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On the Orders of the Generators in the 18-Stem of the Homotopy Groups of Spheres

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§ 1. Introduction

Let π_i^n be the 2-component of $\pi_i(S^n)$. The purpose of this paper is to determine the orders of the generators of the groups π_{n+18}^n for $n=10$, 11 and 12. H. Toda determined π_{n+i}^n for $i\leqq 19$ and all n in [2]. He defined the generators λ'', ξ'' of π_{28}^{10} and λ', ξ' of π_{29}^{11}, making use of Propositions in [2, Chapter 11] which assert the existence of new generators under certain conditions. Thus he obtained the group structures and generators of π_{n+18}^n ($n=10$, 11 and 12) in [2, Theorem 12.22], which states

$$\pi_{28}^{10}\approx Z_8\oplus Z_2\oplus Z_2\colon \textit{ generated by } \lambda'', \xi'' \textit{ and } \eta_{10}\circ\bar{\mu}_{11},$$

$$\pi_{29}^{11}\approx Z_8\oplus Z_4\oplus Z_2\colon \textit{ generated by } \lambda', \xi' \textit{ and } \eta_{11}\circ\bar{\mu}_{12},$$

$$\pi_{30}^{12}\approx Z_{32}\oplus Z_4\oplus Z_4\oplus Z_2\colon \textit{ generated by } \xi_{12}, E\lambda', E\xi' \textit{ and } \eta_{12}\circ\bar{\mu}_{13}.$$

But the orders of λ'', ξ'', λ' and ξ' were not determined in [2]. In this paper, in order to investigate their properties further, we shall define new elements $\bar{\bar{\xi}}''$, $\bar{\bar{\lambda}}''$ and $\tilde{\lambda}''$ of π_{28}^{10} by Toda brackets. Then making use of the various properties of Toda brackets, we shall obtain many relations involving these new elements and the ones defined in [2]. These results will be stated in Propositions 1–4: These relations enable us to determine the orders of λ'', ξ'', λ' and ξ'. As the main results of this paper we shall determine the direct summands of π_{n+18}^n for $n=10$, 11 and 12;

Theorem. *The group* π_{n+18}^n ($n=10$, 11 *and* 12) *has the following direct summands with the generators defined by H. Toda in* [2].

$$\pi_{28}^{10}=Z_8\{\xi''\}\oplus Z_2\{\xi''\pm\lambda''\}\oplus Z_2\{\eta_{10}\circ\bar{\mu}_{11}\},$$

$$\pi_{29}^{11}=Z_8\{\xi'\}\oplus Z_4\{\xi'+\lambda'\}\oplus Z_2\{\eta_{11}\circ\bar{\mu}_{12}\},$$

$$\pi_{30}^{12}=Z_{32}\{\xi_{12}\}\oplus Z_4\{E\xi'+4\xi_{12}\}\oplus Z_4\{E\xi'+E\lambda'\}\oplus Z_2\{\eta_{12}\circ\bar{\mu}_{13}\},$$

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where $Z_2\{\xi''\pm\lambda''\}$ *means that either* $\xi''+\lambda''$ *or* $\xi''-\lambda''$ *is the generator of order* 2.

We remark that K. Ôguchi obtained in [1, (2.19)], the same results as the above theorem. But, as we see by [1, Lemma (2,18)], his generators are different from those defined by H. Toda in [2].

We shall prove the theorem at the last part of this paper. We shall use the notations of [2]. We shall use the results of [2, Chapters 1–4] and many relations listed in [2, Chapter 14] and [1, Table I].

§ 2. Some new elements and relations

In this section we shall define some new elements by Toda brackets. Studying the Hopf invariants of these elements, we shall determine the various relations involving Toda's elements λ'', ξ'', λ', ξ' and others. These results enable us to prove our Theorem in the next section.

