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On the Stable Hurewicz Image of Stunted Quaternionic Projective Spaces

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§ 0. Introduction

Let HP^n $(0 \leqq n \leqq \infty)$ be the quaternionic n-dimensional projective space. We denote the stunted projective space HP^n/HP^{m-1} by HP^n_m $(1 \leqq m \leqq n \leqq \infty)$. For a space X with a base point, $\pi^s_*(X)$ means the stable homotopy groups of the space X.

Let

$$h: \pi^s_{4n}(HP^\infty_m) \longrightarrow H_{4n}(HP^\infty_m; Z) \cong Z$$

be the stable Hurewicz homomorphism. Let $h_{n,m}$ be the index of the subgroup Image h in $H_{4n}(HP^\infty_m)$. Our main interest in this paper is in the following problem.

Problem 1. *Determine the number* $h_{n,m}$.

Notice that the above problem can be stated as follows.

Problem 2. *Determine the stable order of the attaching map* $\varphi_{n,m}$ *of the top cell in the space* HP^n_m.

Therefore the e-invariants of the map $\varphi_{n,m}$ give a lower bound $h^A_{n,m}$, say, for $h_{n,m}$, that is, $h^A_{n,m}$ divides $h_{n,m}$.

There is a folk-lore conjecture which asserts that this lower bound $h^A_{n,m}$ is actually equal to the number $h_{n,m}$. For the case $m=1$, the conjecture was verified by several authors [12] [13] [14], and the case $m=2$ is treated in [7].

Let CP^∞ be the infinite dimensional complex projective space. Using the transfer map $t: HP^\infty \to CP^\infty$ it is easy to see that the odd-primary component of the number $h_{n,m}$ can be determined from the solution of the similar problem for the complex projective space. And the complex case is treated in [4] [5]. So in this paper we consider only the 2-primary

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component of $h_{n,m}$. In fact this paper is an outgrowth of the third author's attempt to apply the methods which were used in [5] to the quaternion case.

Roughly speaking our main theorem of this paper can be stated as follows.

Theorem. *If n is sufficiently large compared with m, then the number $h^A_{n,m}$ is equal to the number $h_{n,m}$.*

The most fundamental difference between the complex case and the quaternionic is that the complex numbers have a commutative multiplication but not the quaternions. Nevertheless they have many similar algebraic properties.

This paper is organized as follows. In Section 1 we give explicit algebraic conditions on the spherical elements in $H_{4n}(HP^\infty_m)$. In Section 2 we see that these algebraic conditions are periodic; and this periodicity is realized geometrically in Section 4. The conditions are reformulated in Section 3 in terms of KO-theory and Adams operations. Section 4 is an application of the theorem of Mahowald about the sphere of origin of the image of J in the stable homotopy groups of spheres. In Section 5 we state and prove our main theorem (Theorem 5.5).

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§ 1. The algebraic conditions

Let j: $HP^\infty \to HP^\infty_m$ be the canonical collapsing map. We denote the modulo torsion index of j_*: $\pi^s_{4n}(HP^\infty) \to \pi^s_{4n}(HP^\infty_m)$ by $d(n, m)$, where $n \geqq m \geqq 1$. Let $h_{n,m}$ be the modulo torsion index of the stable Hurewicz homomorphism h: $\pi^s_{4n}(HP^\infty_m) \to H_{4n}(HP^\infty_m)$. Then clearly we have

$$h_{n,m} \cdot d(n, m) = h_{n,1}.$$

As is well-known, $h_{n,1} = (2n)!/a(n)$ [12] [13] [14], where $a(n) = 1$ if n is even and $= 2$ if n is odd. So in order to determine the number $h_{n,m}$ it is enough to determine the number $d(n, m)$. In this section we shall give an upper bound for the number $d(n, m)$.

Let $\widetilde{KO}_*(X)$ (resp. $\widetilde{KO}^*(X)$) be the reduced real K-homology (resp. cohomology) of a based space X. Recall that

$$H^*(HP^\infty; Z) \cong Z[[x^H]],$$

$$\tilde{H}_*(HP^\infty; Z) \cong Z\{\beta^H_1, \beta^H_2, \beta^H_3, \cdots\},$$

$$KO^*(HP^\infty)\cong \widetilde{KO}^*(S^0)[[x]],$$

$$\widetilde{KO}_*(HP^\infty)\cong \widetilde{KO}_*(S^0)\{\beta_1, \beta_2, \beta_3, \cdots\},$$

where $x^H \in H^4(HP^\infty; Z)$ is the first symplectic Pontrjagin class of the canonical quaternionic line bundle ξ over HP^∞, $\beta_i^H \in H_{4i}(HP^\infty; Z)$ is the dual of $(x^H)^i$, $x \in KO^4(HP^\infty)$ is the KO-theoretic first Pontrjagin class of the bundle ξ and $\beta_i \in \widetilde{KO}_{4i}(HP^\infty)$ is the dual of $x^i \in KO^{4i}(HP^\infty)$.

Let ph_n: $KO^{4*}(HP^\infty)\to H^{4n}(HP^\infty; Q)$ be the $4n$-th component of the Pontrjagin character ph. In order to describe $ph_n(x^s)$ explicitly we need a certain numerical function.

Definition 1.1 [17] (Central factorial numbers of the second kind). *Define numbers $M(n, s)$ by the following equation;*

$$(e^t+e^{-t}-2)^s=\sum_{n\geqq 1}\frac{(2s)!}{(2n)!}M(n, s)t^{2n}.$$

Lemma 1.2. [17]

1) (*Recursive formula*)

$$M(n, 1)=1 \quad \textit{if } n\geqq 1 \textit{ and } M(1, s)=0 \textit{ if } s>1,$$
$$M(n, s)=M(n-1, s-1)+s^2M(n-1, s).$$

In particular, $M(n, s)$ is an integer.

