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# Equivariant Whitehead Groups and *G*-Expansion Categories

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***Dedicated to Professor Minoru Nakaoka on his 60th birthday***

## Introduction

As to the classical theory of Whitehead torsions and Whitehead groups the original works by J.H.C. Whitehead [21], [22], [23] and the expository work by J. Milnor [12], 1966, is celebrated. Since then there appeared about 1970 the trials [5], [17] to define Whitehead group $\mathrm{Wh}(X)$ of a space $X$ and to prove the isomorphism

$$\mathrm{Wh}(X)\cong\mathrm{Wh}(\pi_1 X),$$

where the right hand side is the classical algebraically defined Whitehead group of the fundamental group of $X$.

In 1974 S. Illman [7] defined equivariant Whitehead group of a finite $G$-$CW$-complex $X$, denoted by $\mathrm{Wh}_G(X)$, where $G$ is a compact Lie group, equivariant Whitehead torsion for a $G$-homotopy equivalence $f: X\to Y$ between finite $G$-$CW$-complexes $X$, $Y$ as an element of $\mathrm{Wh}_G(X)$, and described the basic properties of $\mathrm{Wh}_G(X)$. He tried also to decompose $\mathrm{Wh}_G(X)$ and to describe it algebraically for abelian $G$ and proposed to use restricted Whitehead groups $\mathrm{Wh}_G(X, \mathscr{F})$ with respect to families $\mathscr{F}$ of closed subgroups of $G$ in studies of equivariant Whitehead group of $X$.

In 1978 M. Rothenberg [13] defined equivariant Whitehead groups and torsions in another way for finite $G$ and obtained several results making use of them. Among others he obtained a form of equivariant $s$-cobordism theorem for smooth actions of compact Lie groups $G$ in 3.4 by introducing certain other invariants $\mathrm{Wh}^i_H(\iota)$ for closed subgroups $H$ of $G$. In case $G$ is finite and $X^{(H)}$ is connected and simply connected for all subgroups $H$ of $G$ he proved the collection of the invariants $\mathrm{Wh}_H(\iota)$ coincides with his $G$-Whitehead torsion in 3.10. Here we remark that our definition of $G$-$h$-cobordism [3], [9] is different from that of Rothenberg's. Probably, under the assumption of our dimension gap

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conditions of fixed points of closed subgroups [9] the two definitions might coincide and also his invariants $\mathrm{Wh}_H^{i}(\iota)$ might be well-related to direct summands of Illman's $G$-Whitehead torsion, which are anyway open questions at the moment. Rothenberg's definition of equivariant Whitehead groups does not coincide with Illman's in general except under the assumption that $X^H$ is connected and simply connected for all subgroups $H$ of $G$. There are several reasons for us it is preferable to use the definition by Illman (e.g., $G$-expansion categories), so we follow the line of [7].

In 1978 H. Hauschild [6] gave the natural direct sum decomposition

$$\mathrm{Wh}_G(X) \cong \coprod_{(H)} \mathrm{Wh}_G(X, (H)),$$

where $(H)$ runs over all conjugacy classes of closed subgroups of $G$, and tried to describe $\mathrm{Wh}_G(X)$ algebraically based on this decomposition in a way. (Hauschild's notation of restricted Whitehead groups is different from that in [7], but we prefer to use the orginal notation by Illman.)

In 1982–83 Anderson [1] (for finite $G$) and Illman [8] (for compact $G$) studied further direct sum decompositions and reductions of each Hauschild summand of $\mathrm{Wh}_G(X)$ by going the other way than [6] and finally completed to describe $\mathrm{Wh}_G(X)$ as a direct sum of a certain number of algebraically defined Whitehead groups of certain discrete groups.

As remarked by R. Stöcker [17] the Whitehead groups may be defined for any space $X$ by making use of relative finite $CW$-complexes $(V, X)$. In the present expository article the author tried to define $\mathrm{Wh}_G(X)$ for any $G$-space (with $G$ a Lie group) by making use of relative finite $G$-$CW$-complexes $(V, X)$. Even though the whole theory of equivariant Whitehead groups are effective only for compact Lie groups actions, we will eventually meet with non-compact Lie groups actions at a step of our reductions in Section 8.