We shall define three elements of π_{28}^{10} by the following Toda brackets:

$$\bar{\bar{\xi}}'' \in \{\sigma_{10}, \sigma_{17}\circ\eta_{24}^2, \eta_{26}\}_1;\quad \bar{\bar{\lambda}}'' \in \{\sigma_{10}, \varepsilon_{17}\circ\eta_{25}, \eta_{26}\}_1;\quad \tilde{\lambda}'' \in \{\sigma_{10}\circ\nu_{17}^2, \nu_{23}, \eta_{26}\}_1.$$

We remark that $\Delta(\varepsilon_{19})=\sigma_9\circ\eta_{16}\circ\varepsilon_{17}$; $\Delta(\bar{\nu}_{19})=\sigma_9\circ\eta_{16}\circ\bar{\nu}_{17}=\sigma_9\circ\nu_{16}^3$ and $\Delta(\varepsilon_{19}+\bar{\nu}_{19})=\sigma_9^2\circ\eta_{23}^2$ by [2, (7.1)], since $\bar{\nu}_9^2=\varepsilon_9^2=\bar{\nu}_9\circ\varepsilon_{17}=\varepsilon_9\circ\bar{\nu}_{17}=0$ by [1, Proposition (2.8) (2)] and [2, Lemma 12.10 and (7.20)]. Thus the above Toda brackets are well-defined.

Proposition 1.

(1) $H(\bar{\bar{\xi}}'')=\eta_{19}\circ\varepsilon_{20}+\nu_{19}^3$; $H(\bar{\bar{\lambda}}'')=\eta_{19}\circ\varepsilon_{20}$; $H(\tilde{\lambda}'')=\nu_{19}^3$.

(2) $2\tilde{\lambda}''=0$; $2\bar{\bar{\xi}}''\equiv 2\bar{\bar{\lambda}}''\equiv\sigma_{10}\circ\zeta_{17} \bmod 2\sigma_{10}\circ\zeta_{17}$.

(3) $\xi''\equiv\bar{\bar{\xi}}''\equiv\bar{\bar{\lambda}}''+\tilde{\lambda}'' \bmod \sigma_{10}\circ\zeta_{17}, \eta_{10}\circ\bar{\mu}_{11}$.

(4) $2\xi''\equiv\sigma_{10}\circ\zeta_{17} \bmod 2\sigma_{10}\circ\zeta_{17}$.

Proof. (1) We have $H(\bar{\bar{\xi}}'')\in H\{\sigma_{10}, \sigma_{17}\circ\eta_{24}^2, \eta_{26}\}_1=-\Delta^{-1}(\sigma_9^2\circ\eta_{23}^2)\circ\eta_{27}=(\varepsilon_{19}+\bar{\nu}_{19})\circ\eta_{27}=\eta_{19}\circ\varepsilon_{20}+\nu_{19}^3$. Similarly, we have the other results.

(2) $2\bar{\bar{\xi}}''=\bar{\bar{\xi}}''\circ 2\iota_{28}\in\{\sigma_{10}, \sigma_{17}\circ\eta_{24}^2, \eta_{26}\}_1\circ 2\iota_{28}=\sigma_{10}\circ E\{\sigma_{16}\circ\eta_{23}^2, \eta_{25}, 2\iota_{26}\}\ni\sigma_{10}\circ\zeta_{17} \bmod 2\sigma_{10}\circ\zeta_{17}$ by [2, Lemma 9.1], with the indeterminacy of the Toda bracket $\sigma_{10}\circ E\pi_{27}^{16}\circ 2\iota_{28}=\{2\sigma_{10}\circ\zeta_{17}\}$. It follows then that $2\bar{\bar{\xi}}''\equiv\sigma_{10}\circ\zeta_{17} \bmod 2\sigma_{10}\circ\zeta_{17}$. Similarly we have the other results.

(3) We recall that $H(\xi'')=\eta_{19}\circ\varepsilon_{20}+\nu_{19}^3$ by [2, Lemma 12.19]. Then, these elements have the same Hopf invariant and hence

$$\xi''\equiv\bar{\bar{\xi}}''\equiv\bar{\bar{\lambda}}''+\tilde{\lambda}'' \bmod E\pi_{27}^9=\{\sigma_{10}\circ\zeta_{17}, \eta_{10}\circ\bar{\mu}_{11}\}$$

by the exactness of the $EH\Delta$-sequence and the result of $E\pi_{27}^9$ on [2, p. 164].

(4) is the immediate consequence of (2) and (3).

Proposition 2. (1) $\pi_{28}^{10}=Z_8\{\bar{\bar{\lambda}}''\}\oplus Z_2\{\tilde{\lambda}''\}\oplus Z_2\{\eta_{10}\circ\bar{\mu}_{11}\}$.
(2) $E\pi_{28}^{10}=Z_4\{E\bar{\bar{\lambda}}''\}\oplus Z_2\{E\tilde{\lambda}''\}\oplus Z_2\{\eta_{11}\circ\bar{\mu}_{12}\}$.

