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Whitehead Products in Stiefel Manifolds and Samelson Products in Classical Groups

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Dedicated to Professor Itiro Tamura on his 60th birthday

Abstract

The first non zero homotopy group of the Stiefel manifold $O_{n+k,k}$ of orthonormal k-frames in F^{n+k} is generated by the standard embedding $i_{n+k,k}=i^F_{n+k,k}\colon S^{d(n+1)-1}=O_{n+1,1}\to O_{n+k,k}$ where F is the field of the real numbers R, complex numbers C, or quaternions H and d is the dimension of F over R. We study the Whitehead products $[i_{n+k,k}, i_{n+k,k}]$ and $[i^C_{n+k,k}\circ\eta_{2n+1}, i^C_{n+k,k}]$ where $\eta_m\in\pi_{m+1}(S^m)$ is a generator. As consequences we determine the orders of a few Samelson products in the classical groups and obtain a relation between the stable and unstable James numbers of the complex Stiefel manifolds.

§ 1. Introduction and statement of results

Our first result is

Theorem (1.1). *If $n+1$ is not a power of 2, then the Whitehead square $[i_{n+k,k}, i_{n+k,k}]$ is non-zero for every $k\geq 1$ and it is of order 2 except the real case for n even.*

The real case for n even has been solved completely by Mahowald [19, Theorem B] as follows: $[i_{n+k,k}, i_{n+k,k}]$ is infinite cyclic and if $n\neq 2$, 4 or 8, then it generates a direct summand (see also Marcum and Randall [21, Proposition 2.1]).

By (3.6) below, our problem in the real case for $n+1$ a power of 2 is the same as Problem 4 of Jones in [22, p. 436]. In [13] J. R. Hubbuck and the author study the complex and quaternionic cases for $n+1$ a power of 2.

It is well-known that the Whitehead square $w_m=[\iota_m, \iota_m]$ of the generator $\iota_m\in\pi_m(S^m)$ vanishes if and only if $m=1$, 3 or 7. Hence $[i_{n+k,k}, i_{n+k,k}]$ $=i_{n+k,k*}(w_{d(n+1)-1})$ vanishes for every $k\geq 1$ if $n=1$, 3 or 7 in the real case, if $n=0$, 1 or 3 in the complex case, and if $n=0$ or 1 in the quaternionic

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case. As remarked by Marcum and Randall [21, p. 355], if w_n can be halved, then $[i_{n+k,k}, i_{n+k,k}]=0$ for every $k\geq 2$ in the real case. This is the case for $n=15$ [39, (10.10)] and for $n=31$ [27, (8.31)].

Bott [8] has studied Samelson products $\langle x, y\rangle$ in the classical groups where x and y are in stable range. From Theorem (1.1) we shall obtain a few results in unstable range. As it will be recalled in Section 5, the first unstable homotopy groups $\pi_{2n}(U(n))$ and $\pi_{4n+2}(Sp(n))$ are finite cyclic. Let $\alpha\in\pi_{2n}(U(n))$ and $\beta\in\pi_{4n+2}(Sp(n))$ be any generators. Then we shall prove

Theorem (1.2). *If $n+1$ is not a power of 2, then the Samelson products $\langle\alpha, \alpha\rangle\in\pi_{4n}(U(n))$ and $\langle\beta, \beta\rangle\in\pi_{8n+4}(Sp(n))$ are of order 2 and they do not depend on choice of α and β.*

Theorem (1.3). *Let n be even with $n+2$ not a power of 2, and let $\gamma\in\pi_{2n+2}(U(n))\cong Z_{(n+1)!}\oplus Z_2$ be any element of order $(n+1)!$. Then $\langle\gamma, \gamma\rangle\in\pi_{4n+4}(U(n))$ is of order 2 and it does not depend on choice of γ.*

Theorem (1.4). *Let $n\equiv 3 \bmod 4$ with $n+1$ not a power of 2, and let $\delta\in\pi_{n-1}(SO(n))\cong Z_2$ be the generator. Then $\langle\delta, \delta\rangle\in\pi_{2n-2}(SO(n))$ is of order 2.*

Mahowald [19, Theorem A] has determined the order of the Samelson square of a generator of $\pi_{n-1}(SO(n))$ for n even and $n\neq 2$, 4 or 8 (see also Lundell [18, Theorem 5.4]).

Theorem (1.5). *The followings are equivalent:*
(i) $\langle\alpha, \alpha\rangle\circ\eta_{4n}\neq 0$.
(ii) $[i^C_{n+k,k}\circ\eta_{2n+1}, i^C_{n+k,k}]\neq 0$ *for* $k=n+2$.
(iii) $[i^C_{n+k,k}\circ\eta_{2n+1}, i^C_{n+k,k}]\neq 0$ *for every* $k\geq 1$.
(iv) *n is even, positive and $n\neq 2$ or 6.*

Corollary (1.6). *Let n be even and positive. Then $\langle\alpha, \alpha\rangle\in\pi_{4n}(U(n))$ generates a direct summand of order 2, and so does $[i^C_{n+k,k}, i^C_{n+k,k}]$ for every $k\geq 1$.*

First few exceptional cases in (1.2), (1.3), (1.4) can be solved easily as follows: $\langle\alpha, \alpha\rangle=0$ for $n=1, 3$; $\langle\beta, \beta\rangle=0$ for $n=1$; $\langle\gamma, \gamma\rangle=0$ for $n=2$; $\langle\delta, \delta\rangle=0$ for $n=3, 7, 15, 31$. Moreover we can show that $\langle\alpha, \alpha\rangle=0$ for $n=7$ by [13, Proposition 1.2] and (3) of (5.10) below and that $\langle\alpha, \alpha\rangle$ generates a direct summand of order 2 for $n=5$ by ad hoc computations.

As another application of (1.5), we obtain a relation between the stable and unstable James numbers of the complex Stiefel manifolds. The following completes [28, Corollary 2.4] and [29, Theorem 2].

Theorem (1.7). *The quotient* $W\{2n+2, n+2\}/W^s\{2n+2, n+2\}$ *is* 2 *if* $n=2$ *or* 6, *and* 1 *otherwise.*

The contents of the various sections of the paper is as follows. In Section 2, by following James [15], we shall recall some facts about the Stiefel manifolds and their related spaces, and then we shall note that $i_{n+k,k}$ is a homotopy Thom class. In Section 3, by using *EHP* sequences, we shall recall that the Whitehead square of a homotopy Thom class of a vector bundle can be detected by a squaring operation under certain conditions. Then, in Section 4, we shall prove Theorem (1.1) by calculating the actions of squaring operations in a suitable space. We shall prove Theorems (1.2), (1.3) and (1.4) in Section 5. In Section 6, we shall prepare a few lemmas. Using them, we shall prove Theorems (1.5), (1.7) and Corollary (1.6) in Section 7.

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§ 2. The Stiefel manifolds and related spaces

Let E denote the suspension functor. That is, $EX=X\wedge S^1$ is the reduced suspension of a topological space X with a prescribed base point, E: $\pi_i(X)\rightarrow\pi_{i+1}(EX)$ the suspension homomorphism, and E: $\tilde{H}^i(X; M)\rightarrow \tilde{H}^{i+1}(EX; M)$ the suspension isomorphism for an abelian group M. Let E^k denote the k-fold suspension. A *map* means a continuous map which preserves base points if we consider them. We use, for simplicity, the same notation for a map and its homotopy class. Let $\eta_2\in\pi_3(S^2)$, $\nu_4\in\pi_7(S^4)$ and $\sigma_8\in\pi_{15}(S^8)$ be the elements defined and named in [39, Chapter V]; they are essentially the 2-components of the Hopf maps. Let $\eta_n=E^{n-2}\eta_2$, $\nu_n=E^{n-4}\nu_4$, $\sigma_n=E^{n-8}\sigma_8$, $\eta_n^k=\eta_n\circ\eta_{n+1}\circ\cdots\circ\eta_{n+k-1}$, and $\nu_n^2=\nu_n\circ\nu_{n+3}$. Let $Z\{a\}$ be the infinite cyclic group and $Z_m\{b\}$ the cyclic group of order m generated by a and b respectively. If $\theta\in\pi_j(X)$ and $\tau\in\pi_k(X)$, then $[\theta, \tau]\in\pi_{j+k-1}(X)$ denotes the Whitehead product of θ and τ (see [40]). Let $w_m=[\iota_m, \iota_m]\in\pi_{2m-1}(S^m)$ be the Whitehead product of the identity map ι_m of S^m with itself. It is well-known that the order of w is infinite when m is even; two when m is odd and $m\neq 1, 3, 7$; and one, that is $w_m=0$, when $m=1, 3, 7$.

