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On the Image $\rho(BP^*(X)\to H^*(X; Z_p))$

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In this paper we study ways to calculate the Brown-Peterson cohomology $BP^*(X)$ localized at a prime p when the Steenrod algebra action on the ordinary mod p cohomology $H^*(X; Z_p)=HZ_p^*(X)$ is known. One of the most difficult problems is to know which elements in $H^*(X)_{(p)}$ are permanent cycles in the Atiyah-Hirzebruch spectral sequence $H^*(X; BP^*)\Rightarrow BP^*(X)$. This is equivalent to know the image ρ: $BP^*(X)\to H^*(X)_{(p)}$ where ρ is the Thom map.

Cohomology operations on $HZ_p^*(X)$ give some informations about the image. For example if $Q_n x\neq 0$ in $HZ_p^*(X)$, then x is not in Image $\rho(BP^*(X)\to HZ_p^*(X))$, where Q_n is the Milnor primitive operation. We study the above facts in more general situation.

Let: ρ: $h\to k$ be a map of spectra. In Section 1, we note the importance of Image $\rho(h^*(k)\to k^*(k))=\rho(h, k)$, indeed, if an operation θ is in $\rho(h, k)$, then for each $x\in k^*(X)$, $\theta x\in$ Image $\rho(h^*(X)\to k^*(X))$. The image $\rho(P(n), P(m))$ and $\rho(k(n), HZ_p)$ are studied in Section 2. Since $\rho(PB, HZ_p)=0$, we consider $K(Z, n)$ or $K(Z_n, n)$ as k instead of HZ_p in Section 3. Here we introduce the Tamanoi's results. In Section 4, $\rho(BP, K(Z, 3))$ and $BP^*(K(Z, 3))$ are studied. Applications for finite H-spaces are given in Section 5. For example, in the case $p=2$, let X be a simply connected finite associative H-space and let Q^* be the indecomposable elements in $HZ_2^*(X)$. Then

$$(Q^{2n+1})^2\subset \text{Image}\ \rho(BP^*(X)\longrightarrow HZ_2^*(X)/(HZ_2^+(X)^3)).$$

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§ 1. Maps of cohomology theories

Let ρ: $h\to k$ be a map of spectra and let $k=\{k_n\}$ be the Ω-spectrum, i.e., $k^n(X)\simeq[X, k_n]$. For simplicity of notations, let us write Image $\rho(h^*(k)\to k^*(k))$ (resp. Image $\rho(h(k_n)\to k(k_n))$) by $\rho(h, k)$ (resp. $\rho(h, k_n)$).

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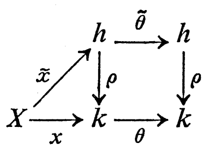
Lemma 1.1. *If $\theta \in \rho(h, k)$ (resp. $\theta \in \rho(h, k_n)$), then for $x \in k^*(X)$ (resp. $x \in k^n(X)$), $\theta x \in$ Image $\rho(h^*(X) \to k^*(X))$.*

Proof. Each element $x \in k^n(X)$ is represented by a map $x: X \to k_n$. Since $\theta \in \rho(h, k_n) = \text{Image}\, \rho(h^*(k_n) \to k^*(k_n))$, there is a map $i: k_n \to h$ such that $\rho i = \theta$. Hence $\theta x = \rho i x$. q.e.d.

It is immediate from the above lemma that if $\theta \in \rho(h, k)$, then $\rho(h, k) \supset \text{Im}\, \theta = \theta k^*(k)$ but in general, $\rho(h, k) \not\supset k^*(k)\theta$.

Lemma 1.2. *If $\theta \in \rho^{*-1}\rho h^*(h)$ and $x \in$ Image $\rho(h^*(X) \to k^*(X))$, then θx is also contained in the* Image ρ.

Proof. Let $\rho^*\theta = \rho\tilde{\theta}$ and $\rho\tilde{x} = x$. Then the following diagram is commutative and we have $\theta x = \rho\tilde{\theta}\tilde{x}$. q.e.d.



