-r-\iota r)$, $r \in D$, for any hyperelliptic involution i that D might have. Then the elements in $\varphi(D)\cap t\varphi(D)$ are the points $\varphi(p')$, $p' \in D$, for which there exists $q' \in D$ with

$$\mathcal{O}_D(p'-q')=\mathcal{O}_D(p-q) \quad \text{modulo } \pi^*E,$$

i.e., $\mathcal{O}_D(p+\iota q+\iota p'+q') \in \pi^* \operatorname{Pic}(E)$.

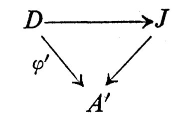
If $p+\iota q+\iota p'+q'$ is not a canonical divisor, by Riemann-Roch $h^0\mathcal{O}_D(p+\iota q+\iota p'+q')=2$ and its class can belong to $\pi^*\operatorname{Pic}(E)$ only if the divisor itself is of the form $\pi^*(x+y)$, $x, y \in E$. In this case $\pi^*(x+y)= p+\iota p+q+\iota q$, and we find $p'=p$, $q'=q$ or $p'=\iota q$, $q'=\iota p$. On the other hand, $h^0(\omega_D)=3$ implies the existence of a canonical divisor of the form $p+\iota q+r+s$, uniquely determined by the assumption on p and q and $r\neq\iota s$. If $p+\iota q+\iota p'+q'$ is this divisor, then $p'=\iota r$, $q'=s$ or $p'=\iota s$, $q'=r$.

Altogether we count four pairs (p', q'). □

D may admit several elliptic involutions inducing several embeddings into abelian surfaces.

(1.10) **Proposition.** *The embedding $D\hookrightarrow A$ is (up to a 2-torsion translation) uniquely determined by the property $i_E=\iota\,|\,D$.*

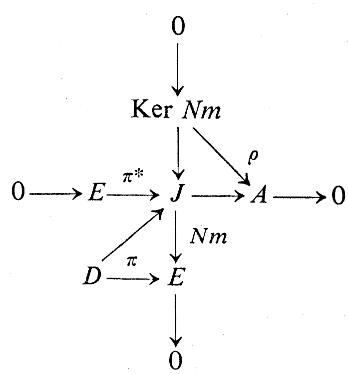
Proof. Let $\varphi'\colon D\to A'$ be another embedding into an abelian surface A' such that i_E extends to an involution ι' of A' with 16 fixed points. By the universality of jacobians there is a unique morphism $J\to A$ such that



commutes. For the same reason the involution i_E extends uniquely to an involution j of J covering ι'. Since $j\,|\,\pi^*E=\mathrm{id}$, π^*E is contained in the kernel of $J\to A'$, i.e., φ' factors through φ. But since $\varphi'(D)\subset A'$ is smooth, the induced epimorphism $A\to A'$ is bijective.

1.4. The surface Prym (D/E)

Let D be a smooth genus-3 curve and $\pi\colon D\to E$ a double covering. As above take some branch point $e_0\in D$ and embed $D\hookrightarrow J$ via $p\to \mathcal{O}_D(p-e_0)$. There is a morphism $Nm\colon J\to E$ defined on the level of divisors by taking the image under π. The surface $P:=(\ker Nm)^0$ is called the *Prym variety* Prym (D/E).



Clearly $Nm\circ\pi^*\colon E\to E$ is multiplication by 2. This situation is considered in great generality by Mumford [M2]. He observes:

(1) The involution i_E on D extends to an involution j on J acting on π^*E as id and on P as multiplication by -1.

(2) $P\cap\pi^*E$ is the 2-torsion subgroup $(\pi^*E)_2$ of π^*E.

(3) The principal polarization of J induces a duality between the maps π^* and Nm and a duality between P and A.

In our case the polarization map $\rho\colon P\to A=P^\vee$ has kernel $(\pi^*E)_2\simeq \boldsymbol{Z}_2\times\boldsymbol{Z}_2$, so the induced polarization on P is of type $(1,2)$. Now A carries the (1, 2)-polarization given by $\mathcal{O}_A(\varphi D)$. These two polarizations coincide in the following sense:

(1.11) **Lemma.** *The image $\rho(P_2)$ of the 2-torsion points in P coincides with the group $T(\mathcal{O}_A(D))$.*

Proof. By (2) both groups have order 4, so it suffices to show $\rho(P_2) \supset T(\mathcal{O}_A(D)) = \{\varphi(e_0), \cdots, \varphi(e_3)\}$, where $e_0, \cdots, e_3$ are the branch points of π.

For $j=1, 2, 3$ consider the curve

$$\Delta_j = \{(p, q) \in D \times D : \mathcal{O}_E(\pi p + \pi q) = \mathcal{O}_E(\pi e_j + \pi e_0)\}.$$

Via the projection $(p, q) \mapsto p$ it is a double cover of D, in fact it is the pullback to D of the double cover $\gamma: E \to \boldsymbol{P}_1$ (given by the linear system $\mathcal{O}_E(\pi e_j + \pi e_0)$) under the map $D \xrightarrow{\pi} E \xrightarrow{\gamma} \boldsymbol{P}_1$. Since the branch points of γ do not all coincide with $\pi e_0, \cdots, \pi e_3$, the curve Δ_j clearly is irreducible.

Now we fix $x_j \in E$, some branch point of γ, and map $\Delta_j \to J$ via $\psi_j: (p, q) \to \mathcal{O}_D(p + q - \pi^* x_j)$. Then

i) $\mathcal{O}_E(2x_j) = \mathcal{O}_E(\pi e_j + \pi e_0)$, hence choosing for p, q the two points over x_j, we have $(p, q) \in \Delta_i$ and $\psi(p, q) = 0$, the origin of J.

ii) $\psi(i_E p, i_E q) + \psi(p, q) = \mathcal{O}_D(\pi^*(\pi p + \pi q) - \pi^*(2x_j)) = 0$, so $\psi(\Delta_j)$ is contained in P.

iii) Putting $(p, q) = (e_j, e_0)$ we find $\psi(e_j, e_0) = \mathcal{O}_D(e_j - e_0) \otimes \mathcal{O}_D(2e_0 - \pi^* x_j)$. So the 2-torsion element $\psi(e_j, e_0)$ in P differs from $\mathcal{O}_D(e_j - e_0)$ only by an element in $\pi^* E$. This proves $\varphi(e_j) \subset \rho(P_2)$. □

(1.12) **Duality theorem.** *Let D be a smooth genus-*3 *curve, $\pi: D \to E$ a double covering over an elliptic curve E, and $D \to A$ the corresponding embedding into an abelian surface A (Section* 1.3). *Then both A and* Prym (D/E) *carry a natural* (1, 2)*-polarization identifying one with the dual of the other.*

This duality theorem is quite basic for my way of explaining Haine's isomorphism (Section 5.3). It is illuminating to compare it with the approach [P] of Pantazis. To do this we perform the following construction:

We denote by $\mathcal{T}$ the line bundle on E responsible for the covering $\pi: D \to E$, i.e., the bundle satisfying $\mathcal{T}^{\otimes 2} = \mathcal{O}_E(\pi e_0 + \cdots + \pi e_3)$ and $\pi^* \mathcal{T} = \mathcal{O}(e_0 + \cdots + e_3)$. Similarly to the situation above we define

$$\Delta = \{(p, q) \in D \times D : \mathcal{O}_E(\pi p + \pi q) = \mathcal{T}\}$$

and observe that Δ is an irreducible curve. Then we put $D^\vee = \Delta/\delta$, where δ is the involution $(p, q) \leftrightarrow (q, p)$. $D^\vee$ is the curve obtained from the tower (τ the map defined by $|\mathscr{T}|$)

$$D \xrightarrow{\pi} E \xrightarrow{\tau} \boldsymbol{P}_1$$

by Pantazis' *bigonal construction* [P, Section 2.3]. The curve $D^\vee$ admits an involution $i_E^\vee$ induced by $(p, q) \leftrightarrow (i_E p, i_E q)$ and the quotient $D^\vee/i_E^\vee$ is an elliptic curve $E^\vee$. The induced map $E^\vee \to \boldsymbol{P}_1$ has the branch points $\tau\pi e_0, \cdots, \tau\pi e_3$. There is one fixed point of $i_E^\vee$ for each branch point $t_i \in E$ for τ. It follows that $D^\vee$ has genus 3 again. Pantazis observes that $D^\vee$ is isomorphic to $P \cap W$, where $W \subset J$ is the theta divisor, i.e., the image of $D \times D$. In fact, upon choosing a theta characteristic—say $\pi^* t_i$—on D one may define a map $D^\vee \to J$ by $\varphi^\vee: (p, q) \to \mathcal{O}_D(p + q - \pi^* t_i)$ and check (as in the proof of Proposition (1.8)) that $\varphi^\vee | D^\vee$ is biregular with $\varphi^\vee(D^\vee) \subset P$. Pantazis proves Prym $(D^\vee/E^\vee)$ is dual to Prym (D/E), which is another proof for the duality theorem (1.12).

§ 2. Polarizations of type (2, 4)

As before let A be an abelian surface and $\mathscr{L}$ on A a line bundle of type (1, 2). Unless mentioned otherwise we exclude the product polarization (1.3).

2.1. The linear system $|\mathscr{L}^{\otimes 2}|$ and its symmetries

By Riemann-Roch $h^0(\mathscr{L}^{\otimes 2}) = 8$.

(2.1) **Proposition** [R]. *The line bundle $\mathscr{L}^{\otimes 2}$ on A is very ample, so the linear system $|\mathscr{L}^{\otimes 2}|$ embeds A into $\boldsymbol{P}_7$ as a smooth surface of degree* 16.

Proof [R]. Let $D \in |\mathscr{L}|$ be some smooth curve and $C \subset A$ any translate of D. Since $(2D - C)^2 = D^2 = 4$, the bundle $\mathcal{O}_A(2D - C)$ is ample and $h^1(\mathscr{L}^{\otimes 2} \otimes \mathcal{O}_A(-C)) = 0$, cf. [M3, Ch. 6, § 2]. Restriction $H^0(\mathscr{L}^{\otimes 2}) \to H^0(\mathscr{L}^{\otimes 2} | C)$ therefore is surjective. Since deg $(\mathscr{L}^{\otimes 2} | C) = 8$, the restricted bundle $\mathscr{L}^{\otimes 2} | C$ is very ample. Moving C one finds that $\mathscr{L}^{\otimes 2}$ itself is very ample. □

$\mathscr{L}^{\otimes 2}$ is of type (2, 4). The translation group $T(\mathscr{L}^{\otimes 2})$ therefore is isomorphic to $T(2, 4) = (\boldsymbol{Z}_2 \times \boldsymbol{Z}_4)^2$ and the representation of $G(\mathscr{L}^{\otimes 2})$ on $H^0(\mathscr{L}^{\otimes 2})$ is isomorphic to the Schroedinger representation of $G(2, 4)$. Since $\mathscr{L}$ is symmetric (Proposition (1.6)), $\mathscr{L}^{\otimes 2}$ is totally symmetric. So also $EG(\mathscr{L}^{\otimes 2})$ acts on $H^0(\mathscr{L}^{\otimes 2})$ as $EG(2, 4)$ on $V(2, 4)$ does, see Section 0.2.

We now specify elements of $EG(\mathscr{L}^{\otimes 2})$ and a basis for $H^0(\mathscr{L}^{\otimes 2})$:

Let σ_1, σ_2 be generators of $Z_2 \times Z_4 = K$ and τ_1, τ_2 be generators of $Z_2 \times Z_4 = \hat{K}$ such that

$$\tau_1(\sigma_1) = -1, \qquad \tau_2(\sigma_2) = i, \qquad \tau_2(\sigma_1) = \tau_1(\sigma_2) = 1.$$

Put $\sigma := \sigma_2^2$, $\tau := \tau_2^2$. They generate the subgroup $T(\mathscr{L}) = 2T(\mathscr{L}^{\otimes 2})$ in $T(\mathscr{L}^{\otimes 2})$. As before denote by ι the involution on A with fixed points $e_1, \cdots, e_{16}$.

For $V := V(2, 4) = \mathrm{Hom}_Z(K, k)$ we take a basis of the form $u_j \cdot v_k$, $j = 0, 1$, $k = 0, 1, 2, 3$, where

$$u_j \cdot v_k(\sigma_1^l, \sigma_2^m) = \delta_{l,m}^{j,k}.$$

Then $EG(2, 4)$ is generated by

$$\begin{array}{ll} \tilde{\sigma}_1 \colon u_j \longrightarrow u_{j+1} & \tilde{\sigma}_2 \colon v_k \longrightarrow v_{k+1} \\ \tilde{\tau}_1 \colon u_j \longrightarrow (-1)^j u_j & \tilde{\tau}_2 \colon v_k \longrightarrow i^k v_k \\ \tilde{\iota} \colon u_j v_k \longrightarrow u_{-j} v_{-k} & \end{array}$$

and we clearly have

$$\begin{array}{ll} [\tilde{\sigma}_1, \tilde{\tau}_1] = -1, & [\tilde{\sigma}_2, \tilde{\tau}_2] = i, \\ \tilde{\iota}\tilde{\sigma}_j^\nu \tilde{\iota} = \tilde{\sigma}_j^{-\nu}, & \tilde{\iota}\tilde{\tau}_j^\nu \tilde{\iota} = \tilde{\tau}_j^{-\nu}. \end{array}$$

Under the isomorphism $EG(2, 4) \simeq EG(\mathscr{L}^{\otimes 2})$ the elements $\tilde{\sigma}_1, \cdots, \tilde{\iota}$ are liftings of $\sigma_1, \cdots, \iota \in ET(\mathscr{L}^{\otimes 2})$. For convenience we drop the tildes on the letters denoting elements from $EG(2, 4)$. It will be clear from the context, whether we are dealing with translations or their liftings.

In the sequel, on $P_7 = P(V)$ we shall use the coordinates

$$\begin{array}{llll} x_1 = u_0(v_0 + v_2) & x_3 = u_0(v_1 + v_3) & x_5 = u_0(v_0 - v_2) & x_7 = u_0(v_1 - v_3) \\ x_2 = u_1(v_0 + v_2) & x_4 = u_1(v_1 + v_3) & x_6 = u_1(v_0 - v_2) & x_8 = u_1(v_1 - v_3) \end{array}$$

In these coordinates the action of the extended Heisenberg group is given by Table 1.

