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The Kervaire Invariant of Some Fiber Homotopy Equivalences

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§ 0. Introduction

Suppose we are given a proper fiber preserving map $\xi: E_+ \to E_-$ of degree d between vector bundles $E_\pm$ over a closed smooth manifold X. Then transversality theorem says that one can convert ξ via a proper homotopy into a map h transverse to the zero section X in E_-. This produces a triple $\kappa = (W, f, \beta)$ called a normal map of degree d; $W = h^{-1}(X)$, $f = h|W: W \to X$ is a degree d map, and $\beta: TW \to f^*(TX + E_+ - E_-)$ is a stable vector bundle isomorphism. We perform surgery on κ via normal cobordism to make f a homology (or homotopy) equivalence with an appropriate coefficient. Unfortunately this is not always possible. We encounter an obstruction called surgery obstruction. Needless to say, to know whether or not the surgery obstruction vanishes is a substantial problem which is studied by many people. Among surgery obstructions the Kervaire invariant (or sometimes called the Kervaire obstruction) has a rich history and is an object with outstanding interest. It is defined with a value in $\boldsymbol{Z}/2 = \{0, 1\}$ provided that the dimension of X is even and d is odd. Since it depends only on the given map ξ, we will denote it by $c(\xi) \in \boldsymbol{Z}/2$.

In this paper we are concerned with the Kervaire invariant obtained in the following way. Let G be a (compact) Lie group. Suppose we are given a principal G bundle $\pi: P \to X$ and a proper G map $\omega: V \to U$ of an odd degree d between G representations. Then we combine them to get a proper fiber preserving map of degree d, written ω_π, between associated vector bundles $P \times_G V$ and $P \times_G U$. Obviously $c(\omega_\pi)$ must be described in terms of the given data ω and π. The purpose of this article is to give an explicit description in some cases.

As a consequence there exists a fiber homotopy equivalence with the Kervaire invariant one over some X. For instance this is the case for $X = P(\boldsymbol{C}^n)$ (n: even) the complex projective space of complex dimension

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$n-1$. The reader should notice that the case $X=S^{2^i-2}$ corresponds to the famous Kervaire invariant conjecture ([B]). It is beyond our scope, but the method used in the proof of Theorem 3.1 may be useful to the conjecture.

This paper is organized as follows. In Section 1 we review the definition of the Kervaire invariant and recall Sullivan's characteristic variety formula. An invariant $K(\omega)$ is introduced in Section 2, which is our main object. Section 3 is the main part of this article in which we consider a special proper S^1 map $\omega^{p,q}$ and compute $K(\omega^{p,q})$. The tools used for the computation are Sullivan's characteristic variety formula and Z_2 surgery theory developed by T. Petrie. Section 4 treats a proper T^n map of degree one. In Section 5 we view $K(\omega)$ via the Burnside ring. Then the results of the previous sections are interpreted in terms of the Burnside ring. In Section 6 we observe how Theorem 3.1 is related to the result of Morita [Mo] and Wood [Wo].

Finally I would like to express my gratitude to T. Petrie who suggested me to study $\omega_n^{p,q}$ of Section 3. It was a starting point of this research. I also thank Y. D. Tsai for kindly sending me a copy of the R. Schultz's letter [S] in which he obtained the same result as Theorem 3.1 by a different method. The sections 5 and 6 are motivated by his observation.

§ 1. The Kervaire invariant and Sullivan's characteristic variety formula

First we shall give the definition of the Kervaire invariant following 13B of [Wa 2]. Let $\xi: E_+\to E_-$ be a proper fiber preserving map over a closed connected smooth manifold X of dimension $2n$. From now on we will assume the fiber degree d of ξ (up to sign) is odd and call such ξ a *mod 2 fiber homotopy equivalence* (the motivation of this term is Dold's theorem which asserts that ξ is a fiber homotopy equivalence iff $d=\pm 1$).

As explained in the Introduction ξ produces a normal map $\kappa=(W, f, \beta)$ of degree d. We perform surgery on κ via normal cobordism to make f a homology equivalence (inducing an isomorphism on the fundamental class) with $Z_{(2)}$ coefficients, where $Z_{(2)}$ denotes the ring obtained by localizing Z by the ideal generated by 2. The surgery obstruction, written σ, lies in an L group $L_{2n}(Z_{(2)}[\pi_1X], w)$. Here $Z_{(2)}[\pi_1X]$ is a group ring of π_1X with coefficient $Z_{(2)}$ and w is a homomorphism from π_1X to $Z_2=\{\pm 1\}$ which assigns 1 (resp. -1) to $g\in\pi_1X$ according as g preserves (resp. reverses) an orientation on the universal cover of X. Since L group is a covariant functor, w induces a homomorphism $w_*: L_{2n}(Z_{(2)}[\pi_1X], w)\to L_{2n}(Z_{(2)}[Z_2], \mathrm{id})$ where id stands for the identity map on Z_2. It is not difficult to see that the latter group is isomorphic to $Z/2=\{0, 1\}$ (cf. § 4 of

[Wa 1] and 13A of [Wa 2]). The Kervaire invariant $c(\xi)$ is then defined as the value of $w_* \sigma$ in $\boldsymbol{Z}/2$ through this isomorphism. In other words the Kervaire invariant is the Arf invariant of a quadratic function defined on the kernel of the induced homomorphism f_* from $H_n(W; \boldsymbol{Z}/2)$ to $H_n(X; \boldsymbol{Z}/2)$ (see § 5 of [BM]). In some cases it is a complete surgery obstruction; for example, when $\pi_1 X$ is trivial and n is odd.