Corollary 1.3. *If $\theta \in \rho(h, k)$, then $\rho(h, k) \supset \theta k^*(k) \cup (\rho^{*-1}\rho h(h))\theta$.*

§ 2. *BP*-module spectra

Let BP be the Brown-Peterson spectrum with the coefficient $BP^* = Z_{(p)}[v_1, \cdots]$. Let k be a complex oriented ring spectrum such that k^* is a $Z_{(p)}$-module. Then from the universal property of BP, there is a map of ring spectra $\rho_k: BP \to k$. Moreover if p is an odd prime number and k^* is a $BP^*/(p, \cdots, v_{n-1})$-module, then there is a map of BP-module spectra $\rho': P(n) \to k$ with $\rho_k = \rho'\rho_{P(n)}$. Here $P(n)$ is the BP-module spectrum with the coefficient $P(n)^* = BP^*/(p, \cdots, v_{n-1})$ [6], [7].

Examples of k such that $k^*(k)$ are known are not so many, e.g., $P(n)$, $k(n)$ and $P(\infty) = HZ_p$ [6], [8], [9]

(2.1) $\quad P(n)^*(P(n)) \simeq P(n)^* \otimes_{BP^*} BP^*(BP) \otimes \Lambda(Q_0, \cdots, Q_{n-1})$

(2.2) $\quad k(n)^*(k(n)) \simeq (k(n)^*\{s_\alpha \mid \alpha_i < p^n\} \oplus B') \otimes \Lambda(Q_0, \cdots, Q_{n-1})$

where B' is some $k(n)^*/(v_n)$-module (for details see [9]).

Lemma 2.3. *When $p \geqq 3$, $h = P(m)$ and $k = P(n)$ for $m < n \leqq \infty$,*

(1) $Q_s\rho = 0$ *for* $m \leqq s$

(2) $\rho^{*-1}\rho P(m)^*(P(m)) = P(n)^*(P(n))$

(3) $\rho(P(m), P(n)) = Q_m \cdots Q_{n-1} P(n)^*(P(n)) = P(n)^*(P(n)) Q_m \cdots Q_{n-1}$.

Proof. From the Sullivan exact sequence

$$\begin{array}{ccc} P(m)^*(X) & \xrightarrow{\ v_m\ } & P(m)^*(X) \\ {\scriptstyle \delta_m}\nwarrow & & \swarrow{\scriptstyle \rho_m} \\ & P(m+1)^*(X) & \end{array}$$

and from the fact $\rho\delta = Q_m$, we get $Q_m\rho = 0$ for $k = P(m+1)$. From (2.1) it is easily seen $Q_s\rho = 0$ for $m \leqq s < n$ and $k = P(n)$. The formula (2) is proved by (1) and (2.1).

We know (Theorem 3.12 in [8]) that $aQ_m\cdots Q_{n-1} = (\pm)Q_m\cdots Q_{n-1}a$ for $a \in P(n)^*(P(n))$. From the definition of the operation Q_i,

$$Q_m\cdots Q_{n-1} = \rho_{n-1}\cdots\rho_m\delta_m\cdots\delta_{n-1} \in \rho(P(m), P(n)).$$

Therefore $\rho(P(m), P(n))$ contains the right hand side module in (3). For each element x not contained in the module (3), there is a with $m \leqq s < n$ such that $Q_s x \neq 0$. Hence the proof is completed. q.e.d.

Next we consider the case $h = k(n)$ and $k = HZ_p$. From the Sullivan exact sequence, $Q_n \in \rho(k(n), HZ_p)$. Moreover we have the following proposition.

Proposition 2.4. *For* $p \geqq 3$, $\rho(k(n), HZ_p) = \operatorname{Im} Q_n = Q_n HZ_p^*(HZ_p)$.

Proof. Consider the Atiyah-Hirzebruch spectral sequence $H^*(HZ_p; k(n)^*) \Rightarrow k(n)^*(HZ_p)$. Since the first differential is given by $d^{2p^n-1}(x\otimes 1) = = Q_n x \otimes v_n$, we have $E_{2p^n}^{*,*} = Q_n\mathscr{A}\otimes k(n)^*/(v_n)$ where $\mathscr{A}$ is the Steenrod algebra of the ordinary mod p cohomology. In particular, $E_{2p^n}^{*,t} = 0$ unless $t = 0$. Hence the spectral sequence collapes; $E_{2p^n}^{*,*} = E_\infty$ and the extension is trivial. Thus we obtain $k(n)^*(HZ_p) \simeq Q_n\mathscr{A}$ and the Thom map $\rho\colon k(n)^*(HZ_p) \to HZ_p^*(HZ_p)$ maps to $Q_n\mathscr{A}$ because ρ coincides with the edge homomorphism of the Atiyah-Hirzebruch spectral sequence. q.e.d.

