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A Characterization of the Kahn-Priddy Map

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Dedicated to Professor Nobuo Shimada on his 60th birthday

§ 1. Introduction and statements of main results

We denote by P^n the real n-dimensional projective space. E^kP^n for $k\geq 0$ denotes the k-fold reduced suspension of P^n and $E^k\phi_n$: $E^{n+k}P^{n-1}\to S^{n+k}$ the k-fold reduced suspension of a mapping ϕ_n: $E^nP^{n-1}\to S^n$. $E^k\phi_n$ for $n\geq 2$ is called a Kahn-Priddy map if the homotopy class of the restriction $E^k\phi_n\,|\,S^{n+k+1}$ generates $\pi_{n+k+1}(S^{n+k})$. We denote by $s(n)$ the number of i such that $0<i\leq n$ and $i\equiv 0$, 1, 2 or 4 mod 8.

By abuse of notation, we often use the same letter for a mapping and its homotopy class. Our first result is the following

Theorem 1.1. *Let* ϕ_{2n+1}: $E^{2n+1}P^{2n}\to S^{2n+1}$ *be a Kahn-Priddy map. Then the order of* $E^k\phi_{2n+1}$ *is* $2^{s(2n)}$ *for* $k\geq 0$.

For a CW-complex K, we put $\pi^n(K)=[K, S^n]$ which is n-th cohomotopy group if $K=EK'$ or dim $K\leq 2n-2$. Let H: $\pi^n(E^nP^{n-1})\to\pi^{2n-1}(E^nP^{n-1})$ be the Hopf homomorphism [10] and p_n: $P^n\to S^n$ the canonical map. Then our second result is the following

Theorem 1.2. ϕ_{2n+1}: $E^{2n+1}P^{2n}\to S^{2n+1}$ *is a Kahn-Priddy map if and only if* $H(\phi_{2n+1})=E^{2n+1}p_{2n}$.

Our basic idea is based on [3]. To prove Theorem 1.1, we shall use the $\widetilde{KO}$-group of P^n [1] and the suspension order of the identity class of $E^{2n}P^{2n}$ [9]. To prove Theorem 1.2, we shall use the essential uniqueness of Kahn-Priddy maps [2] and the EHP-sequence.

The problem determining the order of the Kahn-Priddy map was posed by Goro Nishida who solved it in the case of odd primes [7]. The

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§ 2. The order of the Kahn-Priddy map

By Theorem 7.4 of [1], $\widetilde{KO}(P^n)\approx Z/2^{s(n)}$. The inclusion $P^1\to P^n$ maps $\widetilde{KO}(P^n)$ onto $\widetilde{KO}(P^1)$. As is well known, α is a generator of $\pi_{n+1}(S^n)$ if and only if $\alpha^*: \widetilde{KO}^n(S^n)\to\widetilde{KO}^n(S^{n+1})$ is an epimorphism. So we have the following

Lemma 2.1. *$E^k\phi_n: E^{n+k}P^{n-1}\to S^{n+k}$ for $k\geq 0$ is a Kahn-Priddy map if and only if $(E^k\phi_n)^*: \widetilde{KO}^{n+k}(S^{n+k})\to\widetilde{KO}^{n+k}(E^{n+k}P^{n-1})$ is an epimorphism.*

By Corollary to Theorem 2.8 of [9], the order of $E^k\phi_{2n+1}$ is a divisor of $2^{s(2n)}$. So Lemma 2.1 leads us to Theorem 1.1.

Example. By Theorem 2.3 of [3], the symmetric square of S^n is homeomorphic to the mapping cone $S^n\cup C(E^nP^{n-1})$. We denote by $f_n: E^nP^{n-1}\to S^n$ the attaching map. By Lemma 3.2 of [3], $f_n^*: \widetilde{KO}^n(S^n)\to \widetilde{KO}^n(E^nP^{n-1})$ is onto. So, by Lemma 2.1, f_n is a Kahn-Priddy map.

Remark. The fact that $f_n: E^nP^{n-1}\to S^n$ is a Kahn-Priddy map is directly obtained from inspecting the definition of the symmetric square of S^n ([3] and [5]).

§ 3. Main results used in the proof of Theorem 1.2

Let $\phi, \psi: E^{2n+2}P^{2n}\to S^{2n+2}$ be Kahn-Priddy maps. Then, the following theorem is a direct consequence of Formulation 2.3 ii) of [2] and it shows the essential uniqueness of Kahn-Priddy maps.

Theorem 3.1. *There exists a self-homotopy equivalence ε of $E^{2n}P^{2n}$ such that $\psi=\phi\circ E^2\varepsilon$.*

By Theorem 4.9 of [10], we have the *EHP*-sequence of the following form.