In general it is not so easy to compute the Kervaire invariant directly from the definition. We shall recall a useful formula for the computation called Sullivan's characteristic variety formula. Following [BM], we denote by $Q(S^0)/0$ the classifying space of a fiber preserving map. The connected components of $Q(S^0)/0$ are exactly the classifying space $(Q(S^0)/0)_d$ for fiber preserving maps of a fixed degree d $(d \geqq 0)$ up to sign. The degree of the Whitney sum is the product of the degrees of the factors, hence the Whitney sum corresponds to a pairing of classifying spaces

$$\mu\colon (Q(S^0)/0)_{d_1} \times (Q(S^0)/0)_{d_2} \longrightarrow (Q(S^0)/0)_{d_1 d_2}.$$

Sullivan's characteristic variety formula (see Propositions 5.5, 5.6 and Remark 5.7 of [BM]). *Let Γ be the monoid of odd positive integers and let $(Q(S^0)/0)_\Gamma$ be the union of $(Q(S^0)/0)_d$, $d \in \Gamma$. Let $\alpha(\xi)$ be a map from X to $(Q(S^0)/0)_d$ which classifies a mod 2 fiber homotopy equivalence ξ of degree d over X. Then there is a unique cohomology class k (called the Kervaire-Sullivan class)*

$$k = \sum k_{2i-2}, \quad k_{2i-2} = \sum_{d \in \Gamma} k_{2i-2,d}, \quad k_{2i-2,d} \in H^{2i-2}((Q(S^0)/0)_d; \boldsymbol{Z}/2)$$

such that

$$c(\xi) = \langle v^2(X) \cup \alpha(\xi)^*(k), [X]_2 \rangle \in \boldsymbol{Z}/2$$

where $v(X)$ is the total Wu class, $[X]_2$ is the mod 2 fundamental class of X, and

$$k_{0,d} = \begin{cases} 1 & \text{if } d \equiv \pm 3 \pmod 8 \\ 0 & \text{if } d \equiv \pm 1 \pmod 8). \end{cases}$$

Moreover k is primitive, i.e.,

$$\mu^*(k) = k \otimes 1 + 1 \otimes k$$

with respect to the Whitney sum

$$\mu\colon (Q(S^0)/0)_\Gamma \times (Q(S^0)/0)_\Gamma \longrightarrow (Q(S^0)/0)_\Gamma.$$

The homotopy set $[X, (Q(S^0)/0)_\Gamma]$ turns out to be an abelian group by virtue of μ. Hence if we regard the Kervaire invariant as a function

$c: [X, (Q(S^0)/0)_{\Gamma}] \to Z/2$, then the primitivity of k means that c is a homomorphism.

§ 2. Definition of $K(\omega)$

Let $\pi^{(m)}: EG^{(m)} \to BG^{(m)}$ be a universal G bundle which classifies principal G bundles over simplicial complexes of dimension less than m. Take a large m and let f be a classifying map of a principal G bundle $\pi: P \to X$. Clearly the mod 2 fiber homotopy equivalence ω_π associated with π and ω is induced from $\omega_{\pi^{(m)}}$ via f. This means that

$$\alpha(\omega_\pi)^* k = f^* \alpha(\omega_{\pi^{(m)}})^* k.$$

Therefore by Sullivan's characteristic variety formula it is essentially sufficient to calculate $\alpha(\omega_{\pi^{(m)}})^* k$, because we may assume that the Wu class $v(X)$ and f are understood.

Since m is a sufficiently large integer depending on the dimension of X and k_0 is already known, it is convenient to introduce a notation

$$K(\omega) = \varprojlim \alpha(\omega_{\pi^{(m)}})^* (k - k_0) \in \varprojlim H^*(BG^{(m)}; Z/2) = H^*(BG; Z/2).$$

Our aim is to decide $K(\omega)$ explicitly.

The reader should notice that 2-groups are important because $H^*(BG; Z/2)$ can be detected by 2-subgroups of G in most cases.

§ 3. A special case

About 1970 T. Petrie discovered an interesting proper S^1 map of degree one, written $\omega^{p,q}$ below, between S^1 representations. This section is devoted to the computation of $K(\omega^{p,q})$.

Let t^s denote the s times tensor product of the complex one dimensional standard S^1 module. Let p and q be relatively prime positive integers and choose integers a and b satisfying $-ap+bq=1$. Then we define

$$\omega^{p,q}: t^p + t^q = V^{p,q} \longrightarrow t + t^{pq} = U^{p,q}$$

by

$$\omega^{p,q}(z_1, z_2) = (\bar{z}_1^a z_2^b, z_1^q + z_2^p).$$

It is not difficult to check that $\omega^{p,q}$ is a proper S^1 map of degree one (see Section 2 of [MaP]). The fiber homotopy equivalence associated with the Hopf S^1 bundle $\pi_n: S(C^n) \to P(C^n)$ will be abbreviated by $\omega_n^{p,q}: V_n^{p,q} \to U_n^{p,q}$. With these understood

Theorem 3.1. *Let x be the generator of $H^2(BS^1; Z/2)$. Then*

$$K(\omega^{p,q})=(p^2-1)(q^2-1)/24 \sum x^{2^i-1}.$$

This is equivalent that

$$c(\omega_n^{p,q})=\begin{cases}(p^2-1)(q^2-1)/24 \pmod 2 & \textit{if } n \textit{ is even},\\ 0 & \textit{if } n \textit{ is odd}.\end{cases}$$

Remark 3.2. If n is odd, then $\dim P(C^n)\equiv 0 \pmod 4$. In addition $P(C^n)$ is simply connected. Hence $c(\omega_n^{p,q})=0$ by a general result (see 13A, B of [Wa2]).

The equivalence of the above two statements can be seen as follows. As is well known (see p. 134 of [Mi2]), we have

$$v(P(C^n))=\sum \binom{n-1-j}{j} x^j. \tag{3.3}$$

Write $K(\omega^{p,q})=\sum K_{2^i-2}(\omega^{p,q})x^{2^{i-1}-1}$ $(K_{2^i-2}(\omega^{p,q}) \in Z/2)$ and let $n\equiv 2^{r-1}$ (mod 2^r). Then an elementary calculation yields

$$c(\omega_n^{p,q})=K_{2^r-2}(\omega^{p,q}), \tag{3.4}$$

which verifies the desired equivalence. The check of (3.4) is left to the reader.

By Remark 3.2 we may assume n is even. Our computation consists of two parts.

