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On the Spectra $L(n)$ and a Theorem of Kuhn

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Dedicated to Professor Nobuo Shimada on his 60th birthday

§ 0. Introduction

Let Z_2^n be the elementary abelian 2-group. In [7] Mitchell and Priddy have shown that stably BZ_2^n contains some copies of spectra $M(n)=e_nBZ_2^n$ as a direct summand, where $e_n \in \hat{Z}_2GL_n(F_2)$ is the Steinberg idempotent. It is also shown that there is an equivalence of spectra $M(n)\simeq L(n)\vee L(n-1)$, where $L(n)=\Sigma^{-n}Sp^{2^n}S^0/Sp^{2^n-1}S^0$. In [5], Kuhn has shown that there is a split exact sequence

$$\longrightarrow L(n)\longrightarrow L(n-1)\longrightarrow\cdots\longrightarrow L(0)=S^0$$

extending the Kahn-Priddy theorem [4] and solved the Whitehead conjecture.

In [9], the author determined the structure of the stable homotopy group $\{BZ_2^n, BZ_2^m\}$ and the composition formula. Let $M_{n,m}(F_2)$ be the set of (n, m)-matrices. Then there are inclusions of rings

$$\hat{Z}_2GL_n(F_2)\longrightarrow\hat{Z}_2M_{n,n}(F_2)\longrightarrow\{BZ_2^n, BZ_2^n\}\longrightarrow[QBZ_2^n, QBZ_2^n]$$

where $QBZ_2^n=\Omega^\infty\Sigma^\infty BZ_2^n$ is the infinite loop space.

In this paper, studying the structure of those rings we shall show the following. The Steinberg idempotent $e_n \in \hat{Z}_2GL_n(F_2)$ is decomposed as $e_n=a_n+b_n$ in the bigger rings and a_n, b_n are primitive in $\{BZ_2^n, BZ_2^n\}$. We determine the structure of $\{M(n), M(m)\}$ and $\{L(n), L(m)\}$. Finally we give a simple proof of the theorem of Kuhn.

§ 1. Steinberg idempotents and matrix algebra

Let R be the ring of 2-adic integers $\hat{Z}_2$ or the prime field F_2. Let $M_{n,m}(F_2)$ be the set of all (n, m)-matrices over F_2. We denote by $R\tilde{M}_{n,m}(F_2)$ the free R-module generated by elements of $M_{n,m}(F_2)$ with the relation 0-matrix$=0$. There is an obvious pairing

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$$R\tilde{M}_{n,m}(F_2) \otimes R\tilde{M}_{m,l}(F_2) \longrightarrow R\tilde{M}_{n,l}(F_2).$$

In particular, $R\tilde{M}_{n,n}(F_2)$ is a ring and $R\tilde{M}_{n,m}(F_2)$ is a left $R\tilde{M}_{n,n}(F_2)$ and right $R\tilde{M}_{m,m}(F_2)$-module.

Given a subset S of $M_{n,m}(F_2)$, we denote $\sum_{A\in S} A$ by $\bar{S}$. If S is a subset of $\Sigma_n \subset GL_n(F_2)$, then $\sum_{A\in S}(-1)^{\mathrm{sgn}(A)}A$ is denoted by $\tilde{S}$. Let B_n and U_n be the Borel subgroup and the unipotent subgroup of $GL_n(F_2)$, respectively. Then the Steinberg idempotents are defined [7] by

$$e_n = \bar{B}_n \tilde{\Sigma}_n / [GL_n(F_2)\colon U_n] \in \hat{Z}_2 GL_n(F_2)$$
$$e'_n = \tilde{\Sigma}_n \bar{B}_n / [GL_n(F_2)\colon U_n] \in \hat{Z}_2 GL_n(F_2).$$

Now we fix some notations. Let $C_i = (i, \cdots, n)$ and $C'_i = (1, \cdots, i) \in \Sigma_n$ be the cyclic permutations, and let $T_n = \{C_1, \cdots, C_n\}$ and $T'_n = \{C'_1, \cdots, C'_n\}$. Given a vector $b = (b_1, \cdots, b_{n-1}) \in F_2^{n-1}$, let

$$R_{n-1}(b) = \begin{pmatrix} 1 & & & b_1 \\ & \ddots & 0 & \vdots \\ 0 & & \ddots & \vdots \\ & & 1 & b_{n-1} \end{pmatrix} = (E_{n-1}, b^t) \in M_{n-1,n}(F_2), \quad \text{and}$$

$$L_n(b) = \begin{bmatrix} b_1 \cdots\cdots b_{n-1} \\ 1 & & \\ & \ddots \; 0 & \\ 0 & \ddots & \\ & & 1 \end{bmatrix} = \begin{pmatrix} b \\ \\ E_{n-1} \end{pmatrix} \in M_{n,n-1}(F_2).$$

Let $R_{n-1} = \{R_{n-1}(b)\}_{b\in F_2^{n-1}}$ and $L_n = \{L_n(b)\}_{b\in F_2^{n-1}}$. $R_{n-1}(0)$ and $L_n(0)$ are denoted by J_{n-1} and I_n, respectively.

In the following, the congruence mod 2, $a \equiv b$ mod 2, is denoted simply by $a \equiv b$. The following observation is useful. Let $\phi\colon F_2^r \to M_{n,m}(F_2)$ be an affine map. Then $\overline{\mathrm{Im}(\phi)} \equiv 0$ if and only if the associated linear map of ϕ is not a monomorphism.

Lemma 1.1. (i) $J_{n-1} e_n \equiv J_{n-1} \tilde{T}_n e_n \equiv e_{n-1} J_{n-1} \tilde{T}_n$, $n \geqq 1$.
(ii) $(\bar{L}_n J_{n-1} \tilde{T}_n + J_n \tilde{T}_{n+1} \bar{L}_{n+1}) e_n \equiv e_n$, $n \geqq 1$. *Here* $R\tilde{M}_{n,m}(F_2)$ *stands for the zero ring if* $n=0$ *or* $m=0$.