2) $$\frac{(2s)!}{2}M(n, s)=\sum_{i=0}^{s}(-1)^i\binom{2s}{i}(s-i)^{2n}.$$

3) $$(2s-1)!\,M(n, s)=s^{2n-1}+\sum_{i=1}^{s-1}(-1)^i\left\{\binom{2s-1}{i}-\binom{2s-1}{i-1}\right\}(s-i)^{2n-1}$$

Definition 1.3.

$$d^A(n, m)=\underset{s\geqq m}{\text{g.c.d}}\left\{\frac{(2s)!\,M(n, s)}{a(n)a(n-s)}\right\}.$$

Making use of the integrality of the Pontrjagin character we have the following proposition.

Proposition 1.4. *For $n\geqq m\geqq 1$, the integer $d(n, m)$ is a divisor of the integer $d^A(n, m)$.*

Proof. From the definition 1.1 and a well-known formula for $ph(x)$ we have

$$ph_n(x^s) = \frac{(2s)!}{(2n)!} M(n, s)(x^H)^n.$$

Since the canonical collapsing map $j: HP^\infty \to HP^\infty_m$ induces monomorphisms in both KO-cohomology and ordinary cohomology, using above facts and the integrality of the Pontrjagin character it is easy to see that if $\lambda\beta^H_n \in H_{4n}(HP^\infty_m; Z)$ comes from $\pi^s_{4n}(HP^\infty_m)$ through the Hurewicz homomorphism then the integer λ must satisfy the following divisibility condition:

$$\text{for any } s \geqq m, \quad \lambda \frac{(2s)!}{(2n)!} M(n, s) \in a(n-s)Z.$$

Therefore, setting $\lambda = h_{n,m} = h_{n,1}/d(n, m)$, we see that for any $s \geqq m$ the number

$$\frac{(2s)!\, M(n, s)}{a(n)a(n-s) \cdot d(n, m)}$$

must be an integer. In other words the integer $d^A(n, m)$ is a multiple of the integer $d(n, m)$. q.e.d.

Remark. From the proof of Proposition 1.4, the e-invariant of the attaching map of the top cell in HP^n_m is easily obtained.

§ 2. Some properties of the integer $d^A(n, m)$

In this section we shall study some properties of the integer $d^A(n, m)$. As mentioned in Introduction, we are only interested in the 2-primary component. We use the following notation.

Definition 2.1.

$$d^A_2(n, m) = \nu_2(d^A(n, m)),$$
$$d_2(n, m) = \nu_2(d(n, m)),$$

where $\nu_2(i)$ is the exponent of 2 in the prime decomposition of an integer i.

Lemma 2.2. *For any $n > m \geqq 1$, $d^A_2(n, m) \leqq 2n-3$.*

Proof. From the definition of $d^A(n, m)$ it is obvious that

$$d^A_2(n, m) \leqq d^A_2(n, m+1) \leqq \cdots \leqq d^A_2(n, n-1) \leqq d^A_2(n, n).$$

From 1) of Lemma 1.2, we have

$$M(n, n-1) = (n-1)n(2n-1)/6.$$

So

$$d_2^A(n, n-1) = \min\left\{\nu_2\left(\frac{(2n-2)!\,M(n, n-1)}{a(n)a(1)}\right), \nu_2\left(\frac{(2n)!}{a(n)}\right)\right\}$$

$$= \min\left\{\nu_2\left(\frac{(2n)!\,(n-1)}{a(n)24}\right), \nu_2\left(\frac{(2n)!}{a(n)}\right)\right\}$$

$$= \begin{cases} \nu_2\left(\dfrac{(2n)!}{a(n)}\right) & \text{if } n \equiv 1 \bmod 8, \\ \nu_2\left(\dfrac{(2n)!}{a(n)}\right) + \nu_2(n-1) - 3 & \text{if } n \not\equiv 1 \bmod 8. \end{cases}$$

Let $\alpha(i)$ be the number of 1's in the 2-adic expansion of an integer i. Then, as is well-known,

$$\nu_2(k!) = k - \alpha(k).$$

Then using the above formula it is easy to see that for any $n > m \geqq 1$, $d_2^A(n, m) \leqq 2n-3$. q.e.d.

Let b be a non-negative integer. We denote the number $\max\{2, 2^{b-3}\}$ by $t(b)$.

Lemma 2.3. *If $b \leqq 2n-1$, then for any $s \geqq 1$*

$$\frac{(2s)!\,M(n, s)}{a(n)a(n-s)} \equiv \frac{(2s)!\,M(n+t(b), s)}{a(n+t(b))\,a(n+t(b)-s)} \bmod 2^b.$$

Proof. Note that

$$\frac{(2s)!\,M(m, s)}{a(n)a(n-s)} = \left(\frac{2s}{a(n)a(n-s)}\right)(2s-1)!\,M(n, s)$$

and that $2s/(a(n)a(n-s))$ is an integer. Since $b \leqq 2n-1$, using the formula 3) of Lemma 1.2 and the fact that $(\text{odd})^{2t(b)} \equiv 1 \bmod 2^b$, we have the desired result. q.e.d.

Proposition 2.4. *For any n and m such that $n \geqq m \geqq 1$, we have*

$$d_2^A(n, m) \leqq m^2 - 1.$$

For the proof of Proposition 2.4 we need the following Lemma 2.5. We postpone its proof until Section 3. In this section we assume Lemma 2.5.