Of course we are interested in these symmetries because they induce an action of $(Z_2 \times Z_4)^2 \rtimes Z_2$ on P_7 leaving invariant the embedded surface A.

The subspaces $\{x_7 = x_8 = 0\}$, (resp. $\{x_1 = \cdots = x_6 = 0\}$) consist of invariants, (resp. anti-invariants) for ι. We denote the corresponding subspaces in P_7 by P_5^+, (resp. P_1^-). As ι acts by $+1$ over $e_1, \cdots, e_{16}$, these 16 points are the transversal intersection of $A \subset P_7$ with P_5^+, whereas $A \cap P_1^- = \phi$.

Table 1 (action of $EG(\mathscr{L}^{\otimes 2})$)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
σ_1	x_2	x_1	x_4	x_3	x_6	x_5	x_8	x_7
σ_2	x_3	x_4	x_1	x_2	x_7	x_8	$-x_5$	$-x_6$
τ_1	1	-1	1	-1	1	-1	1	-1
τ_2	x_5	x_6	ix_7	ix_8	x_1	x_2	ix_3	ix_4
$\sigma=\sigma_2^2$	1	1	1	1	-1	-1	-1	-1
$\tau=\tau_2^2$	1	1	-1	-1	1	1	-1	-1
ι	1	1	1	1	1	1	-1	-1

At any point e_i the tangent plane of A is ι-invariant, transversal to $\boldsymbol{P}_5^+$. It follows that this tangent plane is the span of e_i and $\boldsymbol{P}_1^-$.

The following proposition shows in particular that each of the four $T(\mathscr{L})$-orbits in $\{e_1, \cdots, e_{16}\}$ spans a plane in $\boldsymbol{P}_5^+$.

(2.2) **Proposition.** *Let $a_1=te_1 \in A$ and $\{a_1, \cdots, a_4\}$ be the $T(\mathscr{L})$-orbit of a_1. Then*

—if $t \notin T(\mathscr{L}^{\otimes 2})$, the points $a_1, \cdots, a_4$ span a 3-space;

—if $t \in T(\mathscr{L}^{\otimes 2})$, the points $a_1, \cdots, a_4$ span a plane.

Proof. Let $D \in |\mathscr{L}|$ be a smooth curve and put $C=tD$. Then

$$\mathcal{O}_C(C)=\mathcal{O}_C(a_1+\cdots+a_4),$$
$$\mathscr{L}^{\otimes 2}\otimes\mathcal{O}_C(-a_1-\cdots-a_4)=\mathscr{L}^{\otimes 2}\otimes\mathcal{O}_C(-C).$$

This latter bundle of degree 4 is isomorphic with $\omega_C(C)=\mathcal{O}_C(C)$ if and only if $\mathcal{O}_C(2D-2C)$ is trivial, which by $h^1(\mathcal{O}_C(2D-3C))=0$ is equivalent to $\mathcal{O}_A(2D-2C)\simeq\mathcal{O}_A$, i.e, $t \in T(\mathscr{L}^{\otimes 2})$. So, if $t \notin T(\mathscr{L}^{\otimes 2})$ we have

$$h^0(\mathscr{L}^{\otimes 2}\otimes\mathcal{O}_C(-a_1-\cdots-a_4))=2 \quad \text{and} \quad h^0(I_{a_1,\cdots a_4/A}\cdot\mathscr{L}^{\otimes 2})=4$$

This shows that $a_1, \cdots, a_4$ span a 3-space. And if $t \in T(\mathscr{L}^{\otimes 2})$, then $h^0(\mathscr{L}^{\otimes 2}\otimes\mathcal{O}_C(-a_1-\cdots-a_4))=3$, $h^0(I\cdot\mathscr{L}^{\otimes 2})=5$, and $a_1, \cdots, a_4$ span a plane. □

2.2. Quadrics Q_t through A of rank ≤ 4

By a quadric I mean sometimes a quadratic polynomial, sometimes the zero-locus of such a polynomial. However, I try to follow the convention that a polynomial is denoted by a small letter q, the corresponding capital letter Q denoting its zero locus in projective space.

In [M4] Mumford constructed quadratic equations for abelian varieties embedded by a linear system $\mathscr{L}^{\otimes 4}$ with $\mathscr{L}$ ample. The aim of this section is to show that his construction in our case $\mathscr{L}^{\otimes 2}$ works too.

For each $t \in T(A)$ the line bundle $t_*\mathscr{L}\otimes(-t)_*\mathscr{L}$ is isomorphic to $\mathscr{L}^{\otimes 2}$. Let $s_1, s_2 \in H^0(t_*\mathscr{L})$ and $s'_1, s'_2 \in H^0((-t)_*\mathscr{L})$ be generators and put $f_{ij}=s_i \cdot s'_j \in H^0(\mathscr{L}^{\otimes 2})=H^0(\mathscr{O}_{P_7}(1))$.

(2.3) **Lemma.** a) *If $t \notin T(\mathscr{L}^{\otimes 2})$, then $f_{11}, \cdots, f_{22}$ are linearly independent.*

b) *If $t \in T(\mathscr{L}^{\otimes 2})$ then $t_*\mathscr{L}\simeq(-t)_*\mathscr{L}$ and we may take $s'_i=s_i$. In this case f_{11}, f_{22} and $f_{12}=f_{21}$ are linearly independent.*

Proof. Assume $\sum c_{ij}f_{ij}=0$, explicitly

$$c_{11}s_1s'_1+c_{12}s_1s'_2+c_{21}s_2s'_1+c_{22}s_2s'_2=0.$$

We assume $(c_{21}, c_{22})\neq(0, 0)$ without loss of generality. Restricting to the curve C: $s_1=0$ (which we may assume irreducible), because of $s_2|C\neq 0$ we find $(c_{21}s'_1+c_{22}s'_2)|C=0$. This implies $t_*\mathscr{L}\simeq(-t)_*\mathscr{L}$. In this case, if $s_i=s'_i$,

$$c_{11}f_{11}+c_{12}f_{12}+c_{22}f_{22}=c_{11}s_1^2+c_{12}s_1s_2+c_{22}s_2^2=0$$

implies $c_{jj}=0$ when restricting to $s_i=0$ $(i\neq j)$. Afterwards $c_{12}=0$ too. □

By (2.3) the quadric

$$q_t=f_{11}f_{22}-f_{12}f_{21} \in H^0(\mathscr{O}_{P_7}(2)) \tag{2.4}$$

has rank 4, (resp. rank 3 if $t \in T(\mathscr{L}^{\otimes 2})$). It clearly vanishes on A. Varying $t \in T(A)$ we obtain a 2-parameter family of quadrics $Q_t\subset \boldsymbol{P}_7$ through A. In fact, Q_t depends only on the orbit of t under the group $ET(\mathscr{L})$, i.e., on the image of t in $\mathrm{Km}\, A/T(\mathscr{L})=\mathrm{Km}\, A^\vee$. Let $S_t\subset Q_t$ be the singular subspace. It has equations $f_{11}=\cdots=f_{22}=0$, so it is of dimension 3, (resp. 4). This space meets A where $s_1=s_2=0$ or $s'_1=s'_2=0$, i.e., in the base points of $t_*\mathscr{L}$ and $(-t)_*\mathscr{L}$.

For $t \in T(\mathscr{L}^{\otimes 2})$ the quadric Q_t carries two families of 5-planes

$$uf_{11}+vf_{12}=vf_{22}+uf_{21}=0 \qquad (u: v) \in \boldsymbol{P}_1$$
$$uf_{11}+vf_{21}=vf_{22}+uf_{12}=0 \qquad (u: v) \in \boldsymbol{P}_1.$$

The two families cut out on A the

$$\text{base locus of } t_*\mathscr{L} \cup \text{pencil } |(-t)_*\mathscr{L}|$$
$$\text{base locus of } (-t)_*\mathscr{L} \cup \text{pencil } |t_*\mathscr{L}|,$$

i.e., essentially the two pencils $|t_*\mathscr{L}|$ and $|(-t)_*\mathscr{L}|$.

For $t \in T(\mathscr{L}^{\otimes 2})$, the quadric Q_t carries only one family of 5-planes

$$uf_{11}+vf_{12}=vf_{22}+uf_{12}=0$$

cutting out on A the pencil $|t_*\mathscr{L}|$.

Notice that each $f_{ij} \in H^0(\mathscr{L}^{\otimes 2})=H^0(\mathcal{O}_{P_7}(1))$ vanishes at the four base points of $|t_*\mathscr{L}|$ to the second order, i.e., the 6-spaces $f_{ij}=0$ in $\boldsymbol{P}_7$ touch A at these points. For $t=0$ in particular we find:

(2.5) **Lemma.** *The singular* 4*-space* S_0 *of* Q_0 *is spanned by* $e_1, \cdots, e_4$ *and* $\boldsymbol{P}_1^-$.

2.3. Special quadrics Q_{EF} through A of rank 4

If A specializes such that $|\mathscr{L}|$ contains a curve $E+F$ of type c), then another type of rank-4 quadrics through A exists: Take generators $s_1, s_2 \in H^0(\mathcal{O}_A(2E))$ and $s_1', s_2' \in H^0(\mathcal{O}_A(2F))$. Just as above one shows that the four sections $f_{ij}=s_is_j' \in H^0(\mathcal{O}_A(2E+2F))=H^0(\mathscr{L}^{\otimes 2})$ are linearly independent. So

$$q_{EF}=f_{11}f_{22}-f_{12}f_{21} \tag{2.6}$$

is a rank-4 quadric vanishing on A. Its singular 3-plane S_{EF} does not meet A. The two families of 5-planes on Q_{EF} cut out the pencils $|2E|$ and $|2F|$ on A.

The curve $E+F$ is invariant under one nontrivial $t \in T(\mathscr{L}) \subset T_2(A)$, (cf. 1.2). For all $t \in T_2(A)$ we have $\mathcal{O}_A(2E) \simeq \mathcal{O}_A(2(tE))$ and $Q_{tE,tF}=Q_{EF}$. For $t \in T(A)$ we have $\mathcal{O}_A(2(t(E)+t(F))) \simeq \mathscr{L}^{\otimes 2}$ if and only if $t \in T(\mathscr{L}^{\otimes 2})$, so there are $32=(\# T(\mathscr{L}^{\otimes 2}))/2$ translated curves $tE+tF$. But for any $16=\# T_2(A)$ the quadrics $Q_{tE,tF}$ coincide. So there are precisely two quadrics originating from $E+F$ and its translates.

(2.7) **Proposition.** *Let* $Q \subset \boldsymbol{P}_7$ *be a quadric of rank* ≤ 4 *and* S_Q *its singular locus. If* $S_Q \cap A=\phi$, *then* $Q=Q_{t(E),t(F)}$ *for some* $t \in T(\mathscr{L}^{\otimes 2})$, $E+F \in |\mathscr{L}|$.

Proof. Since A spans $\boldsymbol{P}_7$, the rank of Q is bigger than 2. If rank $Q=3$, then the pencil of 5-planes on Q cuts out on A a pencil $|E|$ without base points, but such that $2E$ is a hyperplane section. This would be in conflict with $E^2=0$. So rank $Q=4$. The two pencils of 5-planes on Q cut out on A two pencils without base points. By Lemma (1.1) they are contained in linear systems $|mE|$, (resp. $|nF|$), with E, F elliptic and $m, n \geq 2$. Since $\mathcal{O}_A(mE+nF)=\mathcal{L}^{\otimes 2}$ we have $2mn(E.\ F)=16$. From Table (1.2) and Assumption (1.3) we conclude $m=n=E\cdot F=2$. □

The action of $EG(\mathcal{L}^{\otimes 2})$ on $H^0(\mathcal{O}_{P_7}(1))$ induces an action on $H^0(\mathcal{O}_{P_7}(2))$.

(2.8) **Lemma.** *The quadratic polynomials $q_{t(E),t(F)}$ from above are invariants under $EG(\mathcal{L})$.*

Proof. The curve $E+F$ is invariant under one nontrivial element $\mu \in T(\mathcal{L})$ but not under some other $\nu \in T(\mathcal{L})$, (cf. 1.2). So μ may be lifted to $\mathcal{O}_A(2t(E))$ and $\mathcal{O}_A(2t(F))$ acting trivially on the space of sections, whereas ν lifts with an eigenvector for both eigenvalues ± 1 on $H^0(\mathcal{O}_A(2t(E)))$ and $H^0(\mathcal{O}_A(2t(F)))$. We may take s_1, s_2, s_1', s_2' to be just these eigenvectors. Then our liftings act as in Table 2.

Table 2

	s_1	s_2	s_1'	s_2'	f_{11}	f_{12}	f_{21}	f_{22}	$f_{11}f_{22}-f_{12}f_{21}$
μ	1	1	1	1	1	1	1	1	1
ν	1	-1	1	-1	1	-1	-1	1	1

This proves the assertion for $G(\mathcal{L})\subset EG(\mathcal{L})$, because any lifting of μ, ν as involutions to $\mathcal{L}^{\otimes 2}$ is unique up to the sign, and this sign cancels when squaring the action.

To prove the assertion for ι, consider first the case $E+F \in |\mathcal{L}|$. Then $\iota C=C$ for all $C \in |2E|$ or $|2F|$ and we may lift ι to $\mathcal{O}_A(2E)$ and $\mathcal{O}_A(2F)$ acting trivially on the space of sections, hence on the f_{ij} and on q_{EF}. If $t \in T(\mathcal{L}^{\otimes 2})$, but $t \notin T_2(A)$, then $\iota t(E)=(-t)(E)$ and the same for F. So ι lifts to $\mathcal{O}_A(2E)$ and $\mathcal{O}_A(2F)$ with an eigenvector for both eigenvalues ± 1. Just as above for ν we see that this lifting acts trivially on $q_{t(E)t(F)}$. □

2.4. Quadratic equations for $A\subset P_7$

The aim of this section is to find a basis for the vector space $H^0(I_{A/P_7}(2))$. We use the fact that this space is a representation space for $EG(\mathcal{L}^{\otimes 2})$.