Proof. (i) is an easy calculation, and we prove (ii). Note that $\bar{L}_{n+1} \bar{B}_n = \bar{B}_{n+1} I_{n+1}$. Let $C \in T_{n+1}$ be a non trivial element, then as observed above, $J_n C \bar{B}_{n+1} I_{n+1} \equiv 0$ and hence

$$J_n \tilde{T}_{n+1} \bar{L}_{n+1} B_n = J_n \tilde{T}_{n+1} \bar{B}_{n+1} I_{n+1} \equiv J_n \bar{B}_{n+1} I_{n+1}.$$

Let $B_{n+1} \ni B = (b_{i,j})$. In the summation $J_n \bar{B}_{n+1} I_{n+1} \tilde{\Sigma}_n$, we may assume that $b_{i,n+1} = 0$ for $1 < i < n+1$. Then dividing the summation according to $b_{1,n+1}=0$ or 1, we have

$$J_n\tilde{T}_{n+1}\bar{L}_{n+1}e_n \equiv e_n + \sum J_n B I_{n+1}\tilde{\Sigma}_n,$$

where the summation is taken over all B such that $b_{1,n+1}=0$. Similarly we see that $J_{n-1}\tilde{T}_n\bar{B}_n\tilde{\Sigma}_n \equiv J_{n-1}\bar{B}_n\tilde{\Sigma}_n$ by (i), and easily we have $J_{n-1}\bar{B}_n\tilde{\Sigma}_n \equiv \bar{B}_{n-1}J_{n-1}\tilde{\Sigma}_n$. Then $\bar{L}_n\bar{B}_{n-1}J_{n-1}\tilde{\Sigma}_n = \bar{B}_n I_n J_{n-1}\tilde{\Sigma}_n$. But this is just the latter term of the above equation. This completes the proof.

Lemma 1.2. *Let* $m \leqq n-2$. *Then* $e_n F_2\tilde{M}_{n,m}(F_2) = F_2\tilde{M}_{m,n}(F_2)e_n = 0$.

Proof. For a matrix $B \in B_n$ we write $B=\begin{pmatrix}1 & b\\ 0 & B'\end{pmatrix}$, where $B' \in B_{n-1}$ and $b \in F_2^{n-1}$. Let $A=\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix} \in M_{n,m}(F_2)$, $a_i \in F_2^m$. Let $A'=\begin{pmatrix}a_2\\ \vdots\\ a_n\end{pmatrix}$, then we have $BA=\begin{pmatrix}a_1+bA'\\ B'A'\end{pmatrix}$. If $m \leqq n-2$, then the affine map $f(b)=a_1+bA'$ has a non trivial kernel. Hence $e_nA \equiv \bar{B}_n\tilde{\Sigma}_nA \equiv 0$ and $e_nF_2\tilde{M}_{n,m}(F_2)=0$. The rest is similar.

Lemma 1.3. (i) $e_nF_2\tilde{M}_{n,n-1}(F_2) = \bar{L}_ne_{n-1}F_2GL_{n-1}(F_2)$.
(ii) $F_2\tilde{M}_{n-1,n}(F_2)e_n = F_2GL_{n-1}(F_2)e_{n-1}J_{n-1}\tilde{T}_n$.

Proof. (i) Let $B \in B_n$ and $A \in M_{n,n-1}(F_2)$. Then $BA=\begin{pmatrix}a_1+bA'\\ B'A'\end{pmatrix}$ and if A' is singular, then as in the proof of the above lemma, we have $\bar{B}_nA \equiv 0$. Let $\Sigma_n = \bigcup\begin{pmatrix}1 & \\ & \Sigma_{n-1}\end{pmatrix}C_i$ be the coset decomposition. Then

$$e_nA = \sum \bar{B}_n\begin{pmatrix}1 & \\ & \tilde{\Sigma}_{n-1}\end{pmatrix}C_iA \equiv \sum\begin{pmatrix}x(b)\\ B'T(C_iA)'\end{pmatrix}$$

where $B' \in B_{n-1}$, $T \in \Sigma_{n-1}$ and $x(b)=bT(C_iA)'+$constant vector. Then we see that $e_nA \equiv \sum \bar{L}_ne_{n-1}(C_iA)'$, where $(C_iA)'$ is non singular. Hence $e_nA \in \bar{L}_ne_{n-1}F_2GL_{n-1}(F_2)$. On the other hand, for any $H \in GL_{n-1}(F_2)$, it is easy to see that $I_ne_{n-1}H \in e_nF_2\tilde{M}_{n,n-1}(F_2)$. This completes the proof of (i). The proof of (ii) is similar using Lemma 1.1, (i).

In particular we have

Lemma 1.4. $e_nI_n \equiv \bar{L}_ne_{n-1}$.

Corollary 1.5. $e_nI_ne_{n-1} \equiv e_n\bar{L}_ne_{n-1}$ *and* $e_{n-1}J_{n-1}e_n \equiv e_{n-1}J_{n-1}\tilde{T}_ne_n$.

For the Steinberg idempotent e_n', we have similar results. Replace e_n, $\bar{L}_n$, I_n, J_n and T_n with e_n', $\bar{R}_{n-1}$, J_{n-1}, I_{n+1} and T_n' respectively in the above formulae, and convert the direction of the composition, then all lemmas in this section hold for e'_n. For example

Lemma 1.6. (i) $e'_nI_n \equiv e'_n\tilde{T}'_nI_n \equiv \tilde{T}'_nI_ne'_{n-1}$.
(ii) $e'_n(\tilde{T}'_nI_n\bar{R}_{n-1} + \bar{R}_n\tilde{T}'_{n+1}I_{n+1}) \equiv e'_n$.

