Advanced Studies in Pure Mathematics 9, 1986
Homotopy Theory and Related Topics
pp. 135–148

Characteristic Classes of T^2-bundles

Shigeyuki Morita

§ 1. Introduction

In our previous paper [5], we have proposed the problem to determine characteristic classes of differentiable fibre bundles whose fibres are diffeomorphic to a given closed manifold M, in other words the problem to compute the cohomology group $H^*(B\,\mathrm{Diff}\,M)$. The case when M is a closed orientable surface of genus greater than or equal to two has been treated in [4] [5]. In this paper we consider the case when M is the 2-dimensional torus T^2. Let $\mathrm{Diff}_+ T^2$ be the group of all orientation preserving diffeomorphisms of T^2 equipped with the C^∞ topology. Then our main result is

Theorem 1.1.

$$\dim \tilde{H}^n(B\,\mathrm{Diff}_+ T^2; \boldsymbol{Q}) = \begin{cases} 0 & n \not\equiv 1 \pmod 4 \\ 2m-1 & n = 24m+1 \\ 2m+1 & n = 24m+5,\ 24m+9,\ 24m+13 \\ & \qquad\qquad\qquad \textit{or}\ 24m+17 \\ 2m+3 & n = 24m+21. \end{cases}$$

The first non-trivial group is $H^5(B\,\mathrm{Diff}_+ T^2; \boldsymbol{Q}) \cong \boldsymbol{Q}$ and $\dim H^{4k+1}(B\,\mathrm{Diff}_+ T^2; \boldsymbol{Q})$ is approximately $\frac{1}{3}k$. Obviously the ring structure on $H^*(B\,\mathrm{Diff}_+ T^2; \boldsymbol{Q})$ defined by the cup product is trivial. We can also obtain informations on the torsions and by making use of them we obtain

Theorem 1.2. Mod 2 *and* 3 *torsions, we have*

$$\tilde{H}_n(B\,\mathrm{Diff}_+ T^2; \boldsymbol{Z}) = \begin{cases} \textit{torsion} & n \equiv 0 \pmod 4 \\ \textit{free abelian group of rank} & \\ \textit{indicated in Theorem } 1.1 & n \equiv 1 \pmod 4 \\ 0 & n \equiv 2, 3 \pmod 4. \end{cases}$$

Received January 24, 1985.

Moreover it turns out that p-torsions appear in $H_{4*}(B\,\mathrm{Diff}_+T^2; \boldsymbol{Z})$ for *any* prime p (see Remark 5.2 for more precise statements).

The proof of the above theorems consists of elementary but pleasant computations in linear algebra. Finally we remark that the remaining case when $M=S^2$ should be well-known because of Smale's theorem [7]: $\mathrm{Diff}_+S^2 \simeq SO(3)$.

§ 2. T^2-bundles

Let Diff_0T^2 be the connected component of the identity of Diff_+T^2. Then as is well-known the factor group $\mathrm{Diff}_+T^2/\mathrm{Diff}_0T^2$, which is the mapping class group of T^2, can be naturally identified with $SL_2\boldsymbol{Z}$. Therefore we have a fibration

$$(*)\quad B\,\mathrm{Diff}_0T^2 \longrightarrow B\,\mathrm{Diff}_+T^2 \longrightarrow K(SL_2\boldsymbol{Z}, 1).$$

T^2 acts on itself by "translations" and hence it can be considered as a subgroup of Diff_0T^2. It is easy to see that the action by conjugations of $SL_2\boldsymbol{Z}$ on this subgroup $T^2 \subset \mathrm{Diff}_0T^2$ is the same as the standard one. Now Earle and Eells [3] proved that the inclusion $T^2 \subset \mathrm{Diff}_0T^2$ is a homotopy equivalence so that $B\mathrm{Diff}_0T^2$ has the homotopy type of $K(\boldsymbol{Z}^2, 2)$. Hence if we choose suitable elements $x, y \in H^2(B\mathrm{Diff}_0T^2; \boldsymbol{Z})$, we can write

$$H^*(B\mathrm{Diff}_0T^2; \boldsymbol{Z}) = \boldsymbol{Z}[x, y]$$

on which $SL_2\boldsymbol{Z}$ acts through the automorphism of it given by $\gamma \to {}^t\gamma^{-1}$ ($\gamma \in SL_2\boldsymbol{Z}$).