We follow the notation of James (see [15] for details). We denote the real numbers by R, the complex numbers by C, and the quaternions by H. Let F be one of these fields, and d the dimension of F over R. Let F^n denote the right F-module of n-tuples of elements of F. The standard inner product (;) is defined on F^n by the formula $(x;y)=\sum \bar{x}_i y_i$ where $x=(x_1, \cdots, x_n)$, $y=(y_1, \cdots, y_n)$ and the bar denotes the conjugation in F. The group of automorphisms of F^n which preserves this product is denoted by $G_n=G(F^n)$. As usual, we write $G(R^n)=O(n)$, $G(C^n)=U(n)$, $G(H^n)=Sp(n)$. The group of rotations of R^n is denoted by $SO(n)\subset O(n)$. In F^n the vectors of length 1 form a $(dn-1)$-dimensional sphere $S(F^n)=S^{dn-1}$.

We embed F^n in F^{n+1} by adjoining zero as the last entry. Every complex n-vector $(z_1, \cdots, z_n)$ determines a real $2n$-vector $(x_1, y_1, \cdots, x_n, y_n)$ where $z_r=x_r+\sqrt{-1}\,y_r$, and similarly every quaternionic n-vector determines a complex $2n$-vector. In accordance with these conventions we have the embeddings $G_n\subset G_{n+1}$, $U(n)\subset SO(2n)$, $Sp(n)\subset U(2n)$.

For $n\geq 0$ and $k\geq 1$, $O_{n+k,k}$ denotes the Stiefel manifold of orthonormal k-frames in F^{n+k}. Following tradition we denote $O_{n+k,k}$ by $V_{n+k,k}$ in the real case, by $W_{n+k,k}$ in the complex case, and by $X_{n+k,k}$ in the quaternionic case. We identify $O_{n+k,k}$ with the homogeneous space G_{n+k}/G_n in the usual way. For example we see that $O_{k,k}=G_k$, $O_{k,1}=S(F^k)$, and $V_{n+k,k}=SO(n+k)/SO(n)$ when $n\geq 1$.

Embeddings $i\colon O_{n+k,k}\subset O_{n+k+m,k+m}$ for $m>0$, $i'\colon W_{n+k,k}\subset V_{2n+2k,2k}$, and $i'\colon X_{n+k,k}\subset W_{2n+2k,2k}$ are defined from the above embeddings of the groups. In particular, we write $i_{n+k,k}=i^F_{n+k,k}$ for the embedding $S^{d(n+1)-1}=O_{n+1,1}\subset O_{n+k,k}$.

When $1\leq m<k$, we have the fibration

$$O_{n+m,m}\subset O_{n+k,k}\xrightarrow{p} O_{n+k,k-m}. \tag{2.1}$$

As is easily seen, the embeddings defined above are compatible with each other and with p. In particular we have

Lemma (2.2). *The following diagram is commutative*:

$$\begin{tikzcd}
S^{2n+1}=W_{n+1,1} \arrow[r, "i^C_{n+k,k}"] \arrow[d, "i'"] \arrow[dd, bend right=60, "="'] & W_{n+k,k} \arrow[d, "i'"] \\
V_{2n+2,2} \arrow[r, "i"] \arrow[d, "p"] & V_{2n+2k,2k} \arrow[d, "p"] \\
S^{2n+1}=V_{2n+2,1} \arrow[r, "i^R_{2n+2k,2k-1}"] & V_{2n+2k,2k-1}.
\end{tikzcd}$$

A similar diagram consisting of $X_{*,*}$ and $W_{*,*}$ is also commutative.

First few homotopy groups of $O_{n+k,k}$ are well-known. For example, from the homotopy exact sequences of (2.1), [30, p. 260] and [15, (5.8)], we have

Proposition (2.3). (i) *The space $O_{n+k,k}$ is $(d(n+1)-2)$-connected. Excluding the real case for $n=0$ or n odd and $k\geq 2$, $\pi_{d(n+1)-1}(O_{n+k,k})$ is infinite cyclic, while it is cyclic of order 2 in the exceptional cases. For all cases the group is generated by $i_{n+k,k}$.*

(ii) *The group $\pi_{2n+2}(W_{n+k,k})$ is $Z_2\{i_{n+k,k}\circ\eta_{2n+1}\}$, if $k=1$ or n is even ≥ 2, and 0 otherwise.*

From (2.3) and the homotopy exact sequence of (2.1), we have

Lemma (2.4). *If $k\geq n+2$, then the embedding induces an isomorphism $\pi_j(O_{2n+1,n+1})\cong\pi_j(O_{n+k,k})$ for $j\leq 2d(n+1)-3$. Similarly $\pi_j(O_{2n+2,n+2})\cong \pi_j(O_{n+k,k})$ for $j\leq 2d(n+1)-3+d$ and $k\geq n+3$.*

It follows from (2.4) that we can restrict our attention to the case $k\leq n+1$ when we study $[i_{n+k,k}, i_{n+k,k}]=i_{n+k,k*}(w_{d(n+1)-1})$ and to the case $k\leq n+2$ when we study

$$[i_{n+k,k}\circ\eta_{d(n+1)-1}, i_{n+k,k}]=i_{n+k,k*}(w_{d(n+1)-1}\circ\eta_{2d(n+1)-3}).$$

It follows from (2.2) that $[i^R_{2n+2k,2k-1}, i^R_{2n+2k,2k-1}]=i^R_{2n+2k,2k-1*}w_{2n+1}=(p\circ i'\circ i^C_{n+k,k})_* w_{2n+1}=(p\circ i')_*[i^C_{n+k,k}, i^C_{n+k,k}]$, and similarly that

$$[i^C_{2n+2k,2k-1}, i^C_{2n+2k,2k-1}]=(p\circ i')_*[i^H_{n+k,k}, i^H_{n+k,k}].$$

Hence we have

Lemma (2.5). (i) If $[i^R_{2n+2k,2k-1}, i^R_{2n+2k,2k-1}]\neq 0$, *then* $[i^C_{n+k,k}, i^C_{n+k,k}]\neq 0$.
(ii) *If* $[i^C_{2n+2k,2k-1}, i^C_{2n+2k,2k-1}]\neq 0$, *then* $[i^H_{n+k,k}, i^H_{n+k,k}]\neq 0$.

The converse of (i) does not hold. In fact, if $2\leq k\leq 4$ then $[i^C_{7+k,k}, i^C_{7+k,k}]\neq 0$ by [13, Proposition 1.2] but $[i^R_{14+2k,2k-1}, i^R_{14+2k,2k-1}]=0$ as remarked in the introduction. I know nothing about the converse of (ii).

Next we see that it suffices for our purpose to study Whitehead products in the stunted quasi-projective spaces $Q_{n+k,k}$.

The topological group $S(F)$ acts on $S(F^n)$ and the orbit space $P_n=P(F^n)$ is the projective $(n-1)$-space over F. We embed $P_n\subset P_{n+k}$ in the obvious way. Let $P_{n+k,k}=P_{n+k,k}(F)$ denote the space P_{n+k}/P_n obtained from P_{n+k} by collapsing P_n to a point. When $n=0$, we make the usual convention that $P_{k,k}=P^+_k$, the union of P_k and a point space.

The quasi-projective space $Q_n=Q_n(F)$ is defined to be the image of the map ϕ: $S(F^n)\times S(F)\to G(F^n)$, where $\phi(u, q)$ is defined by $\phi(u, q)v=u(q-1)(u; v)+v$. Equivalently, Q_n may be taken as the identification space obtained from $S(F^n)\times S(F)$ by collapsing $S(F^n)\times\{1\}$ and identifying (u, q) with $(uz, z^{-1}qz)$ for all $z\in S(F)$. From this description we can see that $Q_n=P_n^+$ in the real case, and that $Q_n=EP_n^+$ in the complex case; here we take the added point as the base point of P_n^+. We embed $Q_n\subset Q_{n+k}$ in the obvious way, so that $Q_n=Q_{n+k}\cap G_n$. We define the stunted quasi-projective space $Q_{n+k,k}=Q_{n+k,k}(F)$ to be the space obtained from Q_{n+k} by collapsing Q_n to a point. We also regard $Q_{n+k,k}$ as a subspace of $O_{n+k,k}$ in the obvious way. Note that

$$Q_{n+1,1}=O_{n+1,1}=S^{d(n+1)-1}, \tag{2.6}$$

and that, when $1\leq m<k$, we have the commutative diagram:

$$\begin{array}{ccc} Q_{n+m,m} & \longrightarrow & Q_{n+k,k} \\ \downarrow & & \downarrow \\ O_{n+m,m} & \longrightarrow & O_{n+k,k} \end{array} \tag{2.7}$$

where all maps are the embeddings. We write $i_{n+k,k}=i^F_{n+k,k}$ for the embedding $S^{d(n+1)-1}=Q_{n+1,1}\subset Q_{n+k,k}$.

We shall use the following (see [15, Theorem (3.4)] for its proof).

