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Pluricanonical Systems of Algebraic Varieties of General Type of Dimension $\leqq 5$

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§ 1. Introduction

Let X be a non-singular projective variety of dimension n over an algebraically closed field k of characteristic zero.

Definition. A Cartier divisor D is called *nef and big* if

(i) $(D \cdot C)_X \geqq 0$ for every curve C on X, and

(ii) $D^n > 0$.

In this paper we consider the following problem:

Assume that the canonical divisor K_X is nef and big. Then, find the small integer $m(n)$ depending only on dim X such that the rational map $\Phi_{|mK_X|}$ associated with $|mK_X|$ is a birational map onto its image for every $m \geqq m(n)$.

The following result is one of the partial answers.

Theorem 1. *Assume that the canonical divisor K_X is nef and big, and that $n \leqq 5$. Then there exists a positive integer $m(n)$ such that for every $m \geqq m(n)$, the rational map $\Phi_{|mK_X|}$ associated with $|mK_X|$ is a birational map onto its image. Here $m(n)$ is given as follows:*

$$m(1)=3, \quad m(2)=5, \quad m(3)=8, \quad m(4)=20, \quad m(5)=36.$$

This theorem will be proved in Section 2. Moreover, in Section 3, we will show that improved results will be obtained if we use Miyaoka's inequality explained later. In Section 4, a similar theorem is shown to hold for the anti-pluricanonical systems.

We now recall known results concerning our problem. Matsusaka first proved the existence of $m(n)$ ([8], [9], [5], [4]). Maehara presented a function $m(n)$ for general n ([6]). Wilson gave $m(3)=25$ ([12]). Recently, Benveniste presented $m(3)=8$ ([1]). Matsuki improved his argument and finally showed $m(3)=7$ ([7]). Our result concerns only the values $m(4)$

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and $m(5)$. But our method is applicable to all $n \geqq 6$. In order to get the good estimation of $m(n)$ we have only to solve a certain system of linear inequalities. But this requires a computer and a good software.

§ 2. Proof of Theorem 1

In this section, let X be a non-singular projective variety of dimension n. We denote by $[x]$ the greatest integer which does not exceed x.

Proposition 2. *Assume that K_X is nef and big, and that $n \geqq 4$. Then $|mK_X| \neq \phi$ for any $m \geqq 2 \cdot [n/2]$.*

Proof. We define a polynomial $P(x)$ by $P(m) = \chi(mK_X)$. Let $R_{\boldsymbol{R}} = \{x \in \boldsymbol{R} \mid x > 1/2,\ P(x) = 0\}$, $R_{\boldsymbol{N}} = R_{\boldsymbol{R}} \cap \boldsymbol{N}$, $r_{\boldsymbol{R}}$ be the number of real roots of $P(x)$ in $x > 1/2$ counting the multiplicity precisely, r be the number of integral roots of $P(x)$ in $x > 1/2$ counting the multiplicity precisely, $\alpha = \max\{x \in R_{\boldsymbol{N}}\}$, and $N = [n/2]$. By the Serre duality, $P(1-x) = (-1)^n P(x)$. Thus $r \leqq r_{\boldsymbol{R}} \leqq N$. Since $H^i(X, mK_X) = 0$ for $i \geqq 1$ and $m \geqq 2$, we have $P(m) \geqq 0$ for integers $m \geqq 2$.

(I) Assume $\alpha \geqq 2r+2$. Let $a = \alpha - 2r - 2$. Since $|\alpha K_X| = \phi$, it follows that $|(r+1-b)K_X| = \phi$ or $|(r+1+a+b)K_X| = \phi$ for every $0 \leqq b \leqq r-1$. That is, $P(r+1-b) = 0$ or $P(r+1+a+b) = 0$ for $0 \leqq b \leqq r-1$. Thus $P(x)$ has at least $r+1$ roots in $\boldsymbol{N}$. This contradicts the definition of r, which implies $\alpha \leqq 2r+1$.