First we consider the action on this space of the subgroup $(Z_2)^3\subset EG(\mathcal{L}^{\otimes 2})$ generated by σ, τ, and ι (see Table 1). Writing an arbitrary element of this group as $\sigma^s\tau^t\iota^j$, we may denote the eight characters of this group by

$$0, s, t, j, s+t, s+j, t+j, s+t+j.$$

On the 2×2-blocks of the symmetric 8×8 coefficient matrix (q_{ij}) of a quadric q this group acts by the following characters (see Table 1):

$$\begin{pmatrix} 0 & t & s & s+t+j \\ t & 0 & s+t & s+j \\ s & s+t & 0 & t+j \\ s+t+j & s+j & t+j & 0 \end{pmatrix}.$$

The invariants of this action are the 2×2-block diagonal quadrics

$$\operatorname{diag}(B_1, B_2, B_3, B_4) = \begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & B_3 & \\ 0 & & & B_4 \end{pmatrix}.$$

(2.9) **Lemma.** *$H^0(I_{A/P_7}(2))$ is contained in the 12-dimensional space of 2×2-block diagonal quadrics.*

Proof. The space $H^0(I_{A/P_7}(2))$ is spanned by eigenfunctions q for σ, τ, and ι. If q is not a 2×2-block diagonal matrix, it has the 2×2-block form

$$q = \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{cccc} \quad & i & \quad & j \end{array} \\ \begin{pmatrix} 0 & & & \\ & 0 & & * \\ & & 0 & \\ & * & & 0 \end{pmatrix} \end{array} \qquad 1\le i<j\le 4.$$

Since $H^0(I_{A/P_7}(2))$ does not contain rank-2 quadrics, any such q will have rank 4. Its singular 3-plane is $x_{2i-1}=x_{2i}=x_{2j-1}=x_{2j}=0$. This 3-plane consists of fixed points for σ, τ, or $\sigma\tau$, and thus cannot meet A. By Proposition (2.7) we have $q=q_{t(E)t(F)}$, and this is impossible by Lemma (2.8). □

Now we use the remaining symmetries to decompose further this space of 2×2-block diagonal matrices. First consider the group generated by σ_1 and τ_1 acting on quadratic polynomials as in Table 3.

Table 3

	$x_{2k-1}^2+x_{2k}^2$	$x_{2k-1}^2-x_{2k}^2$	$x_{2k-1}\cdot x_{2k}$
σ_1	1	-1	1
τ_1	1	1	-1

It follows that the space of 2×2-block diagonal matrices

$$q=\operatorname{diag}(B_1, B_2, B_3, B_4)$$

decomposes into a direct sum of three mutually non-isomorphic 4-dimensional spaces:

$$V^1=\left\{\operatorname{diag}\begin{pmatrix}a_1 & 0\\ 0 & a_1\end{pmatrix}, \cdots, \begin{pmatrix}a_4 & 0\\ 0 & a_4\end{pmatrix}: a_1, \cdots, a_4 \in \boldsymbol{k}\right\}$$
$$V^2=\left\{\operatorname{diag}\begin{pmatrix}b_1 & 0\\ 0 & -b_1\end{pmatrix}, \cdots, \begin{pmatrix}b_4 & 0\\ 0 & -b_4\end{pmatrix}: b_1, \cdots, b_4 \in \boldsymbol{k}\right\}$$
$$V^3=\left\{\operatorname{diag}\begin{pmatrix}0 & c_1\\ c_1 & 0\end{pmatrix}, \cdots, \begin{pmatrix}0 & c_4\\ c_4 & 0\end{pmatrix}: c_1, \cdots, c_4 \in \boldsymbol{k}\right\}.$$

On each $V^j\simeq \boldsymbol{k}^4$ the group generated by σ_2 and τ_2 acts as in Table 4.

Table 4

	a_1	a_2	a_3	a_4	a_1+a_3	a_2+a_4	a_2-a_4	a_1-a_3
σ_2	a_2	a_1	a_4	a_3	a_2+a_4	a_1+a_3	a_1-a_3	a_2-a_4
τ_2	a_3	$-a_4$	a_1	$-a_2$	1	1	1	-1

and each V^j decomposes into two isomorphic irreducible representations of dimension 2. Each V^j contains a 1-parameter family $V^j_{\lambda:\mu}$, $(\lambda:\mu)\in \boldsymbol{P}_1$, of isomorphic representations, e.g.

$$V^1_{\lambda:\mu}=\{a_1, \cdots, a_4: \lambda(a_1+a_3)+\mu(a_2-a_4)=\lambda(a_2+a_4)+\mu(a_1-a_3)=0\}.$$

Generators are

$$(2.10)\quad \begin{aligned}
&\text{for } V^1_{\lambda_1:\mu_1}: \begin{cases} q_1=\mu_1(x_1^2+x_2^2)-\lambda_1(x_3^2+x_4^2)+\mu_1(x_5^2+x_6^2)+\lambda_1(x_7^2+x_8^2)\\ q_2=-\lambda_1(x_1^2+x_2^2)+\mu_1(x_3^2+x_4^2)+\lambda_1(x_5^2+x_6^2)+\mu_1(x_7^2+x_8^2)\end{cases}\\
&\text{for } V^2_{\lambda_2:\mu_2}: \begin{cases} q_3=\mu_2(x_1^2-x_2^2)-\lambda_2(x_3^2-x_4^2)+\mu_2(x_5^2-x_6^2)+\lambda_2(x_7^2-x_8^2)\\ q_4=-\lambda_2(x_1^2-x_2^2)+\mu_2(x_3^2-x_4^2)+\lambda_2(x_5^2-x_6^2)+\mu_2(x_7^2-x_8^2)\end{cases}\\
&\text{for } V^3_{\lambda_3:\mu_3}: \begin{cases} q_5=2(\mu_3x_1x_2-\lambda_3x_3x_4+\mu_3x_5x_6+\lambda_3x_7x_8)\\ q_6=2(-\lambda_3x_1x_2+\mu_3x_3x_4+\lambda_3x_5x_6+\mu_3x_7x_8).\end{cases}
\end{aligned}$$

(2.11) **Decomposition theorem.** *There are three projective parameters* $(\lambda_j: \mu_j)\in \boldsymbol{P}_1, j=1, 2, 3,$ *such that*

$$H^0(I_{A/\boldsymbol{P}_7}(2))=V^1_{\lambda_1:\mu_1}\oplus V^2_{\lambda_2:\mu_2}\oplus V^3_{\lambda_3:\mu_3}.$$

Proof. Each V^j contains nontrivial rank-2 quadrics. A cannot lie on such a quadric, so $H^0(I_{A/P_7}(2))$ does not contain a whole space V^j, $j=1,2$, or 3. The assertion follows if we show $h^0(I_{A/P_7}(2))>4$. Now $h^0(\mathcal{O}_{P_7}(2))=36$ and $h^0(\mathcal{L}^{\otimes 4})=32$, so it suffices to prove that restriction

$$H^0(\mathcal{O}_{P_7}(2)) \longrightarrow H^0(A, \mathcal{L}^{\otimes 4})$$

is not surjective. To do this we lift ι to $\mathcal{L}^{\otimes 4}$ and $\mathcal{L}^{\otimes 2}$ such that it acts trivially over $e_1, \cdots, e_{16}$. Then we have eigenspaces

$H^0(\mathcal{L}^{\otimes 4})^-$ of dimension 14	(by 0.2)
$(\mathrm{Sym}^2 H^0(\mathcal{L}^{\otimes 2}))^-$ of dimension 12	(cf. Table 1).

The restriction map is ι-equivariant, but not surjective for anti-invariant sections. □

(2.12) **Proposition.** *The only quadric Q containing A and singular at a point $x \in A$ is the quadric Q_t from* 2.2 *with $te_1=x$.*

Proof. Assume first that $t \notin T(\mathcal{L}^{\otimes 2})$. By Proposition (2.2) the four translates of x under $T(\mathcal{L})$ span a 3-space. Any quadric q singular at x will (by its $T(\mathcal{L})$-invariance, Lemma (2.8)) be singular on this whole 3-space. Q shares this property with Q_t. If $Q \neq Q_t$, then Q does not contain a general P_5 on Q_t (from any of the two pencils). Then Q intersects P_5 in two 4-spaces (coinciding or not).

However P_5 meets A in some smooth $C \in |t_*\mathcal{L}|$ with $\mathcal{L}^{\otimes 2}|C$ very ample, $H^0(\mathcal{L}^{\otimes 2}) \to H^0(\mathcal{L}^{\otimes 2}|C)$ is injective and

$$h^0(\mathcal{L}^{\otimes 2}|C) = (\mathcal{L}^{\otimes 2}\cdot C)+1-g(C)=6.$$

So C spans P_5 and for $Q \neq Q_t$ we arrive at a contradiction.

Next assume $t \in T(\mathcal{L}^{\otimes 2})$. By $G(\mathcal{L}^{\otimes 2})$-invariance we even may assume $x=e_1$. The points $e_1, \cdots, e_4$ span a plane in P_5^+ (Lemma 2.2), on which Q is singular by $T(\mathcal{L})$-invariance. Q contains the tangent plane of A at e_1, so it contains the line P_1^- (cf. Section 2.1), hence the P_4 spanned by $e_1, \cdots, e_4$ and P_1^-.

On the other hand, Q_0 is singular on this 4-space. For $Q \neq Q_0$ we would obtain the same contradiction as before.

(2.13) **Corollary.** *The quadrics Q_t from Section* 2.2 *and $Q_{t(E)t(F)}$ from Section* 2.3 *are the only quadrics of rank ≤ 4 containing A.*

(2.14) **Theorem.** *The quadrics $q_1, \cdots, q_6$ generate the ideal sheaf I_{A/P_7} at each point $x \in P_7$.*

Proof. Any intersection of six hypersurfaces in $\boldsymbol{P}_7$ is connected. So it suffices to show that in each $x \in A$ five linearly independent quadrics intersect transversally, or—what is the same—that there is exactly one quadric singular at x. This is the assertion of Proposition 2.11. □

§ 3. The parameters

3.1. Symplectic automorphisms of $T(2, 4)$

By Mumford's identification the translation group $T(\mathscr{L}^{\otimes 2})$ is isomorphic with $T:=T(2,4)=K\times\hat{K}^{\vee}$. This isomorphism transforms the natural skew form on $T(\mathscr{L}^{\otimes 2})$ into that on T (cf. 0.2). However this isomorphism is not unique! It can be changed by any *symplectic* automorphism φ of T. (φ is called symplectic if $\langle\varphi x, \varphi y\rangle=\langle x, y\rangle$.)

Another such identification induces another identification of $H^0(\mathscr{L}^{\otimes 2})$ with $V(2,4)$ and finally another embedding $A \hookrightarrow \boldsymbol{P}_7$. So the different equivariant embeddings $A \hookrightarrow \boldsymbol{P}_7$ are parametrized by the symplectic automorphisms of T. We have to know these automorphisms.

(3.1) **Proposition.** *The symplectic automorphisms of* $T=T(2,4)$ *form a group of order* $2^9\cdot 3^2$. *Generators are e.g. given in Table* 5.

Table 5

	σ_1	σ_2	τ_1	τ_2
φ_1	1	τ_2	1	σ_2^3
φ_2	1	$\sigma_2\tau_2$	1	1
φ_3	τ_1	1	σ_1	1
φ_4	$\sigma_1\tau_1$	1	1	1
φ_5	$\sigma_1\sigma$	1	1	$\tau_1\tau_2$

Proof. T is filtered by the characteristic subgroups

$$2T \text{ (generators } \sigma, \tau) \quad \subset \quad T_2 \text{ (generators } \sigma_1, \tau_1, \sigma, \tau).$$

The symplectic automorphisms φ_1 and φ_2 generate $\mathrm{Aut}\,(2T)\simeq\mathfrak{S}_3$ whereas φ_3 and φ_4 generate $\mathrm{Aut}\,(T_2/2T)\simeq\mathfrak{S}_3$. So there is an exact sequence

$$1 \longrightarrow L \longrightarrow \mathrm{Aut}\,(T) \longrightarrow \mathrm{Aut}\,(2T)\times\mathrm{Aut}\,(T_2/2T) \longrightarrow 1.$$

Any $\varphi \in L$ has the properties

$$\varphi(\sigma_1) \in \sigma_1\cdot 2T, \qquad \varphi(\tau_1) \in \tau_1\cdot 2T,$$
$$\varphi(\sigma_2) \in \sigma_2\cdot T_2, \qquad \varphi(\tau_2) \in \tau_2\cdot T_2.$$

Hence any symplectic $\varphi \in L$ is of the form shown in Table 6.

Table 6

	σ_1	σ_2	τ_1	τ_2
φ	$\sigma_1\sigma^a\tau^b$	$\sigma_2\sigma_1^d\tau_1^b\sigma^\alpha\tau^\beta$	$\tau_1\sigma^c\tau^d$	$\tau_2\sigma_1^c\tau_1^a\sigma^\gamma\tau^\alpha$

Here all combinations of $a, b, c, d, \alpha, \beta, \gamma \in Z_2$ are possible, so the symplectic automorphisms in L form a (non-abelian) group of order 2^7. This proves the assertion on the order of our group.