Proposition (2.8). *The pair $(O_{n+k,k}, Q_{n+k,k})$ is $(2dn+3d-3)$-connected. Hence the embedding induces an isomorphism $\pi_j(Q_{n+k,k})\cong\pi_j(O_{n+k,k})$ for $j\leq 2dn+3d-4$.*

It follows from (2.6), (2.7) and (2.8) that we can replace $O_{*,*}$ by $Q_{*,*}$ when we study the Whitehead product

$$[i_{n+k,k}\circ x,\ i_{n+k,k}]\in\pi_{2d(n+1)-3+m}(O_{n+k,k})\quad\text{for } x\in\pi_{d(n+1)-1+m}(S^{d(n+1)-1})$$

and $0\leq m\leq d-1$. In particular the statements obtained by replacing $O_{*,*}$ with $Q_{*,*}$ from (2.2) to (2.5) are true.

We now see that $i_{n+k,k}$ is a homotopy Thom class. Recall that the canonical line bundle $L=L_F$ over $P(F^m)$ is obtained from $F\times S(F^m)$ by identifying (u, v) with (uz, vz) for all $z\in S(F)$, where $u\in F$, $v\in S(F^m)$. Let F' be the subspace of elements $u\in F$ such that $\bar{u}=-u$ and let $S(F)$ act on F' so that $z\in S(F)$ sends u into $z^{-1}uz$. The real $(d-1)$-dimensional vector bundle $L'=L'_F$ over P_m is obtained from $F'\times S(F^m)$ by factoring out the diagonal action. Note that $L'_{\boldsymbol{R}}=0$ and $L'_{\boldsymbol{C}}=1$, while $L'_{\boldsymbol{H}}$ is not trivial. We can define canonically a homeomorphism between the Thom space $P_k^{nL\oplus L'}$ of $nL\oplus L'$, over P_k, and $Q_{n+k,k}$ (see [15, Chapter 5]) so that, when $1\leq m<k$, the homeomorphisms make the following diagram commutative:

$$\begin{array}{ccc} P_m^{nL\oplus L'} & \approx & Q_{n+m,m} \\ \downarrow & & \downarrow \\ P_k^{nL\oplus L'} & \approx & Q_{n+k,k} \end{array}$$

where the vertical maps are the embeddings defined in the obvious way. Note that the embedding

$$i_{n+k,k}\colon S^{d(n+1)-1} = P_1^{nL\oplus L'} \subset P_k^{nL\oplus L'}$$

is called the homotopy Thom class of $nL\oplus L'$. Therefore our problem on $[i_{n+k,k}, i_{n+k,k}]$ is a special case of the problem of Marcum and Randall [21].

From now on we shall identify $P_k^{nL\oplus L'}$ with $Q_{n+k,k}$ by the canonical homeomorphism.

§ 3. Hopf invariants

We denote the ordinary cohomology with mod 2 coefficients by $H^*(-)$. The Steenrod squaring operation of degree k is denoted by Sq^k or $Sq(k)$.

We recall some facts about Hopf invariants. Let ξ be an m-dimensional real vector bundle over a connected finite CW-complex B, where $m \geq 2$. As usual, B^ξ denotes the Thom space of ξ, that is, the one point compactification of the total space of ξ. We take the added point as the base point of B^ξ. Using the usual cell structure of B^ξ, we can easily prove

Lemma (3.1). *The Hurewicz homomorphisms* $\pi_m(B^\xi) \to H_m(B^\xi; Z)$ *and* $\pi_{2m+1}(E(B^\xi \wedge B^\xi)) \to H_{2m+1}(E(B^\xi \wedge B^\xi); Z)$ *are isomorphisms and all groups are isomorphic to* Z *or* Z_2 *according as* ξ *is orientable or not.*

Choose any point $*$ of B. Let $i\colon S^m = *^\xi \subset B^\xi$ be the embedding, which is called the homotopy Thom class of ξ. To study $[i, i] = i_* w_m \in \pi_{2m-1}(B^\xi)$, we use EHP-sequences as W. A. Sutherland has done in [35].

The embedding i induces a homomorphism between parts of the exact EHP-sequences (see [40, p. 548]):

(3.2)
$$\begin{array}{ccccccc} & & & & \pi_{2m+1}(E(*^\xi \wedge *^\xi)) & \xrightarrow{P} & \pi_{2m-1}(*^\xi) \\ & & & & \downarrow{\scriptstyle E(i\wedge i)_*} & & \downarrow{\scriptstyle i_*} \\ \pi_{2m}(B^\xi) & \xrightarrow[E]{} & \pi_{2m+1}(EB^\xi) & \xrightarrow[H]{} & \pi_{2m+1}(E(B^\xi \wedge B^\xi)) & \xrightarrow[P]{} & \pi_{2m-1}(B^\xi). \end{array}$$

By (3.1) we define

$$H_2\colon \pi_{2m+1}(EB^\xi) \longrightarrow Z_2$$

to be H or its mod 2 reduction according as ξ is non-orientable or orientable.

Note that $E(*^{\xi}\wedge *^{\xi})=S^{2m+1}$. Since $\pi_{2m+1}(E(B^{\xi}\wedge B^{\xi}), S^{2m+1})=0$ by Blakers-Massey Theorem (see (7.12) or [40] at p. 368), it follows that $E(i\wedge i)_{*}$ is surjective and hence the cyclic group $\pi_{2m+1}(E(B^{\xi}\wedge B^{\xi}))$ is generated by $E(i\wedge i)_{*}\iota_{2m+1}$. On the other hand, from commutativity of (3.2), we have that $[i, i]=P(E(i\wedge i)_{*}\iota_{2m+1})$. Thus we have

Lemma (3.3). *The order of* $[i, i]$ *is odd if and only if there exists a map* $f\colon S^{2m+1}\rightarrow EB^{\xi}$ *with* $H_2(f)\neq 0$. *In particular, when m is odd or* ξ *is non-orientable, the latter is the necessary and sufficient condition for vanishing of* $[i, i]$.

From Theorem 5.14 and Corollary 5.15 of [4], we have

Lemma (3.4). *Let* $f\colon S^{2m+1}\rightarrow EB^{\xi}$ *be a map which induces the zero homomorphism from* $H^{2m+1}(EB^{\xi})$ *to* $H^{2m+1}(S^{2m+1})$, *and* C_f *the mapping cone of f. Then* $H_2(f)$ *is non-zero if and only if* $Sq^{m+1}\colon H^{m+1}(C_f)\rightarrow H^{2m+2}(C_f)$ *is non-zero.*

Next we consider the special case: $B=P_k$, $\xi=nL\oplus L'$ with $d(n+1)\geq 3$. First we prove

Lemma (3.5). *If* $f\colon S^{2d(n+1)-1}\rightarrow EQ_{n+k,k}$ *is a map and* $n\geq 1$, *then* $f^{*}\colon \tilde{H}^{*}(EQ_{n+k,k}; M)\rightarrow \tilde{H}^{*}(S^{2d(n+1)-1}; M)$ *is zero for any abelian group M.*

Proof. Recall from [15, p. 25] that $EQ_{n+k,k}$ has a cell structure of $S^{dn}\cup e^{d(n+1)}\cup\cdots\cup e^{d(n+k)}$. Thus the results for the complex and quaternionic cases follow from dimensional reason.

We consider the real case. By the universal coefficient theorem it suffices to prove the assertion for $M=Z$. It can be shown that $H^{2n}(Q_{n+k,k}; Z)$ $(\cong H^{2n+1}(EQ_{n+k,k}; Z))$ is Z_2 or zero. Thus $f^{*}\colon H^{2n+1}(EQ_{n+k,k}; Z)\rightarrow H^{2n+1}(S^{2n+1}; Z)$ is zero for any map $f\colon S^{2n+1}\rightarrow EQ_{n+k,k}$. This completes the proof.

From (3.3), (3.4) and (3.5), we have

Proposition (3.6). *Let* $d(n+1)\geq 3$. *Then the order of* $[i_{n+k,k}, i_{n+k,k}]\in\pi_{2d(n+1)-3}(Q_{n+k,k})$ *is odd if and only if there exists a map* $f\colon S^{2d(n+1)-1}\rightarrow EQ_{n+k,k}$ *such that* $Sq^{d(n+1)}\colon H^{d(n+1)}(C_f)\rightarrow H^{2d(n+1)}(C_f)$ *is non-zero. In particular, excluding the real case for n even, the latter is the necessary and sufficient condition for vanishing of* $[i_{n+k,k}, i_{n+k,k}]$.

Remark. The "only if" part of (3.6) can be proved by [14, § 3].

§ 4. Proof of Theorem (1.1)

The following implies Theorem (1.1) by the observations in Section 2.

Theorem (4.1). *If $n+1$ is not a power of 2, then the order of $[i_{n+k,k}, i_{n+k,k}] \in \pi_{2d(n+1)-3}(Q_{n+k,k})$ is even or infinite.*

This follows immediately from (3.6) and

Lemma (4.2). *If $n+1$ is not a power of 2, then $Sq^{d(n+1)}: H^{d(n+1)}(C_f) \to H^{2d(n+1)}(C_f)$ is zero for every map $f: S^{2d(n+1)-1} \to EQ_{n+k,k}$.*

The rest of this section is devoted to the proof of (4.2).

Let A and $A(k)$ denote the mod 2 Steenrod algebra [34] and its k-dimensional component respectively. The following is well-known (see [36, (4.2)]).