(II) Assume $\alpha \geqq 2r$ and $r \geqq 2$. Let $c = \alpha - 2r$. For every integer $0 \leqq b \leqq r-2$, we have $P(r-b) = 0$ or $P(r+b+c) = 0$. Thus $P(x) = 0$ has at least $r-1$ roots in $\{m \in \boldsymbol{N} \mid 2 \leqq m \leqq 2r-2+c\}$. Since $\alpha - 1 > 2r-2+c$, it follows that $P(\alpha-1) > 0$ and $P(\alpha+1) > 0$. Hence $P(x) = 0$ has at least two roots counting the multiplicity precisely in the interval $(\alpha-1, \alpha+1)$. This implies $r+1 \leqq r_{\boldsymbol{R}} \leqq N$.

By (I) and (II), we complete the proof. □

Lemma 3 (See Tankeev [11, Lemma 2]). *Let $|M|$ be a complete linear system free from base points, and let D be a divisor with $|D| \neq \phi$. Assume that $|M|$ is not composed of pencil. If $\Phi := \Phi_{|M+D|}$ is not a birational map, then for a general member Y of $|M|$, Φ is not birational on Y.*

Proof. We may assume that D and M are effective and that $\operatorname{Supp}(M) \not\supset Y$. Let $U = X - \operatorname{Supp}(D) - (\operatorname{Supp}(M) - Y)$ and choose a point $x \in U \cap Y$. There exists a point $y \in U$ such that $\Phi(x) = \Phi(y)$. Since x and y belong to the same effective divisor of $|M+D|$, $x, y \in \operatorname{Supp}(y+D)$. Since $x, y \notin \operatorname{Supp}(D) \cup (\operatorname{Supp}(M) - Y)$, it follows that both x and y belong to Y.

□

Lemma 4 (Benveniste [1], see also Matsuki [7]). *Let S be a non-singular projective surface and R a nef and big divisor on S. If an integer $m \geqq 4$ satisfies the following condition* $(*)$, *then $\Phi_{|K_S+mR|}$ is a birational map.*

$(*)$ *There exists a non-empty open set U of S such that any two distinct points x_1, x_2 satisfy the following condition: Let $\pi: S' \to S$ be the blowing up at x_1 and x_2, $L_1 := \pi^{-1}(x_1)$ and $L_2 := \pi^{-1}(x_2)$. Then $|\pi^*(mR) - 2L_1 - 2L_2| \neq \phi$.*

Theorem 5. *Let X be a non-singular projective variety of dimension n and R a nef and big divisor. We put $q := 4$ if $R^n = 1$ and $q := 3$ otherwise. We assume:*

(i) *For each i with $2 \leqq i \leqq n-1$, there exists a natural number r_i such that $\dim \Phi_{|r_iR|}(X) \geqq i$ and that $h^0(X, r_iR) \geqq i+2$.*

(ii) *There exists an integer $r_0 \geqq q$ such that every integer $r \geqq r_0$ satisfies the following condition:*

(ii–α) *There exists a non-empty open set U of X such that any two distinct points x and y of U satisfy the following condition: Let $\mu: \tilde{X} \to X$ be the blowing up at x and y, $E_x = \mu^{-1}(x)$ and $E_y = \mu^{-1}(y)$. Then we have*

$$|rf^*R - 2E_x - 2E_y| \neq \phi.$$

(ii–β) $H^0(X, rR + K_X) \neq 0$.

Then $\Phi_{|K_X+mR|}$ is birational for all $m \geqq r_0 + (r_2 + \cdots + r_{n-1})$.

Remark. The condition (ii–α) follows from $h^0(X, rR) \geqq 2n+3$.

