Note on Stable Homotopy Types of Stunted Quaternionic Spherical Space Forms

Teiichi Kobayashi

§ 1. Introduction

Let $H_m = \{x, y \mid x^{2^{m-1}} = y^2, xyx = y\}$ be the generalized quaternion group of order 2^{m+1} $(m \geq 2)$. An element of H_m is uniquely expressed as $x^u y^v$ for $0 \leq u < 2^m$, $v = 0, 1$. Let $d_1 \colon H_m \to S^3 = Sp(1) = SU(2)$ be the natural inclusion map defined by $d_1(x) = \exp(2\pi i/2^m)$, $d_1(y) = j$. Then H_m acts freely on the unit sphere S^{4n+3} in the quaternion $(n+1)$-space H^{n+1} by the diagonal action $(n+1)d_1 \colon H_m \to Sp(n+1)$. The quotient manifold S^{4n+3}/H_m is called the quaternionic spherical space form and is denoted by $N^n(m)$. If $n \geq 0$, we have the natural inclusion map $N^{k-1}(m) \subset N^{n+k}(m)$, and denote by $N_k^{n+k}(m)$ the quotient space $N^{n+k}(m)/N^{k-1}(m)$.

The purpose of this note is to study the stable homotopy types of the stunted quaternionic spherical space forms $N_k^{n+k}(m)$. We have

Theorem 1.1. *If* $N_j^{n+j}(m)$ *and* $N_k^{n+k}(m)$ *are of the same stable homotopy type, then* $j \equiv k \bmod 2^{2n-2}$.

This is proved in the way of H. Ōshima [10, Theorem 8.4] (cf. [8, Theorem 1.1]), and is a generalization of the Ōshima's result. As for the converse, we obtain

Theorem 1.2. *If* $j \equiv k \bmod 2^{2n+m-2+\varepsilon}$, *then* $N_j^{n+j}(m)$ *and* $N_k^{n+k}(m)$ *are of the same stable homotopy type, where* $\varepsilon = 1$ *if* n *is odd, and* $\varepsilon = 0$ *if* n *is even* > 0.

This is a consequence of the results of M. F. Atiyah [2, Proposition 2.6], H. Ōshima [10, Theorem 2.1 and Proposition 8.2] and [7, Corollary 1.7], and is also a generalization of Theorem 8.3 (iii) in [10].

We recall in Section 2 the representation rings $R_F(H_m)$ of H_m, where F denotes the field R of the real numbers or the field C of the complex numbers, according to [4], [5], [6] and [11]. In Section 3 we study Adams operations [1] in $K_F(N^n(m))$. The proofs of Theorems 1.1 and 1.2 depend

on the orders of the canonical elements of $\tilde{K}_F(N^n(m))$ (cf. [7], [9] and [10]), and are given in Section 4 and Section 5.

§ 2. The representation rings $R_F(H_m)$

We first recall the complex representation ring $R_C(H_m)$ of H_m (cf. [4, § 47], [11, § 1] and [5, § 3]).

Proposition 2.1. $R_C(H_m)$ *is generated as a free abelian group by* 1, a, b, c *and* d_r ($r=1, 2, \cdots, 2^{m-1}-1$) *defined below*

$$\begin{cases} 1(x)=1 \\ 1(y)=1, \end{cases} \quad \begin{cases} a(x)=1 \\ a(y)=-1, \end{cases} \quad \begin{cases} b(x)=-1 \\ b(y)=1, \end{cases} \quad \begin{cases} c(x)=-1 \\ c(y)=-1, \end{cases}$$

$$d_r(x)=\begin{bmatrix} \omega^r & 0 \\ 0 & \omega^{-r} \end{bmatrix}, \quad d_r(y)=\begin{bmatrix} 0 & (-1)^r \\ 1 & 0 \end{bmatrix},$$

where ω *is a primitive* 2^m-*th root* $\exp(2\pi i/2^m)$ *of unity. The characters of these representations are*

$$\chi(1)(x^u y^v)=1, \quad \chi(a)(x^u y^v)=(-1)^v, \quad \chi(b)(x^u y^v)=(-1)^u,$$

$$\chi(c)(x^u y^v)=(-1)^{u+v}, \quad \chi(d_r)(x^u y^v)=(\omega^{ur}+\omega^{-ur})(1-v),$$

where $u=0, 1, \cdots, 2^m-1, v=0, 1$.