Lemma (4.3). *If s and m are non-negative integers, then*

$$Sq(2^{s+1}m+2^s) \equiv 0 \bmod \sum_{i=0}^{s} Sq(2^i)A(2^{s+1}m+2^s-2^i).$$

Remark. We can prove a result stronger than (4.3): if $s \geq 1$ and $m \geq 0$, then

$$Sq(2^{s+1}m+2^s) \equiv Sq(2^s)Sq(2^{s+1}m) \bmod \sum_{i=1}^{2^s-1} A(i)Sq(2^{s+1}m)A(2^s-i).$$

A slightly weaker form of this was shown to the author by W.A. Sutherland.

Proof of (4.2). From (2.4) and (2.5), or correctly speaking, from their corresponding lemmas for $Q_{*,*}$, it suffices to prove (4.2) for $k=n+1$ in the real case. Hence we consider the real case only.

Let $f: S^{2n+1} \to EQ_{2n+1,n+1}$ be a map. Then, since $f^*: H^{2n+1}(EQ_{2n+1,n+1}) \to H^{2n+1}(S^{2n+1})$ is zero by (3.5), it follows from the cohomology exact sequence of the cofibration

$$S^{2n+1} \xrightarrow{f} EQ_{2n+1,n+1} \xrightarrow{i} C_f \xrightarrow{q} S^{2n+2} \longrightarrow \cdots$$

that $q^*: H^{2n+2}(S^{2n+2}) \to H^{2n+2}(C_f)$ and $i^*: H^j(C_f) \to H^j(EQ_{2n+1,n+1})$ for $j \neq 2n+2$ are isomorphisms. Let $x(r) \in H^r(C_f)$ be the unique non-zero element for $n+1 \leq r \leq 2n+2$. Set $n+1 = 2^s(2m+1)$. Then $m \geq 1$ by the hypothesis. I assert that

$$(4.4) \qquad Sq(2^j)x(2n+2-2^j) = 0 \quad \text{for } 0 \leq j \leq s.$$

This implies that $Sq^{n+1}x(n+1)=0$. For we have $Sq^{n+1}=\sum_{j=0}^{s}Sq(2^j)\lambda_j$ for some $\lambda_j \in A(n+1-2^j)$ by (4.3), and so

$$Sq^{n+1}x(n+1)=\sum_{j=0}^{s}Sq(2^j)\lambda_j x(n+1)=0 \quad \text{by (4.4).}$$

Before proving (4.4), we prove

(4.5) $$Sq(2^j)x(2n+2-2^{j+1})=x(2n+2-2^j) \quad \text{for } 0\leq j\leq s.$$

Recall that $Q_{2n+1,2n}=P(R^{2n+1})$, the $2n$-dimensional real projective space, hence $H^*(Q_{2n+1,2n})=Z_2[t]/(t^{2n+1})$, and $Sq^r t^k=C(k-r, r)t^{k+r}$, where the dimension of t is one and $C(a, b)=(a+b)!/(a!\cdot b!)$. Also recall that the quotient map $q\colon Q_{2n+1,2n}\rightarrow Q_{2n+1,n+1}$ induces an isomorphism

$$q^*\colon H^r(Q_{2n+1,n+1})\longrightarrow H^r(Q_{2n+1,2n}) \quad \text{for } n+1\leq r\leq 2n.$$

For simplicity, we denote $q^{*-1}(t^r)$ by $t(r)$ for $n+1\leq r\leq 2n$; they satisfy the equation $Sq^r t(k)=C(k-r, r)t(k+r)$ for $n+1\leq k+r\leq 2n$. We then have

$$\begin{aligned} Sq(2^j)x(2n+2-2^{j+1}) &= Sq(2^j)i^{*-1}Et(2n+1-2^{j+1}) \\ &= i^{*-1}ESq(2^j)t(2n+1-2^{j+1}) \\ &= C(2n+1-2^{j+1}-2^j, 2^j)i^{*-1}Et(2n+1-2^j) \\ &= i^{*-1}Et(2n+1-2^j) \\ &= x(2n+2-2^j). \end{aligned}$$

Here the fourth equality is a consequence of the equation $C(2n+1-2^{j+1}-2^j, 2^j)\equiv 1 \bmod 2$, which follows from [34, Lemma 2.6] and the hypothesis $m\geq 1$.

Now we prove (4.4) by an induction on j. For $j=0$, we have

$$\begin{aligned} Sq^1x(2n+1) &= Sq^1Sq^1x(2n), \quad \text{by (4.5),} \\ &= 0. \end{aligned}$$

Suppose inductively that (4.4) holds with j replaced by any $k<j$. Then

$$\begin{aligned} &Sq(2^j)x(2n+2-2^j) \\ &\quad = Sq(2^j)Sq(2^j)x(2n+2-2^{j+1}), \quad \text{by (4.5),} \\ &\quad = \sum_{r=0}^{j-1} Sq(2^{j+1}-2^r)Sq(2^r)x(2n+2-2^{j+1}), \quad \text{by the Adem relation,} \\ &\quad \equiv 0 \bmod \sum_{r=0}^{j-1}\sum_{i=0}^{r} Sq(2^i)A(2^{j+1}-2^r-2^i)Sq(2^r)x(2n+2-2^{j+1}), \quad \text{by (4.3),} \\ &\quad \equiv 0 \bmod 0, \quad \text{by the inductive hypothesis.} \end{aligned}$$

Therefore we complete the inductive step and hence the proofs of (4.4) and (4.2).

§ 5. Proofs of Theorems (1.2), (1.3) and (1.4)

Before proving the theorems we make preparations.

Let Δ: $\pi_i(B) \to \pi_{i-1}(F)$ denote the boundary homomorphism of a fibration $F \to X \to B$. The following is well-known (see [17, Lemma 1]).

Lemma (5.1). *If $a \in \pi_{i+1}(B)$ and $b \in \pi_j(S^i)$, then $\Delta(a \circ Eb) = \Delta(a) \circ b$.*

Let G be a topological group. We take always the unit element of G as the base point. If $a \in \pi_i(G)$ and $b \in \pi_j(G)$ with $i, j \geq 1$, then $\langle a, b\rangle \in \pi_{i+j}(G)$ denotes the Samelson product of a and b (see [15] or [40]). Soppose given a principal G-bundle $G \to X \to B$. Then Samelson [31, § 7] (see also [3] or [40, p. 476]) has proven

Proposition (5.2). *If $a \in \pi_i(B)$ and $b \in \pi_j(B)$ with $i, j \geq 2$, then $\Delta[a, b] = (-1)^{i-1}\langle \Delta a, \Delta b\rangle$.*

We shall apply (5.2) to the principal G_n-bundle $G_n \subset G_{n+k} \to O_{n+k,k}$. The boundary homomorphism of this bundle is denoted by

$$\Delta_k = \Delta_k^F : \pi_i(O_{n+k,k}) \longrightarrow \pi_{i-1}(G_n).$$

We list up some well-known results on the homotopy groups of the classical groups.

(5.3) ([7]). If $0 \leq j \leq n-2$, then $\pi_j(O(n))$ is Z for $j \equiv 3 \bmod 4$; Z_2 for $j \equiv 0, 1 \bmod 8$; and 0 otherwise.

(5.4) ([6, 7, 38]). If $0 \leq j \leq 2n-1$, then $\pi_j(U(n))$ is Z for $j \equiv 1 \bmod 2$, and 0 for $j \equiv 0 \bmod 2$.

(5.5) ([7]). If $0 \leq j \leq 4n+1$, then $\pi_j(Sp(n))$ is Z for $j \equiv 3 \bmod 4$; Z_2 for $j \equiv 4, 5 \bmod 8$; and 0 otherwise.

(5.6) ([17]). The group $\pi_{n-1}(O(n))$ is $Z\{\Delta_1 \iota_n\} \oplus Z$ for $n \equiv 0 \bmod 4$ and $n \geq 4$; $Z\{\Delta_1 \iota_n\} \oplus Z_2$ for $n \equiv 2 \bmod 8$ and $n \geq 10$; $Z\{\Delta_1 \iota_n\}$ for $n \equiv 6 \bmod 8$; $Z_2\{\Delta_1 \iota_n\} \oplus Z_2$ for $n \equiv 1 \bmod 8$ and $n \geq 9$; $Z_2\{\Delta_1 \iota_n\}$ for $n \equiv 5 \bmod 8$ or $n \equiv 3 \bmod 4$ and $n \geq 11$; $Z\{(\Delta_1 \iota_2)/2\}$ for $n=2$; and 0 for $n=3$ or 7.

(5.7) ([5, 6, 38]). We have that $\pi_{2n}(U(n)) = Z_{n!}\{\Delta_1 \iota_{2+1}\}$. The group $\pi_{2n+1}(U(n))$ is $Z_2\{\Delta_1 \eta_{2n+1}\}$ for $n \equiv 0 \bmod 2$ and $n \geq 2$; and 0 otherwise. The group $\pi_{2n+2}(U(n))$ is $Z_{(n+1)!} \oplus Z_2$ for $n \equiv 0 \bmod 2$ and $n \geq 4$; $Z_{(n+1)!/2}$ for $n \equiv 1 \bmod 2$; and Z_2 for $n=2$.