Lemma 2.5. *For any $n > m \geqq 1$,*

$$d_2^A(n, m+1)-d_2^A(n, m) \leqq 2m+1.$$

Proof of Proposition 2.4.

$$\begin{aligned} d_2^A(n, m)-d_2^A(n, 1) &= (d_2^A(n, m)-d_2^A(n, m-1))+(d_2^A(n, m-1)-d_2^A(n, m-2))+\cdots \\ &\quad +(d_2^A(n, 3)-d_2^A(n, 2))+(d_2^A(n, 2)-d_2^A(n, 1)) \\ &\leqq 2m-1+2m-3+\cdots+3=m^2-1. \end{aligned}$$

Since $d^A(n, 1)=1$, so $d_2^A(n, 1)=0$. Therefore we have the desired result. q.e.d.

Corollary 2.6.
i) *If* $d_2^A(n, m) \geqq b$ *then* $d_2^A(n+t(b), m) \geqq b$.
ii) *If* $d_2^A(n, m)=b$ *then* $d_2^A(n+t(b+1), m)=b$.
iii) *For an integer m fixed, if we regard the integer* $d_2^A(n, m)$ *as the function of n, then the function* $d_2^A(n, m)$ *is periodic.*

Proof. i) and ii) are obvious from Lemmas 2.2–2.3. iii) From Proposition 2.4 we see that the function $d_2^A(n, m)$ is bounded above. Therefore the function $d_2^A(n, m)$ has a maximum b_0. Then put $D(m)=t(b_0)$. From i) and ii) it is easy to see that $D(m)$ is a period. q.e.d.

Let $D(m)$ be the number cited above, that is,

$$D(m)=t(\max_{n \geqq m}(d_2^A(n, m))).$$

By direct verification we have:

Examples.

$$D(2)=2.$$
$$D(3)=16.$$

Remark. The smallest period, $p(m)$ say, is a divisor of $D(m)$. For example, $p(3)=8$. In later sections we show that the period $D(m)$ can be realized geometrically.

§ 3. A geometrical interpretation of $d^A(n, m)$

In this section we shall give a geometrical interpretation of the number $d^A(n, m)$ in terms of KO-theory and the Adams operation. Throughout this section KO-theory is localized at (2).

Proposition 3.1. *Let σ_n be an arbitrary generator of the free part of $\pi^s_{4n}(HP^\infty)$. Then*

$$h(\sigma_n)=\frac{(2n)!}{a(n)}\beta^H_n,$$

$$h^{KO}(\sigma_n)=\sum_{s\geqq 1}\frac{(2s)!M(n,s)}{a(n)a(n-s)}\beta_s,$$

where h is the ordinary Hurewicz homomorphism, h^{KO} is the KO-Hurewicz homomorphism and we identify $\widetilde{KO}_{4(n-s)}(S^0)$ with the integers Z.

Proof. The first assertion is well-known [12] [13] [14]. The second assertion is obtained using the first and methods like those in the proof of Proposition 1.4. q.e.d.

As an immediate corollary we have

Proposition 3.2. *Let j: $HP^\infty \to HP^\infty_m$ be the canonical collapsing map. Then*

$$j_*h^{KO}(\sigma_n)=\sum_{s\geqq m}\frac{(2s)!M(n,s)}{a(n)a(n-s)}\beta_s.$$

Note that the right hand side of the above equation in Proposition 3.2 can be rewritten as $d^A(n,m)x^{KO}_{n,m}$ for some $x^{KO}_{n,m}\in\widetilde{KO}_{4n}(HP^\infty_m)$. Since $\widetilde{KO}_{4n}(HP^\infty_m)$ is torsion free, the element $x^{KO}_{n,m}$ is uniquely determined.

Lemma 3.3. *Let Ψ^3: $\widetilde{KO}_{4n}(HP^\infty_m)\to\widetilde{KO}_{4n}(HP^\infty_m)$ be the stable Adams operation. ($KO_*(\ )$ is localized at (2).) Then kernel (Ψ^3-1) is isomorphic to $Z_{(2)}$ and generated by the element $x^{KO}_{n,m}$ defined above.*

Proof. As is well-known, rank (kernel (Ψ^3-1)) is equal to the rank of $H_{4n}(HP^\infty_m)$. So kernel (Ψ^3-1) has a single generator. As $d^A(n,m)x^{KO}_{n,m}$ is spherical, $d^A(n,m)x^{KO}_{n,m}$ belongs to kernel (Ψ^3-1). On the other hand, from the definition of $d^A(n,m)$, $x^{KO}_{n,m}$ cannot be divisible in $\widetilde{KO}_{4n}(HP^\infty_m)$. Since $KO_{4n}(HP^\infty_m)$ is torsion free, $x^{KO}_{n,m}$ must be a generator of kernel (Ψ^3-1). q.e.d.

Though Lemma 3.3 gives us an interpretation of the number $d^A(n,m)$, this is inconvenient, because Ker (Ψ^3-1) is not a homology theory. Therefore we prefer to use the following theory.