Proof. We prove this by induction on n. If $n=2$, the result follows from Lemma 4. Assuming $n \geqq 3$, we let $f: X' \to X$ be a resolution of the base locus of $|r_2R|$ and $R' := f^*R$. Then denoting by F and Y the fixed part and a general member of the movable part $|M|$ of $|r_2R'|$ respectively, we have $f^*|r_2R| = |M| + F$. By the choice of f, $|M|$ has no base points and so Y is a smooth $(n-1)$-fold by Bertini's theorem. We put $m > r_0 + r_2 + \cdots + r_{n-1}$. The morphism $\Phi_{|K_X+mR|}$ is birational if and only if so is $\Phi_{|K_{X'}+mR'|}$. Furthermore, the birationality of $\Phi_{|K_{X'}+mR'|}$ follows from that of $\Phi_{|K_{X'}+Y+(m-r_2)R'|}$.

We now show that $\Phi_{|K_{X'}+Y+(m-r_2)R'|}$ is a birational map onto its image.

By the Kawamata-Viehweg vanishing theorem ([2], [13]) $H^1(X', K_{X'} + tR') = 0$ for $t \geqq 1$. Thus, we have a surjective homomorphism

$$H^0(X', K_{X'} + Y + (m-r_2)R') \longrightarrow H^0(Y, K_Y + (m-r_2)R''),$$

where $R'' = R'|_Y$. Therefore in order to prove the claim, it suffices to show that $\Phi_{|K_Y+(m-r_2)R''|}$ is birational. Actually by (ii–β) we have $H^0(X',$

$K_{X'}+(m-r_2)R')\neq 0$, hence we can apply Lemma 3.

Letting $r_0'':=r_0$, $r_2'':=r_3, \cdots, r_{n-2}'':=r_{n-1}$, we shall check that Y, R'', $r_2'', \cdots, r_{n-2}''$ and r_0'' satisfy the conditions (i) and (ii). If this is done, then by induction, we conclude that $\Phi_{|K_Y+tR''|}$ is birational for $t:=m-r_2 \geqq r_0+(r_3+\cdots+r_{n-1})$ and complete the proof of the claim.

(i) Since Y is a general member,

$$\dim \Phi_{|r_i''R''|}(Y) \geqq \dim \Phi_{|r_{i+1}R'|}(X')-1 \geqq i$$

for $i \geqq 2$. Especially $h^0(Y, r_i''R'') \geqq i+1$. We shall show $h^0(Y, r_i''R'')>i+1$ by deriving a contradiction, if we assume $h^0(Y, r_i''R'')=i+1$. Let $\Phi= \Phi_{|r_i''R'|}$, $N=h^0(X', r_{i+1}R')-1$, $X_0=\Phi(X')\subset \boldsymbol{P}^N$, $Y_0=\Phi(Y)$, and let H be a hyperplane in $\boldsymbol{P}^N$. Then $h^0(Y_0, H)=i+1$, since

$$i+1 \leqq \dim Y_0+1 \leqq h^0(Y_0, H) \leqq h^0(Y, r_i''R'')=i+1.$$

Thus Y_0 is contained in an i-dimensional linear subspace L_Y of $\boldsymbol{P}^N$. Since $\dim Y_0 \geqq i$, we have $L_Y=Y_0$. Given two general points x and y in X', there exists a non-singular irreducible $Y\in|M|$ passing through both points. Thus for two general points $\Phi(x)$ and $\Phi(y)$ in $\Phi(X_0)$, there exists an i-dimensional linear subspace L_Y which passes through both points and which is contained in X_0. Since $\dim X_0 \leqq \dim Y_0+1=i+1$ and since X_0 is spanned by i-dimensional linear subspaces, X_0 must be an $(i+1)$- or i-dimensional linear subspace of $\boldsymbol{P}^N$. On the other hand, since $\boldsymbol{P}^N$ is the minimal linear space containing X_0, it follows that $N \leqq i+1$. This contradicts $N=h^0(X', r_{i+1}R')-1 \geqq i+2$.