As is well known [7] $\mathrm{Wh}_G(X)$, more generally every $\mathrm{Wh}_G(X, \mathscr{F})$, is a $G$-homotopy functor with respect to $X$. And the above Hauschild decomposition is a natural isomorphism of $G$-homotopy functors [6]. Here we tried to describe every process of reductions and direct sum decompositions after Hauschild's are also natural isomorphisms of functors even though we need certain cares for meaning of naturalities in last three steps (Corollaries 6.2, 7.2, and 8.3). As a corollary of all of these observations we obtained a sufficient condition for a $G$-map $f: X \to Y$ to induce an isomorphism $f_*: \mathrm{Wh}_G(X, (H)) \cong \mathrm{Wh}_G(Y, (H))$ of the corresponding Hauschild summands (Theorem 9.3), which will be used to prove certain equivariant $s$-cobordism theorems for smooth $G$-h-cobordisms [3] for compact $G$. Cf., also [9].

In [18], p. 49, F. Waldhausen introduced the notion of simplicial expansion category of a space $X$ which is related to algebraic $K$-theory $A(X)$ of $X$ by a certain fibration. In a parallel way we define simplicial $G$-expansion category $\mathscr{E}_G(X)_.$ of a $G$-space $X$ which satisfies a functorial isomorphism: $\mathrm{Wh}_G(X)\cong\pi_0\mathscr{E}_G(X)_.$. The merit to use this category is that i) "$X\mapsto\mathscr{E}_G(X)_.$" is a homotopy preserving functor (Theorem 3.2) and ii) Hauschild decompositions are $\pi_0$ phenomena of a natural homotopy decomposition of this functor (Theorem 4.1), of which the proofs are given in [2] by making use of certain ideas of Waldhausen [19]. Further reductions and direct sum decompositions are given by natural equivalences of simplicial categories which contain cases of equivariant Whitehead groups as corollaries.

The contents of the present article are as follows:

1. Relative $G$-$CW$-complexes.
2. Formal $G$-deformations.
3. $G$-expansion categories.
4. Homotopy decompositions of $\mathscr{E}_G(X)_.$.
5. A reduction of $\mathscr{E}_G(X, (H))_.$.
6. Decompositions of $\mathscr{E}_{WH}(X^H, \{1\})_.$.
7. A reduction of $\mathscr{E}_{WH}(WH\cdot X^H{}_\alpha, \{1\})_.$.
8. Passing to universal covering spaces.
9. The final step of reductions of $\mathrm{Wh}_G(X, (H))$.

Sections 1 and 2 are preparatory for materials of subtitles. In Section 3 we define simplicial $G$-expansion categories and state their homotopy properties (Theorem 3.2) mentioned above. In Section 4 we prepare materials for the homotopy decompositions of simplicial $G$-expansion categories and state the homotopy decomposition (Theorem 4.1) mentioned above. Sections 5–8 describe each step of reductions of homotopy direct summand $\mathscr{E}_G(X, (H))_.$ of $\mathscr{E}_G(X)_.$. All materials of Sections 4–8 may be applied also for higher homotopy groups. Section 9 concerns only with $\mathrm{Wh}_G(X)$ and identifies the final pieces of the direct summands of $\mathrm{Wh}_G(X)$ with algebraically defined Whitehead groups of certain discrete groups.

## § 1. Relative $G$-$CW$-complexes

Let $G$ be a Lie group. By a *$G$-space* $X$ we understand a compactly generated Hausdorff space $X$ with a (continuous left) $G$-action such that the orbit space $G\backslash X$ (with quotient space topology) is Hausdorff (which follows automatically in case $G$ is *compact* [10], (1.8)). Then $G\backslash X$ is also compactly generated [16], [20], (I.4.7), each $G$-orbit is closed in $X$ and distinct two $G$-orbits can be separated by $G$-neighborhoods.

The above conditions to $G$-spaces seems to be somewhat restrictive for arbitrary Lie groups actions, but any $G$-$CW$-complex satisfies these conditions as will be seen below and it is necessary for simple $G$-homotopy theory.