§ 2. Splitting of the Steinberg idempotent

We denote $e_nI_ne_{n-1} \in \hat{Z}_2\tilde{M}_{n,n-1}(F_2)$ and $e_nJ_ne_{n+1} \in \hat{Z}_2\tilde{M}_{n,n+1}(F_2)$ by ∂_n and σ_n, respectively. Similarly ∂'_n and σ'_n for $e'_nI_ne'_{n-1}$ and $e'_nJ_ne'_{n+1}$.

Theorem 2.1. *Let $n \geqq 2$. $\sigma_n\partial_{n+1}$ and $\partial_n\sigma_{n-1}$ are orthogonal idempotents in $F_2\tilde{M}_{n,n}(F_2)$ and $e_n \equiv \sigma_n\partial_{n+1} + \partial_n\sigma_{n-1}$. Similarly $\sigma'_n\partial'_{n+1}$ and $\partial'_n\sigma'_{n-1}$ are orthogonal idempotents in $F_2\tilde{M}_{n,n}(F_2)$ and $e'_n \equiv \sigma'_n\partial'_{n+1} + \partial'_n\sigma'_{n-1}$.*

Proof. Let $\bar{\partial}_n = e_n\bar{L}_ne_{n-1}$ and $\bar{\sigma}_n = e_nJ_n\tilde{T}_{n+1}e_{n+1}$. Then by Corollary 1.5, $\partial_n \equiv \bar{\partial}_n$ and $\sigma_n \equiv \bar{\sigma}_n$. Now

$$\begin{aligned}\sigma_n\partial_{n+1} + \partial_n\sigma_{n-1} &\equiv \bar{\sigma}_n\bar{\partial}_{n+1} + \bar{\partial}_n\bar{\sigma}_{n-1}\\ &= e_nJ_n\tilde{T}_{n+1}e_{n+1}\bar{L}_{n+1}e_n + e_n\bar{L}_ne_{n-1}J_{n-1}\tilde{T}_ne_n\\ &\equiv e_n(J_n\tilde{T}_{n+1}\bar{L}_{n+1} + \bar{L}_nJ_{n-1}\tilde{T}_n)e_n \equiv e_n\end{aligned}$$

by Lemma 1.1. Note that $\partial_{n+1}\partial_n \equiv 0$ and $\sigma_{n-1}\sigma_n \equiv 0$ by Lemma 1.2. Hence $\sigma_n\partial_{n+1}$ and $\partial_n\sigma_{n-1}$ are orthogonal idempotents. Similarly for e'_n and this completes the proof.

Theorem 2.2. *There are isomorphisms as vector spaces*

$$e_nF_2\tilde{M}_{n,m}(F_2)e_m \cong \begin{cases} 0, & |n-m| \geqq 2\\ F_2\{\sigma_n\}, & m=n+1\\ F_2\{\sigma_n\partial_{n+1}\} \oplus F_2\{\partial_n\sigma_{n-1}\}, & m=n \geqq 2\\ F_2\{\partial_n\}, & m=n-1\end{cases}$$

and $e_1F_2\tilde{M}_{1,1}(F_2)e_1 \cong F_2\{\sigma_1\partial_2\}$.

Proof. The case of $|n-m| \geqq 2$ is clear from Lemma 1.2. It is known [7] that the Steinberg module $F_2GL_n(F_2)e_n$ is projective and absolutely irreducible as $GL_n(F_2)$-module. Therefore $e_nF_2GL_n(F_2)e_n \cong F_2\{e_n\}$. Then we have dim $e_nF_2\tilde{M}_{n,n-1}(F_2)e_{n-1} = \dim e_nF_2\tilde{M}_{n,n+1}(F_2)e_{n+1} = 1$ by Lemma 1.3. By Lemma 1.4, $\partial_n \equiv \bar{L}_ne_{n-1} \not\equiv 0$ and $\sigma_n \equiv e_nJ_n\tilde{T}_{n+1} \not\equiv 0$ by Lemma 1.1. This shows the cases $m=n \pm 1$. Finally let S_n be the submodule of $F_2\tilde{M}_{n,n}(F_2)$ spanned by all singular matrices. Then $F_2\tilde{M}_{n,n}(F_2) \cong F_2GL_n(F_2) \oplus S_n$ as the both side $GL_n(F_2)$-module. From the above argument we easily see that dim $e_nS_ne_n=1$ and hence dim $e_nF_2\tilde{M}_{n,n}(F_2)e_n=2$. Now $\partial_{n+1}\sigma_n\not\equiv 0$ and $\sigma_n\partial_{n+1}\not\equiv 0$, for $\partial_{n+1}\sigma_n\partial_{n+1}\equiv(e_{n+1}-\sigma_{n+1}\partial_{n+2})\partial_{n+1}\equiv\partial_{n+1}\not\equiv 0$. Then the case $n=m$ is clear from Theorem 2.1.

Corollary 2.3. *The idempotents* $\sigma_n\partial_{n+1}$ *and* $\partial_n\sigma_{n-1}\in F_2\tilde{M}_{n,n}(F_2)$ *are primitive.*

Now consider the reduction ρ: $\hat{Z}_2M_{n,n}(F_2)\to F_2\tilde{M}_{n,n}(F_2)$. Then as is well known [1], there are lifting idempotents. Therefore from Theorem 2.1 and Corollary 2.3, we have

Corollary 2.4. *There are orthogonal primitive idempotents* a_n, $b_n\in\hat{Z}_2\tilde{M}_{n,n}(F_2)$ *such that* $e_n=a_n+b_n$, $a_n\equiv\sigma_n\partial_{n+1}$ mod 2, *and* $b_n\equiv\partial_n\sigma_{n-1}$ mod 2.