Theorem 3.2. *Let K be a finite CW-complex and $r=3m-2-\dim K$. Then the following sequence is exact.*

$$\pi^m(E^rK)\xrightarrow{E}\pi^{m+1}(E^{r+1}K)\xrightarrow{H}\pi^{2m+1}(E^{r+1}K)\xrightarrow{\Delta}\pi^m(E^{r-1}K)$$
$$\longrightarrow\cdots\xrightarrow{\Delta}\pi^m(K)\xrightarrow{E}\pi^{m+1}(EK)\xrightarrow{H}\pi^{2m+1}(EK).$$

We shall need a little generalization of the well-known formulas about H and Δ. Precisely, Propositions 2.5 and 2.6 of [8] are valid in the following forms.

Proposition 3.3. *Let K, L and M be finite CW-complexes. Let $\alpha \in [E^2L, E(M\wedge M)]$ and $\beta \in [K, L]$. Assume that M is $(m-1)$-connected, $\dim K<3m-2$ and $\dim L<3m-2$. Then $\Delta(\alpha\circ E^2\beta)=\Delta\alpha\circ\beta$. In particular, $\Delta(E^2\beta)=[\iota_m, \iota_m]\circ\beta$ if $M=S^m$ and $L=S^{2m-1}$. Here ι_m denotes the identity class of S^m.*

Proposition 3.4. *Let K and L be CW-complexes. Let $\alpha \in \pi^m(K)$, $\beta \in \pi_k(K)$ and $\gamma \in \pi^k(L)$ satisfy the conditions $E(\alpha\beta)=0$ and $\beta\gamma=0$. Then*

$$H\{E\alpha, E\beta, E\gamma\}_1=-\Delta^{-1}(\alpha\beta)\circ E^2\gamma.$$

Proofs of the propositions are completed following faithfully the ones of Propositions 2.5 and 2.6 of [8]. We omit the details.

§ 4. The Hopf invariant of the Kahn-Priddy map

A standard Kahn-Priddy map $g_n\colon E^nP^{n-1}\to S^n$ is given as follows [4]; $O(n)$ denotes the orthogonal group and Ω^nS^n a space consisting of based self-maps of S^n. $k_n\colon O(n)\to\Omega^nS^n$ denotes the canonical injection. $j_n\colon P^{n-1}\to O(n)$ represents a line L through the origin in R^n as the reflection in the hyperplane perpendicular to L. Then g_n is obtained from taking the adjoint of the composition $k_nj_n\colon P^{n-1}\to\Omega^nS^n$.

From the definition, we have

$$g_n\,|\,E^nP^{n-2}=\pm Eg_{n-1}. \tag{4.1}$$

Let $\gamma_n\colon S^n\to P^n$ be the projection, $i_n\colon P^{n-1}\to P^n$ and $p_n\colon P^n\to S^n$ the canonical maps. Then we have a cofibre sequence:

$$S^{n-1}\xrightarrow{\gamma_{n-1}}P^{n-1}\xrightarrow{i_n}P^n\xrightarrow{p_n}S^n\longrightarrow\cdots. \tag{4.2}$$

As is well known, we have

$$p_n\gamma_n=(1+(-1)^{n-1})\iota_n. \tag{4.3}$$

According to Section 9 of [11], the image of the connecting homomorphism $\partial\colon \pi_n(O(n+1), O(n))\to\pi_{n-1}(O(n))$ is generated by $j_n\gamma_{n-1}\colon S^{n-1}\to O(n)$ and $J(j_n\gamma_{n-1})=\pm[\iota_n, \iota_n]$. Here $J\colon \pi_{n-1}(O(n))\to\pi_{2n-1}(S^n)$ denotes the J homomorphism. From the definitions of J and g_n, $J(j_n\gamma_{n-1})=g_n\circ E^n\gamma_{n-1}$. So we have

$$g_n\circ E^n\gamma_{n-1}=\pm[\iota_n, \iota_n]. \tag{4.4}$$

From the definition of the secondary composition (Chap. 1 of [8]), $\{E^n i_{n-1}, E^n \gamma_{n-2}, E^{n-1} p_{n-1}\}_1$ is represented by the identity class ι_{EK_n} for $n \geq 3$, where $K_n = E^{n-1}P^{n-1}$. By Proposition 1.2. iv) of [8] and by (4.1),

$$\pm g_n \in \pm g_n \circ \{E^n i_{n-1}, E^n \gamma_{n-2}, E^{n-1} p_{n-1}\}_1 \subset \{Eg_{n-1}, E^n \gamma_{n-2}, E^{n-1} p_{n-1}\}_1.$$

So we have

(4.5) $$\pm g_n \in \{Eg_{n-1}, E^n \gamma_{n-2}, E^{n-1} p_{n-1}\}_1.$$

By use of (4.2) and (4.3), we have

(4.6) $\pi^{2n-1}(E^n P^{n-1}) = \{E^n p_{n-1}\} \approx Z$ *or* $Z/2$ *according as n is even or odd.*