Recall $BP[m, n]$ be the BP-spectrum such that $BP[m, n] = Z_p[v_m, \cdots, v_n]$. Then by the Sullivan exact sequence

$$Q_m\cdots Q_n \in \rho(BP[m, n], HZ_p).$$

Corollary 2.5. $\rho(BP[m, n], HZ_p) = \operatorname{Im} Q_m\cdots Q_n$.

Proof. From Proposition 2.4,

$$\rho(BP[m, n], HZ_p) = \bigcap_{i=m}^{n} \operatorname{Im} Q_i.$$

It is easily seen the right hand side of the above formula is $\mathrm{Im}\, Q_m \cdots Q_n$ in $\mathscr{A}$. q.e.d.

§ 3. The image $\rho(BP, K(Z, n))$

From Lemma 2.3, we have $\rho(BP, HZ_p)=0$. Hence when $k=BP$ and $h=HZ_p$, we need to consider the Ω-spectrum, that is, k_n is the Eilenberg MacLane space $K(Z_p, n)$ (or $K(Z, n)$). Tamanoi decided $\rho(BP, K(Z_p, n))$ and $\rho(BP, K(Z, n))$ for $p \geqq 3$ completely in [4] by using Wilson and Ravenel-Wilson results.

Theorem 3.1 (S. Wilson [5]). *For* $k \leqq 2(p^n + \cdots + p + 1)$

$$BP^k(X) \cong BP\langle n\rangle^k(X) \times \prod_{j \geqq n+1} BP\langle j\rangle^{k+2(p^j-1)}(X)$$

where $BP\langle n\rangle$ *is the BP-spectrum with the coefficient* $BP\langle n\rangle \cong Z_{(p)}[v_1, \cdots, v_n]$.

Define $\mathscr{S}_n^m$ to be the set of sequences

$$s = \{(s_1, \cdots, s_n) \mid 0 < s_1 < s_2 \cdots < s_n < m,\ s_i \in Z\}$$

and $\dim s = 2(1 + p^{s_1} + \cdots + p^{s_n})$.

Theorem 3.2 (Ravenel-Wilson [3]). *For* $p \geqq 3$ *and* $n \geqq 3$, *there exist* x_s, y_s *with* $|x_s| = |y_s| = \dim s$ *such that*

(1) $K(m)^*(K(Z, n)) \cong K(m)^*[[x_s \mid s \in \mathscr{S}_{n-2}^m]]$

(2) $K(m)^*(K(Z_p, n-1)) \cong K(m)^*[y_s \mid s \in \mathscr{S}_{n-2}^m]/(y_s^{p^{m-3-s_{n-2}}})$.

Theorem 3.3 (Tamanoi [4]). *For* $p \geqq 3$ *and* $n \geqq 3$,

(1) $\rho(BP, K(Z, n)) = Z_p[Q_s \tau \mid s \in \mathscr{S}_{n-2}^\infty]$

(2) $\rho(BP, K(Z_p, n-1)) = Z_p[Q_s Q_0 \iota \mid s \in \mathscr{S}_{n-2}^\infty]$

where τ, ι *are the fundamental classes and* $Q_s = Q_{s_{n-2}} \cdots Q_{s_1}$ *for* $s \in \mathscr{S}_{n-2}^\infty$.

Since Tamanoi's proof is written in Japanese, we introduce its outline here. We prove only the case $X = K(Z, n)$ and the other case is proved by the similar methods.

Outline of the proof of Theorem 3.3. It follows that $\{Q_s \tau \mid s \in \mathscr{S}_{n-2}^\infty\}$ generates a polynomial algebra by some computation of the Steenrod algebra on a product of Lens spaces.

By the inductive definition of Q_n, we can show

$$Q_s \tau = \mathscr{P}_s Q_{n-2} \cdots Q_1 \tau$$

where $\mathscr{P}_s$ is expressed by a sum of reduced powers. From the Sullivan exact sequence, we get $x=Q_{n-2}\cdots Q_1\tau \in \rho(BP\langle n-2\rangle, K(Z, n))$ and let $\rho(\bar{x})=x$. The dimension of x is just $2p^{n-2}+\cdots+2p+2$. Wilson's theorem says $\bar{x} \in$ Image $\rho(BP^*(X)\to BP\langle n-2\rangle^*(X))$. Therefore $x=\rho(\bar{x}) \in \rho(BP, K(Z, n))$. Since $\rho^{*-1}\rho(BP^*(BP))=\mathscr{A}$ for ρ: $BP\to HZ_p$, we get $Q_s\tau \in \rho(BP, K(Z, n))$.