Assertion. (1) $K_2(\omega^{p,q})=(p^2-1)(q^2-1)/24 \pmod 2$.
(2) $K_{2^r-2}(\omega^{p,q})=K_{2^{r-1}-2}(\omega^{p,q})$ for any $r\geqq 3$.

The proof of (1) is not so difficult. The crucial point in our computation is the verification of (2). The outline is as follows. Set $n=2m$ and suppose m is even. Then by (3.4) we have

$$c(\omega_{2m}^{p,q})=K_{2^r-2}(\omega^{p,q}), \qquad c(\omega_m^{p,q})=K_{2^{r-1}-2}(\omega^{p,q}). \tag{3.5}$$

Therefore it suffices to prove $c(\omega_{2m}^{p,q})=c(\omega_m^{p,q})$. For that purpose we equip $\omega_{2m}^{p,q}$ with an involution such that $\omega_m^{p,q}$ appears as a fixed part. Then Z_2 surgery theory allows us to compare their Kervaire invariants and establish the equality of them.

Proof of Assertion (1). By Lemma 14C.1 of [Wa2]

$$K_2(\omega^{p,q})=s_4(\alpha(\omega_n^{p,q})) \pmod 2 \qquad \text{for } n\geqq 3, \tag{3.6}$$

where $s_4(\alpha(\omega_n^{p,q}))$ is a so-called splitting invariant of a normal map $\alpha(\omega_n^{p,q})$. It is the surgery obstruction of a normal map $\kappa=(W, f, \beta)$ obtained from $\omega_3^{p,q}$ and has an expression

$$8s_4(\alpha(\omega_n^{p,q}))=\text{Sign } W-\text{Sign } P(\boldsymbol{C}^3). \tag{3.7}$$

The Hirzebruch Signature theorem together with the stable isomorphism β: $TW\simeq f^*(TP(\boldsymbol{C}^3)+V_3^{p,q}-U_3^{p,q})$ tells us that

$$\begin{aligned}\text{Sign } W-\text{Sign } P(\boldsymbol{C}^3)&=\langle p_1(W)/3, [W]\rangle-\langle p_1(P(\boldsymbol{C}^3))/3, [P(\boldsymbol{C}^3)]\rangle\\ &=\langle p_1(TP(\boldsymbol{C}^3)+V_3^{p,q}-U_3^{p,q})/3, [P(\boldsymbol{C}^3)]\rangle-\langle p_1(P(\boldsymbol{C}^3))/3, [P(\boldsymbol{C}^3)]\rangle\\ &=\langle p_1(V_3^{p,q}-U_3^{p,q})/3, [P(\boldsymbol{C}^3)]\rangle\end{aligned} \tag{3.8}$$

where $p_1(\ )$ denotes the first Pontrjagin class. On the other hand it follows immediately from the definition of $V_3^{p,q}$ and $U_3^{p,q}$ that

$$V_3^{p,q}=\gamma^p\oplus\gamma^q, \qquad U_3^{p,q}=\gamma\oplus\gamma^{pq} \tag{3.9}$$

where γ^s is the s times tensor product bundle of the Hopf line bundle γ over $P(\boldsymbol{C}^3)$. Since $\langle p_1(\gamma^s), [P(\boldsymbol{C}^3)]\rangle=s^2$ and p_1 behaves additively with respect to the Whitney sum, (3.6)–(3.9) prove Assertion (1).

Proof of Assertion (2). Consider an involution on $\boldsymbol{C}^{2m}$ defined by $(z_1, \cdots, z_{2m})\rightarrow(z_1, \cdots, z_m, -z_{m+1}, \cdots, -z_{2m})$ and denote the resulting $\boldsymbol{Z}_2$ space by $\boldsymbol{C}^{m,m}$. Then the Hopf S^1 bundle naturally inherits the involution and $\omega_{2m}^{p,q}$ becomes equivariant, which will be denoted by $\omega_{m,m}^{p,q}$. Here we consider the trivial involution on $U^{p,q}$ and $V^{p,q}$. The fixed point set of $P(\boldsymbol{C}^{m,m})$ consists of two connected components $P(\boldsymbol{C}^m\times 0)$ and $P(0\times\boldsymbol{C}^m)$, and correspondingly the fixed part $(\omega_{m,m}^{p,q})^{\boldsymbol{Z}_2}$ of $\omega_{m,m}^{p,q}$ decomposes into two pieces. In fact

$$\begin{aligned}&(\omega_{m,m}^{p,q})^{\boldsymbol{Z}_2}\,|\,P(\boldsymbol{C}^m\times 0)=\omega_m^{p,q},\\ &(\omega_{m,m}^{p,q})^{\boldsymbol{Z}_2}\,|\,P(0\times\boldsymbol{C}^m)=\begin{cases}\text{id} & \text{if both } p \text{ and } q \text{ are odd,}\\ \varphi_{p,q} & \text{if } p \text{ is even (hence } q \text{ is odd),}\\ \varphi_{q,p} & \text{if } q \text{ is even (hence } p \text{ is odd),}\end{cases}\end{aligned} \tag{3.10}$$

where id stands for the identity map on $P(0\times\boldsymbol{C}^m)$ and $\varphi_{u,v}$ is the v'th power map from γ^u to γ^{uv}.

Lemma 3.11. $c((\omega_{m,m}^{p,q})^{\boldsymbol{Z}_2}\,|\,P(0\times\boldsymbol{C}^m))=0.$

Proof. Obviously $c(\text{id})=0$; so it suffices to prove $c(\varphi_{u,v})=0$ when u is even and v is odd.

First notice that $P(C^{2m})$ admits an involution T defined by $[z_1: \cdots : z_{2m}] \to [-\bar{z}_2: \bar{z}_1: \cdots : -\bar{z}_{2m-1}: \bar{z}_{2m}]$ and it extends to γ^u and γ^{uv}. Then $\varphi_{u,v}$ turns out to be a T map. This means that $\varphi_{u,v}$ is a pullback of $\varphi_{u,v}/T: \gamma^u/T \to \gamma^{uv}/T$ by the projection map $\pi: P(C^{2m}) \to P(C^{2m})/T$, in particular

$$\alpha(\varphi_{u,v})^* k = \pi^* \alpha(\varphi_{u,v}/T)^* k.$$

Claim. $\pi^*: H^{4i+2}(P(C^{2m})/T; Z/2) \to H^{4i+2}(P(C^{2m}); Z/2)$ *is the zero map for any* i.