Let $bo_*(\ )$ be the (-1)-connected cover of $\widetilde{KO}_*(\ )$ and $bspin_*(\ )$ be its 2-connected cover. As is well-known [11] the operation Ψ^3-1: $\widetilde{KO}_*(\ )$

$\to\widetilde{KO}_*(\ )$ can be uniquely lifted as

$$\Psi^3-1: bo_*(\ )\longrightarrow bspin_*(\ ).$$

We denote the fibre theory of this Adams operation by $A_*(\ )$. So there is a long exact sequence:

$$\cdots\longrightarrow bspin_{i+1}(\ )\longrightarrow A_i(\ )\xrightarrow{d_*} bo_i(\ )\xrightarrow{\Psi^3-1} bspin_i(\ )\longrightarrow\cdots$$

There is a Thom map $T: A_*(X)\to\tilde{H}_*(X; Z_{(2)})$ which factors the Hurewicz map and the generator of $A_0(S^0)\cong Z_{(2)}$ defines the Hurewicz map $h^A: \pi_*^s(X)\to A_*(X)$ factoring the KO-theory Hurewicz map. Thus Lemma 3.3 implies:

Lemma 3.4. *The integer $d^A(n, m)$ is the modulo torsion index of j_*; $A_{4n}(HP^\infty)\to A_{4n}(HP_m^\infty)$.*

Proof. Recall that there are canonical isomorphisms: $bo_{4n}(HP^\infty)\cong\widetilde{KO}_{4n}(HP^n)$, $bo_{4n}(HP_m^\infty)\cong\widetilde{KO}_{4n}(HP_m^n)$, $bspin_{4n}(HP^\infty)\cong\widetilde{KO}_{4n}(HP^{n-1})$ and $bspin_{4n}(HP_m^\infty)\cong\widetilde{KO}_{4n}(HP_m^{n-1})$ and that these isomorphisms are compatible with Adams operations. Note that $h^A(\sigma_n)$ is a generator of the free part of $A_{4n}(HP^\infty)\cong Z_{(2)}+$Torsion. Therefore from Lemma 3.3 and the definition of A-theory, Lemma 3.4 follows. q.e.d.

Now we shall prove Lemma 2.5. We need:

Lemma 3.5. *For any $m\geqq 1$, there is a stable self map g of HP^∞, such that*

$$g_*(\beta_n^H)=2^{2m+1}(4^{n-m}-1)\beta_n^H,$$

where $g_*: H_{4n}(HP^\infty; Z)\to H_{4n}(HP^\infty; Z)$ is the homomorphism induced by g.

Proof. From Theorem 1 in [13], there is a stable map $f(0, s): HP^\infty\to HP^\infty$ such that

$$f(0, s)_*\beta_n^H=a(s-1)\left(\sum_{i=0}^{s}(-1)^i\binom{2s}{i}(s-i)^{2n}\right)\beta_n^H.$$

Let $g=f(0, 2)-8\,(4^{m-1}-1)$ id, where id is the identity map. Then the map g has the desired property. q.e.d.

Proof of Lemma 2.5. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
S^{4m} & \longrightarrow & HP_m^\infty & \xrightarrow{j_m} & HP_{m+1}^\infty & \xrightarrow{\partial} & S^{4m+1} \\
\big\downarrow g_1 & & \big\downarrow g & & \big\downarrow g & & \big\downarrow \Sigma g_1 \\
S^{4m} & \longrightarrow & HP_m^\infty & \xrightarrow{j_m} & HP_{m+1}^\infty & \xrightarrow{\partial} & S^{4m+1},
\end{array}$$

where the horizontal sequences are cofibrations, g is the map in Lemma 3.5 and g_1 is induced from g. By Lemma 3.5 g_1 is null homotopic. Let $x_{n,m} \in A_{4n}(HP_m^\infty)$ be an arbitrary generator of the free part of $A_{4n}(HP_m^\infty) \cong Z_{(2)} + \text{Torsion}$. Then applying $A_{4n}(\ )$ to the above diagram we have that through the homomorphism j_{m*} the element $g_* x_{n,m+1}$ comes from some multiple of $x_{n,m}$ up to torsion. It is clear that the modulo torsion index of g_* in A-theory is the same as that in ordinary homology. Thus the modulo torsion index of j_{m*} divides the modulo torsion index of g_*. Combining these facts and Lemma 3.4, we see that the integer $d^A(n, m+1)/d^A(n, m)$ is a divisor of $2^{2m+1}(4^{n-m}-1)$. This completes the proof of Lemma 2.5. q.e.d.

§ 4. The unstable Adams periodicity.

In [9] or [10] Mahowald determined the sphere of origin of the image of J in the stable homotopy groups of spheres. In this section we apply this result.

The following theorem is due to Mahowald [9] [10].

Theorem 4.1. *Let b be an integer such that $b \geqq 1$. Let $t(b) = \max(2, 2^{b-3})$. Let e be* 0, 2, 1 *or* 0 *according as $b = 0, 1, 2$ or 3 mod 4. Then for any $k \geqq 1$, there is an unstable map $f_{k,b}: S^{4kt(b)+2b+e} \to S^{2b+e+1}$ such that the order of $f_{k,b}$ is 2^b, $f_{k,b}$ represents stably an element of order 2^b in the image of J in the $(4kt(b)-1)$-stem.*

Let M_b be the mod 2^b Moore spectrum, that is,

$$M_b = S^0 \cup_{2^b} e^1.$$

We denote the inclusion from S^0 to M_b by i_0 and the projection from M_b to S^1 by π_0. Let

$$\gamma(b) = \begin{cases} 7 & \text{if } b \leqq 3, \\ 2b+2 & \text{if } b \geqq 4 \text{ and } b = 0 \text{ or } 3 \text{ mod } 4, \\ 2b+3 & \text{if } b \geqq 4 \text{ and } b = 2 \text{ mod } 4, \\ 2b+4 & \text{if } b \geqq 4 \text{ and } b = 1 \text{ mod } 4. \end{cases}$$

Proposition 4.2. *For any $b \geqq 1$ and $k \geqq 1$, there exists an unstable map ${}^kB_b: \Sigma^{4kt(b)+\gamma(b)} M_b \to \Sigma^{\gamma(b)} M_b$ such that $\pi_0 \circ {}^kB_b \circ i_0$ represents stably an ele-*

ment of order 2^b in the image of J in the $(4kt(b)-1)$ stem of the stable homotopy groups of spheres.