Proof. (1) As we see by the argument on [2, p. 164], we have to settle the following group extension;

$$0\longrightarrow Z_4\{\sigma_{10}\circ\zeta_{17}\}\oplus Z_2\{\eta_{10}\circ\bar{\mu}_{11}\}\longrightarrow\pi_{28}^{10}\xrightarrow{H}Z_2\{\nu_{19}^3\}\oplus Z_2\{\eta_{19}\circ\varepsilon_{20}\}\longrightarrow 0.$$

Then we have the result by Proposition 1 (1) (2).

(2) Since the kernel of E: $\pi_{28}^{10}\to\pi_{29}^{11}$ is generated by $\Delta(\mu_{21})=2\sigma_{10}\circ\zeta_{17}$ by [2, (12.25)], and $4\bar{\bar{\lambda}}''=2\sigma_{10}\circ\zeta_{17}$ by Proposition 1 (2), we have the result.

Proposition 3. (1) $E^2\tilde{\lambda}''\equiv\varepsilon_{12}^*\circ\eta_{29}$ mod $\eta_{12}\circ\bar{\mu}_{13}$.
(2) $\varepsilon_{13}^*\circ\eta_{30}=4\xi_{13}+4\lambda$; $\tilde{\lambda}''\equiv\lambda''+\xi''$ mod $\sigma_{10}\circ\zeta_{17}$, $\eta_{10}\circ\bar{\mu}_{11}$; $2\lambda''\equiv 2\xi''\equiv\sigma_{10}\circ\zeta_{17}$ mod $2\sigma_{10}\circ\zeta_{17}$.
(3) $H(\lambda'')=\eta_{19}\circ\varepsilon_{20}$.

Proof. (1) Since $\sigma_{12}\circ\nu_{19}=0$ by [2, (7.20)], we have

$$E^2\tilde{\lambda}''\in E^2\{\sigma_{10}\circ\nu_{17}^2,\nu_{23},\eta_{26}\}_1\subset\{\sigma_{12}\circ\nu_{19}^2,\nu_{25},\eta_{28}\}_1=\pi_{29}^{12}\circ\eta_{29}=\{\varepsilon_{12}^*\circ\eta_{29},\eta_{12}\circ\bar{\mu}_{13}\}$$

by [2, Theorem 12.17, (5.9), (7.14) and (12.23)]. Then we may put

$$E^2\tilde{\lambda}''=x\varepsilon_{12}^*\circ\eta_{29}+y\eta_{12}\circ\bar{\mu}_{13}\qquad\textit{for some integers } x \textit{ and } y.$$

Now, we see that $E^2\pi_{28}^{10}=Z_4\{E^2\bar{\bar{\lambda}}''\}\oplus Z_2\{E^2\tilde{\lambda}''\}\oplus Z_2\{\eta_{12}\circ\bar{\mu}_{13}\}$ by Proposition 2(2), for E: $\pi_{29}^{11}\to\pi_{30}^{12}$ is monic by the result on [2, p. 165]. Then we conclude that $x\equiv 1$ mod 2.

(2) and (3). By [2, Lemma 12.19], we may put

(a) $$H(\lambda'')=\nu_{19}^3+x(\nu_{19}^3+\eta_{19}\circ\varepsilon_{20})\quad\textit{for } x=0 \textit{ or } 1$$

We have $H(\lambda'')=H(\tilde{\lambda}'')+xH(\xi'')=H(\tilde{\lambda}''+x\xi'')$ by Proposition 1 (1) and [2, Lemma 12.19]. Then by the exactness of the $EH\Delta$-sequence we have

(b) $\lambda''\equiv\tilde{\lambda}''+x\xi''$ mod $E\pi_{27}^9=\{\sigma_{10}\circ\zeta_{17},\eta_{10}\circ\bar{\mu}_{11}\}$ *for the integer x in* (a).

Applying E^3: $\pi_{28}^{10}\to\pi_{31}^{13}$ to the relation (b) with the facts that $E\lambda''=2\lambda'$ and $E\xi''=2\xi'$ by [2, Lemma 12.19], we have $E^3\tilde{\lambda}''\equiv 2E^2\lambda'-2xE^2\xi'$ mod $\eta_{13}\circ\bar{\mu}_{14}$, since $\sigma_{13}\circ\zeta_{20}=0$ by [2, (12.23)]. Making use of the relations $E^3\tilde{\lambda}''\equiv\varepsilon_{13}^*\circ\eta_{30}$ mod $\eta_{13}\circ\bar{\mu}_{14}$ by (1) of this Proposition; $E^2\lambda'=2\lambda$ and $E^2\xi'=2\xi_{13}$ by [2, Lemma 12.19], we may put