Let $G$-**Top** denote the category of $G$-spaces (in the above sense) and $G$-maps.

Relative $G$-$CW$-complexes can be defined in a parallel way to [20], Chap. II, as follows (cf., also [7]). A *euclidean* $n$-cell $E^n$ is the unit $n$-ball in $\boldsymbol{R}^n$. Let $H$ be a closed subgroup of $G$. The $G$-space $G/H\times E^n$ is called a *euclidean n-G*-cell (of type $(H)$). We allow always the changes of expressions of a euclidean $n$-$G$-cell $G/H\times E^n=G/s^{-1}Hs\times E^n$ by right actions of elements $s\in G$. Let $X$ be a $G$-space and take a collection $\{G/H_\lambda\times E^n, \lambda\in\Lambda\}$ of euclidean $n$-$G$-cells. Let $S^{n-1}=\partial E^n$ and $f_\lambda$: $G/H_\lambda\times S^{n-1}\to X$, $\lambda\in\Lambda$, $G$-maps. The space $Y=X\cup_f(\coprod_\lambda G/H_\lambda\times E^n)$, $f=\coprod_\lambda f_\lambda$, is called an *n-G-cellular extension* of $X$. It is a quotient space of $X\coprod(\coprod_\lambda G/H_\lambda\times E^n)$ on which $G$ acts. Since $G\backslash Y=G\backslash X\cup_{G\backslash f}(\coprod_\lambda E^n)$, an $n$-cellular extension of $G\backslash X$ in the sense of [20], Chap. II, which is Hausdorff. Hence $Y$ belongs to $G$-**Top** and contains $X$ as a closed $G$-subspace. Let $(Y, X)$ be a pair such that $X$ is an object of $G$-**Top** and $Y$ be given with a filtration

$$X\subset Y^0\subset Y^1\subset\cdots\subset Y^n\subset\cdots,\qquad \cup_n Y^n=Y,$$

such that $Y^n$ is an *n-G-cellular* extension of $Y^{n-1}$ for $n\geq 0$ $(Y^{-1}=X)$. Then every $Y^n$ is an object of $G$-**Top** and $Y^{n-1}$ is a closed $G$-subspace of $Y^n$. Giving the limit space topology to $Y$, $Y$ is also an object of $G$-**Top**. We call such pair $(Y, X)$ a *relative G-CW-complex*. When $X=\phi$, $Y$ is a $G$-$CW$-complex in the usual sense [7], [10].

Let $(Y, X)$ be a relative $G$-$CW$-complex. $(Y, X)^n=(Y^n, X)$ is called the *n-skeleton* of $(Y, X)$ for each $n\geq 0$. We regard $(X, X)$ as the $(-1)$-skeleton of $(Y, X)$. $n$-skeletons $(Y, X)^n$ are also relative $G$-$CW$-complexes. Clearly $Y^n-Y^{n-1}=\coprod_\lambda b^n_\lambda$, $b^n_\lambda=G/H_\lambda\times\mathrm{Int}\,E^n$. Each $b^n$ is called an *open n*-$G$-cell (of type $(H_\lambda)$) and $c^n_\lambda=\bar{b}^n_\lambda$ (in $Y$) a *closed n*-$G$-cell of $(Y, X)$ (according to the notation of [7]). Let $b^n=G/H\times\mathrm{Int}\,E^n$ be an open $n$-$G$-cell of $(Y, X)$. Any $G$-map $g$: $G/H'\times E^n\to Y$, $H'\sim H$, is called a *characteristic* $G$-map of $b^n$ if $g\,|\,G/H'\times\mathrm{Int}\,E^n$: $G/H'\times\mathrm{Int}\,E^n\approx b^n$ is a $G$-homeomorphism. We say that $(Y, X)$ is *finite* if it has only finitely many (open) $G$-cells. A subcomplex of a relative $G$-$CW$-complex and a product of a relative $G$-$CW$-complex with a relative $K$-$CW$-complex as a relative $G\times K$-$CW$-complex may be defined in a parallel way to [20], Chap. II.