For $m>n$, consider the diagram

$$\begin{array}{ccc} BP^*(X) & \xrightarrow{\rho_2} & \\ \downarrow \rho_1 & & k(m)^*(X) \xrightarrow{\ l\ } K(m)^*(X). \\ HZ_p^*(X) & \xleftarrow{\rho_3} & \end{array}$$

Here recall $l\rho_2(r_s x)=x_s$ where $r_s \in BP^*(BP)$ is the operation such that $\rho_1(r_s)=\mathscr{P}_s$. From Ravenel-Wilson theorem, for a given $w \in BP^*(X)$ we can take λ^α, v_m^α such that

$$y=w-\sum_{s,\alpha} \lambda^\alpha v_m^\alpha (r_s x)^\alpha \quad \text{and} \quad l\rho_2(y)=0,$$

where $(r_s x)^\alpha$ are mononials in $Z[r_s x]$. Hence $v_m^K\rho_2(y)=0$ for some large K and so $v_m^{K-1}\rho_2(y)$ is v_m-torsion. But non zero element of dimension $<2(p^m-1)$ is v_m-torsion free. Indeed, from the Sullivan exact sequence, if there exists an element of dimension t as above, then there is a non zero element in $HZ_p^{t-2p^m+1}(X)$. Take m to be larger than dim s. Then $\rho_2(y)=0$ and

$$\rho_1(w)=\rho_1(\sum_{s,\alpha} \lambda^\alpha v_m^\alpha (r_s x)^\alpha)=\sum_{s, v_m^\alpha=1} \lambda^\alpha (Q_s \tau)^\alpha. \qquad \text{q.e.d.}$$

Corollary 3.4. *Let $p\geqq 3$ and $BP(S)$ be the spectrum of the coefficient $BP^*/(S)$ where $S=(a_1, \cdots, a_m)$, $a_i \in BP^*$. Then*

(1) $\beta(BP(S), K(Z, n))=\rho(BP, K(Z, n))$,

(2) $\rho(BP(S), K(Z_p, n))=\rho(PB, K(Z_p, n))$.

§ 4. $BP^*(K(Z, 3))$ and its application

In this section we consider the case $K=K(Z, 3)$ more carefully and consider also the case $p=2$. The mod p cohomology of K is well known

(4.1) $$A=HZ_p^*(K(Z, 3))\cong Z_p[b_1, b_2, \cdots]\otimes\Lambda(c_0, c_1, \cdots)$$

where $c_n=\mathscr{P}^{p^{n-1}}\cdots\mathscr{P}\tau$, $\delta c_n=b_n$, $(n\geqq 1)$ and $|c_n|=2p^n+1$. For $p=2$, $A\cong Z_p[c_0, \cdots]$ where $c_n=Sq^{2n}\cdots Sq^2\tau$, $c_0=\tau$. Let $\delta c_n=b_n$. Then $b_n=c_{n-1}^2$ and in order to avoid separating cases, we think (4.1) is the isomorphism of associated graded algebras filtered by the polynomial algebra of b_i. Moreover for $p=2$, let $Q_m=Sq^{\Delta_m}$ be the Milnor basis.

Lemma 4.2. *In* $HZ_p^*(K)$, $Q_m b_n=0$ *and* $Q_m c_m=0$,

$$Q_m c_n = Q_n c_m = (b_{n-m})^{pm} \qquad \textit{for } n>m>0.$$

Proof. See Lemma 3.4.1 in [11]. The similar arguments prove the lemma for $p=2$. q.e.d.