(c) $\varepsilon_{13}^*\circ\eta_{30}=4\lambda-4x\xi_{13}+y\eta_{13}\circ\bar{\mu}_{14}$ *for the integer x in* (a) *and some integer y.*

By [2, Lemmas 12.15, 12.18 and 12.24, Proposition 12.20, Theorem 14.1], we have the following relations in the stable range;

$$\varepsilon^* \circ \eta = \omega \circ \eta^2 = \eta^* \circ \eta^2 = 4\nu^* = 4\xi \quad and \quad E^\infty \lambda = 2\nu^*.$$

Then applying $E^\infty \colon \pi_{31}^{13} \to {}_2\pi_{18}^S = Z_8\{\nu^*\} \oplus Z_2\{\eta \circ \bar{\mu}\} = Z_8\{\xi\} \oplus Z_2\{\eta \circ \bar{\mu}\}$ (by [2, Theorem 12.22 and Lemma 12.24]) to the relation (c), we have

(d) $\qquad x \equiv 1 \bmod 2; \quad y \equiv 0 \bmod 2.$

Then by (a) (b) and (c), we have

(e) $H(\lambda'') = \eta_{19} \circ \varepsilon_{20}$; $\lambda'' \equiv \tilde{\lambda}'' + \xi'' \bmod \sigma_{10} \circ \zeta_{17}, \eta_{10} \circ \bar{\mu}_{11}$; $\varepsilon_{13}^* \circ \eta_{30} = 4\lambda + 4\xi_{13}$.

Now, we see $2\xi'' \equiv \sigma_{10} \circ \zeta_{17} \bmod 2\sigma_{10} \circ \zeta_{17}$ by Proposition 1 (4). It follows then that

$$\tilde{\lambda}'' \equiv \lambda'' - \xi'' \equiv \lambda'' + \xi'' \bmod \sigma_{10} \circ \zeta_{17}, \eta_{10} \circ \bar{\mu}_{11}.$$

Since $2\tilde{\lambda}'' = 0$ by Proposition 1 (2), we have by Proposition 1 (4),

$$2\lambda'' \equiv 2\xi'' \equiv \sigma_{10} \circ \zeta_{17} \bmod 2\sigma_{10} \circ \zeta_{17}.$$

This completes the proof.

Proposition 4.

(1) $\varepsilon_{12}^* \circ \eta_{29} \equiv 2E\lambda' + 2E\xi' \bmod 4E\lambda', 4E\xi'$; $\eta_{11} \circ \varepsilon_{12}^* \equiv 2\lambda' + 2\xi' \bmod 4\lambda', 4\xi'$.

(2) $4(\lambda' + \xi') = 0$; $4\lambda' = 4\xi' = \sigma_{11} \circ \zeta_{18}$.

(3) $H(\lambda') = \varepsilon_{21}$.

Proof. (1) In the exact sequence $\pi_{29}^{11} \xrightarrow{E} \pi_{30}^{12} \xrightarrow{H} \pi_{30}^{23}$, we see $H(\varepsilon_{12}^* \circ \eta_{29}) = H(\varepsilon_{12}^*) \circ \eta_{29} = \nu_{23}^2 \circ \eta_{29} = 0$ by [2, Lemma 12.15], and hence $\varepsilon_{12}^* \circ \eta_{29} \in E\pi_{29}^{11} = \{E\lambda', E\xi', \eta_{12} \circ \bar{\mu}_{13}\}$ (see [2, p. 165]). We may put

(f) $\quad \varepsilon_{12}^* \circ \eta_{29} = xE\lambda' + yE\xi' + z\eta_{12} \circ \bar{\mu}_{13}$ *for some integers* x, y *and* z.

Applying E we have $\varepsilon_{13}^* \circ \eta_{30} = xE^2\lambda' + yE^2\xi' + z\eta_{13} \circ \bar{\mu}_{14} = 2x\lambda + 2y\xi_{13} + z\eta_{13} \circ \bar{\mu}_{14}$ by [2, Lemma 12.19]. On the other hand, we have already shown that $\varepsilon_{13}^* \circ \eta_{30} = 4\lambda + 4\xi_{13}$ by Proposition 3 (2). Thus we have $4\lambda + 4\xi_{13} = \varepsilon_{13}^* \circ \eta_{30} = 2x\lambda + 2y\xi_{13} + z\eta_{13} \circ \bar{\mu}_{14}$ in $\pi_{31}^{13} = Z_8\{\lambda\} \oplus Z_8\{\xi_{13}\} \oplus Z_2\{\eta_{13} \circ \bar{\mu}_{14}\}$ by [2, Theorem 12.22]. It follows then that $x \equiv y \equiv 2 \bmod 4$; $z \equiv 0 \bmod 2$. Then we have the first relation by (f).