Proof. Consider the Serre spectral sequences associated with the following natural two fibrations over the quaternion projective space $P(H^m)$:

$$\begin{array}{ccccc} S^2 & \longrightarrow & P(C^{2m}) & \longrightarrow & P(H^m) \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \| \\ RP^2 & \longrightarrow & P(C^{2m})/T & \longrightarrow & P(H^m). \end{array}$$

Since the natural projection map from S^2 to RP^2 induces the zero map from $H^2(RP^2; Z/2)$ to $H^2(S^2; Z/2)$, π^* is already the zero map in the level of E_2-terms of degree $4i+2$. This proves the claim.

The claim means that $\alpha(\varphi_{u,v})^* k$ vanishes except for the degree 0 term. Thus Sullivan's characteristic variety formula turns into

$$c(\varphi_{u,v}) = k_{0,v} \langle v^2(P(C^{2m})), [P(C^{2m})]_2 \rangle.$$

But it follows from (3.3) that the right hand side vanishes. This completes the proof of Lemma 3.11.

Now let $\kappa = (W, f, \beta)$ $f: W \to P(C^{m,m})$ be a normal map with an involution obtained from $\omega^{p,q}_{m,m}$ through the equivariant transversality. Clearly the gap hypothesis is satisfied for κ, i.e.,

$$2 \dim P(C^{m,m})^{Z_2} < \dim P(C^{2m}), \qquad 2 \dim W^{Z_2} < \dim W.$$

Hence κ is a Z_2 normal map in the sense of [PR].

Suppose $c(\omega^{p,q}_m) = 0$. Then $c((\omega^{p,q}_{m,m})^{Z_2}) = 0$ by (3.10) and Lemma 3.11. Since m is even, the surgery obstruction to converting f^{Z_2} into a Z_2 homology equivalence is exactly $c((\omega^{p,q}_{m,m})^{Z_2})$. By the assumption it vanishes; so one can assume that f^{Z_2} is a Z_2 homology equivalence. Then the secondary surgery obstruction $\sigma(\kappa)$ is defined as the obstruction to converting f into a homotopy equivalence via Z_2 normal cobordism.

Here the benefit from Z_2 surgery theory is that $\sigma(\kappa)$ lies in $L_{4m-2}(Z[Z_2])$. Immediately from the definition of $\sigma(\kappa)$, its image through the forgetting map from $L_{4m-2}(Z[Z_2])$ to $L_{4m-2}(Z[1])$ is exactly $c(\omega_{2m}^{p,q})$. However the forgetting map is the zero map (see 13A of [Wa2]) and hence $c(\omega_{2m}^{p,q})=0$.

Next suppose $c(\omega_m^{p,q})=1$. The following lemma enables us to reduce this case to the preceding one.

Lemma 3.12. *For each even integer $m\neq 2, 4, 8$ there is a Z_2 normal map $\kappa_m=(W_m, f_m, \beta_m)$ f_m: $(W_m, \partial W_m)\to(D(R^{2m-2,2m}), S(R^{2m-2,2m}))$ such that*

(1) *∂f_m is a Z_2 homotopy equivalence,*

(2) *$c(\kappa_m)=1$ (where the involution on κ_m is forgotten),*

(3) *$c(\kappa_m^{Z_2})=1$,*

where $R^{2m-2,2m}$ denotes R^{4m-2} equipped with an involution defined by $(x_1, \cdots, x_{4m-2})\to(x_1, \cdots, x_{2m-2}, -x_{2m-1}, \cdots, -x_{4m-2})$ and $D(R^{2m-2,2m})$ (resp. $S(R^{2m-2,2m})$) is the unit disk (resp. sphere) of $R^{2m-2,2m}$.

Proof. Let δ be a small positive number and define W_m by

$$W_m=\{(z_1, \cdots, z_{2m})\in C^{m,m} \mid z_1^3+z_2^2+\cdots+z_{2m}^2=\delta\}\cap D(C^{m,m}).$$

The boundary ∂W_m is equivariantly homeomorphic to $S(R^{2m-2,2m})$ (see Theorem A of [I]); so f_m is defined by collapsing the complement of a collar boundary in W_m to a point. β_m is defined using an equivariant stable trivialization of TW_m. Then (2) is well known for $m\neq 2, 4$ (see § 8, 9 of [Mi 1] for example). Since $\kappa_m^{Z_2}$ agrees with $\kappa_{m/2}$ by the construction, (3) also follows. These prove the lemma.

Suppose $m\neq 2, 4, 8$. We choose two points; one is in the connected component of W^{Z_2} corresponding to $P(C^m\times 0)$ via f^{Z_2} and the other is in the interior of $D(R^{2m-2,2m})^{Z_2}$. We do equivariant connected sum $\kappa \# \kappa_m$ of κ with κ_m around them. The resulting Z_2 normal map then has the Kervaire invariant equal to zero on the fixed point set because the Kervaire invariant behaves additively with respect to the connected sum. Therefore the preceding argument implies $c(\kappa \# \kappa_m)=0$. Since $c(\kappa_m)=1$, this verifies $c(\kappa)=1$, i.e., $c(\omega_{2m}^{p,q})=1$.

Thus we have established

$$c(\omega_m^{p,q})=c(\omega_{2m}^{p,q}) \qquad \text{for } m\neq 2, 4, 8.$$

For each r there is a sufficiently large m such that $m\equiv 2^{r-2} \pmod{2^{r-1}}$. Hence the above identity together with (3.5) verifies Assertion (2).

§ 4. Proper T^n maps of degree one

In the previous section we considered a special S^1 map $\omega^{p,q}$ of degree one and computed $K(\omega^{p,q})$. But it is not hard to check that a similar argument to Section 3 works for any proper S^1 map of degree one between S^1 representations. Combining this with the results of Meyerhoff-Petrie [MeP] and tom Dieck-Petrie [tDP], we can determine $K(\omega)$ for any proper T^n (n-dimensional toral group) map ω of degree one between T^n representations. This is the purpose of this section.