Theorem 4.1. *Except for the case* $n=4$ *or* 8,

$$H(g_n) \equiv E^n p_{n-1} \bmod 2E^n p_{n-1}.$$

Proof. The assertion for $n=2$ holds trivially. By Proposition 3.3, $\Delta(\iota_{2n-1}) = \pm[\iota_{n-1}, \iota_{n-1}]$. So, by (4.5), (4.4) and Proposition 3.4,

$$\begin{aligned} \pm H(g_n) &\in H\{Eg_{n-1}, E^n \gamma_{n-2}, E^{n-1} p_{n-1}\}_1 \\ &= -\Delta^{-1}(g_{n-1} \circ E^{n-1}\gamma_{n-2}) \circ E^n p_{n-1} \ni \pm E^n p_{n-1}. \end{aligned}$$

The secondary composition $\{Eg_{n-1}, E^n \gamma_{n-2}, E^{n-1} p_{n-1}\}_1$ is a coset of the subgroup $Eg_{n-1} \circ E[K_n, EK_{n-1}] + \pi_{2n-1}(S^n) \circ E^n p_{n-1}$. Therefore, by Proposition 2.2 of [8], $H\{Eg_{n-1}, E^n \gamma_{n-2}, E^{n-1} p_{n-1}\}_1$ is a coset of $H\pi_{2n-1}(S^n) \circ E^n p_{n-1}$. As is well known, $H\pi_{2n-1}(S^n) = (1+(-1)^n)\pi_{2n-1}(S^{2n-1})$ except for the case $n=1$, 2, 4 or 8. This completes the proof.

We shall prove a half assertion of Theorem 1.2.

Lemma 4.2. *Let* $\phi_n \colon E^n P^{n-1} \to S^n$ *be a Kahn-Priddy map and n odd. Then* $H(\phi_n) = E^n p_{n-1}$.

Proof. By Theorem 3.1, there exists a self-homotopy equivalence ε of $K_n = E^{n-1}P^{n-1}$ satisfying $E\phi_n = Eg_n \circ E^2\varepsilon$. By Theorem 3.2, we have an exact sequence for $n \geq 2$:

$$\pi^{2n+1}(E^3 K_n) \xrightarrow{\Delta} \pi^n(EK_n) \xrightarrow{E} \pi^{n+1}(E^2 K_n).$$

By Proposition 3.3, (4.6) and (4.4), $\Delta(E^{n+2} p_{n-1}) = g_n \circ E^n \gamma_{n-1} \circ E^n p_{n-1}$. So, by the above exact sequence, $\phi_n = g_n \circ (E\varepsilon + aE^n(\gamma_{n-1} p_{n-1}))$ for some integer a. Therefore, by Proposition 2.2 of [8], by Theorem 4.1 and (4.3), $H(\phi_n) = H(g_n) \circ (E\varepsilon + aE^n(\gamma_{n-1} p_{n-1})) = E^n p_{n-1} \circ E\varepsilon + aE^n(p_{n-1}\gamma_{n-1} p_{n-1}) = E^n p_{n-1} \circ E\varepsilon$. Since $E\varepsilon$ induces an automorphism $(E\varepsilon)^*$ of $\pi^{2n-1}(E^n P^{n-1}) \approx Z/2$, we have $E^n p_{n-1} \circ E\varepsilon = E^n p_{n-1}$. This completes the proof.

Proof of Theorem 1.2. It suffices to prove that the converse of Lemma 4.2 is true. By Theorem 3.2, we have an exact sequence for $n\geq 3$:

$$\pi^{n-1}(K_n)\xrightarrow{E}\pi^n(EK_n)\xrightarrow{H}\pi^{2n-1}(EK_n).$$

Suppose that $H(\phi_n)=E^n p_{n-1}$ for odd n. Then, by Theorem 4.1 and the above exact sequence, there exists an element $\alpha\in\pi^{n-1}(K_n)$ such that $\phi_n=g_n+E\alpha$. By Lemma 4.2, $E\alpha$ is not a Kahn-Priddy map since $H(E\alpha)=0$. So $\phi_n|S^{n+1}=g_n|S^{n+1}$. Therefore ϕ_n is a Kahn-Priddy map. This completes the proof.

Problem. In $\pi^{2n+1}(E^{2n+1}P^{2n})$, is an element of order $2^{s(2n)}$ a Kahn-Priddy map?

Example. By [6], $\pi_S^0(P^{2n})\approx Z/4$, $Z/8$, $Z/8\oplus Z/2$ or $Z/16\oplus Z/2$ according as $n=1$, 2, 3 or 4. So the above problem is solved affirmatively for $n\leq 4$.

Example. Let n be even. Then, by Theorem 1.1 and (4.1), the order of Eg_n is $2^{s(n-1)}$ if $n\equiv 6$ mod 8. Moreover, by (4.2), (4.3) and (4.4), the order of Eg_n is $2^{s(n-2)}$ or $2^{s(n-2)+1}$ if $n\equiv 0$, 2 or 4 mod 8.

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