Let us now consider the next relation. The element $\eta_{11} \circ \varepsilon_{12}^*$ is in $\pi_{29}^{11} = Z_8 \oplus Z_4\{\lambda', \xi'\} \oplus Z_2\{\eta_{11} \circ \bar{\mu}_{12}\}$ [2, Theorem 12.22]. Then we may put

(g) $\eta_{11}\circ\varepsilon^*_{12}=a\lambda'+b\xi'+c\eta_{11}\circ\bar{\mu}_{12}$ *for some integers a, b and c.*

Applying E^3: $\pi^{11}_{29}\to\pi^{14}_{32}$ to the relation (g), we have

$$\eta_{14}\circ\varepsilon^*_{15}=aE^3\lambda'+bE^3\xi'+c\eta_{14}\circ\bar{\mu}_{15}=2aE\lambda+2b\xi_{14}+c\eta_{14}\circ\bar{\mu}_{15},$$

by [2, Lemma 12.19], where $\eta_{14}\circ\varepsilon^*_{15}=\varepsilon^*_{12}\#\eta_2=\varepsilon^*_{14}\circ\eta_{31}=4E\lambda+4\xi_{14}$ by [2, Proposition 3.1] and Proposition 3 (2). It follows then that

$$4E\lambda+4\xi_{14}=\eta_{14}\circ\varepsilon^*_{15}=2aE\lambda+2b\xi_{14}+c\eta_{14}\circ\bar{\mu}_{15}$$

in $\pi^{14}_{32}=Z_8\{E\lambda\}\oplus Z_8\{\xi_{14}\}\oplus Z_2\{\eta_{14}\circ\bar{\mu}_{15}\}$ by [2, Theorem 12.22]. Thus we have $a\equiv b\equiv 2 \bmod 4$; $c\equiv 0 \bmod 2$. Then we have the second relation by (g).

(2) By (1), we see $4(\lambda'+\xi')=2(\eta_{11}\circ\varepsilon^*_{12})=\eta_{11}\circ 2\varepsilon^*_{12}=0$.

$4\xi'=2E\xi''\equiv\sigma_{11}\circ\zeta_{18} \bmod 2\sigma_{11}\circ\zeta_{18}=0$ by Proposition 1 (4) and [2, (12.23)].

(3) We may put by [2, Lemma 12.19],

(h) $H(\lambda')=\varepsilon_{21}+d(\bar{\nu}_{21}+\varepsilon_{21})$ *for* $d=0$ *or* 1.

We shall now use [2, Proposition 11.13] for $\alpha=\lambda'\in\pi^{11}_{29}$, $\beta=\varepsilon_{16}+d(\bar{\nu}_{16}+\varepsilon_{16})\in\pi^{16}_{24}$ (where d is the integer in (h)), $n=9$, $i=27$, just as in the proof of [2, Lemma 12.19]. Then there exists an element γ of π^{10}_{28} such that $2\alpha=E\gamma$ and $H(\gamma)\equiv E^3(\varepsilon_{16}+d(\bar{\nu}_{16}+\varepsilon_{16}))\circ\eta_{27} \bmod 2E^2\pi^{17}_{26}=0$. Then we have

$$2\lambda'=E\gamma \quad \textit{and} \quad H(\gamma)=\varepsilon_{19}\circ\eta_{27}+d(\bar{\nu}_{19}+\varepsilon_{19})\circ\eta_{27}.$$

We have a relation $E\lambda''=2\lambda'=E\gamma$ in π^{11}_{29} by [2, Lemma 12.19]. Since Ker $(E\colon \pi^{10}_{28}\to\pi^{11}_{29})=\{2\sigma_{10}\circ\zeta_{17}\}$ by [2, (12.25)], we have $\lambda''\equiv\gamma \bmod 2\sigma_{10}\circ\zeta_{17}$, and hence $H(\lambda'')=H(\gamma)$. It follows that $\eta_{19}\circ\varepsilon_{20}=H(\lambda'')=H(\gamma)=\varepsilon_{19}\circ\eta_{27}+d(\bar{\nu}_{19}+\varepsilon_{19})\circ\eta_{27}$ by Proposition 3(3) and (h), or $d(\bar{\nu}_{19}+\varepsilon_{19})\circ\eta_{27}=0$. Then $d=0$ and we have the result. This completes the proof.