Evaluating the characters, one has the following relations.

Proposition 2.2. $a^2=b^2=c^2=1$, $ab=c$, $d_r d_s=d_{r+s}+d_{r-s}$, $bd_r=d_{2m-1-r}$, *where* $d_0=1+a$, $d_{2m-1}=b+c$, $d_{-r}=d_r$, $d_{2m-1+r}=d_{2m-1-r}$.

Define the elements α, β, γ and δ_r of the reduced representation ring $\tilde{R}_C(H_m)$ by

$$\alpha=a-1, \quad \beta=b-1, \quad \gamma=a+b+c-3, \quad \delta_r=d_r-2 \quad (1\leq r<2^{m-1}).$$

Proposition 2.3. $\tilde{R}_C(H_m)$ *is generated as a free abelian group by* α, β, γ *and* $\delta_r(r=1, 2, \cdots, 2^{m-1}-1)$ *with relations*:

$$\alpha^2=-2\alpha, \quad \beta^2=-2\beta, \quad \gamma=\alpha\beta+2\alpha+2\beta, \quad \alpha\delta_1=-2\alpha,$$

$$\beta\delta_1=-2\beta+\delta_{2m-1-1}-\delta_1, \quad \delta_{r+1}+\delta_{r-1}=\delta_1\delta_r+2\delta_r+2\delta_1,$$

where $\delta_{2m-1}=\gamma-\alpha$, $\delta_0=\alpha$, $\delta_{-r}=\delta_r$, $\delta_{2m-1+r}=\delta_{2m-1-r}$. *Thus* $\tilde{R}_C(H_m)$ *is generated by* α, β *and* δ_1 *as a ring*.

Let $R_R(H_m)$ be the real representation ring of H_m (cf. [11, Proposition 1.5]). Since the complexification $c: R_R(H_m) \to R_C(H_m)$ is monomorphic, in what follows we identify $R_R(H_m)$ with the image in $R_C(H_m)$ under c.

Proposition 2.4. *$R_R(H_m)$, considered as the subring $c(R_R(H_m))$ of $R_C(H_m)$, is generated by* $1, a, b, c, d_{2r}$ *($r = 1, 2, \cdots, 2^{m-2} - 1$), $2d_{2r+1}$ ($r = 0, 1, \cdots, 2^{m-2} - 1$).*

§ 3. Adams operations in $R_F(H_m)$ and $K_F(N^n(m))$

Let $\Psi_F^i: R_F(H_m) \to R_F(H_m)$ be the Adams operation, where $F = C$ or R.

Lemma 3.1. *Let i be odd. Then $\Psi_C^i: R_C(H_m) \to R_C(H_m)$ is given as follows. For $\theta = 1, a, b, c \in R_C(N_m)$,*

$$\Psi_C^i(\theta) = \theta.$$

For $d_r \in R_C(H_m)$ ($r = 1, 2, \cdots, 2^{m-1} - 1$),

$$\Psi_C^i(d_r) = d_{ir}.$$

Proof. First, we prove a relation in H_m:

(3.1.1)
$$(x^u y)^{2k+1} = x^{2^{m-1}k + u} y,$$

where k is any non-negative integer. This is clear for $k = 0$. Suppose that this is true for k. Then

$$(x^u y)^{2(k+1)+1} = (x^u y)^{2k+1}(x^u y)(x^u y) = x^{2^{m-1}k + u} y (x^u y x^u) y$$
$$= x^{2^{m-1}k + u} y^2 y = x^{2^{m-1}k + u} x^{2^{m-1}} y = x^{2^{m-1}(k+1) + u} y,$$

since $x^u y x^u = x^{u-1}(xyx)x^{u-1} = x^{u-1} y x^{u-1} = \cdots = y$.

Second, we recall a formula due to J. F. Adams [1, Theorem 4.1 (vi)]:

(3.1.2) $\quad \chi(\Psi_C^i(\theta))(g) = \chi(\theta)(g^i)$, for $\theta \in R_C(H_m)$ and $g \in H_m$.

For the proof of the lemma it suffices to check that in each equation, the characters of the two sides agree. Obvious calculations, based on (3.1.2), (3.1.1) and Proposition 2.1, show that

$$\chi(\Psi_C^i(\theta))(x^u y^v) = \chi(\theta)(x^u y^v), \qquad \text{for } \theta = 1, a, b, c,$$
$$\chi(\Psi_C^i(d_r))(x^u y^v) = \chi(d_{ir})(x^u y^v),$$

where $x^u y^v \in H_m$.   q.e.d.