(ii–α) Let $U_Y=f^{-1}U\cap Y$ and $r \geqq r_0$. Take two distinct points x and y in U_Y. Denoting by $\nu\colon \tilde{X}'\to X'$ the blowing up at x and y, we put $E_x':=\nu^{-1}(x)$, $E_y':=\nu^{-1}(y)$, $\tilde{Y}:=\nu^{-1}(Y)$, $\tilde{R}:=\nu^*R'$. Since for any member D of $|r\tilde{R}-2E_x'-2E_y'|$, $\tilde{Y}\cap D\neq\phi$, it is enough to show $\tilde{Y}\notin Bs|r\tilde{R}-2E_x'-2E_y'|$. If $\tilde{Y}\in Bs|r\tilde{R}-2E_x'-2E_y'|$, $\tilde{Y}$ is not movable. That is, $|\nu^*M-E_x'-E_y'|=|\tilde{Y}|=\tilde{Y}$. Thus $h^0(X', M)\leqq h^0(\tilde{X}', \nu^*M-E_x'-E_y')+2=3$. This contradicts $h^0(X', M)=h^0(X', r_2R')\geqq 4$.

(ii–β) We consider the exact sequence

$$\begin{aligned}0 &\longrightarrow H^0(X', rR'+K_{X'}) \longrightarrow H^0(X', rR'+Y+K_{X'})\\ &\longrightarrow H^0(Y, rR''+K_Y) \longrightarrow 0.\end{aligned}$$

By assumption, $H^0(X', rR'+K_{X'})\neq 0$. Since $h^0(X', \mathcal{O}(Y))\geqq 2$, we have $h^0(X', rR'+K_{X'})<h^0(X', rR'+Y+K_{X'})$. Thus $H^0(Y, rR''+K_Y)\neq 0$. $\square$

We use the following lemmas to find $r_2, \cdots, r_{n-1}$ in Theorem 2.

Proposition 6 (Matsusaka [8], Maehara [6]). *Let R be a nef and big divisor and* $\dim X=n$. *If $h^0(mR)>m^rR^n+r$, then* $\dim \Phi_{|mR|}(X)>r$.

Lemma 7. *Assume* $\dim X=4$ *and suppose that K_X is nef and big. Then*

(1) $H^0(X, mK_X)\neq 0$, *for* $m\geqq 4$.

(2) $\dim \Phi_{|mK_X|}(X)\geqq 2$, *for* $m\geqq 6$. *Moerover if* $K_X^4=1$, *then* $\dim \Phi_{|5K_X|}(X)\geqq 2$.

(3) $\dim \Phi_{|mK_X|}(X)\geqq 3$, *for* $m\geqq 8$.

(4) $h^0(X, mK_X)\geqq 11$, *for* $m\geqq 7$.

(4*) *If* $h^0(X, 6K_X)\leqq 10$, *then* $\dim \Phi_{|2K_X|}(X)\geqq 2$. *Moreover if* $K_X^4\geqq 2$ *and if* $h^0(X, 5K_X)\leqq 10$, *then* $\dim \Phi_{|2K_X|}(X)\geqq 2$.

Proof. Let $P(m):=\chi(\mathcal{O}_X(mK_X))$. By the Riemann-Roch formula, $P(m)=(1/24)\cdot K_X^4\{m^2(m-1)^2+am(m-1)+b\}$, where $a:=(c_2\cdot K_X^2)/K_X^4$, $b:=24\chi(\mathcal{O}_X)/K_X^4$. By the Kawamata-Viehweg vanishing theorem, $H^i(X, mK_X)=0$ for $m\geqq 2$ and $i\geqq 1$. Thus $P(m)=h^0(X, mK_X)\geqq 0$ for $m\geqq 2$. The statement (1) is a direct consequence of Proposition 2.