Lemma 4.3. Ker Q_m *in* A *is isomorphic to*

$$Z_p[b_1, \cdots]\cdot(\mathrm{Im}\, Q_m \oplus \bigotimes_{n=1}^{m} \Lambda(c_{m+n}-b_n^{pm-pm-n}c_{m-n})\otimes\Lambda(c_m)).$$

Proof. The algebra A is a tensor product of subalgebras

$$Z_p[b_n]\otimes\Lambda(c_{m+n}) \qquad \text{if } n>m$$
$$Z_p[b_n]\otimes\Lambda(c_{m+n}, c_{m-n}) \qquad \text{if } n\leqq m$$

and $\Lambda(c_m)$. Here each subalgebra is closed under the action of Q_m. The cohomology of the above subalgebras of the differential Q_m are

$$Z_p[b_n]/(b_n^{pm}),$$
$$Z_p[b_n]/(b_n^{pm-n})\otimes\Lambda(c_{m+n}-b^{pm-pm-n}c_{m-n})$$

and $\Lambda(c_m)$. Therefore $H(A; Q_m)$ is the tensor product of the above cohomology. The lemma is proved from the fact Ker $Q_m=\mathrm{Im}\, Q_m\oplus H(A; Q_m)$. q.e.d.

For each cohomology theory h, let $F_s=\mathrm{Ker}\,(h^*(X)\to h^*(X^{s-1}))$ where X^s is an s-dimensional skeleton of X. We give $h^*(X)$ the topology by this filtration F_s.

Corollary 4.4. $k(n)^*(K)/F_{2p^n-2}$ *is generated by* $b_1, \cdots, b_{n-1}$ *as a* $k(n)^*$*-algebra. In particular* $\rho(BP, K)=Z_p[b_1, \cdots]$.

Proof. Consider the Atiyah-Hirzebruch spectral sequence of $k(n)^*(K)$. The first non zero differential is $d_{2p^n-1}=v_n\otimes Q_n$.

$$E_{2p^n}^{*,*}\simeq k(n)^*\otimes H(A; Q_n)\oplus(k(n)^*/v_n)\otimes \mathrm{Im}\, Q_n.$$

From Lemma 4.3 and from the fact b_i's are permanent cycles, we have the first assertion. Since $\rho(BP, K)\subset\bigcap_n \rho(k(n), K)$, the second assertion is also proved, by using Wilson's theorem. q.e.d.

Proposition 4.5. *For $p\geqq 3$, as a BP-algebra, $BP^{4*}(K)/F_{2p^5+2p^4}$ is generated by $\tilde{b}_1, \cdots, \tilde{b}_5$ with $\rho(\tilde{b}_i)=b_i$.*

Proof. Let $B=Z[b_1, \cdots]$. Then note that $(BP^*\otimes B)^i=0$ if $i\not\equiv 0$ mod 4. If $P(1)^*(K)/F$ is generated by B as a $P(1)^*$-module, then for each BP^*-module generator $x\in BP^*(K)$, we can take $b\in BP^*\otimes B$ so that $x-b\in pBP^*(K)$. Therefore we can take b for x. Hence we need only prove the above proposition for $P(1)^*(K)$.

Assume $0\neq ax\in E_2^{*,*}$ is a permanent cycle for $a\in P(1)^*$, $x\in HZ_p^{4*}(K)$ in the Atiyah-Hirzebruch spectral sequence of $P(1)^*(K)$. The first non zero differential is $d_{2p-1}=v_1\otimes Q_1$. Hence from Lemma 4.3,

$$x\in \Lambda(c_1, c_2-b_1^{p-1}c_0)\otimes B\oplus \text{Image } Q_1.$$

Since $|c_i|=2p^i+1$ and $|x|=4n$, we have $x\in \text{Im } Q_1$ and so $0\neq a\in P(1)^*/v_1$ $P(2)^*$.

Next compare spectral sequences of $P(1)^*(K)$ and $P(2)^*(K)$. Let $\rho: P(1)\to P(2)$ be the natural map. Then $d_{2p^2-1}\rho(ax)=av_2\otimes Q_2x$. The facts that ax is permanent and $0\neq a\in P(2)^*$, implies $Q_2x=0$.

A $4m$-dimensional element which is of the lowest dimensional in Image Q_1 and is not in B is

$$Q_1(c_0c_1c_2c_3c_4).$$

But this element is not in Ker Q_2. It is necessary

$$|x|\geqq |Q_1(c_0c_1c_2c_4c_5)|$$

for $x\in \text{Ker } Q_2$ and $|x|=4n$. q.e.d.

Question 4.6. As a BP^*-algebra $BP^*(K)$ is generated by $\tilde{b}_1, \tilde{b}_2, \cdots$?

We recall the main lemma in [11], which is also proved by Tamanoi using only stable homotopy theories.