Lemma 3.2. $\Psi_F^{2^m+1}$ *is the identity on* $R_F(H_m)$.

Proof. We first consider the case $F=C$. By Lemma 3.1, it remains only to prove that $d_{ir}=d_r$ for $i=2^m+1$. We may assume $r>0$. Then, by Proposition 2.2,

$$d_{2^m r+r}=d_{2^{m-1}+2^{m-1}(2r-1)+r}=d_{2^{m-1}-2^{m-1}(2r-1)-r}$$
$$=d_{2^m(1-r)-r}=d_{2^m(r-1)+r}=\cdots=d_r.$$

The fact that $\Psi_R^{2^m+1}=1$ follows from the following commutative diagram [1, Theorem 4.1 (iv)]:

$$
\begin{array}{ccc}
R_R(H_m) & \xrightarrow{\Psi_R^i} & R_R(H_m) \\
c\downarrow & & \downarrow c \\
R_C(H_m) & \xrightarrow{\Psi_C^i} & R_C(H_m),
\end{array}
$$

since the complexification c is monomorphic.   q.e.d.

Let $\Psi_F^i\colon K_F(N^n(m))\to K_F(N^n(m))$ be the Adams operation, where $F=C$ or R.

Lemma 3.3. $\Psi_F^{2^m+1}$ *is the identity on* $K_F(N^n(m))$.

Proof. The result follows from the following commutative diagram:

$$
\begin{array}{ccc}
R_F(H_m) & \xrightarrow{\xi_F} & K_F(N^n(m)) \\
\Psi_F^i\downarrow & & \downarrow \Psi_F^i \\
R_F(H_m) & \xrightarrow{\xi_F} & K_F(N^n(m)),
\end{array}
$$

where ξ_F is the natural projection (cf. [5, § 4], [6, § 3], [11, Theorem 2.5]).
                          q.e.d.

§ 4. Proof of Theorem 1.1

Let $i\colon N^{k-1}(m)\to N^{n+k}(m)$ be the inclusion and $p\colon N^{n+k}(m)\to N_k^{n+k}(m)=N^{n+k}(m)/N^{k-1}(m)$ be the projection. Consider the Puppe exact sequence:

$$\tilde{K}_C^{-1}(N^{n+k}(m))\xrightarrow{i^*}\tilde{K}_C^{-1}(N^{k-1}(m))\longrightarrow\tilde{K}_C(N_k^{n+k}(m))\xrightarrow{p^*}\tilde{K}_C(N^{n+k}(m)).$$

Using Atiyah-Hirzebruch spectral sequences, we see that $\tilde{K}_C^{-1}(N^1(m))\cong Z$ and that i^* is trivial. Hence p^* is monomorphic on $\mathrm{Tor}\,(\tilde{K}_C(N_k^{n+k}(m)))$. We prove

Lemma 4.1. $\Psi_C^{2^m+1} = (2^m + 1)^t$ *on* $\mathrm{Tor}\,(\tilde{K}_C(S^{2t} N_k^{n+k}(m)))$.

Proof. Consider the following diagram:

$$
\begin{array}{ccccc}
\tilde{K}_C(N^{n+k}(m)) & \xleftarrow{\ p^*\ } & \tilde{K}_C(N_k^{n+k}(m)) & \xrightarrow{\ I^t\ } & \tilde{K}_C(S^{2t} N_k^{n+k}(m)) \\
\downarrow{\scriptstyle \Psi_C^i} & & \downarrow{\scriptstyle \Psi_C^i} & & \downarrow{\scriptstyle \Psi_C^i} \\
\tilde{K}_C(N^{n+k}(m)) & \xleftarrow{\ p^*\ } & \tilde{K}_C(N_k^{n+k}(m)) & \xrightarrow{\ I^t\ } & \tilde{K}_C(S^{2t} N_k^{n+k}(m)),
\end{array}
$$

where I is the Bott isomorphism. Then, by [1, Corollary 5.3], we have $\Psi_C^i I^t = i^t I^t \Psi_C^i$. If $i = 2^m + 1$, $\Psi_C^{2^m+1}$ is the identity on $\mathrm{Tor}\,(\tilde{K}_C(N_k^{n+k}(m)))$ by Lemma 3.3, since p^* is monomorphic on $\mathrm{Tor}\,(\tilde{K}_C(N_k^{n+k}(m)))$. Hence we have the desired result.   q.e.d.