(2) If $P(2)>2K_X^4+1$, then $\dim \Phi_{|2K_X|}(X)\geqq 2$ by Proposition 6. Thus by (1) we have the desired result. So we assume $P(2)\leqq 2K_X^4+1$. This implies

$$2a+b\leqq 44+24/K_X^4. \tag{2.1}$$

By $P(4)\geqq 1$, we have

$$12a+b\geqq 24/K_X^4-144. \tag{2.2}$$

If $P(6)\leqq 6K_X^4+1$, then

$$30a+b\leqq 24/K_X^4-765. \tag{2.3}$$

But there exists no pair (a, b) satisfying (2.1), (2.2) and (2.3). Thus $P(6)>6K_X^4+1$. By Proposition 6, $\dim \Phi_{|6K_X|}(X)\geqq 2$. Similarly, we have $\dim \Phi_{|mK_X|}(X)\geqq 2$ for $m=7, 8, 9$. By (1), $\dim \Phi_{|mK_X|}(X)\geqq 2$ for $m\geqq 6$. If $K_X^4=1$, then $a\equiv 10 \pmod{12}$ since $P(2)=(4+2a)/24+\chi(\mathcal{O}_X)\in \boldsymbol{Z}$. If $P(5)\leqq 5K_X^4+1$, then

$$20a+b\leqq 24/K_X^4-280. \tag{2.4}$$

The set $\{(a, b)\,|\,(a, b)$ satisfies (2.1), (2.2) and (2.4). $a\equiv 10 \pmod{12}\}$ is empty. Thus we have $P(5)\geqq 5K_X^4+1$, by the same argument.

(3) is proved in the same way as (2).

(4) If $P(2)\geqq 11$, using (1) we have $P(m)\geqq 11$ for any $m\geqq 6$. Thus we may assume $P(2)\leqq 10$; hence

(4.1) $$2a+b \leqq 240/K_X^4-4.$$

Since $P(5) \geqq 1$, we have

(4.2) $$20a+b \geqq 24/K_X^4-400.$$

Define $D_\#:=\{(a, b) \mid (a, b)$ satisfies (4.1) and (4.2)$\}$ and $D_m:=\{(a, b) \mid (a, b)$ satisfies $P(m) \leqq 10\}$. It is easy to see that $D_\# \cap D_7=D_\# \cap D_8=D_\# \cap D_9=D_\# \cap D_{10}=D_\# \cap D_{11}=\phi$. Thus $P(m) \geqq 11$ for $7 \leqq m \leqq 11$. By (1) we have $P(m) \geqq 11$ for any $m \geqq 7$.

(4*) We may assume

(4.5) $$P(2) \leqq 2K_X^4+1.$$

The three inequalities (4.5), $P(4) \geqq 1$ and $P(6) \leqq 10$ have no common solution (a, b). Thus $P(6) \geqq 11$. Similarly, (4.5), $P(4) \geqq 1$ and $P(5) \leqq 10$ have no common solution, whenever $K_X^4 \geqq 3$. If $K_X^4=2$, we have $a \equiv 4$ (mod 6) by $P(2) \in \boldsymbol{Z}$. These four equations have no common solution. $\square$

Lemma 8. *Assume* $\dim X=5$ *and suppose that* K_X *is nef and big. Then we have*:

(1) $H^0(X, mK_X) \neq 0$, *for* $m \geqq 4$.

(2) $\dim \Phi_{|mK_X|}(X) \geqq 2$, *for* $m \geqq 6$.

(3) $\dim \Phi_{|mK_X|}(X) \geqq 3$, *for* $m \geqq 8$.

(4) $\dim \Phi_{|mK_X|}(X) \geqq 4$, *for* $m \geqq 15$.

(4*) *If* $K_X^5=2$, *then* $\dim \Phi_{|14K_X|}(X) \geqq 4$.

(5) $h^0(X, mK_X) \geqq 13$, *for* $m \geqq 8$.

(5*) *If* $h^0(X, 7K_X) \leqq 12$, *then* $\dim \Phi_{|2K_X|}(X) \geqq 2$. *Moreover if* $K_X^5>2$ *and if* $h^0(X, 6K_X) \leqq 12$, *then* $\dim \Phi_{|2K_X|}(X) \geqq 2$.