Now let $\{E_r^{s,t}, d_r\}$ be the Serre spectral sequence for cohomology (with coefficients in a commutative ring R) of the fibration $(*)$. Then by the above argument, The E_2-term is given by

$$\bigoplus_{t=0}^{\infty} E_2^{s,t} = H^s(SL_2\boldsymbol{Z}; R[x, y]).$$

As is well-known the abelianization $H_1(SL_2\boldsymbol{Z})$ of $SL_2\boldsymbol{Z}$ is a cyclic group of order 12 and the kernel of the natural surjection $SL_2\boldsymbol{Z} \to H_1(SL_2\boldsymbol{Z})$ is the commutator subgroup of $SL_2\boldsymbol{Z}$, which in turn is isomorphic to a free group of rank 2 (see [6] for example). Hence applying the standard argument of group cohomology (see e.g. Proposition 10.1 of [1]), we obtain

Proposition 2.1. *If* $s \geq 2$, *then* $\bigoplus_{t=0}^{\infty} E_2^{s,t} = H^s(SL_2\boldsymbol{Z}; R[x, y])$ *is annihilated by* 12. *In particular if* $R=\boldsymbol{Q}$ *or* $\boldsymbol{Z}_n$ *with* $(n, 12)=1$, *then*

$$\bigoplus_{t=0}^{\infty} E_2^{s,t} = H^s(SL_2\boldsymbol{Z}; R[x, y]) = 0 \qquad \textit{for } s \geqq 2.$$

Corollary 2.2. *Let $k=\boldsymbol{Q}$ or $\boldsymbol{Z}_p$ (p is a prime different from 2 and 3). Then*

$$H^n(B\mathrm{Diff}_+T^2; k)\cong E_2^{0,n}\oplus E_2^{1,n-1}.$$

§ 3. Lemmas

As is well-known $SL_2\boldsymbol{Z}$ has the following presentation (see [6])

$$SL_2\boldsymbol{Z}=\langle\alpha, \beta; \alpha^4=\alpha^2\beta^{-3}=1\rangle.$$

Here, for the convenience of later computations, we choose two generators $\alpha=\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ and $\beta=\begin{pmatrix} 0 & 1 \\ -1 & 1\end{pmatrix}$. The action of $SL_2\boldsymbol{Z}$ on $H^*(B\mathrm{Diff}_0T^2; \boldsymbol{Z})$ $=\boldsymbol{Z}[x, y]$ is given by

$$\alpha(x)=-y, \qquad \alpha(y)=x$$
$$\beta(x)=x-y, \qquad \beta(y)=x$$

because ${}^t\alpha^{-1}=\alpha$ and ${}^t\beta^{-1}=\begin{pmatrix} 1 & 1 \\ -1 & 0\end{pmatrix}$.

Now for each $q\in \boldsymbol{N}$, let L_q be the submodule of $\boldsymbol{Z}[x, y]$ consisting of homogeneous elements of degree $2q$. We choose a basis $\{x^q, x^{q-1}y, \cdots, xy^{q-1}, y^q\}$ for L_q and let

$$A_q,\ B_q\in SL_{q+1}\boldsymbol{Z}$$

be the matrix representations of the actions of α and β on L_q with respect to the above basis. Let p denotes either a prime or 0. We write $A_q(p)$ and $B_q(p)$ for the corresponding elements of $SL_{q+1}\boldsymbol{Z}_p$ if p is a prime or of $SL_{q+1}\boldsymbol{Q}$ if $p=0$. It is easy to prove

Lemma 3.1. (i) *If q is odd, then $A_q^2=B_q^3=-E$. Moreover the minimal polynomials of A_q and B_q are t^2+1 and t^3+1 respectively.*

(ii) *If q is even, then $A_q^2=B_q^3=E$ and the minimal polynomials of A_q and B_q are t^2-1 and t^3-1 respectively.*

Corollary 3.2. *If q is odd, then both of $A_q(p)+E$ and $B_q(p)-E$ are invertible provided $p\neq 2$. In fact we have*

$$(A_q(p)+E)^{-1}=-\frac{1}{2}(A_q(p)-E) \quad \textit{and}$$

$$(B_q(p)-E)^{-1}=-\frac{1}{2}(B_q^2(p)+B_q(p)+E).$$

Now let $L_q(p)$ be either $L_q\otimes \boldsymbol{Z}_p$ if p is a prime or $L_q\otimes \boldsymbol{Q}$ if $p=0$. $A_q(p)$ and $B_q(p)$ act on $L_q(p)$. We assume q is even and define

$$L_q^-(p)=\{u\in L_q(p);\ A_q(p)u=-u\}$$
$$L_q'(p)=\{u\in L_q(p);\ (B_q^2(p)+B_q(p)+E)u=0\}.$$