Proof. First we prove in case that $b \neq 1$ and $b \neq 3$. From Proposition 1.8 in [16] it is enough to show that the (unstable) Toda bracket [16] $\{2^b, \Sigma f_{k,b}, 2^b\}_1$ contains zero, where $f_{k,b}$ is the element in Theorem 4.1. By Corollary 3.7 in [16], the above bracket contains zero if $b \geqq 2$. Now let $b=1$ or 3. Then it is known that there exists an unstable map A_b: $\Sigma^{15}M_b \to \Sigma^7 M_b$ such that $\pi_0 \circ A_b \circ i_0 = 2^{3-b}\Sigma\sigma'$, where σ' is a generator of $\pi_{14}(S^7)$. Using the structure of $\pi_{16}(S^8)$ ([16]), it is not hard to see that there is a choice of B_b of A_b such that stably $\pi_0 \circ B_b$ lies in the image of the J-map,

$$j_A\colon A^0(\Sigma^7 M_b) \longrightarrow \pi_s^0(\Sigma^7 M_b),$$

where j_A is a map obtained using the solution of the Adams conjecture. (See [5] or [6].) Since j_A commutes with unstable maps, we see that for any $k \geqq 1$, $\pi_0 \circ B_b^k$ lies, stably, in the image of j_A. This implies that $\pi_0 \circ B_b^k \circ i_0$ stably represents an element of order 2^b in the image of J. So we may put ${}^kB_b = B_b^k$. This completes the proof. q.e.d.

The following lemma is well-known [1].

Lemma 4.3. *Let α: $\Sigma^{4kt(b)}M_b \to M_b$ be a stable map such that the Adams e-invariant of $\pi_0 \circ \alpha \circ i_0$ is 2^{-b}. Then $\alpha^*\colon \widetilde{KO}^*(M_b) \to \widetilde{KO}^*(\Sigma^{4kt(b)}M_b)$ is an isomorphism.*

For this reason we call the map ${}^kB_b$ in Proposition 4.2 Adams periodicity. Combining Proposition 4.2 and Lemma 4.3, we have the following important fact.

Proposition 4.4. *Let BSp be the classifying space of virtual symplectic vector bundles. Let $\lambda_i \in \pi_{4i}(BSp)$ be a generator. Then for $b \geqq 1$ and $k \geqq 1$, if $\gamma(b) \leqq 4n-1$ the following diagram commutes up to a unit of $Z/2^b$:*

$$\begin{array}{ccc}
\Sigma^{4(n+kt(b))-1}M_b & \xrightarrow{\ {}^kB_b\ } & \Sigma^{4n-1}M_b \\
\downarrow \pi_0 & & \downarrow \pi_0 \\
S^{4(n+kt(b))} & & S^{4n} \\
& \searrow \lambda_{n+kt(b)} \quad \swarrow \lambda_n & \\
& BSp. &
\end{array}$$

Proof. Since $[\Sigma^{4n-1}M_b, BSp] \cong \widetilde{KO}^4(\Sigma^{4n-1}M_b) \cong Z/2^b$ with generator

$\pi_0^*\lambda_n$, using Proposition 4.2 and Lemma 4.3 we see that the above diagram commutes up to a unit. q.e.d.

Let $\tau: BSp \to \Omega^\infty \Sigma^\infty HP^\infty$ be the Becker-Segal splitting [3] [15]. We choose the splitting map and fix it. Then as a generator of the free part of $\pi_{4n}^s(HP^\infty) \cong \pi_{4n}(\Omega^\infty \Sigma^\infty HP^\infty)$ we can take $\tau_*\lambda_n$ [15]. From now on we denote this element in $\pi_{4n}^s(HP^\infty)$ by σ_n. Let $j_{k,b}$ be an element of order 2^b in the image of J in the $(4kt(b)-1)$-stem of the stable homotopy groups of spheres. Then we have:

Proposition 4.5.

i) *If* $\gamma(b) \leqq 4n-1$, *then* ${}^kB_b^*(\pi^*\sigma_n) = \pi^*\sigma_{n+kt(b)}$ *up to a unit.*

ii) *If* $4n \geqq 2b+e+1$, *then* $\sigma_n \circ j_{k,b} = 0$ *in* $\pi_{4(n+kt(b))-1}^s(HP^\infty)$, *where* e *is the function of* b *in Theorem* 4.1.

Proof. Obvious from the definition of the element of σ_n and Proposition 4.4.

As an easy corollary of the above proposition we have

Theorem 4.6. *Let* $j: HP^\infty \to HP_m^\infty$ *be the canonical collapsing map. Let* $n \geqq m$ *and* b *be a non-negative integer. If* $j_*\sigma_n = 2^b y_{n,m}$ *for some* $y_{n,m} \in \pi_{4n}^s(HP_m^\infty)$ *and if* $\gamma(b) \leqq 4n-1$, *then for any* $k \geqq 1$, $j_*\sigma_{n+kt(b)} = 2^b y_{n+kt(b),m}$ *for some* $y_{n+kt(b),m} \in \pi_{4(n+kt(b))}^s(HP_m^\infty)$. *In particular, if the assumption holds when* $b = d_2^A(n, m)$ *then* $d_2(n+kt(b), m) \geqq b$.