A sketch of the proof.

Let $a=K^5$, $b=K^3c_2(X)/12$, $c=-2\chi(\mathcal{O}_X)$. Then

$$P(m)=m(m-1)(2m-1)(3m^2-3m-1)a/720+m(m-1)(2m-1)b/12$$
$$+(2m-1)c/2 \quad \text{and} \quad P(m)=h^0(X, mK) \geqq 0 \quad \text{for } m \geqq 2.$$

(2) follows from the fact that the three inequalities $P(2) \leqq 2a+1$, $P(4) \geqq 1$ and $P(6) \leqq 6a+1$ have no common solution (a, b, c).

(3) is easily proved only if we check that $P(2) \leqq 4a+2$, $P(5) \geqq 1$ and that $P(8) \leqq 64a+2$ have no common solution.

(4) In this case $P(2) \leqq 8a+3$, $P(6) \geqq 1$ and $P(15) \leqq 3375a+3$ have no common solution. From this, the result follows.

(4*) If $a=2$, then $12b/a$ is an integer and $12b/a \equiv 11$ (mod 12). Under this condition, $P(2) \leqq 19$, $P(6) \geqq 1$ and $P(14) \leqq 5491$ have no common solution.

(5) $P(2)<12$, $P(5)\geqq 1$ and $P(7)<12$ have no common solution.

(5*) If $a\geqq 4$, then $P(2)<12$, $P(5)\geqq 1$ and $P(6)<12$ have no common solution. □

Proof of Theorem 1. If $\dim X\leqq 3$, the results are already known (see Benveniste [1], Matsuki [7]). If $\dim X=4$ or 5, we apply Theorem 5, where $R=K_X$. When $\dim X=4$, we put r_0, r_2, r_3 as follows:

(i) $r_0=6$, $r_2=8$, $r_3=5$ if $K_X^4\geqq 2$ and $\dim \Phi_{|2K_X|}(X)<2$.

(ii) $r_0=5$, $r_2=8$, $r_3=6$ if $K_X^4=1$ and $\dim \Phi_{|2K_X|}(X)<2$.

(iii) $r_0=2$, $r_2=6$, $r_3=7$ if $\dim \Phi_{|2K_X|}(X)\geqq 2$.

This proves Theorem 1. We can also prove the case of $\dim X=5$ in the same way. □

§ 3. Important remark

Recently, Miyaoka discovered the following surprising inequality. His paper will appear in this volume ([10]).

Miyaoka's inequality. Let X be a non-singular projective variety with K_X nef. Then $3c_2(X)-K_X^2$ is pseudo-effective.

If we use this, we have better estimates:

Theorem 1′. *Under the same assumption as in Theorem* 1, *we can choose* $m(3)=7$, $m(4)=16$, $m(5)=29$.

Proof. This follows directly from Theorem 5, Proposition 2′, Lemma 7′ and Lemma 8′. Last three are stated below.

Proposition 2′. *If* K_X *is nef and big, then* $|mK_X|\neq\phi$ *for any* $m\geqq \max\{2\cdot[n/2]-2, 3\}$.

Proof. The case of $n=4$ or 5 will be proved later in the proofs of Lemmas 7′ and 8′. Thus we treat the case $n\geqq 6$. We use the same notation as in the proof of Proposition 2.