Remark 1. Above results hold clearly for $e'_nF_2\tilde{M}_{n,m}(F_2)e'_m$ replacing ∂_n, σ_n with ∂'_n, σ'_n.

Remark 2. Lemma 1.2 holds for $\hat{Z}_2$ coefficient. For $e_n\hat{Z}_2\tilde{M}_{n,m}(F_2)$ is a direct summand of $\hat{Z}_2\tilde{M}_{n,m}(F_2)$. Therefore $\partial_{n+1}\partial_n=0$ and $\sigma_n\sigma_{n+1}=0$ in $\hat{Z}_2$ coefficient. Moreover using the lifting of idempotents [1], we see that Theorem 2.2 holds for $\hat{Z}_2$ coefficient.

Remark 3. In the (non reduced) semigroup ring $RM_{n,n}(F_2)$, the 0-matrix 0 is a central idempotent. Hence in $RM_{n,n}(F_2)$, e_n splits as a sum of three primitive idempotents for $n\geqq 2$. For $n=1$, we have $e_1=E_1=(E_1-0)+0$ is an orthogonal decomposition. We define $RM_{1,0}(F_2)=R\,\mathrm{Hom}\,(F_2, 0)=R$ with basis σ_0, and $RM_{0,1}(F_2)=R\,\mathrm{Hom}(0, F_2)=R$ with basis ∂_1. Then $\partial_1\sigma_0=0$ and we have a decomposition $e_1\equiv\sigma_1\partial_2+\partial_1\sigma_0$ in $\hat{Z}_2M_{1,1}(F_2)$. Thus Theorem 2.1 holds for $n=1$ and

$$\hat{Z}_2M_{1,1}(F)\cong\hat{Z}_2\{\sigma_1\partial_2\}\oplus\hat{Z}_2\{\partial_1\sigma_0\}.$$

§ 3. Splitting spectra and infinite loop spaces

Let Y be a 2-local spectrum of finite type, and let $\Omega^\infty Y$ be the associated infinite loop space. Let $\{Y, Y\}$ be the stable homotopy ring. The unstable homotopy set $[\Omega^\infty Y, \Omega^\infty Y]$ is an abelian group with the composition product which satisfies the condition of a ring structure except the left distribution law. There is a natural "ring" homomorphism j: $\{Y, Y\}\to[\Omega^\infty Y, \Omega^\infty Y]$. Let Y be a suspension spectrum of a 2-local space X. Then $\Omega^\infty Y=QX$ by definition and denoting $\{Y, Y\}$ by $\{X, X\}$, we see that

$$j\colon \{X, X\}\longrightarrow[QX, QX]$$

is a monomorphism.

We call an element $e \in \{Y, Y\}$ an idempotent mod 2 if $e^2 \equiv e$ mod 2. For an element $f \in [\Omega^\infty Y, \Omega^\infty Y]$, let $f_* \in \mathrm{End}(\pi_*(\Omega^\infty Y)) \cong \mathrm{End}(\pi_*^S(Y))$. An element $f \in [\Omega^\infty Y, \Omega^\infty Y]$ is called a π_*-idempotent mod 2 if $f_*^2 \equiv f_*$ in $\mathrm{End}(\pi_*(\Omega^\infty Y))$.

Given $e \in \{Y, Y\}$, the telescope of the sequence $Y \xrightarrow{e} Y \xrightarrow{e} \cdots$ is denoted by eY. Similarly for $f \in [\Omega^\infty Y, \Omega^\infty Y]$, the telescope of the sequence $\Omega^\infty Y \xrightarrow{f} \Omega^\infty Y \xrightarrow{f} \cdots$ is denoted by $f\Omega^\infty Y$. There are natural maps ϕ_e: $Y \to eY$ and ψ_f: $\Omega^\infty Y \to f\Omega^\infty Y$. Let

$$\xi_e = \phi_e \vee \phi_{1-e}: Y \longrightarrow eY \vee (1-e)Y$$

and

$$\eta_f = \psi_f \times \psi_{1-f}: \Omega^\infty Y \longrightarrow f\Omega^\infty Y \times (1-f)\Omega^\infty Y.$$

Proposition 3.1. *Let* $e \in \{Y, Y\}$ *be an idempotent* mod 2. *Then*

(i) ξ_e: $Y \to eY \vee (1-e)Y$ *is a homotopy equivalence.*

(ii) *Let* $e' \in \{Y, Y\}$ *such that* $e' \equiv e$ mod 2. *Then there is a homotopy equivalence* λ: $eY \to e'Y$.

Proposition 3.2. *Let* $f \in [\Omega^\infty Y, \Omega^\infty Y]$ *be a* π_**-idempotent* mod 2. *Then*

(i) η_f: $\Omega^\infty Y \to f\Omega^\infty Y \times (1-f)\Omega^\infty Y$ *is a homotopy equivalence.*

(ii) *Let* $f' \in [\Omega^\infty Y, \Omega^\infty Y]$ *such that* $f_* \equiv f'_*$ mod 2.
Then there is a homotopy equivalence $\lambda: f\Omega^\infty Y \to f'\Omega^\infty Y$.

Proof of Propositions. Since $e^2 \equiv e$ mod 2, e_* is an idempotent in $\mathrm{End}(\pi_*^S(Y) \otimes Z_2)$ and also in $\mathrm{End}(\pi_*^S(Y) * Z_2)$. Then

$$\pi_*^S(eY) \otimes Z_2 \cong e(\pi_*^S(Y) \otimes Z_2) \quad \text{and} \quad \pi_*^S(eY) * Z_2 \cong e(\pi_*^S(Y) * Z_2)$$

and similarly for $1-e$. Therefore $\xi_{e*} \otimes 1_{Z_2}$ and $\xi_{e*} * 1_{Z_2}$ are isomorphisms. Hence ξ_{e*} is an isomorphism and (i) is proved. Now if $e' \equiv e$ mod 2, then we have a homotopy equivalence $\xi_{e'}$: $Y \to e'Y \vee (1-e')Y$ and using ξ_e and $\xi_{e'}$ we can define a natural map λ: $eY \to e'Y$ in an obvious way, and as above we easily see that λ_*: $\pi_*^S(eY) \to \pi_*^S(e'Y)$ is an isomorphism. This shows (ii). Proof of the latter Proposition is similar.