In a parallel way to [20], (II.1.6) and (I.5.9), we obtain

**Theorem 1.1.** *Let $(Y, X)$ be a relative G-CW-complex. Then,*

i) *$(Y, X)$ is a G-NDR-pair (an equivariant version of an NDR-pair). In particular, the inclusion $i: X \subset Y$ is a G-cofibration;*

ii) *$i: X \subset Y$ is a G-homotopy equivalence iff $X$ is a G-deformation retract of $Y$ (in the strong sense).*

Cf., also [7], (I.1.3).

Let $(V, X)$ and $(W, Y)$ be relative $G$-$CW$-complexes. A $G$-map $f: (V, X) \to (W, Y)$ of pairs is called a *G-map* between relative $G$-$CW$-complexes. It is called *cellular* if $f(V^n) \subset W^n$ for each $n \geq 0$. By the usual skeletonwise and $G$-cellwise arguments [15], (VII.6.17), [10], (4.4), there holds

**Theorem 1.2.** ($G$-cellular approximation). *Let $f: (V, X) \to (W, Y)$ be a G-map between relative G-CW-complexes. There exists a cellular G-map $g: (V, X) \to (W, Y)$ which is G-homotopic to $f$* rel $X$.

A $G$-space $X$ is called to be *locally G-0-connected* if $X^H$ is locally path-connected for each closed subgroup $H$ of $G$, and also to be *locally G-1-connected* if it is locally $G$-0-connected and $X^H$ is semi-locally 1-connected for each closed subgroup $H$ of $G$, where we regard the void set (as a space) is always locally path-connected and semi-locally 1-connected.

Let $(V, X)$ be a relative $G$-$CW$-complex. If $X$ is locally $G$-$n$-connected ($n=0$ or 1), then $V$ is also so.

Let $X$ and $Y$ be locally $G$-0-connected. A $G$-map $f: X \to Y$ is called a *weak G-homotopy equivalence* if $f^H: X^H \to Y^H$ is a weak homotopy equivalence for each closed subgroup $H$ of $G$. Again, by the usual skeletonwise and $G$-cellwise arguments [15], (VII. 6.24), [10], (5.3), there holds

**Theorem 1.3.** (equivariant J.H.C. Whitehead's). *Let $X$ be a locally G-0-connected G-space, and $f: (V, X) \to (W, X)$ a G-map between relative G-CW-complexes such that $f|X = 1_X$. If $f: V \to W$ is a weak G-homotopy equivalence, then $f: (V, X) \to (W, X)$ is a G-homotopy equivalence* rel $X$.

Let $P$ be a finite polyhedron. According to [18], p. 48, we define $P$-parametrized relative $G$-$CW$-complex as follows: Let $X$ be a $G$-space and $(W, X \times P)$ be a pair of $G$-spaces such that i) $X \times P$ is closed in $W$, ii) $W$ is endowed with a projection $\pi: W \to P$ ($G$-map with trivial $G$-action on $P$) satisfying $\pi | X \times P = \mathrm{pr}_2$, the 2-nd projection. We call such a pair a *P-parametrized pair* of $G$-spaces. Take a collection $\{G/H_\lambda \times E^n, \lambda \in \Lambda\}$ of euclidean $n$-$G$-cells. Let $f_\lambda: G/H_\lambda \times S^{n-1} \times P \to W$ be a $G$-map which is compatible with projections onto $P$. The $G$-space $V = W \cup_f (\coprod_\lambda G/H_\lambda \times E^n \times P)$, $f = \coprod_\lambda f_\lambda$, is called *P-parametrized n-G-cellular extension* of $W$.

By our choice of $f_\lambda$, $G/H_\lambda \times E^n \times P$ is attached to $W$ so that the 3-rd projection $G/H_\lambda \times E^n \times P \to P$ is compatible with the projection $\pi\colon W \to P$, and we receive a well defined projection $\pi'\colon V \to P$ which extends $\pi$. Thus $(V, X \times P)$ is also a $P$-parametrized pair of $G$-spaces. Let $(V, X \times P)$ be a pair such that $X$ is an object of $G$-**Top** and $V$ be given with a filtration

$$X \times P \subset V^0 \subset V^1 \subset \cdots \subset V^n \subset \cdots, \quad \cup_n V^n = V$$