To give generators, we define φ_{abcd} and $\varphi_{\alpha\beta\gamma}$ in Table 7

Table 7

	σ_1	σ_2	τ_1	τ_2
φ_{abcd}	$\sigma_1\sigma^a\tau^b$	$\sigma_2\sigma_1^d\tau_1^b$	$\tau_1\sigma^c\tau^d$	$\tau_2\sigma_1^c\tau_1^a$
$\varphi_{\alpha\beta\gamma}$	1	$\sigma_2\sigma^\alpha\tau^\beta$	1	$\tau_2\sigma^\gamma\tau^\alpha$

and observe by inspection

$$\varphi_1^{-1}\varphi_{abcd}\varphi_1=\varphi_{badc} \qquad \varphi_1^{-1}\varphi_{\alpha\beta\gamma}\varphi_1=\varphi_{\alpha\gamma\beta}$$
$$\varphi_2^{-1}\varphi_{abcd}\varphi_2=\varphi_{a,a+b,c,c+d} \qquad \varphi_2^{-1}\varphi_{\alpha\beta\gamma}\varphi_2=\varphi_{\alpha+\gamma,\beta+\gamma,\gamma}$$
$$\varphi_3^{-1}\varphi_{abcd}\varphi_3=\varphi_{cdab}$$
$$\varphi_4^{-1}\varphi_{abcd}\varphi_4=\varphi_{a+c,b+d,c,d}.$$

These relations imply that $\varphi_5=\varphi_{1000}$ and $\varphi_{010}=\varphi_2^2$ together with $\varphi_1, \cdots, \varphi_4$ generate the group. □

An easy argument [HM, p. 65] shows that each symplectic automorphism φ is induced by an automorphism γ of $V(2, 4)$ normalizing the action of the Heisenberg group $G(2, 4)$. In fact, φ lifts to an automorphism $\tilde{\varphi}$ of $G(\mathscr{L}^{\otimes 2})$ acting trivially on the center k^* of this group. Denoting by ρ the representation of $G(\mathscr{L}^{\otimes 2})$ on $V(2, 4)$, each φ induces another representation $g \to \rho(\tilde{\varphi}g)$ in which the center k^* acts by multiplication. By the Stone-Neumann theorem [EDAV, Proposition 3] however, such a representation is unique. So there is an isomorphism γ of $V(2, 4)$ with

$$\rho(g)(\gamma v)=\gamma\rho(\tilde{\varphi}g)v \qquad \text{for all } v \in V(2, 4)$$
$$\text{i.e., } \rho(\tilde{\varphi}g)=\gamma^{-1}\rho(g)\gamma.$$

This general argument in our case is not necessary. Table 8 shows explicitly automorphisms $\gamma_1, \cdots, \gamma_5$ of $V(2, 4)$ normalizing the Heisenberg group and inducing on $T(2, 4)$ the symplectic automorphisms $\varphi_1, \cdots, \varphi_5$. Here, as usual, I do not distinguish between σ, $\tilde{\sigma}$, $\rho(\sigma)$, etc. I need this explicit form of $\gamma_1, \cdots, \gamma_5$ because I want to trace the action of the normalizing automorphisms $\gamma_1, \cdots, \gamma_5$ on the parameters $\chi_j = \lambda_j/\mu_j$. (This is given by the last three columns of Table 8).

Table 8

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
γ_1	1	1	x_5	x_6	x_3	x_4	$-i$	$-i$
γ_2	x_5	x_6	$i\sqrt{i}\,x_3$	$i\sqrt{i}\,x_4$	x_1	x_2	$i\sqrt{i}\,x_7$	$i\sqrt{i}\,x_8$
γ_3	x_1+x_2	$-x_1+x_2$	x_3+x_4	$-x_3+x_4$	x_5+x_6	$-x_5+x_6$	x_7+x_8	$-x_7+x_8$
γ_4	1	i	1	i	1	i	1	i
γ_5	1	1	1	1	1	-1	1	-1

	σ_1	σ_2	τ_1	τ_2	$\chi_1 = \frac{\lambda_1}{\mu_1}$	$\chi_2 = \frac{\lambda_2}{\mu_2}$	$\chi_3 = \frac{\lambda_3}{\mu_3}$
γ_1	1	τ_2	1	σ_2	$\frac{\chi_1+1}{\chi_1-1}$	$\frac{\chi_2+1}{\chi_2-1}$	$\frac{\chi_3+1}{\chi_3-1}$
γ_2	1	$i\sqrt{i}\,\tau_2\sigma_2$	1	1	$-i\chi_1$	$-i\chi_2$	$-i\chi_3$
γ_3	τ_1	1	$-\sigma_1$	1	χ_1	χ_3	χ_2
γ_4	$i\tau_1\sigma_1$	1	1	1	χ_2	χ_1	χ_3
γ_5	$\sigma_1\sigma$	1	1	$\tau_1\tau_2$	χ_1	χ_2	$\frac{1}{\chi_3}$

Automorphisms normalizing $G(2, 4)$. The action on σ_i, τ_i is conjugation $\gamma^{-1}\sigma_i\gamma$, $\gamma^{-1}\tau_i\gamma$.

The resulting symmetries acting on the affine parameters $\chi_j = \lambda_j/\mu_j$ can be described more systematically as follows: The symmetry group is generated by

—the symmetric group $\mathfrak{S}_3$ permuting χ_1, χ_2, and χ_3 (generated by γ_3 and γ_4)

—the full octahedral group (of order 48) acting simultaneously on χ_1, χ_2 and χ_3 (generated by γ_1: $\chi \mapsto (\chi+1)/(\chi-1)$, γ_2^3: $\chi \mapsto i\chi$, and $\chi \mapsto 1/\chi$ which is a result of a combination of γ_3, γ_4, and γ_5; the octahedron has the vertices $\chi = 0, \infty, 1, i, -1, -i$).

—three groups of order four acting independently on χ_1, χ_2, and χ_3

generated by γ_5: $\chi_j \mapsto 1/\chi_j$ and $[\gamma_2, \gamma_5]$: $\chi_j \mapsto -\chi_j$.

It is easy to check that the group generated has order $6 \cdot 48 \cdot 4 \cdot 4 = 2^9 \cdot 3^2$.

3.2. Degenerations

Here we consider the variety $A \subset \boldsymbol{P}_7$ defined by the equations $q_1 = \cdots = q_6 = 0$ and specialize the parameters $\lambda_1 : \mu_1$, $\lambda_2 : \mu_2$, $\lambda_3 : \mu_3$ in these equations.

Case 1. $\lambda_1 = \lambda_2 = \lambda$, $\mu_1 = \mu_2 = \mu$, but $\lambda \neq \pm\mu$.

The equations $q_1 = \cdots = q_4 = 0$ are equivalent to

$$E:\ \mu x_1^2 - \lambda x_3^2 + \mu x_5^2 + \lambda x_7^2 = -\lambda x_1^2 + \mu x_3^2 + \lambda x_5^2 + \mu x_7^2 = 0$$
$$E':\ \mu x_2^2 - \lambda x_4^2 + \mu x_6^2 + \lambda x_8^2 = -\lambda x_2^2 + \mu x_4^2 + \lambda x_6^2 + \mu x_8^2 = 0.$$

Since $\lambda \neq \pm\mu$, the varieties E, E' describe smooth elliptic quartic space curves in the 3-spaces $x_2 = x_4 = x_6 = x_8 = 0$ and $x_1 = x_3 = x_5 = x_7 = 0$, respectively. The four equations describe the join J of E and E', i.e., the threefold swept out by all lines joining points of E and E'.

The differentials of the two quadrics defining E vanish along E' and those defining E' vanish along E. The quadrics Q_5 and Q_6 contain E and E'. So A contains two curves along which at most four of the differentials $dq_1, \cdots, dq_6$ are linearly independent. A therefore is *singular* along these curves.

Fix a line $L \subset J$ joining the two points

$$x = (x_1 : 0 : x_3 : 0 : x_5 : 0 : x_7 : 0) \in E, \quad x' = (0 : x_2 : 0 : x_4 : 0 : x_6 : 0 : x_8) \in E'.$$

On it the two quadrics q_5 and q_6 are proportional. If they vanish in one point of L, not on E or E', they vanish on L identically. This shows that $A \subset J$ is swept out by some of these lines. For given $x \in E$ the lines in A through x are those joining x with $x' \in E'$ whenever (x_2, x_4, x_6, x_8) lies in the plane

$$(\mu_3 x_1,\ -\lambda_3 x_3,\ \mu_3 x_5,\ \lambda_3 x_7)\begin{pmatrix} x_2 \\ x_4 \\ x_6 \\ x_8 \end{pmatrix} = 0.$$

This plane meets E' at most in four points. So there are at most four such lines through x. This implies: A is a ruled surface (or consists of components which are ruled).

Case 1′. Assume additionally $\lambda_3 = \lambda$, $\mu_3 = \mu$. For $x \in E$, the two planes

$$\begin{pmatrix} \mu_3 x_1, & -\lambda_3 x_3, & \mu_3 x_5, & \lambda_3 x_7 \\ -\lambda_3 x_1, & \mu_3 x_3, & \lambda_3 x_5, & \mu_3 x_7 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \\ x_6 \\ x_8 \end{pmatrix} = 0$$

now intersect in the tangent line to E' at the point x' with $x'_{2j}=x_{2j-1}$ ($j=1, \cdots, 4$). No tangent of E' meets this curve in a second point, i.e., through $x \in E$ there is just one line in A. It joins x with $x' \in E'$ which corresponds to x under the canonical isomorphism $E \to E'$.

Case 2. $\lambda_1=\lambda_2=\mu_1=\mu_2$, but $\lambda_3 \neq \pm\mu_3$.
We may argue as above, but now

$$E: \ x_1=\pm x_3, \ x_5=\pm i x_7$$
$$E': \ x_2=\pm x_4, \ x_6=\pm i x_8$$

degenerate into four lines

$$L_1: \ x_1=x_3, \ x_5=ix_7 \qquad L_2: \ x_1=x_3, \ x_5=-ix_7$$
$$L_3: \ x_1=-x_3, \ x_5=-ix_7 \quad L_4: \ x_1=-x_3, \ x_5=ix_7$$

forming a circle ($L_j \cap L_{j+1} \neq \phi$, $L_j \cap L_{j+2} = \phi$); similarly for E'. It is easy to check explicitly:

$$x=(a, a, ib, b) \in L_1 \quad \text{is joined to}$$
$$x'=(b, b, ia, a) \in L'_1 \quad \text{in } A \text{ and}$$
$$x''=(b, b, -ia, a) \in L'_3 \quad \text{in } A.$$

The symmetry $\tau_1 \in G(\mathscr{L}^{\otimes 2})$ cyclically permutes the lines L_j and L'_j. So we see: L_j and L'_k are joined by a pencil of lines if $j \equiv k \pmod 2$. The surface A breaks up into eight quadric surfaces S_j (joining L_j with L'_j) and S'_j (joining L_j with L'_{j+2}) which meet in a line according to the following pattern: S_j with S_{j+1}, S'_j with S'_{j+1}, S_j with S'_j and S'_{j+2}. The pattern is described by a real 2-dimensional complex on a real torus:

S_1	S_2	S_3	S_4	Glue ends
S'_1	S'_2	S'_3	S'_4	

Glue top and bottom with a horizontal 2-shift. This is the analog of the picture [HM, p. 80].

Among all these lovely facts in the sequel we shall only need the following consequence of the discussion under Case 1:

(3.2) **Proposition.** *The variety defined by* $q_1=\cdots=q_6=0$ *is not a smooth surface as soon as*

$$r:=r_{12}\cdot r_{23}\cdot r_{31}=0$$

where

$$r_{jk}=r_{jk}(\lambda,\mu):=(\lambda_j^2\mu_k^2-\lambda_k^2\mu_j^2)(\lambda_j^2\lambda_k^2-\mu_j^2\mu_k^2).$$

Proof. The vanishing of r_{jk} is equivalent with $\chi_j=\pm\chi_k$ or $\pm 1/\chi_k$ where $\chi_j=\lambda_j/\mu_j$ $(j=1,2,3)$. The symmetries from Section 3.1 do not change the abstract variety (only its embedding). They can be used to transform any such condition into $\chi_1=\chi_2$. This is considered in the discussion above and found to lead to singular A. □

It will be shown in Section 4.3 that A is smooth if and only if $r_{12}\cdot r_{23}\cdot r_{31}\neq 0$.

3.3. Properties of the expressions r_{jk}

The expressions r_{jk} defined above are equal to one fourth of the minors of the 3×3-matrix

$$M=M(\lambda,\mu)=\begin{pmatrix}(\lambda_1^2+\mu_1^2)^2 & (\lambda_2^2+\mu_2^2)^2 & (\lambda_3^2+\mu_3^2)^2\\ (\lambda_1^2-\mu_1^2)^2 & (\lambda_2^2-\mu_2^2)^2 & (\lambda_3^2-\mu_3^2)^2\\ 4\lambda_1^2\mu_1^2 & 4\lambda_2^2\mu_2^2 & 4\lambda_3^2\mu_3^2\end{pmatrix}$$

which obviously has rank ≤ 2. To be precise, we have

$$M\cdot\begin{pmatrix}r_{23}\\ r_{31}\\ r_{12}\end{pmatrix}=0.$$

We shall often make use of the following fact.

(3.3) *If* $r=r_{12}\cdot r_{23}\cdot r_{31}\neq 0$, *then each row and each column of the matrix* M *contains at most one entry* $=0$.