Lemma 3.2. *If $p\neq 2$ and $q=2r$, then*

$$\dim L_q^-(p)=\begin{cases} r+1 & r\text{: odd}\\ r & r\text{: even.}\end{cases}$$

Proof. It is easy to see that

$$\{x^q-y^q,\ x^{q-1}y+xy^{q-1},\ x^{q-2}y^2-x^2y^{q-2},\ \cdots,\ x^{r+1}y^{r-1}-x^{r-1}y^{r+1},\ x^ry^r\}\quad (r\text{: odd})\ \text{ or}$$

$$\{x^q-y^q,\ x^{q-1}y+xy^{q-1},\ x^{q-2}y^2-x^2y^{q-2},\ \cdots,\ x^{r+1}y^{r-1}+x^{r-1}y^{r+1}\}\quad (r\text{: even})$$

forms a basis of $L_q^-(p)$.

Next we determine $\dim L_q'(p)$. We first consider the case $p=0$.

Lemma 3.3. Trace $B_q=1, 1, 0, -1, -1, 0$ *according as* $q\equiv 0, 1, 2, 3, 4, 5 \pmod 6$.

Proof. Observe that $B_q=(b_{ij}^{(q)})$, where

$$b_{ij}^{(q)}=(-1)^{i+1}\binom{q-j+1}{i-1}\quad (i, j=1, \cdots, q+1).$$

$\left(\text{Here we understand that } \binom{s}{t}=0 \text{ if } t>s\right)$. In other words the j-th column of B_q consists of coefficients of the polynomial $(1-t)^{q-j+1}$. B_q is naturally a minor matrix of B_{q+1} and if we look at the "third quadrant infinite matrix" $B=\lim_{q\to\infty} B_q$ carefully, we find out that

$$\text{Trace } B_q=\text{the coefficient of } t^q \text{ in the power series}$$
$$1+t(1-t)+t^2(1-t)^2+\cdots$$

But we have

$$\sum_{n=0}^{\infty}(t(1-t))^n=\frac{1}{1-t+t^2}$$
$$=\frac{1}{(t-\omega)(t-\overline{\omega})}$$

where $\omega=\exp(2\pi i/6)$. From this we conclude

$$\text{Trace } B_q=\frac{1}{3}(\omega^q-\omega^{q+2}+\omega^{5q}-\omega^{5q+4}).$$

Then the desired result follows from a direct computation.

Lemma 3.4. *If q is even, then*

$$\operatorname{rank}(B_q^2+B_q+E)=2k+1 \qquad \textit{for } q=6k,\ 6k+2 \textit{ or } 6k+4.$$

Proof. According to Lemma 3.1 (ii), the characteristic polynomial of B_q is

$$(t-1)^a(t^2+t+1)^b$$

for some $a, b \in N$. But clearly

$$a+2b=q+1 \quad \text{and} \quad a-b=\text{Trace } B_q.$$

A simple computation using Lemma 3.3 implies the result.

Next we show that the above lemma also holds even if we replace B_q by $B_q(p)$ $(p\neq 3)$.

Lemma 3.5. *Let $B_q=(b_{ij}^{(q)})$ and define $C_q=(c_{ij}^{(q)})$ by*

$$c_{ij}^{(q)}=b_{q+2-i,q+2-j}^{(q)}.$$

Then we have $C_q=B_q^{-1}$. In other words, B_q and B_q^{-1} are mutually symmetric with respect to the "center" of them.

Proof. We use induction on q. If $q=1$, then it is easy to check that $B_1C_1=E$. We assume that $B_iC_i=E$ for $i=1, \cdots, q-1$. Now let $b_i^{(q)}$ be the i-th row of B_q and let $c_j^{(q)}$ be the j-th column of C_q. We can write

$$B_q=\begin{pmatrix} * & B_{q-1} \\ (-1)^q & \mathbf{0} \end{pmatrix}, \qquad C_q=\begin{pmatrix} \mathbf{0} & c_{q+1}^{(q)} \\ C_{q-1} & \end{pmatrix}.$$

Hence by the induction assumption, it suffices to prove

$$b_i^{(q)}c_{q+1}^{(q)}=\delta_{i,q+1}$$

for $i=1, \cdots, q+1$. Now it is easy to check that

$$\sum_{k=1}^{i} b_{kj}^{(q)}=b_{i,j+1}^{(q)}=b_{ij}^{(q-1)}$$

for any i, j $(j \leqq q)$. Hence we have

$$b_1^{(q)}+b_2^{(q)}+\cdots+b_i^{(q)}=(b_i^{(q-1)}\ 1) \quad (i=1, \cdots, q) \quad \text{and}$$
$$b_1^{(q)}+b_2^{(q)}+\cdots+b_{q+1}^{(q)}=(\mathbf{0}\ 1).$$