Note that if $n \geqq m+1$, then the assumption that $\gamma(b) \leqq 4n-1$ is always satisfied (See Lemma 2.2.). As an application of the above theorem we have

Corollary 4.7. [7] $d_2(n, 2) = d_2^A(n, 2) = 3$ *if* n *is even and* $=1$ *if* n *is odd. Moreover* $j_*\sigma_n$ *is divisible by* 2 *in* $\pi_{4n}^s(HP_2^\infty)$.

Proof. Easy computations in the spectral sequence:

$$H_*(HP_*^\infty; \pi_*^s(S^0)) \Longrightarrow \pi_*^s(HP^\infty)$$

tell us that $j_*(\sigma_2)$ is divisible by 8 in $\pi_8^s(HP_2^\infty)$ and $j_*(\sigma_3)$ is divisible by 2 in $\pi_{12}^s(HP_2^\infty)$. On the other hand by direct calculation it is easy to see that $d_2^A(n, 2) = 3$ if n is even and $=1$ if n odd. Therefore, applying Theorem 4.6 we have the desired results. q.e.d.

Remark. 1) Let $HP^{m-1} \to HP^\infty \to HP_m^\infty \xrightarrow{\partial} \Sigma HP^{m-1}$ be the cofibre sequence. If the assumption of Theorem 4.6 holds, that is, if $j_*\sigma_n = 2^b y_{n,m}$ for some $y_{n,m} \in \pi_{4n}^s(HP_m^\infty)$ then there is an element

$$y_{n+kt(b),m} \in \pi^s_{4(n+kt(b))}(HP^\infty_m)$$

such that

$$j_* \sigma_{n+kt(b)} = 2^b y_{n+kt(b),m},$$

and

$$\partial y_{n+kt(b),m} \in \langle \partial y_{n,m}, 2^b, j_{k,b} \rangle,$$

where $\langle\ ,\ ,\ \rangle$ is the (stable) Toda bracket. (Cf. [7]).

2) From computations of the above spectral sequence unless $n=14$ mod 16, we can show that $d_2(n, 2)=d_2^A(n, 3)$, where $d_2^A(n, 3)=3$ if n is odd, $=4$ if $n=0$ mod 4, $=5$ if $n=2$ mod 8 and 7 if $n=-2$ mod 8. The difficulty in the case that $n=14$ mod 16 is that we do not know whether $j_*\sigma_{14}$ is divisible by 2^7 in $\pi^s_{56}(HP^\infty_3)$ or not. In other cases $j_*\sigma_n$ is divisible by $2^{d_2^A(n,3)}$.

§ 5. The canonical Adams periodicity

In this section we shall show that there is a stable Adams periodicity map which has a certain nice property and using this Adams periodicity obtain our main theorem.

Proposition 5.1. *Let $b \geq 1$ and $k \geq 1$. Let ${}^k\tilde{B}_b$: $\Sigma^{4kt(b)-1}M_b \to M_b$ be any stable Adams periodicity map. Let $\sigma_n \in \pi^s_{4n}(HP^\infty)$ be the generator of the free part which is obtained by the Becker-Segal splitting. Then for any $b \geq 1$ and $k \geq 1$, if $4n \geq 2b+e+1$, there exists an element $\sigma'_{n+kt(b)} \in \pi^s_{4(n+kt(b))}(HP^\infty)$ such that $\sigma'_{n+kt(b)}$ is a generator of the free part of the 2-component of $\pi^s_{4(n+kt(b))}(HP^\infty)$ and $\sigma_n \circ \pi_0 \circ {}^k\tilde{B}_b = \sigma'_{n+kt(b)} \circ \pi_0$, where e is the function of b stated in Theorem* 4.1.

Proof. Since ${}^k\tilde{B}_b$ is an Adams periodicity map, stably $\pi_0 \circ {}^k\tilde{B}_b \circ i_0 = j_{k,b}$. So from Proposition 4.5,

$$\sigma_n \circ \pi_0 \circ {}^k\tilde{B}_b \circ i_0 = \sigma_n \circ j_{n,k} = 0.$$

Therefore there is an element $\sigma'_{n+kt(t)} \in \pi^s_{4(n+kt(b))}(HP^\infty)$ such that $\sigma_n \circ \pi_0 \circ {}^k\tilde{B}_b = \sigma'_{n+kt(b)} \circ \pi_0$. Consider the induced homomorphism

$$\sigma'^*_{n+kt(b)} : \widetilde{KO}^4(HP^\infty) \longrightarrow \widetilde{KO}^4(S^{4(n+kt(b))}).$$

Let $\iota_i \in \widetilde{KO}^4(S^{4(i+1)})$ be a generator and $x \in \widetilde{KO}^4(HP^\infty)$ be the first Pontrjagin class (see § 1). Then

$$\begin{aligned}\pi_0^*(\sigma'^*_{n+kt(b)}(x)) &= {}^k\widetilde{B}_b^*(\pi_0^*(\sigma_n^*(x)))\\ &= {}^k\widetilde{B}_b^*(\pi_0^*(\iota_{n-1})) \quad \text{(By Proposition 3.1)}\\ &= \pi_0^*(\iota_{n+kt(b)-1}). \quad \text{(By Lemma 4.3)}\end{aligned}$$

This implies that $\sigma'_{n+kt(b)}$ is a generator of the free part of the 2-component of $\pi^s_{4(n+kt(b))}(HP^\infty)$. q.e.d.