(5.8) ([10]). The group $\pi_{4n+2}(Sp(n))$ is $Z_{(2n+1)!}\{\Delta_1 \iota_{4n+3}\}$ for $n \equiv 0 \bmod 2$;

and $Z_{(2n+1)!2}\{\Delta_1\iota_{4n+3}\}$ for $n\equiv 1 \bmod 2$.

If m is odd, then $\pi_r(S^m)$ is finite for all $r>m$, as shown by Serre [33, p. 494]. From the homotopy exact sequences of the fibrations $G_{u-1}\to G_u\to S^{du-1}$, $2\leq u\leq n$, it follows that

(5.9) $\pi_{dn+i}(G_n)$ is finite for all $i\geq 0$ unless the case is real.

Let $\alpha\in\pi_{2n}(U(n))$, $\beta\in\pi_{4n+2}(Sp(n))$ and $\delta\in\pi_{n-1}(O(n))$ be generators; where we assume that $n\equiv 3 \bmod 4$ in the real case.

Lemma (5.10).

(1) *$\langle\alpha,\alpha\rangle=\langle\Delta_1^C\iota_{2n+1},\Delta_1^C\iota_{2n+1}\rangle$ and $\langle\beta,\beta\rangle=\langle\Delta_1^H\iota_{4n+3},\Delta_1^H\iota_{4n+3}\rangle$.*

(2) *If $2k\geq 2n+2+i$ and $i\geq 0$, then restricted to the torsion subgroup, $\Delta_k\colon\pi_{4n+i+1}(W_{n+k,k})\to\pi_{4n+i}(U(n))$ is injective.*

(3) *If $x\in\pi_{2n+j}(S^{2n})$, then $\Delta_k^C[i_{n+k,k}\circ Ex, i_{n+k,k}]=\langle\alpha,\alpha\rangle\circ E^{2n}x$.*

Proof. Unless the case is real, the order of $w_{d(n+1)-1}$ is at most 2, and hence so is $\langle\Delta_1\iota_{d(n+1)-1},\Delta_1\iota_{d(n+1)-1}\rangle$ by (5.2). Therefore (1) follows from (5.7) and (5.8).

From (5.4), (5.9) and the homotopy exact sequence of $U(n)\subset U(n+k)\to W_{n+k,k}$, (2) follows.

It follows that $\Delta_k^C[i_{n+k,k}, i_{n+k,k}]=\Delta_k i_{n+k,k*}w_{2n+1}=\Delta_1 w_{2n+1}=\langle\Delta_1\iota_{2n+1},\Delta_1\iota_{2n+1}\rangle=\langle\alpha,\alpha\rangle$ from (5.2) and (1), and that $[i_{n+k,k}\circ Ex, i_{n+k,k}]=[i_{n+k,k}, i_{n+k,k}]\circ E^{2n+1}x$ for $x\in\pi_{2n+j}(S^{2n})$ from (8.18) at p. 484 of [40]. Hence $\Delta_k[i_{n+k,k}\circ Ex, i_{n+k,k}]=\Delta_k[i_{n+k,k}, i_{n+k,k}]\circ E^{2n}x=\langle\alpha,\alpha\rangle\circ E^{2n}x$ by (5.1) and (1). Thus (3) follows.

Now we begin to prove the theorems. The above (1) implies the second half of Theorem (1.2). The first half of (1.2) in the complex case follows from (2), (3) and Theorem (1.1).

By the same argument as the complex case, I could not obtain any result for the real and quaternionic cases except the following: if $n\equiv 3 \bmod 4$ and $k\geq n+1$, then $\pi_{2n-1}(O(n+k))=0$ by (5.3) so that

$$\Delta_k\colon\pi_{2n-1}(V_{n+k,k})\longrightarrow\pi_{2n-2}(O(n))$$

is injective. Since $\Delta_k[i_{n+k,k}, i_{n+k,k}]=\langle\Delta_1\iota_n,\Delta_1\iota_n\rangle$ by the same proof as (3) above, it follows that $\langle\Delta_1\iota_n,\Delta_1\iota_n\rangle\neq 0$ if and only if $[i_{n+k,k}, i_{n+k,k}]\neq 0$ for some, hence every, $k\geq n+1$. Therefore Theorem (1.1) implies Theorem (1.4), since $\Delta_1\iota_n=\delta$ by (5.6).

To settle the quaternionic case, we consider the commutative diagram:

$$\begin{array}{ccccc} Sp(n) & \longrightarrow & Sp(n+1) & \xrightarrow{p} & S^{4n+3} \\ \downarrow & & \downarrow & & \downarrow = \\ U(2n+1) & \longrightarrow & U(2n+2) & \xrightarrow{p} & S^{4n+3} \end{array}$$

where the unlabeled maps are the embeddings defined in the obvious way. From this we obtain the commutative diagram:

$$\begin{tikzcd} \pi_{8n+5}(S^{4n+3}) \arrow[r, "\Delta_1^H"] \arrow[d, "="] & \pi_{8n+4}(Sp(n)) \arrow[d] \\ \pi_{8n+5}(S^{4n+3}) \arrow[r, "\Delta_1^C"] & \pi_{8n+4}(U(2n+1)). \end{tikzcd}$$

It follows from (5.2) and (1) of (5.10) that the right vertical map sends $\langle\beta, \beta\rangle$ into $\langle\Delta_1^C\iota_{4n+3}, \Delta_1^C\iota_{4n+3}\rangle$. Thus, if the latter is non-zero, then so is $\langle\beta, \beta\rangle$. Therefore Theorem (1.1) and the first half of (1.2) in the complex case imply the first half of (1.2) in the quaternionic case. This completes the proof of Theorem (1.2).

Next we prove Theorem (1.3). Let $n\geq 1$. Consider the exact sequence:

$$\pi_{2n+3}(S^{2n+1})\xrightarrow{\Delta_1}\pi_{2n+2}(U(n))\xrightarrow{i_*}\pi_{2n+2}(U(n+1))\longrightarrow$$
$$\pi_{2n+2}(S^{2n+1})\xrightarrow{\Delta_1}\pi_{2n+1}(U(n)).$$

Since $\pi_{2n+3}(S^{2n+1})=Z_2\{\eta_{2n+1}^2\}$ and $\pi_{2n+2}(S^{2n+1})=Z_2\{\eta_{2n+1}\}$ (see [39] for example), it follows from (5.7) i_* is surjective if and only if n is even, and that $\Delta_1\eta_{2n+1}^2$ generates a direct summand for n even≥ 4.

Suppose that n is even with $n+2$ not a power of 2. Let $\gamma\in\pi_{2n+2}(U(n))\cong Z_{(n+1)!}\oplus Z_2$ be any element of order $(n+1)!$. Then $i_*\gamma$ generates $\pi_{2n+2}(U(n+1))\cong Z_{(n+1)!}$ as seen above. Thus $\langle i_*\gamma, i_*\gamma\rangle=i_*\langle\gamma, \gamma\rangle$ is non-zero by Theorem (1.2) and so is $\langle\gamma, \gamma\rangle$. By (2.2), i': $S^{2n+3}=X_{n/2+1,1}\subset W_{n+2,2}$ is a cross-section of p: $W_{n+2,2}\to W_{n+2,1}=S^{2n+3}$, so that $\pi_{2n+3}(W_{n+2,2})$ is a direct sum of the images of i_* and the upper i'_* in the following commutative diagram:

$$\begin{tikzcd} Z_2\{\eta_{2n+1}^2\}=\pi_{2n+3}(W_{n+1,1}) \arrow[r, "i_*"] \arrow[d, "\Delta_1^C"] & \pi_{2n+3}(W_{n+2,2}) \arrow[d, "\Delta_2^C"] & \pi_{2n+3}(X_{n/2+1,1})=Z\{\iota_{2n+3}\} \arrow[l, "i'_*"'] \arrow[d, "\Delta_1^H"] \\ Z_{(n+1)!}\oplus Z_2\cong\pi_{2n+2}(U(n)) \arrow[r, equal] & \pi_{2n+2}(U(n)) \arrow[d] & \pi_{2n+2}(Sp(n/2)) \arrow[l, "i'_*"'] \\ & \pi_{2n+2}(U(n+2))=0. & \end{tikzcd}$$

Since $\Delta_1^C\eta_{2n+1}^2$ generates a direct summand of order 2 as seen above, $\gamma_0=\Delta_2^C i'_*\iota_{2n+3}$ is of order $(n+1)!$. By (5.8), $\Delta_1^H\iota_{2n+3}$ generates $\pi_{2n+2}(Sp(n/2))$. Thus, by Theorem (1.2), $\langle\Delta_1^H\iota_{2n+3}, \Delta_1^H\iota_{2n+3}\rangle$ is of order 2, and it is sent into $\langle\gamma_0, \gamma_0\rangle$ under the lower i'_*, so that the order of $\langle\gamma_0, \gamma_0\rangle$ is at most 2 and hence just 2.