such that $V^n$ is a $P$-parametrized $n$-$G$-cellular extension of $V^{n-1}$ for $n \geq 0$ $(V^{-1} = X \times P)$. Then $V^n$ is an object of $G$-**Top** endowed with a $P$-parametrization $\pi_n$ compatible with the inclusions, i.e., $\pi_{n+1} | V^n = \pi_n$, and $V^n$ is a closed $G$-subspace of $V^{n+1}$. Thus, giving the limit space topology to $V$, it is also an object of $G$-**Top** with a projection $\pi\colon V \to P$ such that $\pi | V^n = \pi_n$. Such a pair $(V, X \times P)$ is called *$P$-parametrized relative $G$-$CW$-complex.*

Let $(V, X \times P)$ be a $P$-parametrized relative $G$-$CW$-complex defined as above. *Skeletons* are also defined similarly to relative $G$-$CW$-complexes. $V^n - V^{n-1} = \coprod_\lambda b^n_\lambda$, $b^n = G/H_\lambda \times \mathrm{Int}\, E^n \times P$. Each $b^n_\lambda$ is called a ($P$-parametrized) *open* $n$-$G$-cell (of type $(H_\lambda)$) and $c^n_\lambda = \bar{b}^n_\lambda$ (in $V$) a ($P$-parametrized) *closed* $n$-$G$-cell of $(V, X \times P)$. Let $b^n = G/H \times \mathrm{Int}\, E^n \times P$ be a $P$-parametrized open $n$-$G$-cell of $(V, X \times P)$. Any $G$-map $g\colon G/H' \times E^n \times P \to V$, $H' \sim H$, is called a *characteristic $G$-map* of $b^n$ if it is compatible with projections onto $P$ and $g | G/H' \times \mathrm{Int}\, E^n \times P\colon G/H' \times \mathrm{Int}\, E^n \times P \approx b^n$ is a $G$-homeomorphism. For each $t \in P'$ define $V_t = \pi^{-1}(t)$, $t \in P$, then $(V_t, X)$, $X = X \times \{t\}$, is a relative $G$-$CW$-complex. We say that $(V, X \times P)$ is *finite* if $V - X \times P$ contains only finitely many open $G$-cells. In the present article we are interested only in $P$-parametrized finite relative $G$-$CW$-complexes.

We may define the notions: *$P$-parametrized $G$-maps* between $P$-parametrized relative $G$-$CW$-complexes, *$P$-parametrized $G$-homotopies* between $P$-parametrized $G$-maps, *$P$-parametrized $G$-homotopy equivalences*, *$P$-parametrized $G$-deformation retracts* and *$P$-parametrized $G$-cofibrations.* And we have the $P$-parametrized version of Theorem 1.1 as follows:

**Theorem 1.4.** *Let $(V, X \times P)$ be a $P$-parametrized relative $G$-$CW$-complex. Then*

i) *$i\colon X \times P \subset V$ is a $P$-parametrized $G$-cofibration;*

ii) *$i\colon X \times P \subset V$ is a $P$-parametrized $G$-homotopy equivalence iff $X \times P$ is a $P$-parametrized $G$-deformation retract of $V$.*

## § 2. Formal $G$-deformations

From now on, we assume *all ($P$-parametrized) relative $G$-$CW$-complexes considered are finite.* Let $(V, X \times P)$ be a $P$-parametrized relative

$G$-$CW$-complex and $G/H\times E^n$ a euclidean $n$-$G$-cell (of type $(H)$). Express $\partial E^n=S^{n-1}=E_+^{n-1}\cup E_-^{n-1}$ as a union of the upper hemisphere and lower one so that $E_+^{n-1}\cap E_-^{n-1}=S^{n-2}$. Let $f$: $G/H\times E_-^{n-1}\times P\to V$ be a $P$-parametrized $G$-map such that $f(G/H\times S^{n-2}\times P)\subset V^{n-2}$ and $f(G/H\times E_-^{n-1}\times P)\subset V^{n-1}$. Identifying $(E_+^{n-1}, S^{n-2})$ with $(E^{n-1}, S^{n-2})$ by a standard homeomorphism, we see that