Now we recall the structure of the stable homotopy group $\{BZ_2^n, BZ_2^m\}$. Let V be a subgroup of Z_2^n and let f: $V \to Z_2^m$ be a homomorphism. Define an element $u_{V,f} \in \{BZ_2^n, BZ_2^m\}$ by the composition

$$BZ_2^n \xrightarrow{\ \tau\ } BV \xrightarrow{\sigma(Bf)} BZ_2^m$$

where τ is the transfer of the covering $BV \to BZ_2^n$ and σ denotes the suspension functor. In the sequel, $\sigma(Bf)$ is denoted simply by f. Then in [9] followings are shown.

Theorem 3.3. *There is an isomorphism*

$$\{BZ_2^n, BZ_2^m\} \cong \oplus \hat{Z}_2\{u_{V,f}\}$$

where the sum is taken over all (V, f), $f \neq 0$.

Theorem 3.4. *Let* $V \subset Z_2^n$ *and* $W \subset Z_2^m$ *be subgroups and let* $f\colon V \to Z_2^m$ *and* $g\colon W \to Z_2^l$ *be homomorphisms. Let*

$$U = f^{-1}(W) \subset V \quad \text{and} \quad [Z_2^m : f(V)W] = 2^a.$$

Then

$$u_{W,g} u_{V,f} = 2^a u_{U,gf}.$$

Now we have an inclusion of rings

$$i\colon \hat{Z}_2 \tilde{M}_{m,n}(F_2) \longrightarrow \{BZ_2^n, BZ_2^m\}$$

defined by $i(f) = u_{Z_2^n, f}$. It is clear that i is compatible with compositions. In [9] we have also shown the following

Lemma 3.5. *A primitive idempotent in* $\hat{Z}_2 \tilde{M}_{n,n}(F_2)$ *is primitive in* $\{BZ_2^n, BZ_2^n\}$.

Now we recall the Mitchell-Priddy splitting. For $n \geqq 2$, the spectrum $e_n BZ_2^n$ is denoted by $M(n)$. We put $M(1) = BZ_2 \vee S^0 = e_1((BZ_2)_+)$. For $n \geqq 2$, let $a_n, b_n \in \hat{Z}_2 \tilde{M}_{n,n}(F_2)$ be idempotents in Corollary 2.4. By the remark of Section 2, we may define $a_1, b_1 \in \hat{Z}_2 M_{1,1}(F_2) \subset \{(BZ_2)_+, (BZ_2)_+\}$. Define spectra $M_a(n) = a_n BZ_2^n$ and $M_b(n) = b_n BZ_2^n$ for $n \geqq 2$, and $M_a(1) = a_1(BZ_{2+})$ and $M_b(1) = b_1(BZ_{2+})$. Then we have

Theorem 3.6. *The spectra* $M_a(n)$ *and* $M_b(n)$ *are indecomposable and there is a stable splitting*

$$M(n) \simeq M_a(n) \vee M_b(n), \quad n \geqq 1.$$

Proof. Since $e_n = a_n + b_n$ (orthogonal decomposition), $M(n) \simeq M_a(n) \vee M_b(n)$ is clear. The indecomposability of $M_a(n)$ and $M_b(n)$ follows from Corollary 2.3 and Lemma 3.5.

In [7], it is shown that there are spectra $L(n)$, $n \geqq 0$, $L(0) = S^0$, $L(1) = BZ_2$, and a splitting $M(n) \simeq L(n) \vee L(n-1)$, $n \geqq 1$. In [9] we have shown that the splitting of BZ_2^n by indecomposable spectra is essentially unique. Thus we have

Corollary 3.7. *$L(n)$, $n\geqq 0$, is indecomposable and $M_a(n)\simeq L(n)$ and $M_b(n)\simeq L(n-1)$.*

§ 4. Equivariant stable cohomotopy

Let G be a finite group. For G-space X and Y, $\{X_+, Y_+\}_G$ denotes the stable G-homotopy group, where X_+ is the based G-space with the disjoint base point. Let H be a subgroup of G, and let $N(H)$ be the normalizer of H. $N(H)/H$ is denoted by $W(H)$. Then the Segal-tom Dieck and Hauschild theorems are stated as follows.

Theorem 4.1 ([2], [3]). *Lex X be a finite CW-complex with the trivial G-action. Then there are isomorphisms*

$$\xi\colon \bigoplus_{(H)} \{X_+, EW(H)_+\}_{W(H)} \longrightarrow \{X_+, S^0\}_G$$

and

$$\lambda_H\colon \{X_+, EW(H)_+\}_{W(H)} \longrightarrow \{X_+, BW(H)_+\}$$

where the sum is taken over the conjugacy classes of subgroups of G and $EW(H)$ is a free contractible $W(H)$-space.