At two points we shall meet the problem to find parameters $\lambda^\vee$, $\mu^\vee$ such that for some constants $c_1, c_2, c_3\in \boldsymbol{k}$ we have:

$$\begin{aligned}
&((\lambda_1^\vee)^2+(\mu_1^\vee)^2, (\lambda_2^\vee)^2+(\mu_2^\vee)^2, (\lambda_3^\vee)^2+(\mu_3^\vee)^2)=c_1(\lambda_1^2+\mu_1^2, \lambda_1^2-\mu_1^2, 2\lambda_1\mu_1)\\
(3.4)\quad &((\lambda_1^\vee)^2-(\mu_1^\vee)^2, (\lambda_2^\vee)^2-(\mu_2^\vee)^2, (\lambda_3^\vee)^2-(\mu_3^\vee)^2)=c_2(\lambda_2^2+\mu_2^2, \lambda_2^2-\mu_2^2, 2\lambda_2\mu_2)\\
&(2\lambda_1^\vee\mu_1^\vee, 2\lambda_2^\vee\mu_2^\vee, 2\lambda_3^\vee\mu_3^\vee)=c_3(\lambda_3^2+\mu_3^2, \lambda_3^2-\mu_3^2, 2\lambda_3\mu_3).
\end{aligned}$$

Because of

$$((\lambda_j^\vee)^2+(\mu_j^\vee)^2)^2=((\lambda_j^\vee)^2-(\mu_j^\vee)^2)^2+(2\lambda_j\mu_j)^2$$

the constants c_1, c_2, c_3 must satisfy

$$M(\lambda,\mu)\cdot\begin{pmatrix}-c_1^2\\ c_2^2\\ c_3^2\end{pmatrix}=0.$$

i.e., up to a common factor

$$c_1=\pm\sqrt{r_{32}} \qquad c_2=\pm\sqrt{r_{31}} \qquad c_3=\pm\sqrt{r_{12}}.$$

Putting $c_j=\sqrt{r_{kl}}$ for some chosen square root, one now may easily solve the (compatible) equations

$$\text{(3.5)}\quad \begin{aligned}
(\lambda_1^\vee)^2&=\frac{1}{2}(\sqrt{r_{32}}(\lambda_1^2+\mu_1^2)+\sqrt{r_{31}}(\lambda_2^2+\mu_2^2)) \\
(\mu_1^\vee)^2&=\frac{1}{2}(\sqrt{r_{32}}(\lambda_1^2+\mu_1^2)-\sqrt{r_{31}}(\lambda_2^2+\mu_2^2)) & \qquad \lambda_1^\vee\mu_1^\vee&=\sqrt{r_{12}}(\lambda_3^2+\mu_3^2) \\
(\lambda_2^\vee)^2&=\frac{1}{2}(\sqrt{r_{32}}(\lambda_1^2-\mu_1^2)+\sqrt{r_{31}}(\lambda_2^2-\mu_2^2)) \\
(\mu_2^\vee)^2&=\frac{1}{2}(\sqrt{r_{32}}(\lambda_1^2-\mu_1^2)-\sqrt{r_{31}}(\lambda_2^2-\mu_2^2)) & \qquad \lambda_2^\vee\mu_2^\vee&=\sqrt{r_{12}}(\lambda_3^2-\mu_3^2) \\
(\lambda_3^\vee)^2&=\frac{1}{2}(\sqrt{r_{32}}\,\lambda_1\mu_1+\sqrt{r_{31}}\,\lambda_2\mu_2) \\
(\mu_3^\vee)^2&=\frac{1}{2}(\sqrt{r_{32}}\,\lambda_1\mu_1-\sqrt{r_{31}}\,\lambda_2\mu_2) & \qquad \lambda_3^\vee\mu_3^\vee&=\sqrt{r_{12}}\,\lambda_3\mu_3
\end{aligned}$$

which determine $\lambda_j^\vee$, $\mu_j^\vee$ uniquely up to a common factor ± 1, $j=1,2,3$. With these parameters $\lambda^\vee$, $\mu^\vee$ we may form the matrix $M^\vee=M(\lambda^\vee,\mu^\vee)$ and its minors $r_{jk}^\vee=r_{jk}(\lambda^\vee,\mu^\vee)$. Squaring the entries in the matrix equation (3.4) we find

$$r_{12}^\vee=r_{23}^\vee=r_{31}^\vee=r.$$

So we have in particular

(3.6) *If* $\lambda_1,\cdots,\mu_2$ *satisfy* $r_{12}\cdot r_{23}\cdot r_{31}\neq 0$, *then also* $r_{12}^\vee\cdot r_{23}^\vee\cdot r_{31}^\vee\neq 0$.

3.4. Theta-Null values

The aim of this section is to determine the variety in $\boldsymbol{P}_7$ swept out by the 2-torsion points $e_1,\cdots,e_{16}\in A$ when the parameters λ_j/μ_j vary. These 16 points are the intersection of A with the 5-space $x_7=x_8=0$. For fixed

λ_j, μ_j their equations therefore are

$$(3.7)\qquad \begin{aligned} &\mu_1(x_1^2+x_2^2)-\lambda_1(x_3^2+x_4^2)+\mu_1(x_5^2+x_6^2)=0\\ -&\lambda_1(x_1^2+x_2^2)+\mu_1(x_3^2+x_4^2)+\lambda_1(x_5^2+x_6^2)=0\\ &\mu_2(x_1^2-x_2^2)-\lambda_2(x_3^2-x_4^2)+\mu_2(x_5^2-x_6^2)=0\\ -&\lambda_2(x_1^2-x_2^2)+\mu_2(x_3^2-x_4^2)+\lambda_2(x_5^2-x_6^2)=0\\ &\mu_3x_1x_2-\lambda_3x_3x_4+\mu_3x_5x_6=0\\ -&\lambda_3x_1x_2+\mu_3x_3x_4+\lambda_3x_5x_6=0. \end{aligned}$$

These three pairs of linear equations (for $x_1^2+x_2^2$, $x_3^2+x_4^2$ etc.) are equivalent with the following proportionalities

$$(3.8)\qquad \begin{aligned} &(x_1^2+x_2^2,\ x_3^2+x_4^2,\ x_5^2+x_6^2)=c_1(\lambda_1^2+\mu_1^2,\ 2\lambda_1\mu_1,\ \lambda_1^2-\mu_1^2)\\ &(x_1^2-x_2^2,\ x_3^2-x_4^2,\ x_5^2-x_6^2)=c_2(\lambda_2^2+\mu_2^2,\ 2\lambda_2\mu_2,\ \lambda_2^2-\mu_2^2)\\ &(2x_1x_2,\ 2x_3x_4,\ 2x_5x_5)=c_3(\lambda_3^2+\mu_3^2,\ 2\lambda_3\mu_3,\ \lambda_3^2-\mu_3^2) \end{aligned}$$

with c_1, c_2, $c_3 \in \boldsymbol{k}$. Using the relation

$$(\lambda_1^2+\mu_1^2)^2=(\lambda_1^2-\mu_1^2)^2+(2\lambda_1\mu_1)^2$$

we obtain for the variety of theta-null values the equations

$$(3.9)\qquad x_1^2x_2^2=x_3^2x_4^2+x_5^2x_6^2 \qquad x_1^4+x_2^4=x_3^4+x_4^4+x_5^4+x_6^4.$$

Conversely, given $x_1, \cdots, x_6$ satisfying these relations, one may solve equations (3.8), putting $c_1=c_2=c_3=1$, to obtain

$$\begin{aligned} \lambda_1&=\pm\sqrt{x_1^2+x_2^2+x_5^2+x_6^2} & \mu_1&=\pm\sqrt{x_1^2+x_2^2-x_5^2-x_6^2}\\ \lambda_2&=\pm\sqrt{x_1^2-x_2^2+x_5^2-x_6^2} & \mu_2&=\pm\sqrt{x_1^2-x_2^2-x_5^2+x_6^2}\\ \lambda_3&=\pm\sqrt{x_1x_2+x_5x_6} & \mu_3&=\pm\sqrt{x_1x_2-x_5x_6}. \end{aligned}$$

The signs have to be adjusted in each row to satisfy

$$2\lambda_1\mu_1=x_3^2+x_4^2, \qquad 2\lambda_2\mu_2=x_3^2-x_4^2, \qquad 2\lambda_3\mu_3=x_3x_4.$$

The point $(x_1 : \cdots : x_6)$ therefore belongs to the variety A with these parameters $\lambda_j : \mu_j$. Consequently we have:

(3.10) **Proposition.** *Equations* (3.9) *define the threefold of theta-null values.*

(3.11) **Proposition.** *The variety of theta-null values* (3.9) *is irreducible.*

Remark. The proposition implies that it is impossible to distinguish one of the 16 torsion points $e_1, \cdots, e_{16}$ in an algebraic way. This is the reason for the convention adopted in 0.2, namely to define our abelian surfaces as homogeneous spaces and not as groups with a distinguished origin.

Proof of the proposition. As a complete intersection of two hypersurfaces, the theta-null variety is connected in codimension 2 [Ha 1], so if it decomposes, the irreducible components have to meet in surfaces, and not in curves only. It therefore suffices to prove that the singularities of (3.9) lie on a finite set of curves.

The condition for x on (3.9) to be singular is that the two rows

$$\begin{matrix} -x_1x_2^2 & -x_1^2x_2 & x_3x_4^2 & x_3^2x_4 & x_5x_6^2 & x_5^2x_6 \\ -x_1^3 & -x_2^3 & x_3^3 & x_4^3 & x_5^3 & x_6^3 \end{matrix}$$

are dependent. If $x_1\cdot \cdots \cdot x_6 \neq 0$ this leads to

$$x_2 = i^k x_1, \quad x_4 = i^l x_3, \quad x_6 = i^m x_5, \quad k \equiv l \equiv m(2)$$
$$x_1^4 = x_3^4 + x_5^4,$$

which defines 16 (Fermat-) quartic curves. If $x_1\cdot \cdots \cdot x_4 \neq 0$ but $x_5 \cdot x_6 = 0$, we find

$$x_5 = x_6 = 0$$
$$x_2 = i^k x_1, \quad x_4 = i^l x_3 \quad k \equiv l(2)$$
$$x_1^4 = x_3^4$$

which describes points on these 16 curves. If $x_3 \cdot x_4 = x_5 \cdot x_6 = 0$, then $x_1 \cdot x_2 = 0$ too. This leads to 8 Fermat curves obtained from

$$x_1 = x_3 = x_5 = 0 \qquad x_2^4 = x_4^4 + x_6^4$$

by permutation of coordinates. □

If the parameters λ_j, μ_j belong to an abelian surface, $A \cap \boldsymbol{P}_5^+$ consists of 16 points.

(3.12) **Lemma.** *Whenever $r \neq 0$, $A \cap \boldsymbol{P}_5^+$ consists of exactly 16 points forming an orbit under the group $T_2(A)$ of 2-torsion translations.*

Proof. For given $\lambda_1, \cdots, \mu_6$ we have to find c_1, c_2, c_3 and $x_1, \cdots, x_6$ solving the equations (3.8).

But this is exactly the system (3.4) with

$$x_1=\lambda_1^\vee,\quad x_2=\mu_1^\vee,\quad x_3=\lambda_3^\vee,\quad x_4=\mu_3^\vee,\quad x_5=\lambda_2^\vee,\quad x_6=\lambda_3^\vee.$$

Up to a common constant factor we have a choice of the sign for each c_i and for each pair x_{2j-1}, x_{2j}. This leads to exactly 16 different points x.

The translations $\sigma, \tau \in T_2(A)$ change the signs for the x_j, whereas σ_1, (resp. τ_1) correspond to the sign changes for c_2, (resp. c_3). □

§ 4. Kummer surfaces

In this section $\lambda_1 : \mu_1, \lambda_2 : \mu_2, \lambda_3 : \mu_3 \in \boldsymbol{P}_1$, are arbitrary parameters, $q_1, \cdots, q_6$ denote the quadrics (2.10) formed with these parameters, and $A \subset \boldsymbol{P}_7$ is the variety they define.

4.1. The hypernet

The six quadrics $q_1, \cdots, q_6$ span a *hypernet of quadrics* $\sum_1^6 z_j q_j$ in $\boldsymbol{P}_7$. The action of the Heisenberg group on $\boldsymbol{P}^7$ induces the action on the parameter space $\boldsymbol{P}_5$ of this hypernet of Table 9.

Table 9

	z_1	z_2	z_3	z_4	z_5	z_6
σ_1	1	1	-1	-1	1	1
σ_2	z_2	z_1	z_4	z_3	z_6	z_5
τ_1	1	1	1	1	-1	-1
τ_2	1	-1	1	-1	1	-1

Just as the q_j do, the hypernet comes in symmetric 2×2-blocks

$$\sum_1^6 z_j q_j = \begin{pmatrix} B_1(z) & & & 0 \\ & B_2(z) & & \\ & & B_3(z) & \\ 0 & & & B_4(z) \end{pmatrix}$$

$$B_1(z)=\begin{pmatrix} \mu_1 z_1-\lambda_1 z_2+\mu_2 z_3-\lambda_2 z_4 & \mu_3 z_5-\lambda_3 z_6 \\ \mu_3 z_5-\lambda_3 z_6 & \mu_1 z_1-\lambda_1 z_2-\mu_2 z_3+\lambda_2 z_4 \end{pmatrix}$$

$$B_2(z)=\begin{pmatrix} -\lambda_1 z_1+\mu_1 z_2-\lambda_2 z_3+\mu_2 z_4 & -\lambda_3 z_5+\mu_3 z_6 \\ -\lambda_3 z_5+\mu_3 z_6 & -\lambda_1 z_1+\mu_1 z_2+\lambda_2 z_3-\mu_2 z_4 \end{pmatrix}$$

$$B_3(z)=\begin{pmatrix} \mu_1 z_1+\lambda_1 z_2+\mu_2 z_3+\lambda_2 z_4 & \mu_3 z_5+\lambda_3 z_6 \\ \mu_3 z_5+\lambda_3 z_6 & \mu_1 z_1+\lambda_1 z_2-\mu_2 z_3-\lambda_2 z_4 \end{pmatrix}$$

$$B_4(z)=\begin{pmatrix} \lambda_1 z_1+\mu_1 z_2+\lambda_2 z_3+\mu_2 z_4 & \lambda_3 z_5+\mu_3 z_6 \\ \lambda_3 z_5+\mu_3 z_6 & \lambda_1 z_1+\mu_1 z_2-\lambda_2 z_3-\mu_2 z_4 \end{pmatrix}.$$

The discriminant $\prod_1^4 \det B_j(z)=0$ therefore decomposes into four rank-3 cones:

$$K_1:\ (\mu_1 z_1-\lambda_1 z_2)^2-(\mu_2 z_3-\lambda_2 z_4)^2-(\mu_3 z_4-\lambda_3 z_6)^2$$
$$K_2:\ (\lambda_1 z_1-\mu_1 z_2)^2-(\lambda_2 z_3-\mu_2 z_4)^2-(\lambda_3 z_5-\mu_3 z_6)^2$$
$$K_3:\ (\mu_1 z_1+\lambda_1 z_2)^2-(\mu_2 z_3+\lambda_2 z_4)^2-(\mu_3 z_5+\lambda_3 z_6)^2$$
$$K_4:\ (\lambda_1 z_1+\mu_1 z_2)^2-(\lambda_2 z_3+\mu_2 z_4)^2-(\lambda_3 z_5+\mu_3 z_6)^2.$$

These cones are invariant under σ_1 and τ_1 but under σ_2 and τ_2 they are permuted transitively

$$\sigma_2\colon K_1\leftrightarrow K_2,\quad K_3\leftrightarrow K_4,\qquad \tau_2\colon K_1\leftrightarrow K_3,\quad K_2\leftrightarrow K_4.$$

We denote by $S_1, \cdots, S_4\subset \boldsymbol{P}_4$ the singular planes of the cones $K_1, \cdots, K_4$. Then we observe:

$$\begin{aligned}\dim S_1\cap S_2&=\dim S_3\cap S_4\\ &=-1+\#\{\text{terms } \lambda_1^2-\mu_1^2,\ \lambda_2^2-\mu_2^2,\ \lambda_3^2-\mu_3^2 \text{ that vanish}\}\\ \dim S_1\cap S_3&=\dim S_2\cap S_4\\ &=-1+\#\{\text{terms } \lambda_1\mu_1,\ \lambda_2\mu_2,\ \lambda_3\mu_3 \text{ that vanish}\}\\ \dim S_1\cap S_4&=\dim S_2\cap S_3\\ &=-1+\#\{\text{terms } \lambda_1^2+\mu_1^2,\ \lambda_2^2+\mu_2^2,\ \lambda_3^2+\mu_3^2 \text{ that vanish}\}\end{aligned}$$

(Here $\dim=-1$ means the set is empty). The intersection of three singular planes always is empty.