Let $x_{n,m} \in A_{4n}(HP^\infty_m)$ be a generator of the free part of $A_{4n}(HP^\infty_m)$ ($x_{n,1}=h^A(\sigma_n)$ by Lemma 3.3), $d_*: A_*(HP^\infty_m) \to bo_*(HP^\infty_m)$ be the natural homomorphism in the long exact sequence in Section 3 and $x^{KO}_{n,m} \in \widetilde{KO}_{4n}(HP^\infty_m)$ be the element introduced in Section 3.

Lemma 5.2. *Let* $n \geq m \geq 1$.

1) $A_{4n}(HP^\infty_m) \cong Z_{(2)} + Z/2 + \cdots + Z/2$.

2) $d_*(x_{n,m}) = x^{KO}_{n,m}$ *and* $\partial x_{n,m}$ *is independent of the choice of* $x_{n,m}$ *and of order* $2^{d^A_2(n,m)}$, *where we identify* $bo_{4n}(HP^\infty_m)$ *with* $\widetilde{KO}_{4n}(HP^n_m) \subset \widetilde{KO}_{4n}(HP^\infty_m)$.

3) $j_* h^A(\sigma_n) = 2^{d^A_2(n,m)} x_{n,m}$.

Proof. Note that $bspin_q(X) \cong \operatorname{Im}\{\widetilde{KO}_q(X^{(q-3)}) \to \widetilde{KO}_q(X^{(q-2)})\}$ and $bo_q(X) \cong \operatorname{Im}\{KO_q(X^{(q)}) \to \widetilde{KO}_q(X^{(q+1)})\}$, where $X^{(q)}$ is the q-th skeleton of a complex X. Now consider the following commutative diagram;

$$\begin{array}{ccccccc}
bspin_{4n+1}(HP^\infty) & \longrightarrow & A_{4n}(HP^\infty) & \xrightarrow{d_*} & bo_{4n}(HP^\infty) & \xrightarrow{\Psi^3-1} & bspin_{4n}(HP^\infty)\\
\downarrow j_* & & \downarrow j_* & & \downarrow j_* & & \downarrow j_*\\
bspin_{4n+1}(HP^m_\infty) & \longrightarrow & A_{4n}(HP^\infty_m) & \xrightarrow{d_*} & bo_{4n}(HP^\infty_m) & \xrightarrow{\Psi^3-1} & bspin_{4n}(HP^\infty_m)\\
\downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
bspin_{4n}(HP^{m-1}) & \longrightarrow & A_{4n}(HP^{m-1}) & \xrightarrow{d_*} & bo_{4n-1}(HP^{m-1}) & \xrightarrow{\Psi^3-1} & \\
\downarrow & & \downarrow & & \downarrow & & \\
bspin_{4n}(HP^\infty) & \longrightarrow & A_{4n-1}(HP^\infty) & \xrightarrow{d_*} & bo_{4n-1}(HP^\infty) & \xrightarrow{\Psi^3-1}, &
\end{array}$$

where all straight sequences are exact. Note that $j_*: bspin_{4n+1}(HP^\infty) \to b\,\mathrm{spin}_{4n+1}(HP^\infty_m)$ and $j_*: bo_{4n}(HP^\infty) \to bo_{4n}(HP^\infty_m)$ are epic. Also remark that $bo_{4n-1}(HP^\infty)$ and $bo_{4n-1}(HP^{m-1})$ are zero. Then by chasing the above diagram 1) and, 2) easily follow. In general, by Lemma 3.4, $j_*(h^A\sigma_n) = 2^{d^A_2(n,m)}x_{n,m} + \text{torsion}$. Using 1) and Corollary 4.7, 3) follows. q.e.d.

Let $\pi^l_s(X; Z/2^b)$ be the stable cohomotopy theory with mod 2^b-coefficients, that is, $\pi^l_s(X; Z/2^b) \cong \{X, \Sigma^l M_b\}$. Similarly let $A^l(X; Z/2^b)$ be A-cohomology with mod 2^b-coefficients. Any Adams periodicity map acts on $\pi_s^l(X; Z/2^b)$ and $A^l(X; Z/2^b)$ as an operator.

In [6] canonical stable periodicity operators ${}^k\tilde{B}_b$ are constructed which have the following nice properties.

Theorem 5.3. [6] *Let X be a finite complex. For any $b \geqq 1$ and $k \geqq 1$, there exists a stable Adams periodicity map ${}^k\tilde{B}_b$: $\Sigma^{4kt(b)} M_b \to M_b$ which has the following property. Assume $x \in$ kernel$(h^A: \pi_s^l(X; Z/2^b) \to A^l(X; Z/2^b))$. If there exists an integer k such that $4kt(b) \geqq \dim X - l + 3$ and $\Sigma^{4kt(b)-l} X$ is a triple suspension of some space, then ${}^k\tilde{B}_b(x) = 0$.*

As an application of the above theorem we have

Theorem 5.4. *Let $n > m > 1$ and $b \geqq 1$. If $d_2^A(n, m) \geqq b$, then for any k such that $kt(b) \geqq n - m + 1$ there exists some generator $\sigma'_{n+kt(b)}$ (in the 2-component) of the free part of $\pi^s_{4(n+kt(b))}(HP^\infty)$ such that $j_*(\sigma'_{n+kt(b)}) = 2^b y$ for some $y \in \pi^s_{4(n+kt(b))}(HP^\infty_m)$, and in particular $d_2(n + kt(b), m) \geqq b$, where j: $HP^\infty \to HP^\infty_m$ is the canonical collapsing map.*

Proof. Let ξ be the canonical symplectic line bundle over HP^{n-m}. Let M be some multiple of J-order of ξ. Then as is well-known ([8] or [2]) the stunted quaternionic quasi projective space $Q_{M-m, n-m+1}$ is S-dual to HP^n_m. Also there is an S-duality map $S^1 \to M_b \wedge M_b$.