We have

$$\begin{aligned}\langle \Delta_1^C \eta_{2n+1}^2, \Delta_1^C \eta_{2n+1}^2 \rangle &= \Delta_1^C [\eta_{2n+1}^2, \eta_{2n+1}^2], \quad \text{by (5.2)},\\ &= \Delta_1^C (w_{2n+1} \circ \eta_{4n+1}^4), \quad \text{by [40, p. 484] and [2, (2.12)]},\\ &= 0, \quad \text{since } \eta_{4n+1}^4 = 0 \text{ by [39]},\end{aligned}$$

and, by [15, (15.2)],

$$\langle \gamma_0, \Delta_1^C \eta_{2n+1}^2 \rangle = -\langle \Delta_1^C \eta_{2n+1}^2, \gamma_0 \rangle.$$

Therefore the biadditivity of the Samelson product implies

$$\langle \gamma_0 + \Delta_1^C \eta_{2n+1}^2, \gamma_0 + \Delta_1^C \eta_{2n+1}^2 \rangle = \langle \gamma_0, \gamma_0 \rangle.$$

Since any γ of order $(n+1)!$ is an odd multiple of either γ_0 or $\gamma_0 + \Delta_1^C \eta_{2n+1}^2$, we have $\langle \gamma, \gamma \rangle = \langle \gamma_0, \gamma_0 \rangle$. This completes the proof of Theorem (1.3).

§ 6. Lemmas

We prepare several lemmas which will be used in the proofs of (1.5), (1.6) and (1.7) in Section 7.

Let $\tilde{i}: G_n \to O(dn)$ denote the identity map in the real case and the embeddings $i': U(n) \subset O(2n)$ and $i'': Sp(n) \subset U(2n) \subset O(4n)$ in the other case. Let $J: \pi_r(O(n)) \to \pi_{r+n}(S^n)$ be the J-homomorphism (see [39] for example). James and Whitehead [16, pp. 200–201] has shown

$$(6.1) \qquad w_{d(n+1)-1} = (-1)^d E^{d-1} J \tilde{i}_* \Delta_1 \iota_{d(n+1)-1}.$$

We then have

Lemma (6.2). *Exclude the real case for n even. If $x \in \pi_{d(n+1)-2}(G_n)$ and moreover if x is a multiple of $\Delta_1 \iota_n$ in the real case, then*

$$\langle x, x \rangle = x \circ E^{d-2} J \tilde{i}_* x$$

where $E^{-1} y$ means an element z with $Ez = y$.

Proof. Unless the case is real, $\pi_{d(n+1)-2}(G_n)$ is a cyclic group generated by $\Delta_1 \iota_{d(n+1)-1}$ by (5.7) and (5.8). Thus $x = k \Delta_1 \iota_{d(n+1)-1}$ for some integer k in any case.

In the real case, since n is odd, w_n is desuspendable by [1, Corollary 1.3].

For simplicity, set $\iota = \iota_{d(n+1)-1}$. Then

$$\begin{aligned}\langle x, x\rangle &= k^2\Delta_1 w_{d(n+1)-1}\\ &= (-1)^d k^2\Delta_1 E E^{d-2} J\tilde{i}_*\Delta_1\iota, \quad \text{by (6.1)},\\ &= (-1)^d k^2(\Delta_1\iota\circ E^{d-2}J\tilde{i}_*\Delta_1\iota), \quad \text{by (5.1)},\\ &= ((-1)^d k\Delta_1\iota)\circ E^{d-2}J\tilde{i}_*(k\Delta_1\iota), \quad \text{by (8.3) at p. 479 of [40]},\\ &= x\circ E^{d-2}J\tilde{i}_* x, \quad \text{since } -\Delta_1^R\iota=\Delta_1^R\iota \quad \text{by (5.6)}.\end{aligned}$$

This completes the proof.

Although the following is well-known, I could not find its precise proof anywhere, so we give it for completeness.

Lemma (6.3). *If n is even, then $HJi'_*\Delta_1^C\iota_{2n+1}=\eta_{4n-1}$ and $HJi''_*\Delta_1^H\iota_{4n+3} \equiv \nu_{8n-1} \bmod 2\pi_{8n+2}(S^{8n-1})$. Here H denotes the generalized Hopf homomorphism, as usual.*

Proof. By EHP sequences, we see that the kernel of the suspension $E: \pi_{4n}(S^{2n})\to\pi_{4n+1}(S^{2n+1})$ is generated by $w_{2n}\circ\eta_{4n-1}$ which is sent, under $H: \pi_{4n}(S^{2n})\to\pi_{4n}(S^{4n-1})$, into zero, since $H(w_{2n}\circ\eta_{4n-1})=H(w_{2n})\circ\eta_{4n-1}=(\pm 2\iota_{4n-1})\circ\eta_{4n-1}=0$. Thus $H\tau$ is the same element for all τ with $E\tau=w_{2n+1}$. On the other hand there exists a of order 2 such that $Ea=w_{2n+1}$ and $Ha=\eta_{4n-1}$ by [39, Proposition 11.10]. Therefore $HJi'_*\Delta_1^C\iota_{2n+1}=Ha=\eta_{4n-1}$ by (6.1).

For the quaternionic case, consider the diagram:

$$\begin{tikzcd}
\pi_{8n+4}(S^{8n+1}) \arrow[d, "P"] & \pi_{8n+5}(S^{8n+3}) \arrow[d, "P"] & \pi_{8n+6}(S^{8n+5}) \arrow[d, "P"] & \\
\pi_{8n+2}(S^{4n}) \arrow[r, "E"] \arrow[d, "H"] & \pi_{8n+3}(S^{4n+1}) \arrow[r, "E"] & \pi_{8n+4}(S^{4n+2}) \arrow[r, "E"] & \pi_{8n+5}(S^{4n+3}) \\
\pi_{8n+2}(S^{8n-1}). & & &
\end{tikzcd}$$

We first determine $H(\text{Kernel }(E^3))$. Since the kernel of each E is the image of the appropriate P, it is generated by $w_{4n}\circ g_{8n-1}$, $w_{4n+1}\circ\eta^2_{8n+1}$ and $w_{4n+2}\circ\eta_{8n+3}$ respectively. Here $g_4: S^7\to S^4$ is the Hopf map and $g_m=E^{m-4}g_4$. By the first part of this proof and [26, 4.19] (see also [9]), there exist $a\in\pi_{8n}(S^{4n})$ and $b\in\pi_{8n-1}(S^{4n-3})$ such that $Ea=w_{4n+1}$, $Ha=\eta_{8n-1}$ and $E^5b=w_{4n+2}\circ\eta_{8n+3}$. Thus the kernel of E^3 is generated by

$$w_{4n}\circ g_{8n-1},\ a\circ\eta^2_{8n},\ E^3b$$

which are sent, under H, into

$$\pm 2g_{8n-1},\ \eta^3_{8n-1}=4\nu_{8n-1},\ 0$$

respectively. Since g_{8n-1} generates $\pi_{8n+2}(S^{8n-1})$ and $g_{8n-1}\equiv\nu_{8n-1}$ mod $2\pi_{8n+2}(S^{8n-1})$, we then have $H(\text{Kernel}(E^3))=2\pi_{8n+2}(S^{8n-1})$.

On the other hand, by [37, 3.13], there exists $c\in\pi_{8n+2}(S^{4n})$ of order 8 such that $E^3c=w_{4n+3}$ and $Hc=\nu_{8n-1}$. Hence $Ji''_*\Delta_1^H\iota_{4n+3}-c$ is contained in Kernel (E^3) by (6.1). Therefore

$$\begin{aligned}HJi''_*\Delta_1^H\iota_{4n+3}&\equiv Hc \bmod H(\text{Kernel}(E^3))\\&\equiv\nu_{8n-1} \bmod 2\pi_{8n+2}(S^{8n-1}).\end{aligned}$$

This completes the proof of (6.3).

Lemma (6.4). *Let $n\geq 1$ and exclude the real case for n even. Then $Ep_*\Delta_1^F w_{d(n+1)-1}$ is equal to 0, $[(n+1)\eta_{2n}^2, \iota_{2n}]$ or $[(n+1)\nu_{4n}^2, \iota_{4n}]$ according as F is R, C or H. Here p: $G_n\to O_{n,1}=S^{dn-1}$ is the canonical fibration.*

Proof. We have the commutatve diagram:

$$\begin{array}{ccc}\pi_{2d(n+1)-3}(S^{d(n+1)-1}) & \xrightarrow{\Delta_2^F} & \pi_{2d(n+1)-4}(S^{dn-1})\\ \Delta_1^F\downarrow & & \downarrow =\\ \pi_{2d(n+1)-4}(G_n) & \xrightarrow{p_*} & \pi_{2d(n+1)-4}(S^{dn-1}).\end{array}$$

Consider first the real case. From [30], $\pi_{n-1}(V_{n+1,2})$ is Z if n is odd, and Z_2 if n is even. It then follows from the homotopy exact sequence of $S^{n-1}\to V_{n+1,2}\to S^n$ that $\Delta_2\iota_n$ is $\pm 2\iota_{n-1}$ if n is even, and 0 if n is odd. Hence $p_*\Delta_1 w_n=\Delta_2 w_n=\Delta_2\iota_n\circ E^{-1}w_n=0$ for n odd.