Next, we recall the order of the canonical element of $\tilde{K}_C(N^n(m))$.

Let λ be the canonical complex plane bundle over the quaternion projective space $HP^n = S^{4n+3}/S^3$, and let $\pi: N^n(m) = S^{4n+3}/H_m \to HP^n$ be the natural projection. Let $\xi_C: \tilde{R}_C(H_m) \to \tilde{K}_C(N^n(m))$ be the natural projection defined in [5, § 4] and put $\delta = \xi_C(\delta_1)$. Then we have

$$\delta = \pi^* \lambda - 2$$

(cf. [5, Lemma 4.4]). The order $\#\delta^i$ of $\delta^i \in \tilde{K}_C(N^n(m))$ is determined by H. Ōshima in [10, Proposition 5.2] and by T. Mormann in [9, Chapter 2, Theorem 4.52] as follows.

Proposition 4.2. $\#\delta^i = 2^{2n+m+1-2i}$ $(1 \leq i \leq n)$.

Proof of Theorem 1.1. If $N_j^{n+j}(m)$ is stably homotopy equivalent to $N_k^{n+k}(m)$, we may assume that there exists a homotopy equivalence

$$g: S^{2t} N_j^{n+j}(m) \longrightarrow S^{2t+4j-4k} N_k^{n+k}(m)$$

for some integer $t \geq 0$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
\tilde{K}_C(S^{2t+4j-4k} N_k^{n+k}(m)) & \xrightarrow{\ g^*\ } & \tilde{K}_C(S^{2t} N_j^{n+j}(m)) \\
\downarrow{\scriptstyle \Psi_C^i} & & \downarrow{\scriptstyle \Psi_C^i} \\
\tilde{K}_C(S^{2t+4j-4k} N_k^{n+k}(m)) & \xrightarrow{\ g^*\ } & \tilde{K}_C(S^{2t} N_j^{n+j}(m)),
\end{array}
$$

where $i = 2^m + 1$. Then, by Lemma 4.1,

$$(2^m+1)^{t+2j-2k} g^* = (2^m+1)^t g^* \qquad \text{on } \mathrm{Tor}\,(\tilde{K}_C(S^{2t+4j-4k} N_k^{n+k}(m))),$$

and so, for any torsion element $\alpha \in \tilde{K}_C(S^{2t} N_j^{n+j}(m))$, $(2^m+1)^{t+2j-2k}\alpha = (2^m+1)^t \alpha$. Now we have

$$\tilde{K}_C(S^{2t}N_j^{n+j}(m)) \cong \tilde{K}_C(N_j^{n+j}(m)) \qquad \text{(by Bott isomorphism)}$$

$$\cong \tilde{K}_C((N^n(m))^{j\tau\pi^*\lambda}) \qquad \text{(by Ōshima [10, Theorem 2.1])}$$

$$\cong K_C(N^n(m)) \qquad \text{(by Thom isomorphism [3, Theorem 7.2])},$$

where X^ζ denotes the Thom complex of a vector bundle ζ over X. Hence, by Proposition 4.2, there is an element of order 2^{2n+m-1} in $\tilde{K}_C(S^{2t}N_j^{n+j}(m))$. Therefore

$$(2^m+1)^{t+2j-2k} - (2^m+1)^t \equiv 0 \mod 2^{2n+m-1},$$

that is, $(2^m+1)^{2j-2k} - 1 \equiv 0 \mod 2^{2n+m-1}$. Set $2j-2k = u \cdot 2^v$, where u is odd. Then, according to [8, Lemma 3.1], $(2^m+1)^{u \cdot 2^v} - 1 \equiv u \cdot 2^{v+m} \mod 2^{v+m+1}$. Thus $v+m \geq 2n+m-1$, and so $v \geq 2n-1$. Hence we have $j-k \equiv 0 \mod 2^{2n-2}$.   q.e.d.