For any quadric $q(z)$ in the hypernet we have

$$\operatorname{rank} q(z)=8-\#\{j\colon z\in K_j\}-\#\{j\colon z\in S_j\}.$$

To discuss the different possibilities we need:

(4.1) **Lemma.** *On each singular plane S_j the other cones cut out a pencil of conics with four distinct base points, provided* $r\neq 0$ (cf. Section 3.4).

Proof. In view of the symmetries permuting the cones we may consider S_1 only, Because of the symmetry γ_5 from Table 2, we may assume $\mu_1\cdot\mu_2\cdot\mu_3\neq 0$ without loss of generality. Then we eliminate

$$z_1=\frac{\lambda_1}{\mu_1}z_2,\qquad z_3=\frac{\lambda_2}{\mu_2}z_4,\qquad z_5=\frac{\lambda_3}{\mu_3}z_6$$

and find

$$K_2 \cap S_1:\ \left(\frac{\lambda_1^2-\mu_1^2}{\mu_1}\right)^2 z_2^2-\left(\frac{\lambda_2^2-\mu_2^2}{\mu_2}\right)^2 z_4^2-\left(\frac{\lambda_3^2-\mu_3^2}{\mu_3}\right)^2 z_6^2$$

$$K_3 \cap S_1:\ \lambda_1^2 z_2^2-\lambda_2^2 z_4^2-\lambda_3^2 z_6^2$$

$$K_4 \cap S_1:\ \left(\frac{\lambda_1^2+\mu_1^2}{\mu_1}\right)^2 z_2^2-\left(\frac{\lambda_2^2+\mu_2^2}{\mu_2}\right)^2 z_4^2-\left(\frac{\lambda_3^2+\mu_3^2}{\mu_3}\right)^2 z_6^2.$$

These three conics clearly belong to the same pencil. The pencil has four distinct base points, unless one coordinate z_{2k} of a base point vanishes. But then

$$\lambda_m^2\mu_m^2(\lambda_l^2-\mu_l^2)^2-\lambda_l^2\mu_l^2(\lambda_m^2-\mu_m^2)^2=r_{lm}=0$$

($\{k, l, m\}=\{1, 2, 3\}$), a contradiction. □

(4.2) **Proposition.** *Assume* $r \neq 0$. *Then*

rank $q(z) \leq 3$ *iff* z *belongs to a singular plane and the three other cones*
rank $q(z) \leq 4$ *iff* $z \in K_1 \cap \cdots \cap K_4$ *or* z *belongs to two singular planes.*

Further, the number of intersecting singular planes equals 2ν *where* ν *is the number of products* $\prod_1^3 \lambda_j\mu_j$, $\prod_1^3(\lambda_j^2+\mu_j^2)$, $\prod_1^3(\lambda_j^2-\mu_j^2)$, *that vanish, i.e.,* $\nu=0$, 1, 2, *or* 3.

The proof is obvious by Lemma (4.1).

4.2. The dual Kummer surface $K^\vee=\mathrm{Km}\, A^\vee$

In this section we assume $r \neq 0$. The variety

$$K^\vee=K_1 \cap K_2 \cap K_3 \cap K_4 \subset \boldsymbol{P}_5$$

parametrizes quadrics $q(z)$ of rank ≤ 4 in the hypernet. The four cones are obviously linearly dependent. They are linear combinations of, e.g.,

$$L_1=\frac{1}{2}(K_1+K_3)=\mu_1^2 z_1^2+\lambda_1^2 z_2^2-\mu_2^2 z_3^2-\lambda_2^2 z_4^2-\mu_3^2 z_5^2-\lambda_3^2 z_6^2$$

$$L_2=\frac{1}{2}(K_2+K_4)=\lambda_1^2 z_1^2+\mu_1^2 z_2^2-\lambda_2^2 z_3^2-\mu_2^2 z_4^2-\lambda_3^2 z_5^2-\mu_3^2 z_6^2$$

$$L_3=\frac{1}{4}(K_3-K_1)=\frac{1}{4}(K_4-K_2)=\lambda_1\mu_1 z_1 z_2-\lambda_2\mu_2 z_3 z_4-\lambda_3\mu_3 z_5 z_6.$$

To study $K^\vee$ we first consider the net of quadrics in $\boldsymbol{P}_5$ spanned by $K_1, \cdots, K_4$ i.e., the net $\sum_1^3 t_k L_k$, $(t_1: t_3: t_3) \in \boldsymbol{P}_3$.

(4.3) **Lemma.** a) *The only quadrics* $\sum t_k L_k$ *in the net having rank* ≤ 3 *are the four cones* $K_1, \cdots, K_4$.

b) *Let* H: $z_{2j-1}=z_{2j}=0, j=1, 2,$ *or* 3, *be a coordinate* 3-*plane. The four cones are the only quadrics in the net restricting to* H *such that*

—*the restriction has* rank ≤ 2

—*the singular locus of the restricted quadric meets* $K^\vee$.

Proof. Again the net comes in symmetric 2×2-blocks:

$$\sum t_k L_k = \begin{pmatrix} B_1(t) & & \\ & -B_2(t) & \\ & & -B_3(t) \end{pmatrix}$$

$$B_j(t)=\begin{pmatrix} \mu_j^2 t_1+\lambda_j^2 t_2 & \lambda_j\mu_j t_3 \\ \lambda_j\mu_j t_3 & \lambda_j^2 t_1+\mu_j^2 t_2 \end{pmatrix}$$

$$\det B_j(t)=\lambda_j^2\mu_j^2(t_1^2+t_2^2-t_3^2)+(\lambda_j^4+\mu_j^4)t_1t_2.$$

If $\det B_k(t)=\det B_l(t)=0$, then $r_{kl}\neq 0$ implies

$$t_1\cdot t_2=t_1^2+t_2^2-t_3^2=0,$$

i.e.,

$$t=(1:0:\pm 1) \text{ or } (0:1:\pm 1).$$

These are the parameters for the four cones.

Proof of a): We have to exclude $B_k(t)=0$, $\det B_l(t)=0$ for $k\neq l$. But $B_k(t)=0$ implies

$$\lambda_k\mu_k t_3=\mu_k^2 t_1+\lambda_k^2 t_2=\lambda_k^2 t_1+\mu_k^2 t_2=0.$$

If $\lambda_k\mu_k=0$, then $\lambda_k^4\neq\mu_k^4$ so $t_1=t_2=0$ and $\det B_l(t)=\lambda_l\mu_l t_3^2\neq 0$ by (3.3).

If $t_3=0$, then $(t_1, t_2)\neq 0$ and $\mu_k^2 t_1+\lambda_k^2 t_2=0$. Together with

$$\det B_l(t)=(\lambda_l^2 t_1+\mu_l^2 t_2)(\mu_l^2 t_1+\lambda_l^2 t_2)=0$$

this contradicts $r_{kl}\neq 0$.

Proof of b): We have to exclude $B_k(t)=0$ and $z\in K^\vee$ for some z with $z_{2m-1}=z_{2m}=z_{2j-1}=z_{2j}=0$ ($\{j, k, m\}=\{1, 2, 3\}$).

If $\lambda_k\mu_k=0$, then $\mu_k^4\neq\lambda_k^4$ and the vanishing of $L_1(z)$, $L_2(z)$ would lead to the contradiction $z_{2k-1}=z_{2k}=0$.

If $t_3=0$, then $\mu_k^4=\lambda_k^4$, hence $\lambda_k\mu_k\neq 0$, $z_{2k-1}\cdot z_{2k}=0$ by $L_3(z)=0$, and $z_{2k-1}=z_{2k}=0$ by $L_1(z)=L_2(z)=0$. □

Now we finally can prove:

(4.4) **Proposition.** *Under the assumption* $r\neq 0$, *the variety* $K^\vee$ *is* *an irreducible surface of degree* 8, *smooth except for* 16 *isolated normal singularities* (*in each of the four singular planes the four base points of the pencil of conics from Lemma* (4.1)). *Away from these singularities,* L_1, L_2 *and* L_3 *intersect transversally.*

Proof. Let $u \in K^\vee$, but $u \notin S_j$ for $j=1, \cdots, 4$. The three quadrics L_1, L_2, L_3 meet transversally at u unless there is some quadric $L=\sum t_k L_k$ singular at u. By the invariance of L under the involutions σ_1 and τ_1, this L would be singular at the three points $u, \sigma_1 u, \tau_1 u$. If they span a plane, then rank $L \leq 3$, so L would be some K_j by Lemma (4.3) a) and $u \in S_j$, contradicting the choice of u. If they span a line, then $u_{2j-1}=u_{2j}=0$ for some j and rank $(L \mid z_{2j-1}=z_{2j}=0) \leq 2$, with Lemma (4.3) b) another contradiction. If $u=\sigma_1 u=\tau_1 u$, then for some permutation $\{j, k, l\}=\{1, 2, 3\}$ we would have $u_{2j-1}=u_{2j}=u_{2k-1}=u_{2k}=0$. Hence

$$\mu_l^2 u_{2l-1}^2 - \lambda_l^2 u_{2l}^2 = \lambda_l^2 u_{2l-1}^2 - \mu_l^2 u_{2l}^2 = \lambda_l \mu_l u_{2l-1} u_{2l} = 0,$$

which is impossible.

This proves that $K^\vee$ is smooth away from the singular planes. The 16 points $z \in S_j \cap K, j=1, \cdots, 4$, are isolated surface singularities. At each of them, by Lemma (4.1), two cones K_l, K_m meet transversally. The threefold $K_l \cap K_m$ is smooth near z. On it K_j cuts out an isolated surface singularity. By [Ha 2, Proposition 8.23, p. 186] it will be normal. □

It will be shown below (Section 4.3) that $r \neq 0$ implies A is an abelian surface. In this case consider the incidence variety

$$\Gamma=\left\{(x, z) \in A \times \boldsymbol{P}_5 : \sum_1^6 z_j dq_j(x)=0\right\}$$

parametrizing points $x \in A$ and quadrics singular at these points. By Proposition (2.12) the projection $\Gamma \rightarrow A$ is bijective, hence an isomorphism. The projection $\Gamma \rightarrow \boldsymbol{P}_5$ maps Γ into the variety of points z parametrizing quadrics of rank ≤ 4. This means

—either $z \in K^\vee$

—or $z \in S_j \cap S_k$ for some $j \neq k$,

(cf. Proposition (4.2)). Since $K^\vee$ is an irreducible surface (Proposition (4.4)), and since the quadrics q_t form a 2-parameter family, the image of Γ in $\boldsymbol{P}_5$ is this surface $K^\vee$. It follows that

—the quadrics q_t are exactly the quadrics $q(z)$, $z \in K^\vee$,

—the quadrics $q_{tE,tF}$ are exactly the quadrics $q(z)$, $z \in S_j \cap S_k$.

If Q_t is singular at $x \in A$, then its singular space S_t meets A exactly in the $EG(\mathscr{L})$-orbit of x (cf. Section 2.2). It follows that the map $A=\Gamma \rightarrow K^\vee$ induces a bijective map

$$A/EG(\mathscr{L})=\mathrm{Km}\,(A/T(\mathscr{L}))=\mathrm{Km}\,(A^\vee)$$

and by normality of both surfaces, this map is an isomorphism.

(4.5) **Theorem.** *Mapping $x \in A$ to the quadric Q_t singular at x, one obtains an isomorphism* $\mathrm{Km}\,(A^\vee)\to K^\vee$ *and an identification of $A^\vee$ with the double cover of $K^\vee$ parametrizing the pencils of $\boldsymbol{P}_5$'s contained in the quadrics Q_t.*

4.3. The Kummer surface $K=\mathrm{Km}\,A$

Denote by $\pi\colon \boldsymbol{P}_7\to\boldsymbol{P}_5^+$ the projection

$$(x_1:\cdots:x_8)\longrightarrow(x_1:\cdots:x_6)$$

with center $\boldsymbol{P}_1^-$. Since $A\cap\boldsymbol{P}_1=\phi$, the map $\pi\,|\,A$ is well defined. If $x, y \in A$ with $\pi x=\pi y$, from the equations $q_1=\cdots=q_6=0$ one deduces

$$x_7^2+x_8^2=y_7^2+y_8^2,\quad x_7^2-x_8^2=y_7^2-y_8^2,\quad x_7x_8=y_7y_8.$$

So $y=x$ or $y=\iota x$, i.e., π induces a bijective map of A/ι onto $K:=\pi A\subset\boldsymbol{P}_5^+$. Since K is a surface for general choice of the $\lambda_1,\cdots,\mu_3$, it has dimension ≥ 2 for all $\lambda_1,\cdots,\mu_3$. If $r\neq 0$, by Lemma (3.12) we have $A\not\subset\boldsymbol{P}_5^+$, so the map $A\to K$ is a double cover. The quadrics

$$\mu_1q_1-\lambda_1q_2\colon (\mu_1^2+\lambda_1^2)(x_1^2+x_2^2)-2\lambda_1\mu_1(x_3^2+x_4^2)+(\mu_1^2-\lambda_1^2)(x_5^2+x_6^2)$$
$$\mu_2q_3-\lambda_2q_4\colon (\mu_2^2+\lambda_2^2)(x_1^2-x_2^2)-2\lambda_2\mu_2(x_3^2-x_4^2)+(\mu_2^2-\lambda_2^2)(x_5^2-x_6^2)$$
$$\frac{1}{2}(\mu_3q_5-\lambda_3q_6)\colon (\mu_3^2+\lambda_3^2)x_1x_1-2\lambda_3\mu_3x_3x_4+(\mu_3^2-\lambda_3^2)x_5x_6$$

are singular along the line $\boldsymbol{P}_1^-$, hence come from quadrics

$$Q_1''=\pi(\mu_1Q_1-\lambda_1Q_2),\quad Q_2''=\pi(\mu_2Q_3-\lambda_3Q_4),\quad Q_3''=\pi(\mu_3Q_5-\lambda_3Q_6)$$

in $\boldsymbol{P}_5^+$ vanishing on K.