Next we consider the complex case. The result for $n=1$ is true, since $w_3=0$ and $[2\eta_2^2, \iota_2]\in 2\pi_5(S^2)=0$ by [39]. Hence we assume that $n\geq 2$. By (5.4) and (5.7), we see easily that $\pi_{2n}(W_{n+1,2})$ is Z_2 if n is odd ≥ 3, and 0 otherwise. It then follows from the homotopy exact sequence of $S^{2n-1}\to W_{n+1,2}\to S^{2n+1}$ that

$$\Delta_2\iota_{2n+1}=(n+1)\eta_{2n-1}=\begin{cases}\eta_{2n-1} & \text{if } n \text{ is even}\\ 0 & \text{if } n \text{ is odd.}\end{cases}\tag{6.5}$$

Thus we have

$$\begin{aligned}Ep_*\Delta_1 w_{2n+1}&=E\Delta_2 w_{2n+1}\\&=E\Delta_2 EJi'_*\Delta_1\iota_{2n+1}, &&\text{by (6.1)},\\&=E(\Delta_2\iota_{2n+1}\circ Ji'_*\Delta_1\iota_{2+1}), &&\text{by (5.1)},\\&=E\Delta_2\iota_{2n+1}\circ w_{2n+1}, &&\text{by (6.1)},\\&=(n+1)\eta_{2n}\circ w_{2n+1}, &&\text{by (6.5)},\end{aligned}$$

$$\begin{aligned}&=[(n+1)\eta_{2n}, (n+1)\eta_{2n}]\\&=[(n+1)^2\eta_{2n}^2, \iota_{2n}], \quad \text{by [40, p. 484] and [2, (2.12)],}\\&=[(n+1)\eta_{2n}^2, \iota_{2n}], \quad \text{since } 2\eta_{2n}^2=0.\end{aligned}$$

Finally we consider the quaterinionic case. The result for $n=1$ is true, since $w_7=0$ and $[2\nu_4^2, \iota_4] \in 2\pi_{13}(S^4)=0$ by [39]. Thus we assume that $n\geq 2$. From [24], $\pi_{4n+1}(Sp(n-1))$ is $Z_{(24,n+1)}\oplus Z_2$ if n is odd, and $Z_{(24,n+1)}$ if n is even, where (s, t) denotes the greatest common divisor of s and t. By using exact homotopy sequences of reasonable fibrations, we can see that $\pi_{4n+2}(X_{n+1,2})\cong Z_{(24,n+1)}$ and $\Delta_2\iota_{4n+3}=(24, n+1)rg_{4n-1}$ for some r with $(r, 2)=(r, 3)=1$. Note from [39] that $g_{4n-1}\circ g_{4n+2}$ is a generator of $\pi_{4n+5}(S^{4n-1})=Z_2\{\nu_{4n-1}^2\}$ and hence $g_{4n-1}\circ g_{4n+2}=\nu_{4n-1}^2$. By the similar method as the complex case, we then have $Ep_*\Delta_1 w_{4n+3}=[(24, n+1)\nu_{4n}^2, \iota_{4n}]=[(n+1)\nu_{4n}^2, \iota_{4n}]$. This completes the proof of (6.4).

Lemma (6.6). *Let* $n\geq 2$. *Then* $[\eta_n^3, \iota_n]\neq 0$ *if and only if* $n\equiv 0 \bmod 4$ *and* $n\neq 4$ *or* 12.

Proof. By using [39], we can show that $[\eta_n^3, \iota_n]=w_n\circ\eta_{2n-1}^3=0$ for $n=2$, 3 or 4. We omit the details.

Let $n\geq 5$. Then $\eta_n^3=4\nu_n$ by [39, (5.5)], and so $[\eta_n^3, \iota_n]=4[\nu_n, \iota_n]$. Suppose first that n is odd. From the observations in Chapter XI of [39], $[\nu_n, \iota_n]$ is contained in the image of

$$\pi_{n+3}(P_{n+1,1})\xrightarrow{i_*}\pi_{n+3}(P_{n+k,k})\xrightarrow{E^{n-1}}\pi_{2n+2}(E^{n-1}P_{n+k,k})\xrightarrow{p_k}\pi_{2n+2}(S^n)$$

for every $k\geq 1$, where $P_{n+r,r}=P_{n+r,r}(R)$ (see [39] for the details). Since $\pi_{n+3}(P_{n+2,2})\cong\pi_{n+3}(V_{n+2,2})$ by (2.8), and since the latter is $Z_2\oplus Z_2$ by [30], we have $2[\nu_n, \iota_n]=0$. Hence $[\eta_n^3, \iota_n]=0$. The order of $[\nu_n, \iota_n]$ has been determined by several persons but we shall not need it.

Suppose next that n is even ≥ 6. When $n\equiv 2 \bmod 4$, the order of $[\nu_n, \iota_n]$ is 4 by [11, (3)], so that $[\eta_n^3, \iota_n]=0$. When $n\equiv 0 \bmod 4$ and $n\neq 12$, 20 or 28, Theorem C of [20] implies that the order of $[\nu_n, \iota_n]$ is 8 so that $[\eta_n^3, \iota_n]\neq 0$. On the other hand, by [39, Theorem 10.3], [23, Theorem B] and [27, Theorem 3, (b)], we can see that the order of $[\nu_n, \iota_n]$ is 4, 8 or 8 according as n is 12, 20 or 28. Hence $[\eta_{12}^3, \iota_{12}]=0$ and $[\eta_n^3, \iota_n]\neq 0$ for $n=20$ or 28. This completes the proof of (6.6).

Remark. One can prove non-triviality of $4[\nu_{4m}, \iota_{4m}]$, at least for $m\geq 8$ and $m\not\equiv 15 \bmod 16$, by using [25, 26] instead of [20]. The author is grateful to Professor Y. Nomura for informing him of this.

§ 7. Proofs of Theorems (1.5), (1.7) and Corollary (1.6).

In this section we consider the complex case only.

It follows that (i), (ii) of (1.5) are equivalent from (2), (3) of (5.10), and that so are (ii), (iii) of (1.5) from (2.4).

We prove that (i), (iv) of (1.5) are equivalent at least when $n\neq 6$. The case $n=6$ will be studied later. Throughout the proof we assume that $n\geq 1$. From (1) of (5.10), it suffices to study the case $\alpha=\Delta_1\iota_{2n+1}$. Then $\langle\alpha, \alpha\rangle\circ\eta_{4n}=\Delta_1[\eta_{2n+1}, \iota_{2n+1}]$ by (3) of (5.10). It follows from [12, (4.4) and (4.11)] that $[\eta_{2n+1}, \iota_{2n+1}]=0$ if and only if n is odd. Thus $\langle\alpha, \alpha\rangle\circ\eta_{4n}=0$ for n odd. Next we assume that n is even. Let p: $U(n)\rightarrow W_{n,1}=S^{2n-1}$ be the usual map. Then

$$\begin{aligned} Ep_*(\langle\alpha, \alpha\rangle\circ\eta_{4n}) &= Ep_*\langle\alpha, \alpha\rangle\circ\eta_{4n+1} \\ &= Ep_*\Delta_1 w_{2n+1}\circ\eta_{4n+1}, \quad \text{by (3) of (5.10)}, \\ &= [\eta_{2n}^2, \iota_{2n}]\circ\eta_{4n+1}, \quad \text{by (6.4)}, \\ &= [\eta_{2n}^3, \iota_{2n}], \quad \text{by (8.18) at p. 484 of [40].} \end{aligned}$$

Thus (6.6) implies that $\langle\alpha, \alpha\rangle\circ\eta_{4n}\neq 0$ when n is even and $n\neq 2$ or 6. Since the 2-torsion element $\langle\Delta_1\iota_5, \Delta_1\iota_5\rangle\circ\eta_8$ is contained in $\pi_9(U(2))\cong\pi_9(S^3)\cong Z_3$, [39], it is zero. Therefore (i) and (iv) of (1.5) are equivalent when $n\neq 6$.