First we shall remember a result of Meyerhoff-Petrie, which decides a (stable) pair of complex T^n representations admitting a proper T^n map of degree one. To state it we need some notations. Let $\boldsymbol{P}=\{p_1, \cdots, p_l\}$ be a set of relatively prime integers with $l\geqq 2$ and let $\boldsymbol{P}=\prod(\Psi^{p_i}-1)$ be the associated operation in the complex representation ring of T^n, where Ψ^s ($s\in \boldsymbol{Z}$) is the Adams operation.

Theorem 4.1 (Theorem 6.3 of [MeP]). *Let U and V be complex T^n representations. Then there is a proper T^n map of degree one from $V\oplus S$ to $U\oplus S$ with some T^n representation S iff*

$$U-V=\sum_{\chi}\sum_{\boldsymbol{P}} a_{\boldsymbol{P},\chi}\boldsymbol{P}\chi$$

where χ is an irreducible representation of T^n and $a_{\boldsymbol{P},\chi}$ is a non-negative integer.

Remark. Of course $\omega^{p,q}$: $V^{p,q}\to U^{p,q}$ is a special case of this theorem. As a matter of fact, the case where

$$\chi=t,\quad p_1=p,\quad p_2=q,\quad a_{\boldsymbol{P},\chi}=1$$

corresponds to $\omega^{p,q}$.

As indicated before $K(\omega)$ is invariant under stabilization of ω. Moreover $K(\omega)$ behaves additively with respect to the Whitney sum by the primitivity of the Kervaire-Sullivan class k. On the other hand it is known in [tDP] that a proper T^n map between (a pair of) T^n representations is stably unique up to a proper T^n homotopy. These mean that it suffices to treat the case where $U-V=\boldsymbol{P}\chi$. Furthermore if we regard χ as a homomorphism from T^n to S^1, then $U-V=\chi^*\boldsymbol{P}t$. Thus we have only to treat the case $U-V=\boldsymbol{P}t$.

Theorem 4.2. *Let $U-V=\boldsymbol{P}t$ and let ω: $V\oplus S\to U\oplus S$ be a proper S^1 map of degree one where S is a complex S^1 representation. Then*

$$K(\omega)=\left\{\prod_{i=1}^{l}(p_i^2-1)\right\}\Big/24\sum x^{2r-1},$$

where x is the generator of $H^2(BS^1; Z/2)$ *as before. In particular* $K(\omega)=0$ *in case* $l \geqq 3$.

Proof. We repeat the argument done in Section 3 for this ω. Then one can observe that Assertion (1) of Section 3 should be replaced by

$$K_2(\omega)=\{\prod (p_i^2-1)\}/24 \pmod 2$$

and that the essentially same proof as Assertion (2) works in this case. This verifies the former statement of the theorem. Since $\{p_i\}$ are relatively prime integers, $\prod (p_i^2-1)$ is divisible by 48 if $l \geqq 3$. This verifies the latter statement. Q.E.D.

This theorem together with the above observation establishes

Corollary 4.3. *Let* $\omega: V \to U$ *be a proper* T^n *map between* T^n *representations. Suppose the degree of* ω *is one. Then, if we express*

$$U-V=\sum_{\chi}\sum_{P} a_{P,\chi} P\chi$$

by Theorem 4.1, *then*

$$K(\omega)=\sum_{\chi}\sum_{P} a_{P,\chi}\{\prod (p_i^2-1)\}/24\, \chi^* \left(\sum x^{2^r-1}\right)$$

where χ *is an irreducible complex* T^n *representation,* χ^* *is the homomorphism from* $H^*(BS^1; Z/2)$ *to* $H^*(BT^n; Z/2)$ *induced by regarding* χ *as a homomorphism from* T^n *to* S^1, *and* $a_{P,\chi}$ *is a non-negative integer.*

§ 5. Further calculations

Let $\omega: V \to U$ be a proper G map of degree odd between G representations. Since $H^*(BG; Z/2)$ can be detected by 2-subgroups in most cases, the computation of $K(\omega)$ reduces essentially to the case where G is a 2-group. Among 2-groups Z_2 toral groups are especially important. For example, in case $G=0(n)$, $U(n)$ or T^n, $H^*(BG; Z/2)$ can be detected by its maximal Z_2 toral subgroup as is well known. If G is a Z_2 toral, then Smith theory tells us that V must be isomorphic to U (note that the degree of ω is assumed to be odd). Hence ω determines an element of odd degree in the equivariant stable homotopy ring ω_G^0. In turn ω_G^0 is isomorphic to the Burnside ring $A(G)$ and $A(G)$ is better understood than ω_G^0. Thus we are led to view $K(\omega)$ as a map from a subset of $A(G)$ to $H^*(BG; Z/2)$. Then Theorems 3.1, 4.2 and Corollary 4.3 can be interpreted from this point of view.

Let us explain these in more details. In the following G will denote a compact Lie group for a while. The equivariant stable homotopy ring ω_G^0 is defined by

$$\omega_G^0=\varinjlim_V [S(V),\ S(V)]_G$$

where V is a real G representation and $[S(V),\ S(V)]_G$ denotes the G homotopy set of G selfmaps on $S(V)$. The multiplication is induced by the join operation. The degree of a G selfmap on $S(V)$ is invariant by stabilization, so the degree map $D\colon \omega_G^0\to \boldsymbol{Z}$ is well defined. We set

$${}_{od}\omega_G^0=\{\omega\in\omega_G^0 \mid D(\omega) \text{ is odd}\}.$$

Since D preserves the multiplication (in fact D is a ring homomorphism), ${}_{od}\omega_G^0$ is closed under the multiplication.