(4.6) **Proposition.** *If $r\neq 0$, these three quadrics generate the ideal sheaf of an irreducible surface $Q_1''\cap Q_2''\cap Q_3''\subset\boldsymbol{P}_5^+$ of degree* 8, *which is smooth except for* 16 *normal singularities. This surface then is the variety K.*

Proof. We use the coincidence (for which I have no explanation) that the three quadrics Q_1'', Q_2'', Q_3'' are obtainable from

$$\frac{1}{2}(K_1+K_2+K_3+K_4)\colon (\lambda_1^2+\mu_1^2)(z_1^2+z_2^2)-(\lambda_2^2+\mu_2^2)(z_3^2+z_4^2)-(\lambda_3^2+\mu_3^2)(z_5^2+z_6^2)$$

$$\frac{1}{2}(K_1-K_2+K_3-K_4)\colon (\mu_1^2-\lambda_1^2)(z_1^2-z_2^2)-(\mu_2^2-\lambda_2^2)(z_2^2-z_4^2)-(\mu_3^2-\lambda_3^2)(z_5^2-z_6^2)$$

$$2L_3\colon 2\lambda_1\mu_1z_1z_2-2\lambda_2\mu_2z_3z_4-2\lambda_3\mu_3z_5z_6$$

by the following procedure:

1. the coordinate transformation

$$x_1=z_1,\quad x_2=z_2,\quad x_3=z_5,\quad x_4=z_6,\quad x_5=iz_3,\quad x_6=iz_4$$

2. *transposing the* 3×3 *coefficient matrix.*

Step 1 does not change the assertion. Step 2 is performed by passing to the dual parameters $\lambda_j^\vee$, $\mu_j^\vee$, see Section 3.3. If $r\neq 0$, then also $r^\vee\neq 0$ by (3.6). So Proposition (4.6) follows from Proposition (4.4). □

This proposition is the main tool to prove smoothness of A if $r\neq 0$:

(4.7) **Lemma.** *Let* $x\in A$ *such that* $x\notin \boldsymbol{P}_5^+$ *and* $\pi x\in K$ *is smooth. Then* A *is smooth at* x.

Proof. By assumption the three quadrics q_1'', q_2'', q_3'' meet transversally at πx. In view of the assumption $(x_7, x_8)\neq(0,0)$ the matrix of derivatives

	q_1	q_2	q_3	q_4	q_5	q_6
∂_7	λ_1x_7	μ_1x_7	λ_2x_7	μ_2x_7	λ_3x_8	μ_3x_8
∂_8	λ_1x_8	μ_1x_8	$-\lambda_2x_8$	$-\mu_2x_8$	λ_3x_7	μ_3x_7

obviously has rank 2. □

(4.8) **Lemma.** *Whenever* $x\in A\cap\boldsymbol{P}_5^+$, *its image* $\pi x=x$ *on* K *is a singularity there.*

Proof. If K is smooth at x the double cover $A\to K$ will split near x, i.e., A is locally reducible near x. Now any locally reducible point on A must lie on $\boldsymbol{P}_5^+$, because any $x\in A$, $x\notin\boldsymbol{P}_5^+$ is either smooth or isomorphic to one of the (normal) singularities of K. But in this situation consider the elements of order 4 in $T(\mathscr{L}^{\otimes 2})$. They leave A unchanged, but transform $\boldsymbol{P}_5^+$ into the 5-spaces $x_{2j-1}=x_{2j}=0$, $j=1,2,3$. Their common intersection with $\boldsymbol{P}_5^+$ is empty. Any locally reducible point of A would have to lie in this intersection. □

So we conclude, that for all 16 points of $A\cap\boldsymbol{P}_5^+$ (cf. Lemma (3.12)) the image on K is singular. By Lemma (4.7) A is smooth outside of $\boldsymbol{P}_5^+$. Using again $\bigcap\{t\boldsymbol{P}_5^+ : t \in T(\mathscr{L}^{\otimes 2})\}=\phi$, we see that A is smooth everywhere.

(4.9) **Theorem.** *For* $(\lambda_1 : \mu_1)$, $(\lambda_2 : \mu_2)$, $(\lambda_3 : \mu_3) \in \boldsymbol{P}_1^3$ *the following properties are equivalent*:

i) $r \neq 0$

ii) *the quadrics* $q_1, \cdots, q_6$ *generate the ideal sheaf of a smooth abelian surface* A (*of degree* 16, *with a* (2, 4)-*polarization*) *in* $\boldsymbol{P}_7$.

Proof. The implication ii)$\Rightarrow$i) has been shown in Proposition (3.2). If conversely $r \neq 0$, by the above $q_1, \cdots, q_6$ generate the ideal of a smooth surface $A \subset \boldsymbol{P}_7$. The singularities of K then are quotients of smooth points by an involution with isolated fixed point, i.e., ordinary nodes. By Brieskorn's simultaneous resolution theorem the minimal resolution $\tilde{K}$ of K is a smooth $K3$-surface, because smooth intersections of three quadrics in $\boldsymbol{P}_5$ are such surfaces. The double cover $A \rightarrow K$ is connected, and from Nikulin's theorem [N] it follows that A is an abelian surface with $K=\operatorname{Km} A$.

A is invariant under $T(\mathscr{L}^{\otimes 2})$, by Lemma (3.12) the 2-torsion elements act freely on the 16 points $A \cap \boldsymbol{P}_5^+$. Since $\deg A = 2 \deg K = 16$, the line bundle $\mathcal{O}_A(1)$ defines a polarization of type (1, 8) or (2, 4) on A. The case (1, 8) is excluded because the bundle is invariant under all 2-torsion translations. □

Altogether we have a commutative diagram (Diagram 1):

$$\begin{array}{ccccc}
 & \nearrow \boldsymbol{P}_7 & & & \\
A = Q_1 \cap \cdots \cap Q_6 & & \longrightarrow & A^\vee & \\
\downarrow & \nearrow \boldsymbol{P}_5^+ & & \downarrow & \nearrow \boldsymbol{P}_5(Q_1, \cdots, Q_6) \\
K = \operatorname{Km} A = Q_1'' \cap Q_2'' \cap Q_3'' & & \longrightarrow & \operatorname{Km} A^\vee = K^\vee &
\end{array}$$

Diagram 1

Here $\boldsymbol{P}_5(Q_1, \cdots, Q_6)$ is the hypernet considered in Section 4.1. The $8:1$ map $A \rightarrow \operatorname{Km} A^\vee$ is performed by associating with $x \in A$ the unique quadric $q_t = \sum_1^6 z_j q_j$ singular at x. It induces the $4:1$ map $\operatorname{Km} A \rightarrow \operatorname{Km} A^\vee$ sending $\pi x \in K$ to the unique quadric $\sum_1^3 t_j q_j''$ singular at πx.

The three quadrics q_1'', q_2'' and q_3'' span a plane

$$\boldsymbol{P}_2(Q_1'', Q_2'', Q_3'') = \{z : \lambda_1 z_2 + \mu_1 z_2 = \lambda_2 z_3 + \mu_2 z_4 = \lambda_3 z_5 + \mu_3 z_6 = 0\}$$

in $P_5(Q_1, \cdots, Q_6)$. This is exactly the singular plane S_4 of the cone K_4 from Section 4.1. The nodes of $\mathrm{Km}\, A^\vee$ lie by fours in the four singular planes $S_1, \cdots, S_4$. The plane S_4 is distinguished, because the four nodes it contains are the images of the 16 nodes of $\mathrm{Km}\, A$, i.e., the images of the 16 points $e_1, \cdots, e_{16} \in A$ modulo $T(\mathscr{L})$.

§ 5. Haine's theorem

In this section $A \subset P_7$ is assumed smooth, i.e., $r \neq 0$. Additionally we assume $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \neq 0$ w.l.o.g. (cf. Section 3.1).

5.1. The pencil of odd sections

The sections $x_7, x_8 \in H^0(\mathcal{O}_{P_7}(1)) = H^0(\mathscr{L}^{\otimes 2})$ span the two-dimensional vector space of ι-anti-invariants. Here we consider the pencil of curves

$$C: \alpha x_7 + \beta x_8 = 0, \quad \alpha : \beta \in P_1,$$

on A. Its base points are $\{e_1, \cdots, e_{16}\} = A \cap P_5^+$. We observed in Section 2.1 that at each e_k the tangent plane to A is the join of e_k with P_1^-. Any hyperplane $\alpha x_7 + \beta x_8 = 0$ intersects it transversally. So all C are smooth at $e_1, \cdots, e_{16}$. By Bertini's theorem the general curve C is smooth and connected. Since $C^2 = 16$, its genus is 9.

The rational function x_7/x_8 is invariant under σ, τ, and ι. The pencil $|C|$ on A therefore descends to pencils

$|D|$ on $A^\vee = A/\langle \sigma, \tau \rangle$
$|I|$ on $\mathrm{Km}\, A = A/\iota$
$|E|$ on $\mathrm{Km}\, A^\vee = A/\langle \sigma, \tau, \iota \rangle$.

We have Diagram 2

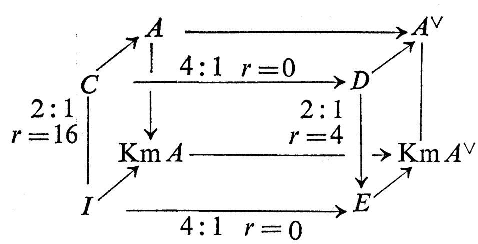


Diagram 2

where r denotes the number of ramification points. Therefore, for smooth C the image curves D, I, E are smooth of genus

$$g(D) = 3, \qquad g(I) = g(E) = 1.$$

The pencil $|D|$ on $A^\vee$ defines a polarization of type (1, 2).

The pencil $|I|$ of elliptic curves on $\mathrm{Km}\,A$ passes through the 16 nodes of this surface. On each of these curves the map $I \to E$ is the quotient with respect to a translation group $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$. So I and E are isomorphic as abstract elliptic curves. The pencil $|E|$ on $\mathrm{Km}\,A^\vee$ passes through the 4 nodes of $\mathrm{Km}\,A^\vee$ lying in the singular plane S_4 (cf. end of Section 4.3).

To find equations for I and E we abbreviate

$$x_0 := \alpha x_7 + \beta x_8.$$

The relation

$$x_0^2 = \frac{\alpha^2+\beta^2}{2}(x_7^2+x_8^2) + \frac{\alpha^2-\beta^2}{2}(x_7^2-x_8^2) + 2\alpha\beta x_7 x_8$$

implies

$$q_4' := \frac{\alpha^2+\beta^2}{2\lambda_1} q_1 + \frac{\alpha^2-\beta^2}{2\lambda_2} q_3 + \frac{\alpha\beta}{\lambda_3} q_5 = q_4'' + x_0^2$$

with

$$\begin{aligned} q_4'' = {} & \frac{\alpha^2+\beta^2}{2\lambda_1}(\mu_1(x_1^2+x_2^2) - \lambda_1(x_3^2+x_4^2) + \mu_1(x_5^2+x_6^2)) \\ & + \frac{\alpha^2-\beta^2}{2\lambda_2}(\mu_2(x_1^2-x_2^2) - \lambda_2(x_3^2-x_4^2) + \mu_2(x_5^2-x_6^2)) \\ & + \frac{2\alpha\beta}{\lambda_3}(\mu_3 x_1 x_2 - \lambda_3 x_3 x_4 + \mu_3 x_5 x_6) \\ = {} & \left(\frac{\mu_1\lambda_2+\mu_2\lambda_1}{2\lambda_1\lambda_2}\alpha^2 + \frac{\mu_1\lambda_2-\mu_2\lambda_1}{2\lambda_1\lambda_2}\beta^2\right)(x_1^2+x_5^2) \\ & + \left(\frac{\mu_1\lambda_2-\mu_2\lambda_1}{2\lambda_1\lambda_2}\alpha^2 + \frac{\mu_1\lambda_2+\mu_2\lambda_1}{2\lambda_1\lambda_2}\beta^2\right)(x_2^2+x_6^2) \\ & + \frac{\mu_3}{\lambda_3} \cdot 2\alpha\beta(x_1x_2+x_5x_6) - (\alpha x_3 + \beta x_4)^2 \end{aligned}$$

a quadric depending on $x_1, \cdots, x_6$ only. This means: Q_4'' considered as cone in $\boldsymbol{P}_7$ touches A along C. The quadric $Q_4'' \subset \boldsymbol{P}_5^+$ touches $\mathrm{Km}\,A$ along I. It follows that $q_1'', \cdots, q_4''$ are quadratic equations for I, providing this curve with a nonreduced (double) structure.

5.2. The complete intersection surface $A_C \subset \boldsymbol{P}_6$

We now project $\boldsymbol{P}_7$ to $\boldsymbol{P}_6$ from the point P_0: $\boldsymbol{P}_1^- \cap \{x_0=0\}$ via

$$p\colon (x_1, \cdots, x_8) \longmapsto (x_0 = \alpha x_7 + \beta x_8, x_1, \cdots, x_6).$$

We denote the image surface $pA \subset \boldsymbol{P}_6$ by A_C. There is the factorisation

$$\pi\colon A \xrightarrow{p} A_C \longrightarrow \mathrm{Km}\, A.$$

So for $x \neq y \in A$ we have $px = py$ if and only if $y = \iota x$ and the secant L of A joining x with y passes through P_0. This is equivalent to $x_0 | L = 0$, i.e., $x \in C$. Hence $p|A$ is bijective on $A \backslash C$, but it is a $2:1$ map of C onto $I = \boldsymbol{P}_5^+ \cap A_C$. On A_C this curve $I = A_C \cap \{x_0 = 0\}$ appears as a double curve. In particular we have $\deg A_C = \deg A = 16$.