Theorem 4.7. *Let $\sum v_jb_j=0$ in $BP^*(X)$. Then there is $y\in HZ_p^*(X)$ such that $Q_j(Y)=\rho(b_j)$.*

Remark. The above theorem is valid also for $p=2$.

Proposition 4.8. *The relations in $BP^{4*}(K)/F_{2p^5+2p^4}$ are given by*

(1) $p\tilde{b}_n+\sum_{i=1}^{n-1} v_i\tilde{b}_{n-i}^{p^i}+\sum_{i=1} v_{n+i}\tilde{b}_i^{p^n} \quad \text{mod } (p, v_1, \cdots)^2,$

(2) *relations in* $(p, v_1, \cdots)^2$.

Moreover we have

(3) $\tilde{b}_i=-r_{p\Delta_{i-1}}\tilde{b}_1 \text{ mod } (p, \cdots)^2.$

Proof. Since Ker $\rho(BP^{4*}(K)\to HZ_p^{4*}(K))/F_{2(p^5+p^4)}=\text{Ideal}\,(p, v_1, \cdots)$, that $pb_n=0$ in $E_\infty^{*,*}$ in the spectral sequence of $BP^*(K)$ implies that there is a relation

$$p\tilde{b}_n+\cdots=0.$$

The element a with $Q_0a=b_n$ is uniquely determined by c_n. From Theorem 4.7, we have the relations (1). This fact also follows (1) and (2) generate relations.

Operate the Quillen operation $r_{p\Delta_{i-1}}$ on $v_1\tilde{b}_1+v_2\tilde{b}_2+\cdots=0$. The fact $r_{p\Delta_{i-1}}v_j=v_1$ if $i=j$ and $=0$ mod $(p, \cdots)^2$ if $i\neq j$, implies the formula (3).

q.e.d.

Theorem 4.9. *Let* $x\in H^3(X; Z)$. *Then for* mod p *reduction* $\bar{x}$, $Q_i(\bar{x})\in$ Image $\rho(BP^*(X)\to HZ_p^*(X))$ *and* $\sum_{i\geqq 1} v_i\tilde{b}_i=0$ *with* $\rho(\tilde{b}_i)=Q_i(\bar{x})$. *Moreover if* $p\geqq 3$, *there are relations* (1)–(3) *in Proposition* 4.8.

§ 5. Finite *H*-spaces

In this section we always assume X to be a simply connected finite associative H-space.

Consider the case $p\geqq 3$. Assume that $Q^{2n}\neq 0$ for at most two n's where $Q^*=HZ_p^*(X)/HZ_p^+(X)\cdot HZ_p^+(X)$. (All known examples hold the above fact.) Then Kane's theorem says [1]

$$|Q^{\text{even}}|=(2p+2) \quad \text{or} \quad (2p+2,\, 2p^2+2)$$

and for each $b_1\in Q^{2p+2}$ (resp. $b_2\in Q^{2p^2+2}$) there are $b_2\in Q^{2p^2+2}$ (resp. $b_1\in Q^{2p+1}$) and $x\in Q^3$ such that $Q_1x=b_1$ and $Q_2x=b_2$. Therefore we have the following theorem from Theorem 4.9.

Theorem 5.1. *If* $p\geqq 3$ *and* $Q^{2n}\neq 0$ *for at most two* n*'s, then* $Q^{\text{even}}\subset$ Image $\rho(BP^*(X)\to Q^*)$ *and for each* $b_1\in Q^{2p+2}$ *there are* $\tilde{b}_1$ *and* $\tilde{b}_2\in BP^*(X)$ *such that* $v_1\tilde{b}_1+v_2\tilde{b}_2=0$, *moreover* $\tilde{b}_2=-r_{p\Delta_1}\tilde{b}_1$ *modulo* $(p, \cdots)^2\cup F_{2(p^5+p^4)}$.

When $p=2$, we consider elements in Q^{2^n+1}. By Lin [2] $Q^{2^n+1}=Sq^{2^{n-1}}Q^{2^{n-1}+1}$. Then it is easily seen $(Q^{2^n+1})^2=Q_{n-1}Q^3$. We also have the following theorem from Theorem 4.9.