Using the above theorem, we show an equivariant version of the Barratt-Quillen theorem. Let E be a G-space. By a (G, E)-covering over X, we mean a pair of G-maps $(p,f)=(X\overset{p}{\leftarrow}\tilde{X}\overset{f}{\rightarrow}E)$, where $p\colon \tilde{X}\rightarrow X$ is a finite covering. Let $(X\overset{p'}{\leftarrow}\tilde{X}'\overset{f'}{\rightarrow}E)$ be another pair. We call (p,f) and (p',f') equivalent if there is an equivalence of coverings $\phi\colon \tilde{X}\rightarrow\tilde{X}'$ such that $f'\phi\sim_G f$. The set of equivalence classes of (G, E)-coverings over X is denoted by $C_G(X, E)$. By the disjoint sum, $C_G(X, E)$ is an abelian monoid. If $E=*$ or $G=\{e\}$, $C_G(X, E)$ is denoted by $C_G(X)$ or $C(X, E)$ respectively. Given a pair (p,f) we define a stable G-map $\omega(p,f)$ by the composition

$$X_+ \overset{\tau}{\longrightarrow} \tilde{X}_+ \overset{\sigma(f_+)}{\longrightarrow} E_+$$

where τ is the equivariant transfer [8]. Then we have a homomorphism

$$\omega\colon C_G(X, E) \longrightarrow \{X_+, E_+\}_G.$$

Then the following is shown in [9].

Lemma 4.2. *There are isomorphisms of monoids*

$$\tilde{\xi}\colon \prod_{(H)} C_{W(H)}(X, EW(H)) \longrightarrow C_G(X)$$

$$\tilde{\lambda}_H: C_{W(H)}(X, EW(H)) \longrightarrow C(X, BW(H))$$

and the following diagram is commutative;

$$\begin{array}{ccccc} C_G(X) & \xleftarrow{\tilde{\xi}} & \prod C_{W(H)}(X, EW(H)) & \xrightarrow{\prod \tilde{\lambda}_H} & \prod C(X, BW(H)) \\ \omega \downarrow & & \downarrow \prod \omega & & \downarrow \prod \omega \\ \{X_+, S^0\}_G & \xleftarrow[\xi]{} & \prod \{X_+, EW(H)_+\}_{W(H)} & \xrightarrow[\prod \lambda_H]{} & \prod \{X_+, BW(H)_+\}. \end{array}$$

Let h and h' be monoid valued contravariant homotopy functor on the category of CW-complexes. We suppose that h' is represented by a grouplike H-space. A natural homomorphism $\psi: h \to h'$ is called a group completion (in the sense of Segal) if the following universal property holds. For any grouplike H-space B and a natural homomorphism $\gamma: h \to [\ , B]_*$, there is a unique natural homomorphism $\gamma': h' \to [\ , B]_*$ such that $\gamma'\psi = \gamma$. Then by a result of [3], we immediately obtain the following

Theorem 4.3. *In the diagram of Lemma* 4.2, *every vertical maps are group completions as functors on X.*

Now by the Segal-tom Dieck isomorphism, we identify $\{X_+, S^0\}_G$ with $\oplus\{X_+, BW(H)_+\}$, when X is finite. We call the summand $\{X_+, BG_+\}$ corresponding to $H=\{e\}$ the free part of $\{X_+, S^0\}_G$. Let G' be another finite group and let

$$\gamma: \{X_+, S^0\}_G \longrightarrow \{X_+, S^0\}_{G'}$$

be a natural transformation of functors on X. We call γ admissible if γ preserves the free part, i.e., $\gamma(\{X_+, BG_+\}) \subset \{X_+, BG'_+\}$. Then we may consider $\gamma \in [Q(BG_+), Q(BG'_+)]$. Moreover if there is a relation among admissible natural transformations, then it gives the same relation in $[Q(BG_+), Q(BG'_+)]$.

We give some examples. First let $f: G' \to G$ be a homomorphism. Any stable G-map is regarded as a stable G'-map *via f*. This gives a stable (hence additive) natural transformation

$$f^*: \{X_+, S^0\}_G \longrightarrow \{X_+, S^0\}_{G'}.$$

Proposition 4.4. (i) *f^* is admissible if and only if f is a monomorphism.*

(ii) *If f is an inclusion $G' \subset G$, then the stable map $f^* \in \{BG_+, BG'_+\}$ is the transfer.*

(iii) *If f is an isomorphism, then $f^* = \sigma Bf^{-1}$.*

Proof is easy from Theorem 4.3.

Next we consider the power operation. Let $f\colon X_+\to S^0$ be a stable G-map. The smash product $f\wedge f\colon (X\times X)_+\to S^0$ can be regarded as a stable $\Sigma_2\int G$-map, where $\Sigma_2\int G$ is the wreath product. Let $\Delta(G)\subset G\times G$ be the diagonal. Then $Z_2\times G\cong Z_2\times\Delta(G)\subset\Sigma_2\int G$. Let $d\colon X\to X\times X$ be the diagonal map. Then we have a stable $Z_2\times G$-map $(f\wedge f)d\colon X_+\to S^0$, and this defines a natural transformation

$$P\colon \{X_+, S^0\}_G\longrightarrow\{X_+, S^0\}_{Z_2\times G}.$$

For a finite G-covering $p\colon \tilde{X}\to X$, $p\times p\colon \tilde{X}\times\tilde{X}\to X\times X$ is regarded as a $\Sigma_2\int G$-covering. Restricting to $d(X)\subset X\times X$, we have a $Z_2\times G$-covering over X and thus we have a natural transformation

$$P'\colon C_G(X)\longrightarrow C_{Z_2\times G}(X),$$

then the following lemma is easily verified from the property of transfers.