$$W=V\cup_f G/H\times E^n\times P=V\cup b^{n-1}\cup b^n$$

is a $P$-parametrized $G$-space (extending the parametrization of $V$) obtained by attaching two $P$-parametrized closed $G$-cells $\bar{b}^{n-1}$ and $\bar{b}^n$ successively to $V$, and $(W, X\times P)$ is also a $P$-parametrized relative $G$-$CW$-complex. Remark that both $G$-cells $b^{n-1}$ and $b^n$ have the same type $(H)$. In this situation we say that $(W, X\times P)$ is a *(P-parametrized) elementary G-expansion* of $(V, X\times P)$ *of type* $(H)$ (and of *dimension* $n$) [7], and write $(V, X\times P)\nearrow_e(W, X\times P)$ according to the notion of [4]. A deformation retraction $E^n\to E_-^{n-1}$ yields a $P$-parametrized $G$-deformation retraction of $(W, X\times P)$ to $(V, X\times P)$.

A finite sequence of elementary $G$-expansions

$$(V, X\times P)\nearrow_e(V_1, X\times P)\nearrow_e\cdots\nearrow_e(V_n, X\times P)=(W, X\times P)$$

(up to $G$-isomorphisms rel $X\times P$) is called a *(P-parametrized) formal G-expansion* denoted by $(V, X\times P)\nearrow(W, X\times P)$. Applying the above $P$-parametrized $G$-deformation retraction to each elementary $G$-expansion in the sequence to express $(V, X\times P)\nearrow(W, X\times P)$ successively, we see that $(V, X\times P)$ is a $P$-parametrized $G$-deformation retract of $(W, X\times P)$.

In the sequence of elementary $G$-expansions to present a formal $G$-expansion, the pairs of $G$-cells in each elementary $G$-expansion must not be replaced by other pairs. There is only one commutation rule of the order of elementary $G$-expansions in a formal $G$-expansion as follows: Suppose $(V, X\times P)\nearrow_e(W_1, X\times P)$, $W_1=V\cup b^{n-1}\cup b^n$, and $(V, X\times P)\nearrow_e(W_2, X\times P)$, $W_2=V\cup b^{m-1}\cup b^m$, be two elementary $G$-expansions, then we get two sequences

$$(V, X\times P)\nearrow_e(W_1, X\times P)\nearrow_e(W_1\cup_V W_2, X\times P)$$

and

$$(V, X\times P)\nearrow_e(W_2, X\times P)\nearrow_e(W_1\cup_V W_2, X\times P).$$

We regard the two formal $G$-expansions $(V, X\times P)\nearrow(W_1\cup_V W_2, X\times P)$ represented by the above two sequences are the same (*elementary commutation rule*).

When $P=pt$, we get the notions of (unparametrized) *elementary G-*

*expansion* $(V, X)\nearrow_e(W, X)$ and *formal G-expansion* $(V, X)\nearrow(W, X)$ of relative $G$-$CW$-complexes. When $(V, X)\nearrow_e(W, X)$ (of type $(H)$ and of dimension $n$), we say that $(V, X)$ is an *elementary G-collapse* of $(W, X)$ (of type $(H)$ and of dimension $n$), and write $(W, X)\searrow^e(V, X)$. A finite sequence (up to $G$-isomorphisms rel $X$) of elementary $G$-collapses is called a *formal G-collapse* denoted by $(W, X)\searrow(V, X)$. A finite sequence of formal $G$-expansions and $G$-collapses from $(V, X)$ to $(W, X)$ is called a *formal G-deformation*, denoted by $(V, X)\diagup\!\!\searrow(W, X)$ according to [4].

Let $(V, X)\nearrow(W, X)$. Then $(V, X)$ is $G$-isomorphic to a $G$-subcomplex of $(W, X)$. Sometimes we regard the formal $G$-expansion as the inclusion $G$-map $i: (V, X)\subset(W, X)$. Now $V$ is a $G$-deformation retract of $W$ as remarked already, and any $G$-retraction $r: W\to V$ is a $G$-homotopy inverse of $i$. Sometimes we regard the formal $G$-collapse $(W, X)\searrow(V, X)$ as the $G$-retraction $r: (W, X)\to(V, X)$. Since any two $G$-retractions $W\to V$ are $G$-homotopic rel $V$, the choice of $G$-retractions $r: (W, X)\to(V, X)$ is irrelevant in simple $G$-homotopy theory.