Let $[\omega]$ be an element of ${}_{od}\omega_G^0$. By the definition one can represent it as a G selfmap on $S(V)$ of an odd degree for some V. Extending it radially, we get a proper G map $\omega\colon V\to V$ of an odd degree. Hence $K(\omega)$ is defined. As is easily seen $K(\omega)$ does not depend on a choice of a representative ω of $[\omega]$. On the other hand there is a ring isomorphism $\psi\colon \omega_G^0\to A(G)$ (see [P]). Set $A_{od}(G)=\psi({}_{od}\omega_G^0)$. Since the Burnside ring $A(G)$ is better understood than ω_G^0, it is convenient to view K as a map

$$K\colon A_{od}(G)\longrightarrow H^*(BG;\ \boldsymbol{Z}/2)$$

through ψ. Here, in case G is finite, $A(G)$ is defined as the ring obtained by the Grothendick construction from the semi-ring consisting of finite G sets. The sum is the disjoint union and the multiplication is the direct product. It is easily observed that $A(G)$ is additively generated by the G sets G/H where H is a subgroup of G and that $A_{od}(G)$ is exactly the subset of $A(G)$ given by

$$\{[S_1]-[S_2]\in A(G) \mid S_i\ (i=1,2) \text{ are finite } G \text{ sets and the difference of their cardinalities is odd}\}$$

where [ ] stands for the class in $A(G)$ of a finite G set. We refer the reader to [tD] in case G is not finite. Henceforth $A_{od}(G)$ will be identified with ${}_{od}\omega_G^0$ via ψ unless otherwise stated.

Lemma 5.1. (1) *Let $\varphi\colon H\to G$ be a homomorphism between Lie groups. Then*

$$K(\varphi^*u)=\varphi^*K(u)\qquad \text{for } u\in A_{od}(G).$$

(2) *$K(uv)=K(u)+K(v)$ for $u,\ v\in A_{od}(G)$ where uv is the multiplication of u and v.*

Proof. (1) is clear from the definition of K. (2) is an immediate consequence of the primitivity of the Kervaire-Sullivan class k (see § 1).

Q.E.D.

From now on G will denote a Z_2 toral $(Z_2)^n$ of rank n unless otherwise stated and an element $a[G/G]$ $(a \in Z)$ of $A(G)$ will be abbreviated by a.

Lemma 5.2. *Let u be an element of $A_{od}(G)$. Then $K(u)$ decomposes into a sum $\sum K(a_i+b_i[G/G_i])$ where G_i is a subgroup of G, a_i is an odd integer and b_i is an integer.*

Proof. We number proper subgroups G_i of G so that

$$|G_1| \geqq |G_2| \geqq \cdots$$

where $|G_i|$ is the order of G_i. Since $A(G)$ is additively generated by 1 and $[G/G_i]$, u has a unique expression

$$u=c+\sum c_i[G/G_i]$$

where $c_i \in Z$ and c is odd as u belongs to $A_{od}(G)$. Multiply both sides of this identity by $(c+c_1|G/G_1|)-[G/G_1]$. The term $[G/G_1]$ then disappears in the right hand side. Note that $c+c_1|G/G_1|$ is odd as c is odd and $|G/G_1|$ is even. We repeat this process inductively to conclude that

$$u \prod (a_i+b_i[G/G_i])=d$$

for some odd d. It follows from (2) of Lemma 5.1 that

$$K(u)+\sum K(a_i+b_i[G/G_i])=K(d).$$

Here $K(d)=0$ by (1) of Lemma 5.1 because it is a pullback image by the homomorphism from G to the trivial group. This proves the lemma.

Q.E.D.

Since $a_i+b_i[G/G_i]$ is a pullback image by the projection homomorphism from G to G/G_i and G/G_i is again a Z_2 toral, Lemma 5.2 tells us that it is sufficient to treat elements of the form $a+b[G]$.

Let $e_1, \cdots, e_n$ be generators of $H^1(BG; Z/2)$ such that $H^*(BG; Z/2) =Z/2[e_1, \cdots, e_n]$. Since $a+b[G]$ is invariant by the general linear group $GL(n, Z/2)$ (this is easily seen if we identify $(Z_2)^n$ with $(Z/2)^n$), $K(a+[G])$ is contained in the invariants

$$H^*(BG; Z/2)^{GL(n,Z/2)}=Z/2[e_1, \cdots, e_n]^{GL(n,Z/2)}$$

by (1) of Lemma 5.1. On the other hand the invariants are decided by Dickson. Let $D_i(n)$ $(1 \leqq i \leqq n)$ be the determinant of the matrix

$$\begin{pmatrix} e_1 & e_2 & \cdots\cdot & e_n \\ e_1^2 & e_2^2 & \cdots\cdot & e_n^2 \\ \vdots & \vdots & & \vdots \\ e_1^{2^{i-1}} & e_2^{2^{i-1}} & \cdots\cdot & e_n^{2^{i-1}} \\ e_1^{2^{i+1}} & e_2^{2^{i+1}} & \cdots\cdot & e_n^{2^{i+1}} \\ \vdots & \vdots & & \vdots \\ e_1^{2^n} & e_2^{2^n} & \cdots\cdot & e_n^{2^n} \end{pmatrix}.$$

Lemma 5.3 (Dickson, see p. 58 of [MM]). *For $i<n$ $D_n(n)$ divides $D_i(n)$ and $Z/2[e_1, \cdots, e_n]^{GL(n,Z/2)}$ is the polynomial subalgebra with generators*

$$D_{n-1}(n)/D_n(n),\ D_{n-2}(n)/D_n(n),\ \cdots,\ D_1(n)/D_n(n),\ D_n(n)$$

where degree $D_i(n)/D_n(n)=2^n-2^i$, degree $D_n(n)=2^n-1$.

Remember that the degree of a factor of $K(a+b[G])$ is of the form 2^i-2.

Lemma 5.4. *$K(a+b[G])$ is contained in the ideal of $H^*(BG; Z/2)^{GL(n,Z/2)}$ generated by $D_n(n)^2$ and $D_1(n)/D_n(n)$.*

Proof. This follows by comparing their degrees. Q.E.D.