Lemma 4.5. *The following diagram is commutative*:

$$\begin{array}{ccc} C_G(X) & \xrightarrow{P'} & C_{Z_2\times G}(X) \\ \downarrow{\scriptstyle\omega_G} & & \downarrow{\scriptstyle\omega_{Z_2\times G}} \\ \{X_+, S^0\}_G & \xrightarrow[P]{} & \{X_+, S^0\}_{Z_2\times G}. \end{array}$$

Now let $G=Z_2^{n-1}$. Let $b\in F_2^{n-1}$ and let $R_{n-1}(b)\in M_{n-1,n}(F_2)$. Then we have a natural transformation

$$R_{n-1}(b)^*\colon \{X_+, S^0\}_{Z_2^{n-1}}\longrightarrow\{X_+, S^0\}_{Z_2^n}.$$

Both P and $R_{n-1}(b)^*$ are not admissible, but we have

Lemma 4.6. $\sum_b R_{n-1}(b)^*-P$ *is admissible.*

Proof. Define a natural transformation

$$\theta\colon \{X_+, S^0\}_{Z_2^{n-1}}\times\{X_+, S^0\}_{Z_2^{n-1}}\longrightarrow\{X_+, S^0\}_{Z_2^n}$$

by $\theta(x, y)=P(x+y)-P(x)-P(y)$. For a finite coverings, this is given by $\theta(\tilde{X}, \tilde{X}')=\tilde{X}\cdot\tilde{X}'\coprod\tilde{X}'\cdot\tilde{X}$, where $\tilde{X}\cdot\tilde{X}'=\tilde{X}\times\tilde{X}'\,|\,d(X)$. Then we easily see that θ is admissible. Consider the composition

$$q_H\colon \{X_+, BZ_{2+}^{n-1}\}\subset\{X_+, S^0\}_{Z_2^{n-1}}\xrightarrow{\sum R_{n-1}(b)^*-P}\{X_+, S^0\}_{Z_2^n}\xrightarrow{p_H}\{X_+, BW(H)_+\}$$

where p_H is the projection. To prove the lemma, it suffices to show that $q_H=0$ for all $H\neq\{e\}$. But by the above observation we see that q_H is additive, for

$$\begin{aligned} q_H(x+y)&=p_H(\sum R_{n-1}(b)^*(x+y)-P(x+y)) \\ &=p_H(\sum R_{n-1}(b)^*(x)+\sum R_{n-1}(b)^*(y)-P(x)-P(y)+\theta(x,y)) \\ &=q_H(x)+q_H(y). \end{aligned}$$

For a free Z_2^{n-1}-set S, we easily see that

$$S\times S=\text{free}+\coprod_b R_{n-1}(b)^*(S)$$

as $Z_2^n=Z_2^{n-1}\times Z_2$ set. Hence this holds for finite coverings. Then $q_H=0$ by Theorem 4.3.

§ 5. Structure of $\{L(n), L(m)\}$ and the theorem of Kuhn

First we consider $\{M(n), M(m)\}=e_m\{BZ_2^n, BZ_2^m\}e_n$. Let $\text{Mon}(n, m)\subset M_{n,m}(F_2)$ be the set of all monomorphisms. For an $A\in\text{Mon}(n, m)$ we have defined a stable map $A^*\in\{BZ_2^n, BZ_2^m\}$. Therefore for any $a\in\hat{Z}_2\text{Mon}(n, m)$ we can define a^*, for example e_n^*, $e_n'^*$, ∂_n^* and $\partial_n'^*$. By Proposition 4.4, $e_n^*=e_n'$ and $e_n'^*=e_n$.

Lemma 5.1. *Suppose that $m\leqq n-2$. Then*

$$(\hat{Z}_2\,\text{Mon}(n, m))^*e_n\equiv 0 \bmod 2$$

and if $m=n-1$, then

$$(\hat{Z}_2\,\text{Mon}(n, n-1))^*e_n\equiv\hat{Z}_2 GL_{n-1}(F_2)e_{n-1}(\tilde{T}_n'I_n)^*.$$

Proof is clear from the fact $(AB)^*=B^*A^*$ and Lemmas 1.2 and 1.6.

Lemma 5.2. *Let*

$$\theta\colon e_m\hat{Z}_2\tilde{M}_{m,n-1}(F_2)e_{n-1}\longrightarrow\{M(n), M(m)\}$$

be a homomorphism defined by $\theta(a)=a\partial_n'^*$. Then θ is a monomorphism.

Proof. Note that $e_mxe_{n-1}\partial_n'^*=e_mxe_{n-1}I_n^*e_n=e_mxe_{n-1}(\tilde{T}_nI_n)^*$. Let $C, C'\in T_n$. If $C\neq C'$ then $\text{Im}(CI_n)$ and $\text{Im}(C'I_n)$ are different. Then the lemma follows from Theorem 3.3.

Theorem 5.3. *$\{M(n), M(m)\}$ is a free $\hat{Z}_2$-module with the following basis.* (i) 0 *if $m\leqq n-3$ or $m\geqq n+2$;* (ii) $\hat{Z}_2\{\sigma_{n-2}\partial_n^*\}$ *if $m=n-2$;* (iii)

$\hat{Z}_2\{\sigma_{n-1}, \partial_{n-1}\sigma_{n-2}\partial_n'^*, \sigma_{n-1}\partial_n\partial_n'^*\}$ *if* $m=n-1$; (iv) $\hat{Z}_2\{\partial_n\sigma_{n-1}, \sigma_n\partial_{n+1}, \partial_n\partial_n'^*\}$ *if* $m=n$; (v) $\hat{Z}_2\{\partial_{n+1}\}$ *if* $m=n+1$.

Proof. By Theorem 3.3 and Lemma 5.1, we have $\{M(n), M(m)\}$ $e_m\hat{Z}_2\tilde{M}_{m,n}(F_2)e_n\oplus\mathrm{Im}(\theta)$. Then the result follows from Lemma 5.2.

Recall that $M(n)\simeq L(n)\vee L(n-1)$. Then by the dimensional reason we immediately obtain

Corollary 5.4. *There are isomorphisms*

$$\begin{aligned}\{L(n), L(m)\} &\cong \hat{Z}_2, \quad \text{if } m=n \text{ or } m=n-1\\ &\cong 0, \quad \text{otherwise.}\end{aligned}$$

A generator of $\{L(n), L(n-1)\}\cong\hat{Z}_2$ is denoted by h_n. Note that $h_n\vee h_{n-1}\colon L(n)\vee L(n-1)\to L(n-1)\vee L(n-2)$ is equivalent mod 2 to $\partial_n'^*\colon M(n)\to M(n-1)$.