From this we can deduce

$$b_i^{(q)}=(b_i^{(q-1)}\ 1)-(b_{i-1}^{(q-1)}\ 1) \quad (i=2, \cdots, q).$$

Similarly we have

$$c_{q+1}^{(q)}=c_q^{(q)}-\begin{pmatrix} c_q^{(q-1)} \\ 0 \end{pmatrix}.$$

Now it is easy to see that

$$b_1^{(q)}c_{q+1}^{(q)}=0 \quad \text{and} \quad b_{q+1}^{(q)}c_{q+1}^{(q)}=1.$$

On the other hand if $2 \leqq i \leqq q$, then

$$\begin{aligned} b_i^{(q)}c_{q+1}^{(q)} &= b_i^{(q)}\left(c_q^{(q)}-\begin{pmatrix} c_q^{(q-1)} \\ 0 \end{pmatrix}\right) \\ &= -b_i^{(q)}\begin{pmatrix} c_q^{(q-1)} \\ 0 \end{pmatrix} \\ &= ((b_{i-1}^{(q-1)}\ 1)-(b_i^{(q-1)}\ 1))\begin{pmatrix} c_q^{(q-1)} \\ 0 \end{pmatrix} \\ &= 0 \end{aligned}$$

by the induction assumption (the first equality follows from the fact that $b_i^{(q)}c_q^{(q)}=b_i^{(q-1)}c_q^{(q-1)}$). This completes the proof.

Lemma 3.6. *For each q, let $B_{q,s}^{(r)}$ $(1 \leqq r \leqq q+1,\ 1 \leqq s \leqq q+2-r)$ be the matrix defined by*

$$B_{q,s}^{(r)}=\begin{pmatrix} b_{1\,s}^{(q)} & b_{1\,s+1}^{(q)} \cdots b_{1\,s+r-1}^{(q)} \\ b_{r\,s}^{(q)} & b_{r\,s+1}^{(q)} \cdots b_{r\,s+r-1}^{(q)} \end{pmatrix}.$$

Then we have $\det B_{q,s}^{(r)}=1$ *for all r, s.*

Proof. First observe that $B_{q,s}^{(r)}=B_{q-s+1,1}^{(r)}$. Hence we may assume that $s=1$ and we simply wirte $B_q^{(r)}$ instead of $B_{q,1}^{(r)}$. If $r=q+1$, then $\det B_q^{(q+1)}=\det B_q=1$. So assume that $r<q+1$. As in the proof of Lemma 3.5, we have

$$\sum_{k=1}^{i} b_{kj}^{(q)}=b_{ij}^{(q-1)}$$

for any i, j $(j \leqq q)$. Hence if we define $\bar{B}_q^{(r)}$ to be the matrix obtained from $B_q^{(r)}$ by the following rule:

$$\text{the } i\text{-th row of } \bar{B}_q^{(r)} = \sum_{k=1}^{i} (\text{the } k\text{-th row of } B_q^{(r)}),$$

then we have

$$\bar{B}_q^{(r)} = B_{q-1}^{(r)}$$

and clearly $\det B_q^{(r)} = \det \bar{B}_q^{(r)} = \det B_{q-1}^{(r)}$. Hence inductively we have

$$\det B_q^{(r)} = \det B_{q-1}^{(r)} = \cdots = \det B_{r-1}^{(r)} = \det B_{r-1} = 1.$$

This completes the proof.

Lemma 3.7. *Assume that q is even and $p \neq 3$. Then we have*

$$\operatorname{rank}(B_q^2(p) + B_q(p) + E) = 2k+1 \qquad \textit{if } q = 6k,\ 6k+2 \textit{ or } 6k+4.$$

Proof. Clearly we have

$$\operatorname{rank}(B_q^2(p) + B_q(p) + E) \leqq \operatorname{rank}(B_q^2 + B_q + E).$$