Let $(V, X)\diagup\!\!\searrow(W, X)$. Taking the composition of inclusions and $G$-retractions we regard sometimes the formal $G$-deformation $(V, X)\diagup\!\!\searrow(W, X)$ as a $G$-map $d: (V, X)\to(W, X)$ rel $X$, whose $G$-homotopy class rel $X$ is independent of the choices to define $d$. A $G$-map $f: (V, X)\to(W, X)$ such that $f|X=1_X$ is called a *simple G-homotopy equivalence* rel $X$ if there exists a formal $G$-deformation $d: (V, X)\to(W, X)$ rel $X$ such that $f\simeq_G g$ rel $X$. Obviously every simple $G$-homotopy equivalence rel $X$ is a $G$-homotopy equivalence rel $X$.

Here we remark that the most properties of formal $G$-deformations described in [7], Chap. II, are valid for our present version to use relative $G$-$CW$-complexes. So we use them simply quoting from [7]. In simple homotopy theory the use of mapping cylinders is very important as in [4], Section 5. In our version we use *reduced* mapping cylinders instead, i.e., let $f: (V, X)\to(W, X)$ be a cellular $G$-map such that $f|X=1_X$, let $M_f$ be the mapping cylinder of $f$ and put $\overline{M}_f=M_f\cup_\pi X$, where $\pi: X\times I\to X$ is the 1-st projection, by which the reduced mapping cylinder $\overline{M}_f$ of $f$ is defined, then $(\overline{M}_f, X)$ is a relative $G$-$CW$-complex. By making use of reduced mapping cylinders in place of unreduced ones, the most of basic properties of mapping cylinders concerning formal $G$-deformations are retained in our version of simple $G$-homotopy theory.

Finally, by a family $\mathscr{F}$ of (closed) subgroups of $G$ we understand a collection of *closed* subgroups $H$ of $G$ such that $H\in\mathscr{F}$ implies $(H)\subset\mathscr{F}$, where $(H)$ denotes the conjugacy class of $H$ [6], [7].

Let $(V, X\times P)$ be a $P$-parametrized relative $G$-$CW$-complex. We say $(V, X\times P)$ is *of type* $\mathscr{F}$ if all types of $G$-cells in $V-X\times P$ are contained in $\mathscr{F}$ [6], [7].

Let $(V, X\times P)\nearrow(W, X\times P)$. When all types of $P$-parametrized elementary $G$-expansions appearing in this formal $G$-expansion are contained in $\mathscr{F}$, then we say it is *of type* $\mathscr{F}$, denoted by $(V, X\times P)\nearrow_{\mathscr{F}}(W, X\times P)$.

Similarly we define a formal $G$-collapse $(W, X)\searrow(V, X)$ *of type* $\mathscr{F}$, denoted by $(W, X)\searrow^{\mathscr{F}}(V, X)$ and a formal $G$-deformation $(V, X)\diagup\!\diagdown\!\searrow(W, X)$ *of type* $\mathscr{F}$, denoted by $(V, X)\underset{\mathscr{F}}{\diagup\!\diagdown\!\searrow}(W, X)$ [6].

## § 3. *G*-expansion categories

Let $X$ be a $G$-space and $\mathscr{F}$ a family of subgroups of $G$. By $\Delta^k$, $k\geq 0$, we denote the standard ordered $k$-simplex. We say "$k$-parametrized" in place of "$\Delta^k$-parametrized" for simplicity.