Now consider the following commutative diagram;

$$\begin{array}{ccc}
\{\Sigma^{4n-1} M_b, HP^\infty_m\} & \xrightarrow{h^A} & \{\Sigma^{4n-1} M_b, HP^\infty_m \wedge A\} \\
\uparrow \cong & & \uparrow \cong \\
\{\Sigma^{4n-1} M_b, HP^n_m\} & \xrightarrow{h^A} & \{\Sigma^{4n-1} M_b, HP^n_m \wedge A\} \\
\downarrow \cong \ (S\text{-dual}) & & \downarrow \cong \ (S\text{-dual}) \\
\{Q_{M-m, n-m+1}, \Sigma^{4(M-n)-1} M_b\} & \xrightarrow{h^A} & \{Q_{M-m, n-m+1}, \Sigma^{4(M-n)-1} M_b \wedge A\},
\end{array}$$

where homomorphisms in the vertical direction are all isomorphic. Let $z = j \circ \sigma_n \circ \pi_0 \in \{\Sigma^{4n-1} M_b, HP^\infty_m\}$ and $x \in \{Q_{M-m, n-m+1}, \Sigma^{4(M-n)-1} M_b\}$ be the element corresponding to z under the isomorphisms. Then the assumption that $d_2^A(n, m) \geqq b$ and 3) of Lemma 5.2 imply that $h^A(z) = 0$. Let $X = Q_{M-m, n-m+1}$ and $l = 4(M-n) - 1$. Then x belongs to the kernel of h^A: $\pi_s^l(X; Z/2^b) \to A^l(X; Z/2^b)$. It is easy to see that if $kt(b) \geqq n + m + 1$ then $4kt(b) \geqq \dim X - l + 3$. Since X is a Thom complex of a certain real $4(M-n) - 1$ dimensional vector bundle over HP^{n-m}, so from the obstruction theory X is a $(4(M + m - 2n) - 1)$-fold suspension of a space Y. Thus $\Sigma^{4kt(b)-l} X = \Sigma^{4(kt(b)-n+m)} Y$. Therefore, applying Theorem 5.3, we see that ${}^k\tilde{B}_b(x) = 0$ and $z \circ {}^k\tilde{B}_b = 0$. Using Proposition 5.1 it follows easily that there exists an element $\sigma'_{n+kt(b)} \in \pi^s_{4(n+kt(b))}(HP^\infty)$ such that $j \circ \sigma'_{n+kt(b)} \circ \pi_0 = 0$. This completes the proof of Theorem 5.4. q.e.d.

As a corollary we have the following theorem.

Theorem 5.5. *Let* $m \geqq 1$. *Then for any n such that* $n \geqq 2D(m)+m$, $d_2(n, m) = d_2^A(n, m)$, *where* $D(m)$ *is the integer mentioned in Corollary* 2.6.

Proof. We may assume that $m \geqq 2$. Under the assumption clearly there is an integer $k \geqq 1$ such that $n-m+1 \leqq 2kD(m) \leqq 2n-2m$. Since $D(m)$ is a period of $d_2^A(n, m)$, $d_2^A(n, m) = d_2^A(n-kD(m), m)$. Let $d_2^A(n, m) = b$. Since $kD(m) = klt(b)$ for some $l \geqq 1$ and since $kD(m) \geqq n-kD(m)-m+1$, by Theorem 5.4, we have the desired result. q.e.d.

References

[1] J. F. Adams, On the groups $J(X)$-IV, Topology, **5** (1966).
[2] M. F. Atiyah, Thom complexes, Proc. London Math. Soc., **11** (1961).
[3] J. C. Becker, Characteristic classes and K-theory, Lecture Notes in Math., **428** (1978), 132–143.
[4] M. C. Crabb and K. H. Knapp, James numbers and codegree of vector bundles I, preprint.
[5] ——, James numbers and codegree of vector bundles II, preprint.
[6] ——, On Adams periodicity in stable homotopy, preprint.
[7] M. Imaoka and K. Morisugi, On the stable Hurewicz image of some stunted projective spaces, I, Publ. RIMS, Kyoto University, **20**–4 (1984).
[8] I. M. James, The topology of Stiefel manifolds, London Math Soc. Lecture note series, **24** (1976).
[9] M. Mahowald, Descriptive homotopy of the elements in the image of the J-homomorphism, Proc. Tokyo conference on Manifolds 1973, 255–263.
[10] ——, The image of J in the EHP sequence, Ann. of Math., **116** (1982), 65–112.
[11] P. May, E_∞-ring spaces and E_∞-ring spectra, Lecture Notes in Math., Springer **577** (1977).
[12] C. A. McGibbon, Self maps of projective spaces, Trans. Amer. Math. Soc., **271** No. 1 (1982), 326–346.
[13] K. Morisugi, Stable self maps of the quaternionic (quasi-) projective spaces, Publ. RIMS, Kyoto University, **20**–5 (1984).
[14] D. M. Segal, On the stable homotopy of quaternionic and complex projective spaces, Proc. Amer. Math. Soc., **25** (1970), 838–841.
[15] G. B. Segal, The stable homotopy of complex projective space, Quart. J. Math. Oxford (2), **24** (1973) 1–5.
[16] H. Toda, Composition methods in homotopy groups of spheres, Annals of Math. Studies, **49** (1962), Princeton Univ. Press.
[17] G. Walker, Estimates for the complex and quaternionic James numbers, Quart. J. Math. Oxford (2), **32** (1981) 467–489.

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