Hence, in view of Lemma 3.4, we have only to show the existence of a minor determinant of $(B_q^2 + B_q + E)$ of degree $2k+1$ (for $q = 6k$, $6k+2$ or $6k+4$), which is a power of 3. Now observe that if $i+j > q+2$, then

$$b_{ij}^{(q)} = 0.$$

We are assuming that q is even so that $B_q^2 = B_q^{-1}$ (see Lemma 3.1 (ii)). Hence by Lemma 3.5, if $i+j < q+2$, then

$$c_{ij}^{(q)} = 0.$$

Therefore the (i, j)-component of $B_q^2 + B_q + E$ coincides with that of B_q if (i, j) belongs to the set

$$K = \{(i, j);\ i+j < q+2 \text{ and } j > i\}.$$

If $q = 6k+2$ or $6k+4$, then it is easy to see that the minor matrix $B_{q,2k+2}^{(2k+1)}$ of B_q is completely contained in the region of B_q corresponding to K so that $B_{q,2k+2}^{(2k+1)}$ can also be considered to be a minor matrix of $B_q^2 + B_q + E$. But we have

$$\det B_{q,2k+2}^{(2k+1)} = 1$$

by Lemma 3.6. Now if $q = 6k$. then the bottom elements of the first and the last columns of $B_{q,2k+1}^{(2k+1)}$ are not contained in the region of B_q corresponding to K. If we denote $D_{q,2k+1}^{(2k+1)}=(d_{ij})$ for the corresponding minor matrix of $B_q^2+B_q+E$, then all the entries of $D_{q,2k+1}^{(2k+1)}$ coincide with those of $B_{q,2k+1}^{(2k+1)}$ except the following two components:

$$d_{2k+1,1}=b_{2k+1,2k+1}^{(q)}+1$$
$$d_{2k+1,2k+1}=b_{2k+1,4k+1}^{(q)}+1=2.$$

Here we have used Lemma 3.5 to deduce the second equality. Then by Lemma 3.6, we conclude that

$$\det D_{q,2k+1}^{(2k+1)}=3.$$

This completes the proof.

§ 4. $H^*(SL_2\boldsymbol{Z}; k[x, y])$

In this section we compute $H^*(SL_2\boldsymbol{Z}; k[x, y])$ for $k=\boldsymbol{Q}$ or $\boldsymbol{Z}_p$ $(p\neq 2, 3)$.

Recall that we denote $L_q(p)$ for $L_q\otimes\boldsymbol{Z}_p$ if p is a prime or for $L_q\otimes\boldsymbol{Q}$ if $p=0$. Now let $Z^1(SL_2\boldsymbol{Z}: L_q(p))$ be the set of all 1-cocycles of $SL_2\boldsymbol{Z}$ with values in $L_q(p)$, namely it is the set of all crossed homomorphisms

$$f\colon SL_2\boldsymbol{Z}\longrightarrow L_q(p).$$

Since $SL_2\boldsymbol{Z}$ is generated by two elements α and β, a crossed homomorphism $f\colon SL_2\boldsymbol{Z}\to L_q(p)$ is completely determined by two values $f(\alpha)$ and $f(\beta)$. Moreover the two relations $\alpha^4=1$ and $\alpha^2=\beta^3$ imply

$$(A_q^3(p)+A_q^2(p)+A_q(p)+E)f(\alpha)=0$$
$$(A_q(p)+E)f(\alpha)=(B_q^2(p)+B_q(p)+E)f(\beta).$$

Conversely if two elements $f(\alpha)$ and $f(\beta)$ of $L_q(p)$ satisfy the above two equations, then there is defined the associated crossed homomorphism $f\colon SL_2\boldsymbol{Z}\to L_q(p)$ with prescribed values at α, β. If we combine the above argument with Lemma 3.1, we can conclude

Lemma 4.1. (i) *If q is odd, then*

$$Z^1(SL_2\boldsymbol{Z}; L_q(p))=\{(u, v)\in L_q(p)\times L_q(p);\ (A_q(p)+E)u =(B_q^2(p)+B_q(p)+E)v\}.$$

(ii) *If q is even, then*

$$Z^1(SL_2\boldsymbol{Z}; L_q(p))=\{(u, v)\in L_q(p)\times L_q(p);\ (A_q(p)+E)u=0,\ (B_q^2(p)+B_q(p)+E)v=0\}.$$

Now let

$$\delta\colon L_q(p) \longrightarrow Z^1(SL_2Z;\ L_q(p))$$

be the homomorphism defined by

$$\delta(u)(\gamma) = (\gamma - 1)u \quad (u \in L_q(p),\ \gamma \in SL_2Z).$$

Then by the definition of cohomology of groups, we have

$$\begin{aligned} H^0(SL_2Z;\ L_q(p)) &= \operatorname{Ker}\delta \\ &= \{u \in L_q(p);\ A_q(p)u - u = B_q(p)u - u = 0\} \quad \text{and} \\ H^1(SL_2Z;\ L_q(p)) &= \operatorname{Cok}\delta. \end{aligned}$$

Proposition 4.2. $H^0(SL_2Z;\ Q[x, y]) = Q$.

Proof. It suffices to prove that the only polynomials in $Q[x, y]$ which are left invariant under the action of SL_2Z are constants. This follows from a direct computation details of which are omitted.