§ 3. Proof of Theorem

We shall use the results on [2, pp. 164–167]. To prove the result of π^{10}_{28}, we have to settle the group extension

$$0\longrightarrow E\pi^9_{27}\longrightarrow\pi^{10}_{28}\xrightarrow{H}Z_2\{\nu^3_{19}\}\oplus Z_2\{\eta_{19}\circ\varepsilon_{20}\}\longrightarrow 0$$

where $E\pi^9_{27}=Z_4\{\sigma_{10}\circ\zeta_{17}\}\oplus Z_2\{\eta_{10}\circ\bar{\mu}_{11}\}$ by [2, p. 164]. Since $H(\xi'')=\nu^3_{19}+\eta_{19}\circ\varepsilon_{20}$ by [2, Lemma 12.19], $H(\lambda'')=\eta_{19}\circ\varepsilon_{20}$ by Proposition 3(3) and $2\lambda''\equiv 2\xi''\equiv\sigma_{10}\circ\zeta_{17} \bmod 2\sigma_{10}\circ\zeta_{17}$ by Proposition 3(2), we have the result of π^{10}_{28}.

Let us consider the exact sequence

$$0 \longrightarrow E\pi_{28}^{10} \longrightarrow \pi_{29}^{11} \xrightarrow{H} Z_2\{\bar{\nu}_{21}\} \oplus Z_2\{\varepsilon_{21}\} \longrightarrow 0.$$

The relation $2\lambda'' \equiv 2\xi'' \equiv \sigma_{10} \circ \zeta_{17} \bmod 2\sigma_{10} \circ \zeta_{17}$ implies $4\lambda'' = 4\xi'' = 2\sigma_{10} \circ \zeta_{17}$. Then by [2, (12.25)], we have

$$E\pi_{28}^{10} = Z_4\{E\lambda''\} \oplus Z_2\{E\xi'' + E\lambda''\} \oplus Z_2\{\eta_{11} \circ \bar{\mu}_{12}\}.$$

Since $H(\xi') = \bar{\nu}_{21} + \varepsilon_{21}$ and $H(\lambda') = \varepsilon_{21}$ with the relations $2\lambda' = E\lambda''$ and $2\xi' = E\xi''$ by [2, Lemma 12.19] and Proposition 4(3), we have the result of π_{29}^{11}.

Let us consider the exact sequence

$$\text{(i)} \qquad 0 \longrightarrow \pi_{29}^{11} \xrightarrow{E} \pi_{30}^{12} \xrightarrow{H} Z_{16}\{\sigma_{23}\} \longrightarrow 0.$$

Since $H(\xi_{12}) \equiv \sigma_{23} \bmod 2\sigma_{23}$ by [2, Lemma 12.14], we have only to settle the group extension. The fact $H(\Delta(\sigma_{25})) = \pm 2\sigma_{23}$ by [2, Propositions 2.5, 2.2 and 2.7] implies

$$4\Delta(\sigma_{25}) \notin E\pi_{29}^{11} \quad \textit{and} \quad 8\Delta(\sigma_{25}) \in E\pi_{29}^{11}$$

by the exactness of the sequence (i). It follows that

$$\operatorname{Ker}(E\colon E\pi_{29}^{11} \to \pi_{31}^{13}) = \{8\Delta(\sigma_{25})\} = \{16\xi_{12}\} \cong Z_2.$$

We see $4E\lambda' = 4E\xi' \neq 0$ in $E\pi_{29}^{11}$ by (i) and the result of π_{29}^{11}. Since $E(4E\lambda') = 4E^2\lambda' = 8\lambda = 0$ by [2, Lemma 12.19], and $\sigma_{12} \circ \zeta_{19} = 8\Delta(\sigma_{25}) = 16\xi_{12}$ as is shown on [2, p. 166], we conclude that

$$8\Delta(\sigma_{25}) = 16\xi_{12} = 4E\xi' = 4E\lambda' = \sigma_{12} \circ \zeta_{19}.$$

Then $Z_4\{E\xi' + 4\xi_{12}\}$ is one of the direct summands. Thus we have the result of π_{30}^{12}. This completes the proof of the theorem.

References

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