§ 5. Proof of Theorem 1.2

Let $r: \tilde{K}_C(X) \to \tilde{K}_R(X)$ and $c: \tilde{K}_R(X) \to \tilde{K}_C(X)$ be the real restriction and the complexification, respectively. Let $\xi_R: \tilde{R}_R(H_m) \to \tilde{K}_R(N^n(m))$ be the natural projection defined in [6, (3.9)] (or in [11, Theorem 2.5]). Then, for the elements v and z of $\tilde{R}_R(H_m)$ defined by $c^{-1}(2\delta_1) = v$ and $c^{-1}(\delta_1^2) = z$ (cf. § 2), we have

$$\xi_R v = r(\pi^*\lambda - 2) \quad \text{and} \quad \xi_R z = c^{-1}((\pi^*\lambda - 2)^2)$$

(cf. [6, Lemma 3.10]), since δ_1 is self-conjugate and $cr = 1 + \text{conjugation}$. For simplicity we write v and z instead of $\xi_R v$ and $\xi_R z$. Then, for the complexification $c: \tilde{K}_R(N^n(m)) \to \tilde{K}_C(N^n(m))$, we have

$$c(v) = 2\delta \quad \text{and} \quad c(z) = \delta^2.$$

The orders of the canonical elements z^i and vz^i of $\tilde{K}_R(N^n(m))$ are determined in [7, Theorems 1.5 and 1.6]. As a corollary, the order of v is determined in [7, Corollary 1.7].

Proposition 5.1. *For $v \in \tilde{K}_R(N^n(m))$,*

$$\sharp v = 2^{2n+m-2+\varepsilon}, \qquad \text{where } \varepsilon = 1 \text{ if } n \text{ is odd}, \varepsilon = 0 \text{ if } n \text{ is even} > 0.$$

Proof of Theorem 1.2. Let $v = \xi_R v = r(\pi^*\lambda - 2) \in \tilde{K}_R(N^n(m))$. If $j-k \equiv 0 \mod 2^{2n+m-2+\varepsilon}$, then, by Proposition 5.1, $J((j-k)v) = 0$, where $J: \tilde{K}_R(N^n(m)) \to \tilde{J}(N^n(m))$ is the J-homomorphism. Thus $J(jr(\pi^*\lambda - 2)) = J(kr(\pi^*\lambda - 2))$. According to [2, Proposition 2.6], this implies that the Thom complexes $(N^n(m))^{j\tau\pi^*\lambda}$ and $(N^n(m))^{k\tau\pi^*\lambda}$ are of the same stable

homotopy type. On the other hand, by [10, Theorem 2.1], there are natural homeomorphisms:

$$N_j^{n+j}(m) \approx (N^n(m))^{jr\pi^*\lambda}, \qquad N_k^{n+k}(m) \approx (N^n(m))^{kr\pi^*\lambda},$$

and hence $N_j^{n+j}(m)$ and $N_k^{n+k}(m)$ are of the same stable homotopy type.

q.e.d.

References

[1] J. F. Adams, Vector fields on spheres, Ann. of Math., **75** (1962), 603–632.

[2] M. F. Atiyah, Thom complexes, Proc. London Math. Soc., (3) **11** (1961), 291–310.

[3] R. Bott, Lectures on $K(X)$, Benjamin, New York, 1969.

[4] C. W. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Pure Appl. Math., **11**, Wiley, New York, London, 1962.

[5] K. Fujii, On the K-ring of S^{4n+3}/H_m, Hiroshima Math. J., **3** (1973), 251–265.

[6] ——, On the KO-ring of S^{4n+3}/H_m, Hiroshima Math. J., **4** (1974), 459–475.

[7] T. Kobayashi, Nonimmersions and nonembeddings of quaternionic spherical space forms, Trans. Amer. Math. Soc., **279** (1983), 723–728.

[8] T. Kobayashi and M. Sugawara, On stable homotopy types of stunted lens spaces, Hiroshima Math. J., **1** (1971), 287–304.

[9] T. Mormann, Topologie Sphärischer Raumformen, Dissertation, Universität Dortmund, 1978.

[10] H. Ōshima, On stable homotopy types of some stunted spaces, Publ. RIMS, Kyoto Univ., **11** (1976), 497–521.

[11] D. Pitt, Free actions of generalized quaternion groups on spheres, Proc. London Math. Soc., (3) **26** (1973), 1–18.

Department of Mathematics
Faculty of Science
Kochi University
Akebono 2–5–1
Kochi 780, Japan