Remark 4.3.*) According to a classical result of Dickson [2] (see also Tezuka [8]), the subring of $Z_p[x, y]$ consisting of those elements which are invariant by the action of SL_2Z, namely $H^0(SL_2Z; Z_p[x, y])$, is the polynomial ring generated by the following two elements

$$x^p y - x y^p \quad \text{and} \quad \frac{x^{p^2} y - x y^{p^2}}{x^p y - x y^p} \equiv y^{p(p-1)} + (x^p - x y^{p-1})^{p-1}.$$

Hence if we write $d_q(p)$ for $\dim H^0(SL_2Z;\ L_q(p))$, then we have

$$\sum_{q=0}^{\infty} d_q(p) t^q = \frac{1}{(1 - t^{p+1})(1 - t^{p(p-1)})}.$$

Proposition 4.4. *If q is odd and $p \neq 2$, then*

$$H^0(SL_2Z;\ L_q(p)) = H^1(SL_2Z;\ L_q(p)) = 0.$$

Proof. According to Corollary 3.2, $B_q(p) - E$ is invertible and so the homomorphism $\delta\colon L_q(p) \to Z^1(SL_2Z;\ L_q(p))$ is injective. Hence $H^0(SL_2Z;\ L_q(p)) = 0$. Next let $(u, v) \in Z^1(SL_2Z; L_q(p))$ be any element (see Lemma 4.1 (i)) so that

$$(A_q(p) + E)u = (B_q^2(p) + B_q(p) + E)v.$$

*) I owe this remark to M. Tezuka. I would like to express my hearty thanks to him.

By Corollary 3.2, we have

$$u=-\frac{1}{2}(A_q(p)-E)(B_q^2(p)+B_q(p)+E)v.$$

Since $B_q(p)-E$ is invertible, there is an element $w \in L_q(p)$ such that $v=(B_q(p)-E)w$. Then

$$u=(A_q(p)-E)w.$$

Therefore

$$(u, v)=((A_q(p)-E)w, (B_q(p)-E)w)=\delta w$$

and hence $H^1(SL_2\boldsymbol{Z}; L_q(p))=0$. This completes the proof.

Henceforth we assume that q is even and consider $H^1(SL_2\boldsymbol{Z}; L_q(p))$. According to Lemma 4.1 (ii), we have an identification

$$Z^1(SL_2\boldsymbol{Z}; L_q(p))=L_q^-(p)\oplus L_q'(p) \qquad (p\neq 2)$$

where $L_q^-(p)$ and $L_q'(p)$ have been defined in Section 3.

Proposition 4.5. *If q is even, then*

$$\dim H^1(SL_2\boldsymbol{Z}; L_q(0))=\begin{cases} 2m-1 & q=12m \\ 2m+1 & q=12m+2,\ 12m+4,\ 12m+6, \\ & \qquad\qquad \text{or } 12m+8 \\ 2m+3 & q=12m+10. \end{cases}$$

Proof. We know that the homomorphism δ; $L_q(0)\to Z^1(SL_2\boldsymbol{Z}; L_q(0))$ is injective (Proposition 4.2). Hence we have

$$\begin{aligned}\dim H^1(SL_2\boldsymbol{Z}; L_q(0)) &=\dim Z^1(SL_2\boldsymbol{Z}; L_q(0))-(q+1) \\ &=\dim L_q^-(0)+\dim L_q'(0)-(q+1).\end{aligned}$$

Then the result follows from Lemma 3.2 and Lemma 3.4.

Proposition 4.6. *Assume q is even and let $d_q(p)=\dim H^0(SL_2\boldsymbol{Z}; L_q(p))$ (see Remark 4.3). Then for $p\neq 2, 3$, we have*

$$\dim H^1(SL_2\boldsymbol{Z}; L_q(p))=\dim H^1(SL_2\boldsymbol{Z}; L_q(0))+d_q(p).$$

Proof. By a similar argument as in the proof of Proposition 4.5, we have

$$\dim H^1(SL_2\boldsymbol{Z}; L_q(p))=\dim L_q^-(p)+\dim L_q'(p)-(q+1)+d_q(p).$$

Then the result follows because we have

$$\dim L_q^-(p) = \dim L_q^-(0) \qquad (p \neq 2)$$

by Lemma 3.2 and also we have

$$\dim L_q'(p) = \dim L_q'(0) \qquad (p \neq 3)$$

by Lemma 3.4 and Lemma 3.7. This completes the proof.