(III) Assume $\alpha\geqq 2N-2$, $N\geqq 3$, and $r\geqq N-1$. Let $d=\alpha-2r+2$. By the same argument as in (II) in the proof of Proposition 2, we conclude that $P(x)=0$ has $N-2$ roots $x_1,\cdots,x_{N-2}$ in $\{m\in N\,|\,2\leqq m\leqq 2N-4+d\}$, and that $P(x)$ has at least two roots x_{N-1} and x_N in the interval $(\alpha-1, \alpha+1)$ counting the multiplicity precisely. Thus

$$P(x)=(1/n!)\cdot(K_X^n)_X(x-1/2)^{n-2N}(x-x_1)\cdots(x-x_N)\cdot(x+x_1-1)\cdots(x+x_N-1).$$

Since $x_i\geqq 2$ for every $0\leqq i\leqq N$, it follows that the coefficient p_{n-2} of

x^{n-2} in $P(x)$ is negative. But $p_{n-2}=(K_X^n+K_X^{n-2}c_2(X))/12\cdot(n-2)!>0$, by Miyaoka's inequality, which is a contradiction.

By (I) in the proof of proposition 2 and (III) we complete the proof. □

Lemma 7′. *Assume* dim $X=4$ *and suppose* K_X *is nef and big. Then*
(1′) $h^0(X, mK_X)\geqq 2$, *for* $m\geqq 3$.
(2′) $\dim \Phi_{|mK_X|}(X)\geqq 2$, *for* $m\geqq 4$.
(3′) $\dim \Phi_{|mK_X|}(X)\geqq 3$, *for* $m\geqq 6$.
(4′) $h^0(X, mK_X)\geqq 11$, *for* $m\geqq 5$.

Proof. By Miyaoka's inequality, we have $a\geqq 1/3$. By $P(2)\geqq 0$, we have $2a+b\geqq -4$. Thus

$$(\sharp)\qquad P(m)\geqq (K_X^4/24)\left\{m^2(m-1)^2+\frac{1}{3}(m-2)(m+1)-4\right\}.$$

(1′) $P(m)\geqq 2$ for $m\geqq 3$, by ($\sharp$).
(2′) $P(m)\geqq mK_X^4+1$ for $m\geqq 4$, by ($\sharp$).
(3′) $P(m)\geqq m^2K_X^4+2$ for $m\geqq 7$, by ($\sharp$). We shall derive the same inequality for $m=6$. If $K_X^4>1$, then $P(6)>37.7K_X^4>36K_X^4+2$. Thus we may assume $K_X^4=1$. Since $P(2)=(4+2a)/24+\chi(\mathcal{O}_X)\in \boldsymbol{Z}$, it follows that $a\equiv 10 \pmod{12}$. Since $a\geqq 1/3$, we have $a\geqq 10$. Thus $P(6)\geqq 49>36K_X^4+2$.
(4′) If $m\geqq 5$, then $P(m)>10$ by ($\sharp$). □

Lemma 8′. *Assume* dim $X=5$ *and suppose* K_X *is nef and big. Then*
(1′) $h^0(X, mK_X)\geqq 2$, *if* $m\geqq 3$.
(2′) $\dim \Phi_{|mK_X|}(X)\geqq 2$, *if* $m\geqq 4$.
(3′) $\dim \Phi_{|mK_X|}(X)\geqq 3$, *if* $m\geqq 6$.
(4′) $\dim \Phi_{|mK_X|}(X)\geqq 4$, *if* $m\geqq 13$.
(5′) $h^0(X, mK_X)\geqq 13$, *if* $m\geqq 5$.

Proof. Let $a=K^5$, $b=K^3c_2(X)/12$, $c=-2\chi(\mathcal{O}_X)$. Then

$$P(m)=m(m-1)(2m-1)(3m^2-3m-1)a/720 +m(m-1)(2m-1)b/12+(2m-1)c/2.$$

Miyaoka's inequality implies $b\geqq a/36$. Since $P(2)\geqq 0$, it follows that $a/30+b/2+3c/2\geqq 0$. Thus

$$(*)\qquad P(m)\geqq (2m-1)\{(3m^4-6m^3+2m^2+m-8)a/720+(m^2-m-2)b/12\}$$
$$\geqq a(2m-1)(9m^4-18m^3+11m^2-2m-34)/2160.$$

Note that a is a positive even integer. The proof is completed in view of the following inequalities in the following cases.