Lemma 5.5. *If b is even, then $K(a+b[G])$ is divisible by $D_n(n)^2$.*

Proof. Let G' be a Z_2 toral of rank $n+1$ containing G and let φ be the inclusion from G to G'. Put $b=2c$. Then $K(a+b[G])=\varphi^*K(a+c[G'])$ by (1) of Lemma 5.1. By Lemma 5.4 $K(a+c[G'])$ is contained in the ideal generated by $D_{n+1}(n+1)^2$ and $D_1(n+1)/D_{n+1}(n+1)$. On the other hand one can see directly from the definition that

$$\varphi^*(D_{n+1}(n+1))=0, \qquad \varphi^*(D_1(n+1)/D_{n+1}(n+1))=D_n(n)^2.$$

This proves the lemma. Q.E.D.

Lemma 5.6. (1) $K(a+b[G])=0$ *if* $b\equiv 0$ (mod 4).
(2) $K(a+b[G])=K(1+2[G])$ *if* $b\equiv 2$ (mod 4).

Proof. (1) This follows by using the argument of Lemma 5.5 twice.
(2) We have

$$(a+b[G])(1+2[G])=a+(2a+b+2^{n+1}b)[G]$$

and the assumption ensures $2a+b+2^{n+1}[G]\equiv 0 \pmod 4$ as a is odd. Applying the functor K to both sides of the above identity, (2) follows from (2) of Lemma 5.1 and the above result (1). Q.E.D.

If $n\geqq 2$, then Lemma 5.6 is improved as follows.

Theorem 5.7. *Let G be a Z_2 toral of rank n ($n\geqq 2$). Then*
(1) *$K(a+b[G])=0$ if b is even,*
(2) *$K(a+b[G])=K(1+[G])$ if b is odd.*

Proof. (1) By (2) of Lemma 5.1 we have

$$0=K((1+[G])^2)=K(1+(2+2^n)[G]).$$

Here $2+2^n\equiv 2 \pmod 4$ by the assumption $n\geqq 2$. This fact and (2) of Lemma 5.6 verify (1).

(2) We have

$$(a+b[G])(1+[G])=a+(a+b+2^n b)[G]$$

and $a+b+2^n b$ is even as a and b are both odd. Hence (2) follows from (2) of Lemma 5.1 and the above (1). Q.E.D.

Observing these results the reader may suspect that $K(a+b[G])$ always vanishes. But this is not the case. As a matter of fact Theorem 3.1 offers a non-trivial example. We shall distinguish three cases by the rank n.

The case where $n=1$ (i.e., $G=Z_2$). Let j be the inclusion from Z_2 to S^1. We denote the restriction of $\omega^{p,q}$ (see § 3) to Z_2 by $j^!\omega^{p,q}\colon j^!V^{p,q}\to j^!U^{p,q}$. As is easily seen $j^!V^{p,q}=j^!U^{p,q}$ and hence $j^!\omega^{p,q}$ determines an element of $A_{od}(G)$.

Lemma 5.8.

$$j^!\omega^{p,q}=\begin{cases} 1 & \textit{if } p \textit{ and } q \textit{ are both odd,} \\ q-(q-1)/2[G] & \textit{if } p \textit{ is even (hence } q \textit{ is odd),} \\ p-(p-1)/2[G] & \textit{if } q \textit{ is even (hence } p \textit{ is odd).} \end{cases}$$

proof. We know that

$$\text{degree } j^!\omega^{p,q}=1$$

$$\text{degree } (j^!\omega^{p,q})^G=\begin{cases} 1 & \text{if } p \text{ and } q \text{ are both odd,} \\ q & \text{if } p \text{ is even and } q \text{ is odd,} \\ p & \text{if } q \text{ is even and } p \text{ is odd.} \end{cases}$$

On the other hand any element of ω_G^0 is detected by the total degree and the fixed point degree. The total degree (resp. the fixed point degree) corresponds to the number of points (resp. the number of fixed points) through the isomorphism ψ (see [P]). These imply the lemma. Q.E.D.

Set

$$\varepsilon(s)=\begin{cases}1 & \text{if } s\equiv 1 \text{ or } 2 \pmod 4,\\ 0 & \text{if } s\equiv 0 \text{ or } 3 \pmod 4.\end{cases}$$

Then Theorem 3.1 and Lemma 5.8 show

$$K(2s+1-s[G])=\varepsilon(s)\sum_{i>1} e^{2i-2}$$

where e is the generator of $H^1(BG; Z/2)$. Moreover the identity $(2s+1-s[G])(1+s[G])=2s+1$ yields

$$K(1+s[G])=K(2s+1-s[G])=\varepsilon(s)\sum e^{2i-2}. \tag{5.9}$$

Since any element $a+b[G]$ splits into $(a-(a-1)/2[G])(1+(a-1+2b)/2[G])$, (5.9) establishes

Theorem 5.10. $K(a+b[Z_2])=\{\varepsilon((a-1)/2)+\varepsilon((a-1+2b)/2)\}\sum e^{2i-2}$.

The reader should observe that this theorem is restatement of Theorem 3.1 and that the result of the T^n case in Section 4 can be deduced from this theorem.

The case where $n=2$ (i.e., $G=Z_2\times Z_2$). By Theorem 5.7 it is sufficient to investigate $K(1+[G])$. Let V be the complex plane equipped with a G action generated by two elements: one is the multiplication by -1 and the other is the complex conjugation. Let $p_d\colon V\to V$ be the d'th power map. Then P_d is equivariant and hence P_d determines an element of ω_G^0. By a similar observation to Lemma 5.8 we can deduce

$$P_d=1+(d-1)/4[G] \qquad \text{if } d\equiv 1 \pmod 4.$$

On the other hand Brumfiel-Madsen [BM] calculates

$$K(P_d)=\begin{cases}0 & \text{if } d\equiv \pm 1 \pmod 8,\\ \sum_{r>1}\sum_{i+j=2r-2} e_1^i e_2^j & \text{if } d\equiv \pm 3 \pmod 8.\end{cases}$$

Putting $d=5$, we get

Theorem 5.11. $K(1+[Z_2\times Z_2])=\sum_{r>1}\sum_{i+j=2r-2} e_1^i e_2^j$.