Finally we give a proof of the Kuhn's theorem [5]. A sequence $\to X_{n+1}\xrightarrow{d} X_n\xrightarrow{d}\cdots$ of stable maps of 2-local spectra is called (half stable) split exact if $d\circ d=0$ and there are maps $s\colon \Omega^\infty X_n\to\Omega^\infty X_{n+1}$ for all n such that $d_*s_*+s_*d_*\equiv 1$ mod 2 in $\mathrm{End}(\pi_*(\Omega^\infty X_n))=\mathrm{End}(\pi_*^S(X_n))$ for all n. Then the sequence

$$\longrightarrow\pi_*^S(X_{n+1})\xrightarrow{d_*}\pi_*^S(X_n)\xrightarrow{d_*}\cdots$$

is clearly split exact. Let $u_n=\Omega^\infty(d)\circ s\colon \Omega^\infty X_n\to\Omega^\infty X_n$ and $v_n=s\circ\Omega^\infty(d)\colon \Omega^\infty X_n\to\Omega^\infty X_n$, then clearly u_n and v_n are π_*-idempotent mod 2 and $u_{n*}+v_{n*}\equiv 1$ mod 2. Then by Proposition 3.2 we have $\Omega^\infty X_n\simeq u_n\Omega^\infty X_n\times v_n\Omega^\infty X_n$, and easily we see that $v_n\Omega^\infty X_n\simeq u_{n-1}\Omega^\infty X_{n-1}$. In Section 3, we have shown that the sequence

$$\longrightarrow M(n+1)\xrightarrow{\sigma_n} M(n)\xrightarrow{\sigma_{n-1}} M(n-1)\longrightarrow\cdots$$

is (stable) split exact. Now the Kuhn's theorem asserts the following.

Theorem 5.5. *The sequence*

$$\longrightarrow M(n+1)\xrightarrow{\partial_{n+1}'^*} M(n)\xrightarrow{\partial_n'^*} M(n-1)\longrightarrow\cdots\longrightarrow M(1)$$

is split exact.

Proof. For any $a\in\hat{Z}_2\tilde{M}_{n,m}(F_2)$, we can define a natural transformation

$$a^*\colon \{X_+, S^0\}_{Z_2^n}\longrightarrow\{X_+, S^0\}_{Z_2^m}.$$

The relations in Section 1 and Section 2 hold for a^* as such natural transformations. Define

$$s_{n-1}=e_n'^*(\bar{R}_{n-1}^*-P)e_{n-1}'^*=e_n(\bar{R}_{n-1}^*-P)e_{n-1}\colon \{X_+, S^0\}_{Z_2^{n-1}}\longrightarrow\{X_+, S^0\}_{Z_2^n},$$

where P is the power operation. Put $\alpha_n=\partial_{n+1}'^*s_n$ and $\beta_n=s_{n-1}\partial_n'^*$. By Lemma 4.6, s_n is admissible and hence so are α_n and β_n, and $s_n\in [Q(BZ_{2+}^n), Q(BZ_{2+}^{n+1})]$. To prove the theorem it suffices to show that $\alpha_{n*}+\beta_{n*}\equiv e_{n*}$ mod 2 regarding α_n and β_n as maps in $[Q(BZ_{2+}^n), Q(BZ_{2+}^n)]$. Now we show that for any reduced element $x\in\{S^q, S^0\}_{Z_2^n}\subset\{S_+^q, S^0\}_{Z_2^n}$,

$$\alpha_n(x)+\beta_n(x)\equiv e_n(x) \bmod 2.$$

Note that $\partial_n'\equiv e_n'T_n'I_ne_{n-1}'$. Then by Lemma 1.6, we have

$$\begin{aligned}\alpha_n+\beta_n&\equiv e_n((\tilde{T}_{n+1}'I_{n+1})^*\bar{R}_n^*+\bar{R}_{n-1}^*(\tilde{T}_n'I_n)^*)e_n\\&\quad+e_n((\tilde{T}_{n+1}'I_{n+1})^*P+P(\tilde{T}_n'I_n)^*)e_n\\&\equiv e_n+e_n((\tilde{T}_{n+1}'I_{n+1})^*P+P(\tilde{T}_n'I_n)^*)e_n.\end{aligned}$$

Now let $C_i'=(1,\cdots,i)\in T_n'$, then $C_i'I_n$ is regarded as a standard inclusion $Z_2^{i-1}\times 0\times Z_2^{n-i}\to Z_2^n$. Then $(C_i'I_n)^*\colon \{X_+, S^0\}_{Z_2^n}\to\{X_+, S^0\}_{Z_2^{n-1}}$ is given by forgetting i-th Z_2-action in Z_2^n. Then by definition $I_{n+1}^*P(x)=x^2$, the cup product. Also we easily see that $(C_{i+1}'I_{n+1})^*P=P(C_i'I_n)^*$ for $i>0$. Thus we easily see

$$((\tilde{T}_{n+1}'I_{n+1})^*P+P(\tilde{T}_n'I_n)^*)(x)=x^2$$

and if $x\in\{S^q, S^0\}_{Z_2^n}$, $q>0$, then $x^2=0$ and hence $(\alpha_n+\beta_n)(x)\equiv e_n(x)$. This completes the proof.

Corollary 5.6. *The sequence*

$$\longrightarrow L(n)\xrightarrow{h_n}L(n-1)\longrightarrow\cdots\longrightarrow L(1)\xrightarrow{h_1}L(0)=(S^0)_{(2)}$$

is split exact.

Remark. As is well known (Kahn-Priddy [4]), there is a split exact sequence $L(1)\xrightarrow{h_1}L(0)\xrightarrow{h_0}HQ_{(2)}$.

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