Theorem 5.2. *For* $p=2$,

$$(Q^{2^n+1})^2\subset \text{Image}\ \rho(BP^*(X)\longrightarrow HZ_2^*(X)/HZ_2^+(X)^3)$$

and for each $x\in Q^3$ *there are* $\tilde{b}_i$ *such that* $\sum v_i\tilde{b}_i=0$ *and* $\rho(\tilde{b}_i)\in(Q^{2^i+1})^2$.

As an example we consider the exceptional Lie group E_8. The mod 3 cohomology is

$$HZ_3^*(E_8)\cong Z_3[x_8, x_{20}]/(x_8^3, x_{20}^3)\otimes\Lambda(x_3, \cdots).$$

Hence there are $\tilde{b}_1$, $\tilde{b}_2$ in $BP^*(E_8)$ such that $\rho(\tilde{b}_1)=x_8$, $\rho(\tilde{b}_2)=x_{20}$

$$v_1\tilde{b}_1+v_2\tilde{b}_2=0, \quad r_{p\Delta_1}(\tilde{b}_1)=-\tilde{b}_2 \bmod (p, \cdots)^2.$$

The mod 2 cohomology of E_8 is

$$HZ_2^*(E_8)\cong Z_2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4)\otimes\Lambda(x_{17}, \cdots).$$

Then there are $\tilde{b}_i$, $1\leqq i\leqq 3$ such that

$$v_1\tilde{b}_1+v_2\tilde{b}_2+v_3\tilde{b}_3=0 \bmod (v_4, v_5, \cdots)$$

with $\rho(\tilde{b}_1)=x_3^2$, $\rho(\tilde{b}_2)=x_5^2$, $\rho(\tilde{b}_3)=x_9^2$.

By Using Theorem 4.9 and arguments similar to [10], we can prove the following theorem. (While $P(n)^*(X)$, $K(n)^*(X)$ have not good commutative product, we use the associated graded algebras filtered by F_s, which have good product.)

Theorem 5.3. *There are BP^*-module isomorphisms for $p=2$*

(1) $BP^*(G_2)\cong BP^*\{1, 2x_3, x_3^3x_5\}\oplus BP^*\{x_3^3, x_3^2x_5\}/(2x_3^3+v_1x_3^2x_5)$
$\oplus BP^*/(2, v_1)\{x_3^2\}$.

(2) $BP^*(F_4)\cong BP^*(G_2)\otimes\Lambda(x_{15}, x_{23})$,

(3) $BP^*(E_6)\cong BP^*(F_4)\otimes\Lambda(x_9, x_{17})$.

Proof. The cohomology of the exceptional Lie group G_2 is

$$HZ_2^*(G_2)\cong Z_2[x_3]/(x_3^4)\otimes\Lambda(x_5).$$

Using the Atiyah-Hirzebruch spectral sequence, we can prove the theorem.
q.e.d.

References

[1] R. Kane, *BP*-torsion in finite *H*-spaces, Trans. Amer. Math. Soc., **264** (1981), 473–497.

[2] J. Lin, On the Hopf algebra structure of the mod 2 cohomology of a finite *H*-space, Publ. RIMS Kyoto Univ., **20** (1984), 877–892.

[3] D. Ravenel and S. Wilson, The Morava *K*-theories of Eilenberg MacLane spaces and the Conner-Floyd conjecture, Amer. J. Math., **102** (1980), 697–748.

[4] H. Tamanoi, On BP $(K(\pi, n))$ and its image in $HZ_p(K(\pi, n))$ for $\pi=Z$ and Z_p, preprint 1983.

[5] S. Wilson, The Ω-spectrum for Brown-Peterson cohomology II, Amer. J. Math., **97** (1975), 101–123.
[6] U. Würgler, Cobordism the theories of unitary manifolds with singularities and formal group laws, Math. Z., **150** (1976), 239–260.
[7] ——, On a class of 2-periodic cohomology theories, Math. Ann., **267** (1984), 251–269.
[8] N. Yagita, On the algebraic structure of cobordism operations with singularities, J. London Math. Soc., **16** (1977), 131–141.
[9] ——, On the Steenrod algebra of Morava K-theory, J. London Math. Soc., **22** (1980), 423–438.
[10] ——, Brown-Peterson cohomology groups of exceptional Lie groups, J. Pure App. Algebra, **17** (1980), 223–226.
[11] ——, On relations between Brown-Peterson cohomology and the ordinary mod p cohomology theory, Kodai Math. J., **7** (1984), 273–285.

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