The case where $n\geqq 3$. Again it is sufficient to investigate $K(1+[G])$

by Theorem 5.7. Lemma 5.4 tells us that $K_2(1+[G])$ and $K_6(1+[G])$ vanish if $n\geqq 3$ and other low degree terms of $K(1+[G])$ also vanish as n rises. Hence it seems plausible to conjecture

$$K(1+[G])=0 \qquad \text{if } n\geqq 3.$$

But the author has no clue to solve it. Note that this conjecture (together with the above observation) is equivalent that

$$K(\sum a_i[G/G_i])=K(\sum b_i[G/G_i]) \qquad \text{if } a_i|G/G_i|\equiv b_i|G/G_i| \pmod 8$$

for any i. In particular it asserts that $K(\sum a_i[G/G_i])$ does not depend on the coefficients a_i such that the order $|G/G_i|$ is divisible by 8.

§ 6. Relation of Theorem 3.1 to the result of Morita-Wood

In [Mo], [Wo] Morita and Wood calculated independently that the Kervaire invariant of hypersurface of degree d (d: odd) in $P(\boldsymbol{C}^n)$ (n: even) is equal to 1 iff $d\equiv\pm 3 \pmod 8$. On the other hand our Theorem 3.1 asserts that $c(\omega_n^{p,q})=1$ iff p is even and $q\equiv\pm 3 \pmod 8$ (or q is even and $p\equiv\pm 3 \pmod 8$). This resemblance seems to suggest a rather deeper relation between them. In this section we will clarify their relationship. In fact we can see through Sullivan's characteristic variety formula that those problems are essentially equivalent.

First we shall state precisely the result of Morita and Wood. Following [Mo] let $V^{n-1}(d)$ be the degree d hypersurface of $P(\boldsymbol{C}^{n+1})$ defined by

$$V^{n-1}(d)=\{[z_0:\cdots:z_n]\in P(\boldsymbol{C}^{n+1})\,|\,z_0^d+\cdots+z_n^d=0\}.$$

Suppose n is even. As is well known, the projection map $f\colon V^{n-1}(d)\to P(\boldsymbol{C}^n)$ defined by $f([z_0:z_1:\cdots:z_n])=[z_1:\cdots:z_n]$ induces an isomorphism from

$$H_i(V^{n-1}(d);\boldsymbol{Z})\longrightarrow H_i(P(\boldsymbol{C}^n);\boldsymbol{Z}) \qquad \text{for } i<n-1.$$

For $i=n-1$, $H_{n-1}(V^{n-1}(d);\boldsymbol{Z})$ is non-trivial but $H_{n-1}(P(\boldsymbol{C}^n);\boldsymbol{Z})$ is trivial as n is even. It is not difficult to see that a cycle of $H_{n-1}(V^{n-1}(d);\boldsymbol{Z})$ can be represented by an embedding of the n-dimensional sphere. Thus a quadratic form is defined on $H_{n-1}(V^{n-1}(d);\boldsymbol{Z}/2)$ (in case $n\neq 2, 4, 8$) similarly to the usual surgery theory and hence the Kervaire invariant of $V^{n-1}(d)$, written $c(V^{n-1}(d))$, is defined as its Arf invariant (see [Wo] for the details). With these understood

Theorem 6.1 (Morita [Mo], Wood [Wo]). *Let n be an even integer distinct from* 2, 4 *and* 8. *Then*

$$c(V^{n-1}(d))=\begin{cases}1 & \text{if } d\equiv\pm 3 \pmod 8,\\ 0 & \text{if } d\equiv\pm 1 \pmod 8.\end{cases}$$

In our situation $c(V^{n-1}(d))$ can be interpreted as follows. Notations in Section 3 will be used freely in the following. Consider the d'th power map $\omega^d\colon t\to t^d$. Clearly ω^d is an S^1 map; so we can associate a mod 2 fiber homotopy equivalence $\omega_n^d\colon \gamma\to\gamma^d$ of degree d over $P(C^n)$. We define a proper homotopy h_s ($s\in[0,1]$) from γ to γ^d by

$$h_s([z_1\colon \cdots\colon z_n\colon z])=[z_1\colon \cdots\colon z_n\colon s(z_1^d+\cdots+z_n^d)+z^d].$$

Obviously $h_0=\omega_n^d$ and h_1 is transverse to the zero section $P(C^n)$ of γ^d. Hence h_1 produces a normal map $\kappa=(W,f,\beta)f\colon W\to P(C^n)$, where $W=h_1^{-1}(P(C^n))=V^{n-1}(d)$. Here note that $c(V^{n-1}(d))=c(\omega_n^d)$. In fact their quadratic forms on $H_{n-1}(V^{n-1}(d);Z/2)$ are the same by their definitions.

Now Sullivan's characteristic variety formula implies that Theorems 3.1 and 6.1 determine $K(\omega^{p,q})$ and $K(\omega^d)$ respectively and vice versa; so a relation between those theorems may be described by $K(\omega^{p,q})$ and $K(\omega^d)$.

Claim. $K(\omega^{p,q})=K(\omega^d)$ when p is even and $q=d$ (or q is even and $p=d$).

The reader should note that Theorems 3.1 and 6.1 together with Sullivan's characteristic variety formula imply this claim but the proof given below does not require those theorems. Namely the proof reveals a relationship between those theorems.

Proof of Claim. Let j be the inclusion map from Z_2 to S^1 as before. The restrictions $j^!\omega^{p,q}$ and $j^!\omega^d$ to Z_2 represent elements of $A(Z_2)$. Suppose p is even and q is odd. Then by Lemma 5.8

$$j^!\omega^{p,q}=q-(q-1)/2[Z_2].$$

A similar observation shows

$$j^!\omega^d=1+(d-1)/2[Z_2].$$

Hence, if $q=d$, then (5.9) means that

$$K(j^!\omega^{p,q})=K(j^!\omega^d).$$

This together with (1) of Lemma 5.1 shows

$$j^*K(\omega^{p,q})=j^*K(\omega^d).$$

Since j^* is injective, this proves the claim. Q.E.D.

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