Now we prove Corollary (1.6) when $n\neq 6$. Let n be even ≥ 2. First assume that $n\neq 2$ or 6. If $\langle\alpha, \alpha\rangle$ did not generate a direct summand in $\pi_{4n}(U(n))$, then there would exist an element $x\in\pi_{4n}(U(n))$ such that $\langle\alpha, \alpha\rangle=2x$ and so $\langle\alpha, \alpha\rangle\circ\eta_{4n}=2x\circ\eta_{4n}=x\circ 2\eta_{4n}=0$, and hence we should obtain a contradiction to Theorem (1.5). Therefore $\langle\alpha, \alpha\rangle$ generates a direct summand, and hence so does $[i_{n+k,k}, i_{n+k,k}]$, since $\Delta_k[i_{n+k,k}, i_{n+k,k}]=\langle\alpha, \alpha\rangle$ by (3) of (5.10), and since $[i_{n+k,k}, i_{n+k,k}]$ is of oder 2 by (1.1). Next we consider the case $n=2$. Since the embedding i': $S^3=Sp(1)\subset U(2)$ induces an isomorphism $\pi_r(S^3)\cong\pi_r(U(2))$ for $r\geq 2$, it suffices to show that $\langle\eta_3, \eta_3\rangle\in\pi_8(S^3)$ generates a direct summand. This is the case, because $\pi_8(S^3)\cong Z_2$ by [39], and $i'_*\langle\eta_3, \eta_3\rangle=\langle i'_*\eta_3, i'_*\eta_3\rangle\neq 0$ by Theorem (1.2). This completes the proof of Corollary (1.6) for $n\neq 6$.

Note that $\langle\iota_3, \iota_3\rangle$ is a generator of $\pi_6(S^3)$ as shown by Samelson [32, §§ 7 and 9], so the Samelson products in S^3 can be determined up to reasonable dimension by [15, (15.4)], [39] and others.

To study the case $n=6$ in (1.5), and to prove Theorem (1.7), we need some facts about the James numbers. We recall them briefly. See [29] for the details.

By the James number $W\{n+k, k\}$ and the stable James number $W^s\{n+k, k\}$, we denote the order of the cokernel of

$$p_*: \pi_{2(n+k)-1}(W_{n+k,k}) \longrightarrow \pi_{2(n+k)-1}(W_{n+k,1}) \cong Z,$$
$$p_*: \pi^s_{2(n+k)-1}(W_{n+k,k}) \longrightarrow \pi^s_{2(n+k)-1}(W_{n+k,1}) \cong Z$$

respectively, where $\pi^s_*(-)$ is the stable homotopy theory.

Using the stunted quasi-projective spaces $Q_{n+k,k}$ instead of $W_{n+k,k}$, we can define the numbers $Q\{n+k, k\}$ and $Q^s\{n+k, k\}$. We have always $Q^s\{n+k, k\} = W^s\{n+k, k\}$. We have also that the stable number is a divisor of the unstable one, and that if $k \leq n+1$ then $Q^s\{n+k, k\} = Q\{n+k, k\} = W^s\{n+k, k\} = W\{n+k, k\}$.

Consider the case $k = n+2$. We shall compute $W\{2n+2, n+2\}/W^s\{2n+2, n+2\}$. Since the pair $(W_{2n+2,n+2}, Q_{2n+2,n+2})$ is $(4n+3)$-connected by (2.8), it follows that

$$Q\{2n+2, n+2\} = W\{2n+2, n+2\}.$$

For simplicity, we denote $Q_{2n+2,n+2}$ by Q, and $i_{2n+2,n+2}: S^{2n+1} \rightarrow Q_{2n+2,n+2}$ by i. We denote the torsion subgroup of a group by Tor. Since $\pi^s_{4n+3}(Q)/\mathrm{Tor} \cong Z$, it sufficies for (1.7) to see the homomorphism $\pi_{4n+3}(Q)/\mathrm{Tor} \rightarrow \pi^s_{4n+3}(Q)/\mathrm{Tor}$ induced by the stabilization homomorphism E^∞. It is not difficult to show that E^∞ induces an isomorphism

$$\pi_{4n+4}(EQ)/\mathrm{Tor} \cong \pi^s_{4n+3}(Q)/\mathrm{Tor}.$$

Thus the order of the cokernel of the homomorphism

$$\tilde{E}: \pi_{4n+3}(Q)/\mathrm{Tor} \longrightarrow \pi_{4n+4}(EQ)/\mathrm{Tor} \cong Z$$

induced by E is equal to

$$Q\{2n+2, n+2\}/Q^s\{2n+2, n+2\} = W\{2n+2, n+2\}/W^s\{2n+2, n+2\}.$$

Since $p: U(2) \rightarrow S^3$ has a cross section $S^3 = Sp(1) \subset U(2)$, we have that $W\{2, 2\} = W^s\{2, 2\} = 1$. Thus we assume that $n \geq 1$. Then we have the commutative diagram of the exact EHP sequences:

$$\begin{tikzcd}
 & \pi_{4n+4}(S^{2n+2}) \arrow[r, "H"] \arrow[d, "Ei_*"] & \pi_{4n+4}(S^{4n+3}) \arrow[r, "P"] \arrow[d, "E(i\wedge i)_*"] & \pi_{4n+2}(S^{2n+1}) \arrow[d, "i_*"] \\
\pi_{4n+3}(Q) \arrow[r, "E"] & \pi_{4n+4}(EQ) \arrow[r, "H"] & \pi_{4n+4}(E(Q\wedge Q)) \arrow[r, "P"] & \pi_{4n+2}(Q).
\end{tikzcd}$$

It follows that $E(i\wedge i)_*$ is surjective, so that $\pi_{4n+4}(E(Q\wedge Q))$ is generated by $E(i\wedge i)_*\eta_{4n+3}$ and

(7.1) $$\pi_{4n+4}(E(Q\wedge Q)) \cong Z_2 \quad \text{or } 0.$$

By chasing the diagram, we see that, when n is odd, $\tilde{E}$ is surjective and hence

$$W\{2n+2, n+2\}/W^s\{2n+2, n+2\}=1 \qquad \text{for } n \text{ odd.}$$

Let n be even. By (7.1), the cokernel of E is Z_2 or 0, and hence $W\{2n+2, n+2\}/W^s\{2n+2, n+2\}$ is 2 or 1. Suppose that it is 2. Then H is surjective onto $\pi_{4n+4}(E(Q\wedge Q))\cong Z_2$, so $P(E(i\wedge i)_*\eta_{4n+3})=0$. Hence $[i, i]\circ\eta_{4n+1}=i_*(w_{2n+1}\circ\eta_{4n+1})=i_*P\eta_{4n+3}=P(E(i\wedge i)_*\eta_{4n+3})=0$. It then follows from the observations in Section 2 that $[i_{2n+2,n+2}, i_{2n+2,n+2}]\circ\eta_{4n+1}$ vanishes in $\pi_{4n+2}(W_{2n+2,n+2})$, and

$$\langle\alpha, \alpha\rangle\circ\eta_{4n}=\Delta_{n+2}([i_{2n+2,n+2}, i_{2n+2,n+2}]\circ\eta_{4n+1})=0.$$

Therefore, from Theorem (1.5) for $n\neq 6$, we have

$W\{2n+2, n+2\}/W^s\{2n+2, n+2\}=1$ if n is even and $n\neq 2$ or 6. On the other hand, we have proved in [28] that

$$W\{2n+2, n+2\}/W^s\{2n+2, n+2\}=2 \qquad \text{for } n=2 \text{ or } 6.$$

This completes the proofs of Theorems (1.5) and (1.7).

To prove Corollary (1.6) for $n=6$, we consider the homomorphisms:

$$\pi_{4n}(U(n))\xrightarrow{p_*}\pi_{4n}(S^{2n-1})\xrightarrow{H}\pi_{4n}(S^{4n-3}).$$

Lemma (7.2). *If n is even, then $Hp_*\langle\alpha, \alpha\rangle=\eta^3_{4n-3}$.*

Proof. We have

$$\begin{aligned}
Hp_*\langle\alpha, \alpha\rangle &=Hp_*\langle\Delta_1\iota_{2n+1}, \Delta_1\iota_{2n+1}\rangle, && \text{by (1) of (5.10)},\\
&=Hp_*(\Delta_1\iota_{2n+1}\circ Ji'_*\Delta_1\iota_{2n+1}), && \text{by (6.2)},\\
&=H(\eta_{2n-1}\circ Ji'_*\Delta_1\iota_{2n+1}), && \text{by (6.5)},\\
&=E(\eta_{2n-2}\wedge\eta_{2n-2})\circ HJi'_*\Delta_1\iota_{2n+1}, && \text{by [39, Proposition 2.2]},\\
&=\eta^3_{4n-3}, && \text{by [2, (2.12)] and (6.3).}
\end{aligned}$$

Note that this lemma gives an alternative proof of non-triviality of $\langle\alpha, \alpha\rangle$ for n even.

Now we prove Corollary (1.6) for $n=6$. Consider the sequence:

$$\pi_{24}(U(6))\xrightarrow{p_*}\pi_{24}(S^{11})\xrightarrow{H}\pi_{24}(S^{21}).$$

By (7.2), $Hp_*\langle\alpha, \alpha\rangle=\eta^3_{21}\neq 0$. On the other hand, by [39], $\pi_{24}(S^{11})\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_3$. Therefore, $\langle\alpha, \alpha\rangle$ can not be halved, so it generates a direct summand. This completes the proof of Corollary (1.6).

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