(1′) $P(m)>0$ for $m\geqq 3$, by $(*)$.
(2′) $P(m)>ma+1$ for $m\geqq 4$, by $(*)$.
(3′) $P(m)>m^2a+2$ for $m\geqq 6$, by $(*)$.
(4′) $P(m)>m^3a+3$ for $m\geqq 13$.
(5′) $P(m)>13$ for $m\geqq 5$. □

§ 4. Anti-pluricanonical system

Theorem 9. *Assume that X is a non-singular complete variety of dimension $n\leqq 4$ and that $-K_X$ is nef and big. Then $\Phi_{|-mK_X|}$ is birational for $m\geqq l(n)$, where $l(n)$ is given by $l(2)=3$, $l(3)=5$, $l(4)=12$.*

Proof. Let $R=-K_X$. We shall find $r_0, r_2, \cdots, r_{n-1}$ satisfying the conditions of Theorem 5.

(2) Case $n=2$. We have $r_0=4$ immediately.

(3) Case $n=3$. Let $a=(-K_X)^3$. Then

$$\chi(-mK_X)=P(m)=m(m+1)(2m+1)a/12+(2m+1).$$

Note that a is a positive even integer, and that $H^i(X, -mK_X)=0$ for $i>0$ and $m\geqq 0$; hence $P(m)=h^0(X, -mK_X)$. If $m\geqq 2$, then $P(m)\geqq 1$. If $m\geqq 3$, then $P(m)>am+1$. Thus $r_2=3$. If $m\geqq 3$, then $P(m)\geqq 9$. Thus $r_0=3$.

(4) Case $n=4$. Let $a=(-K_X)^4$, $b=K_X^2c_2(X)$. Then

$$\chi(-mK_X)=P(m)=\{m^2(m+1)^2a+m(m+1)b\}/24+1.$$

$P(m)=h^0(X, -mK_X)\geqq 0$ for $m\geqq 0$.

Claim 1. $P(m)>0$ for $m\geqq 3$. If $a\neq 2$, then $P(2)>0$. If $a=1$, then $P(1)>0$.

Indeed, if $P(1)>0$ then $P(m)>0$ for any $m\geqq 1$. Thus we assume $P(1)=0$, that is, $b=-12-2a$. Then $P(2)=a-2$, $P(3)=5a-5$. If $a\geqq 2$, then $0<P(3)<P(4)<\cdots$. If $a\geqq 3$, then $P(2)>0$. If $a=1$, then $P(2)<0$. Thus $P(1)>0$ if $a=1$.

Claim 2. $P(4)>4a+1$.

Indeed, by $P(3)\geqq 1$, we have $12a+b\geqq 0$. Thus $P(4)=(100a+5b)/6+1\geqq(100a+5(-12a))/6+1>4a+1$; hence $r_2=4$.

Claim 3. $P(5)>25a+2$.

Indeed, since $P(1)\geqq 0$, we have $b\geqq -2a-12$. If $a\geqq 3$, then $P(5)=25a+(50a+5b)/4+1\geqq 25a+\{50a+5(-2a-12)\}/4+1=25a+(10a-14)\geqq 25a+2$. If $a=2$, then $b\geqq -2a-12=-16$. Thus $P(5)\geqq 56>25a+2$. If $a=1$, then $b\geqq -2$ by $P(1)\geqq 1$. Thus $P(5)\geqq 36>25a+2$. This establishes $r_3=5$.

Claim 4. $P(m)\geqq 11$ for $m\geqq 4$.

To show this, assume $a\geqq 2$. $b\geqq -2a-12$ since $P(1)\geqq 0$. Thus

$P(m) \geqq \{m^2(m+1)^2a+m(m+1)(-2a-12)\}/12+1$. Therefore $P(m) \geqq 11$ for $m \geqq 4$. Assume $a=1$. $b \geqq -2$ since $P(1) \geqq 1$. Thus $P(m) \geqq 11$ for $m \geqq 4$. □

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