§ 5. Proof of Theorems

Theorem 1.1 follows from Corollary 2.2, Proposition 4.2 and Proposition 4.5. Also, if $p \neq 2, 3$, Corollary 2.2, Proposition 4.4 and Proposition 4.6 imply

$$\dim H^n(B\,\mathrm{Diff}_+ T^2; Z_p) = \begin{cases} d_q(p) & n = 2q \ (q\text{: even}) \\ \dim H^n(B\,\mathrm{Diff}_+ T^2; Q) + d_q(p) & n = 2q+1 \\ & (q\text{: even}) \\ 0 & n \equiv 2, 3 \pmod 4. \end{cases}$$

Hence if $n \equiv 2, 3 \pmod 4$, then

$$H_n(B\,\mathrm{Diff}_+ T^2; Z) = 0 \qquad \text{mod } 2, 3 \text{ torsions}$$

by the universal coefficient theorem. Similarly it is easy to deduce that $H_n(B\,\mathrm{Diff}_+ T^2; Z)$ has no p-torsions ($p \neq 2, 3$) if $n \equiv 1 \pmod 4$. This completes the proof of Theorem 1.2.

Remark 5.1. $H_*(B\,\mathrm{Diff}_+ T^2; Z)$ has actually 2 and 3 torsions. This follows from the following argument. The projection $B\,\mathrm{Diff}_+ T^2 \to K(SL_2Z, 1)$ has a right inverse because SL_2Z can be naturally considered as a subgroup of $\mathrm{Diff}_+ T^2$. Hence the homology

$$H_*(SL_2Z; Z) \cong H_*(K(Z_{12}, 1); Z)$$

embeds into $H_*(B\,\mathrm{Diff}_+ T^2; Z)$ as a direct summand. It is easy to check that $H_1(B\,\mathrm{Diff}_+ T^2; Z) \cong Z_{12}$ and $H_2(B\,\mathrm{Diff}_+ T^2; Z) = 0$.

Remark 5.2. By Theorem 1.1 and Theorem 1.2, we have an isomorphism

$$H^{4k}(B\,\mathrm{Diff}_+ T^2; Z_p) \cong \mathrm{Hom}\,(H_{4k}(B\,\mathrm{Diff}_+ T^2; Z), Z_p) \qquad (p \neq 2, 3).$$

On the other hand we have

$$H^{4k}(B\,\mathrm{Diff}_+ T^2; Z_p) \cong L_{2k}(p)^{SL_2Z}$$

by Corollary 2.2, where the right hand side denotes the subspace of $L_{2k}(p)$ consisting of those elements which are left invariant by the action of $SL_2\boldsymbol{Z}$. Then in view of Remark 4.3, we can conclude that the p-primary part of $H_{4k}(B\,\mathrm{Diff}_+ T^2; \boldsymbol{Z})$ is non-trivial provided $2k$ can be expressed as a linear combination of $p+1$ and $p(p-1)$ with coefficients in non-negative integers. Also it can be shown that mod 2 and 3 torsions we have an isomorphism

$$H_{4k}(B\,\mathrm{Diff}_+ T^2; \boldsymbol{Z}) \cong L_{2k}/K_{2k}$$

where K_{2k} denotes the submodule of L_{2k} generated by elements $\gamma(u)-u$ ($u \in L_{2k}$, $\gamma \in SL_2\boldsymbol{Z}$).

Example 5.3. We construct an element of $H_5(B\,\mathrm{Diff}_+ T^2; \boldsymbol{Z})$ which has infinite order. First it can be shown by a direct computation that the crossed homomorphism

$$f\colon SL_2\boldsymbol{Z} \longrightarrow L_2(0)$$

given by $f(\alpha)=x^2-y^2$ and $f(\beta)=0$ represents a non-zero element of $H^1(SL_2\boldsymbol{Z}; L_2(0)) \cong \boldsymbol{Q}$ (see Proposition 4.5). We write $[f] \in H^5(B\,\mathrm{Diff}_+ T^2; \boldsymbol{Q})$ for the corresponding element (see Corollary 2.2). Now let η be the canonical line bundle over $\boldsymbol{CP}^2$ and let $T^2 \to E(k, l) \to \boldsymbol{CP}^2$ be the T^2-bundle associated to the complex 2-plane bundle $\eta^k \oplus \eta^l$ on $\boldsymbol{CP}^2$ ($k, l \in \boldsymbol{Z}$). Let $T^2 \to E'(k, l) \to \boldsymbol{CP}^1$ be the restriction of $E(k, l)$ to $\boldsymbol{CP}^1 \subset \boldsymbol{CP}^2$. Then we can write