(5.1) **Proposition.** *$A_C \subset \boldsymbol{P}_6$ is the complete intersection of the four quadrics $Q_1', \cdots, Q_4'$, where q_1', q_2', q_3' are the quadrics q_1'', q_2'', q_3'' from* (4.3), *but considered as quadric cones in $\boldsymbol{P}_6$, and $q_4' = q_4'' + x_0^2$.*

Proof. $Q_1' \cap Q_2' \cap Q_3' \subset \boldsymbol{P}_6$ is the cone over $\mathrm{Km}\, A$. For each $x'' \in \mathrm{Km}\, A$ there are not more than two points $x' = (x_0, x'')$ on which q_4' vanishes. This shows that $Q_1' \cap \cdots \cap Q_4'$ is a surface, of degree ≤ 16. This surface contains the degree-16 surface A_C. It follows that $A_C = Q_1' \cap \cdots \cap Q_4'$. □

The quadrics $q_1', \cdots, q_4'$ span a web $\sum_1^4 t_j q_j'$. We denote it by $\boldsymbol{P}_3 = \boldsymbol{P}_3\,(q_1', \cdots, q_4')$. It is contained in the hypernet $\boldsymbol{P}_5 = \boldsymbol{P}_5\,(q_1, \cdots, q_6)$.

(5.2) **Proposition.** *The 3-space $\boldsymbol{P}_3(q_1', \cdots, q_4') \subset \boldsymbol{P}_5(q_1, \cdots, q_6)$ lies on the cone K_4 and contains the singular plane S_4.*

Proof. The quadrics q_1', q_2', q_3' lie on the singular plane of the cone

$$K_4\colon (\lambda_1 z_1 + \mu_1 z_2)^2 - (\lambda_2 z_3 + \mu_2 z_4)^2 - (\lambda_3 z_5 + \mu_3 z_6)^2.$$

We have $q_4' = \sum_1^6 z_j q_j$ with

$$z_1 = \frac{\alpha^2 + \beta^2}{2\lambda_1}, \quad z_3 = \frac{\alpha^2 - \beta^2}{2\lambda_2}, \quad z_5 = \frac{\alpha\beta}{\lambda_3}, \quad z_2 = z_4 = z_6 = 0,$$

showing $q_4' \in K_4$. □

We observe that $\alpha : \beta$ can be viewed as the parameter for the 1-parameter family of 3-spaces on K_4.

5.3. The elliptic curve $E \subset \boldsymbol{P}_3$

The web $\sum_1^4 t_j q_j'$ inherits its 2×2-block form from the hyperweb, the fourth 2×2-block degenerating. It follows that the *discriminant surface* in $\boldsymbol{P}_3$ of the web is $K_1' \cup K_2' \cup K_3' \cup S_4$, where $K_j' = \boldsymbol{P}_3 \cap K_j$. The rank of the cone K_j' can drop to 2 only if S_j meets S_4. In general this is not the case (cf. Section 4.1). The cones $K_1, \cdots, K_4$ are dependent, hence because of $\boldsymbol{P}_3 \subset K_4$ the cones K_1', K_2', K_3' are dependent. Their common intersection is the curve $K^\vee \cap \boldsymbol{P}_3$.

(5.3) **Proposition.** *The curve $K^\vee \cap \boldsymbol{P}_3$ is just the image $E \subset K^\vee$ of C. In particular, for general choice of the α, β it is smooth elliptic. Its degree in $\boldsymbol{P}_3$ is* 4.

Proof. $K^\vee \cap \boldsymbol{P}_3$ parametrizes precisely the rank ≤ 4 quadrics $\sum_1^6 z_j q_j$ which are singular at $P_0 \in \boldsymbol{P}_1^-$. We have to show that these are precisely the rank ≤ 4 quadrics singular at some point of C:

i) Assume Q_t is singular at some point $x = te_1 \in C$. If $t \notin T(\mathscr{L}^{\otimes 2})$ the four translates of x under $T(\mathscr{L})$ span the singular 3-space S_t of Q_t (Proposition (2.2)). It follows that $S_t \subset \{x_0 = 0\}$ and since S_t is ι-invariant, $S_t \not\subset \boldsymbol{P}_5^+$, we have $S_t \cap \boldsymbol{P}_1^- \neq \phi$.

ii) Assume $P_0 \in S_t$ and $t \notin T(\mathscr{L}^{\otimes 2})$. Then $q_t = \sum_1^6 z_j q_j$ with $z \notin S_4$, hence q_t does not vanish identically on $\boldsymbol{P}_1^-$. For y and ιy from $S_t \cap A$ the secant L joining y and ιy is ι-invariant, $L \not\subset \boldsymbol{P}_5^+$. So L meets $\boldsymbol{P}_1^-$, hence $P_0 \in L$ and $L \subset \{x_0 = 0\}$. In this way one proves $S_t \cap A \subset C$, i.e., q_t is singular at points of C.

This proves the assertion for all $t \in T(A)$ except for $t \in T(\mathscr{L}^{\otimes 2})$. But for them it follows by continuity.

The 3-spaces in K_4 cut out on $K^\vee$ the pencil $|E|$. So the union $E \cup E'$ of any two such curves is a hyperplane section of $K^\vee$. This proves $\deg E = 4$. □

Because of $q_4' = q_4'' + x_0^2$, a quadric $\sum_1^4 t_j q_j''$ is singular at $x'' \in \boldsymbol{P}_5^+$ if and only if $\sum_1^4 t_j q_j'$ is singular at the two points $(x'', \pm x_7, \pm x_8)$ over it. It follows that the map $I \to E$, induced from $A \to \mathrm{Km}\, A^\vee$ is the same map that sends $x'' \in I$ to the unique quadric in $\boldsymbol{P}_3(q_1'', \cdots, q_4'')$ singular at x'' ("generalized Gauß map" in [M5]). As long as E is smooth, it is the intersection of four quadratic cones in $\boldsymbol{P}_3$. Three of them are K_1', K_2', K_3'. The fourth one, K_0', is distinguished by its absence.

The double cover $D \to E$ is ramified at the four nodes of $K^\vee$ contained in E, i.e., at the four points $\{P_1, \cdots, P_4\} = E \cap S_4$. Any double cover of E ramified over the four points of a plane section is determined by these four points and a class $\mathscr{D} \in \mathrm{Pic}(E)$ with $\mathscr{D}^{\otimes 2} = \mathcal{O}_E(1)$. There are four such classes $\mathscr{D}_i$ corresponding to the four cones $K_0', \cdots, K_3'$: The linear series $|\mathscr{D}_i|$ is cut out on E by the ruling of the cone K_i'. The following is easy to guess:

(5.4) **Lemma.** *The double cover π: $D \to E$ corresponds to the absent cone K_0'.*

Proof. If $\tilde{P}_1, \cdots, \tilde{P}_4 \in D$ are the points over $P_1, \cdots, P_4 \in E$, then the class $\mathscr{D}_i$ defining the covering is distinguished by $\pi^* \mathscr{D} \simeq \mathcal{O}_D(\tilde{P}_1 + \cdots + \tilde{P}_4) = \omega_D$. For $i = 1, 2, 3$ the linear series $\pi^* |\mathscr{D}_i|$ on D is cut out by the pullbacks to $A^\vee$ of the ruling solids in K_i. The linear system $|D|$ is the pullback to $A^\vee$ of the ruling of K_4. It follows that the rulings of the K_i, $i=1, 2, 3$, pull back to $A^\vee$ as the linear systems $|t_iD|$ where $t_i \in T_2(A^\vee)$ does not belong to $T(\mathcal{O}_{A^\vee}(D))$. Hence we have $\mathcal{O}_{A^\vee}(t_iD) \neq \mathcal{O}_{A^\vee}(D)$. So $h^0(\mathcal{O}_{A^\vee}(D-t_iD))=0$ and by Riemann Roch $h^1(\mathcal{O}_{A^\vee}(D-t_iD))=0$. This implies $h^0(\mathcal{O}_D(D-t_iD))=0$, i.e., $\mathcal{O}_D(t_iD)$ is not isomorphic to $\omega_D \simeq \mathcal{O}_D(D)$. □

Diagram 3 summarizes all the varieties and maps we are considering.

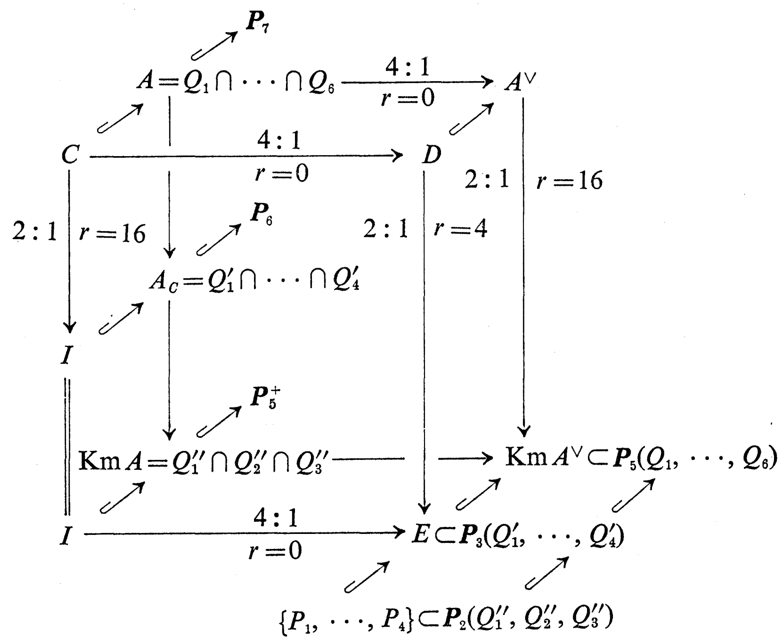


Diagram 3: the characters in the play

In [H] Haine considers exactly this situation: He starts with the elliptic curve $I \subset \boldsymbol{P}_5$ and its associated curve $E \subset \boldsymbol{P}_3$. He writes I: $Q_1 = \cdots = Q_4 = 0$ and defines A_C by $Q_1 = c_1x_0^2, \cdots, Q_4 = c_4x_0^2$. The coefficients c_i determine the plane $\sum_1^4 c_it_i = 0$ in $\boldsymbol{P}_3$ (our plane S_4) which meets E in four points. Forming explicitly the double cover $D \to E$ he proves by computing the period matrices:

(5.5) **Haine's Theorem.** *There is an isomorphism of A with* $\mathrm{Prym}(D/E)$.

This assertion is clear in our setup, because by (1.12) $\mathrm{Prym}(D/E)$ is dual to $A^\vee$, the surface containing D. We have however an explicit geometric description of this isomorphism: The points $\tilde{P} \in D$ parametrize the pencils of P_5's contained in the quadrics $Q_t \supset A$. Each pencil cuts out some class $\mathcal{O}_A(tD)$. Mapping $\tilde{P}$ to this class we obtain a map $D \to A^\vee$ (of course our embedding $D \subset A^\vee$) inducing a map $\mathrm{Pic}^0(D) \subset A^\vee$. Dualizing this map one obtains the identification $A \to \mathrm{Prym}(D/E) \subset \mathrm{Pic}^0(D)$.

§ 6. Comments

(6.1) If a parameter λ_j/μ_j takes one of the 6 values $0, \infty, \pm 1, \pm i$, by Sections 4.1 and 2.3 the surface A is isogenous to a product and $|\mathscr{L}|$ contains a curve $E+F$ of type c) in Table (1.2). This may happen three times, e.g. $\lambda_1/\mu_1=0$, $\lambda_2/\mu_2=1$, $\lambda_3/\mu_3=i$ and $r \neq 0$. This corresponds to six distinguished abelian surfaces. Under the symmetries from Section 3.1 all six surfaces are equivalent. By some tedious computations I checked that the surface is $E_i \times E_i$ with $E_i = \boldsymbol{C}/(\boldsymbol{Z}+i\boldsymbol{Z})$. A polarization of type (1, 2) is given on this surface e.g. by $E+F$, E diagonal, F=graph of multiplication by i.

(6.2) It seems to be the general philosophy that the coordinates of the theta-null values give the moduli. After a question of H. Morikawa I did the simple computations in Section 3.4 and found to my own surprise that these coordinates (up to some ambiguity due to the symplectic automorphisms of $T(2, 4)$ and the monodromy acting on the 2-torsion points) are the parameters λ_j/μ_j for the dual surface $A^\vee$. This must have some reason, but I do not know it.

(6.3) The degenerate surfaces appearing in the family can be considered a little more closely than it is done in Section 3.2. The ruled surface appearing as the general degeneration should be the compactification of a quasi-abelian surface which is a $\boldsymbol{C}^*$-bundle over the elliptic base curve E. Such surfaces are parametrized by points in the group $T(E)$. I expect the surface to be a "translation scroll", i.e., after fixing a parameter $t \in T(E)$ points $x \in E$, $x' \in E'$ are joined if $x' = \pm tx$. Unless t is a 2-torsion element the surface then should be reducible. This is very similar to the degenerations of Horrocks-Mumford surfaces [HM] in $\boldsymbol{P}_4$, as computations of R. Moore seem to indicate. Also the surface which is broken into 8 quadrics is very similar to a degeneration in [HM]. One might expect under mild assumptions on a very ample polarization (of type (d_1, d_2)) that the general degeneration always is a kind of translation scroll, which then further degenerates into a surface broken into $d_1 \times d_2$ quadrics.

(6.4) The Kummer surfaces $\mathrm{Km}\,A \subset \boldsymbol{P}_5$ are intersections of three quadrics, but unlike in the principal polarization case they have 16 nodes and are not smooth. The most interesting surface $\mathrm{Km}(E_i \times E_i) \subset \boldsymbol{P}_5$ implicitly appears in [BH]. It is the modular surface $S(4)$ embedded with the linear system $I+2F$.

(6.5) Haine [H] uses totally different parameters: The modulus of the elliptic curve $E=K_1'\cap K_2'\cap K_3'\subset \boldsymbol{P}_3$ with its level-4-structure and the equation of the plane $S_4\subset \boldsymbol{P}_3$. I tried to transform these into my parameters, but although it seems possible, the computations became too tedious.

(6.6) The question was posed by van Moerbeke to classify all complete intersections in $\boldsymbol{P}_5$ having as affine part the affine part of a smooth abelian surface. At present this is not known, and in this generality I do not see any approach. However in the examples [AM2] always three of the four equations come from the Kummer surface in $\boldsymbol{P}_5$. So under some additional assumptions perhaps a reasonable classification will be possible.

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