$$E'(k.\, l) = D^2 \times S^1 \times S^1 \bigcup_{g_{k,l}} D^2 \times S^1 \times S^1$$

where the pasting map $g_{k,l}\colon \partial D^2 \times S^1 \times S^1 \to \partial D^2 \times S^1 \times S^1$ is given by

$$g_{k,l}(z_1, z_2, z_3) = (z_1^{-1}, z_1^k z_2, z_1^l z_3)$$

($z_1 \in \partial D^2$, $z_2, z_3 \in S^1$). Now for an element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\boldsymbol{Z}$, let $h_\gamma\colon D^2 \times S^1 \times S^1 \to D^2 \times S^1 \times S^1$ be the diffeomorphism defined by

$$h_\gamma(z_1, z_2, z_3) = (z_1, z_2^a z_3^b, z_2^c z_3^d)$$

($z_1 \in D^2$, $z_2, z_3 \in S^1$). It is easy to show that if two relations:

$$ak+bl=k \quad \text{and} \quad ck+dl=l$$

are satisfied, then h_γ extends to a diffeomorphism $h'_\gamma\colon E'(k, l) \to E'(k, l)$ which is an automorphism as a T^2-bundle. Then since $\pi_3(\mathrm{Diff}_+ T^2)=0$, we can extend h'_γ to an automorphism H_γ; $E(k, l) \to E(k, l)$. H_γ is nothing but the automorphism of $E(k, l)$ as a *principal* T^2-bundle defined by the automorphism of T^2 given by γ. Let $M_\gamma(k, l)$ be the mapping torus of H_γ. The natural projection

$$M_\gamma(k, l) \longrightarrow S^1 \times CP^2$$

has the structure of a T^2-bundle. Clearly the classifying map of this T^2-bundle is given by

$$\begin{tikzcd} CP^2 \arrow[r] \arrow[d, "i_0"'] & S^1 \times CP^2 \arrow[r] \arrow[d, "i"'] & S^1 \arrow[d, "\bar{i}"'] \\ B\,\mathrm{Diff}_0 T^2 \arrow[r] & B\,\mathrm{Diff}_+ T^2 \arrow[r] & K(SL_2 Z, 1) \end{tikzcd}$$

where i_0 is characterized by the induced map $i_0^*: H^2(B\,\mathrm{Diff}_0 T^2; Z) \to H^2(CP^2; Z)$ which is given by $i_0^*(x) = k\iota$, $i_0^*(y) = l\iota$ ($\iota \in H^2(CP^2; Z)$ is the first Chern class of η) and the map $\bar{i}$ represents $\gamma^{-1} \in \pi_1(K(SL_2 Z, 1)) = SL_2 Z$. Therefore we conclude that

$$\langle [S^1 \times CP^2], i^*([f]) \rangle = i_0^*(f(\gamma^{-1})) \in H^4(CP^2; Q) \cong Q.$$

If we choose $\gamma = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ and $k = l = 1$, then $\gamma = \beta^{-1}\alpha\beta^{-1}$ so that $f(\gamma^{-1}) = y^2 - 2xy$ and hence $i_0^*(f(\gamma^{-1})) = -\iota^2$. This proves that the corresponding T^2-bundle represents a non-zero element of $H_5(B\,\mathrm{Diff}_+ T^2; Q)$. Similarly we can construct non-zero elements of $H_{4k+1}(B\,\mathrm{Diff}_+ T^2; Q)$ $(k > 1)$ explicitly, but we stop here.

References

[1] Brown, K. S., Cohomology of Groups, Graduate Texts in Mathematics, **87**, Springer Verlag 1982.

[2] Dickson, L., A fundamental system of invariants of the general linear group with a solution of the form problem, Trans. Amer. Math. Soc., **12** (1911), 75–98.

[3] Earle, C. J. and Eells, J., A fibre bundle description of Teichmüller theory, J. Differential Geom., **3** (1969), 19–43.

[4] Morita, S., Characteristic classes of surface bundles, Bull. Amer. Math. Soc., **11** (1984), 386–388.

[5] ——, Characteristic classes of surface bundles, preprint.

[6] Serre, J. P., Arbres, amalgames, SL_2, Astérisque, **46** (1977).

[7] Smale, S., Diffeomorphisms of the 2-sphere, Proc. Amer. Math. Soc., **10** (1959), 621–626.

[8] Tezuka, M., The cohomology of $SL_2(F_p)$ and the Hecke algebra actions, preprint.

Department of Mathematics
College of Arts and Sciences
University of Tokyo
Tokyo 153, Japan

Current Address
Department of Mathematics
Faculty of Sciences
Tokyo Institute of Technology
Tokyo 152, Japan