For each $k\geq 0$ we define the following category $\mathscr{E}_G(X, \mathscr{F})_k$: obj $\mathscr{E}_G(X, \mathscr{F})_k$ consists of all $k$-parametrized finite relative $G$-$CW$-complexes $(V, X\times\Delta^k)$ of type $\mathscr{F}$ such that the inclusions $i$: $X\times\Delta^k\subset V$ are $G$-homotopy equivalences, and Hom $\mathscr{E}_G(X, \mathscr{F})_k$ consists of all $k$-parametrized formal $G$-expansions between our objects. We call $\mathscr{E}_G(X, \mathscr{F})_k$ the *k-parametrized G-expansion category of X* rel $\mathscr{F}$. Because of our finiteness assumption for objects, $\mathscr{E}_G(X, \mathscr{F})_k$ is a small category. When $\mathscr{F}=\{\text{all}\}=\{\text{all closed subgroups of } G\}$, we simply write $\mathscr{E}_G(X)_k=\mathscr{E}_G(X, \{\text{all}\})_k$ and call it the *k-parametrized G-expansion category* of $X$.

Remark that the $G$-homotopy equivalence $i$: $X\times\Delta^k\subset V$ in the above definition implies that $X\times\Delta^k$ is actually a $k$-parametrized $G$-deformation retract of $V$ even though we do not give the proof here.

Let $(V_1, X\times\Delta^k)$ and $(V_2, X\times\Delta^k)$ be objects of $\mathscr{E}_G(X, \mathscr{F})_k$. Since $X\times\Delta^k$ is a $k$-parametrized $G$-deformation retract of $V_1$ and $V_2$ respectively, $X\times\Delta^k$ is also a $k$-parametrized $G$-deformation retract of $W=V_1\cup_{X\times\Delta^k}V_2$ and $(W, X\times\Delta^k)$ is an object of $\mathscr{E}_G(X, \mathscr{F})_k$. Defining the sum

$$(V_1, X\times\Delta^k)+(V_2, X\times\Delta^k)=(W, X\times\Delta^k),\quad W=V_1\cup_{X\times\Delta^k}V_2,$$

on objects, we see easily that $\mathscr{E}_G(X, \mathscr{F})_k$ is a symmetric monoidal category with $(X\times\Delta^k, X\times\Delta^k)$ as the 0-object. Thus $B\mathscr{E}_G(X, \mathscr{F})_k$ is a homotopy commutative $H$-space and $\pi_0 B\mathscr{E}_G(X, \mathscr{F})_k$ is an abelian monoid, where $B\mathscr{E}_G(X, \mathscr{F})_k$ is the classifying space of $\mathscr{E}_G(X, \mathscr{F})_k$ [14] endowing discrete topologies to obj $\mathscr{E}_G(X, \mathscr{F})_k$ and Hom $\mathscr{E}_G(X, \mathscr{F})_k$ respectively.

Now consider the case $k=0$, which we may regard as the unparametrized $G$-expansion category $X$ rel $\mathscr{F}$, and let $[V, X]$ denote the element of $\pi_0 B\mathscr{E}_G(X, \mathscr{F})_0$ represented by an object $(V, X)$. Then, $0=[X, X]$ and $[V, X]=[W, X]$ iff $(V, X)$ is connected to $(W, X)$ by an edge path in $B\mathscr{E}_G(X, \mathscr{F})_0$, which is a finite sequence of arrows ($=$formal $G$-expansions of type $\mathscr{F}$) and inverted arrows ($=$formal $G$-collapses of type $\mathscr{F}$), i.e.,

$[V, X]=[W, X]$ iff $(V, X)\underset{\mathscr{F}}{\diagup\!\!\searrow}(W, X)$. Thus $\pi_0 B\mathscr{E}_G(X, \mathscr{F})_0$ coincides with the equivariant Whitehead group of $X$ rel $\mathscr{F}$ in the sense of [17], p. 42, or [6].

In fact, by a parallel argument to [7], (II.2.3), with a little care for the family $\mathscr{F}$, we see that every element of $\pi_0 B\mathscr{E}_G(X, \mathscr{F})_0$ is invertible, i.e., $\pi_0 B\mathscr{E}_G(X, \mathscr{F})_0$ is an *abelian group*. We define

$$\mathrm{Wh}_G(X, \mathscr{F})=\pi_0 B\mathscr{E}_G(X, \mathscr{F})_0$$

using the notation of [7], p. 42, and call it the *G-Whitehead group of X* rel $\mathscr{F}$.

Let $a$; $[l]\rightarrow[k]$ be an order-preserving (or weakly monotonic) map, where $[k]=\{0<1<2<\cdots<k\}$, the totally ordered set. Let $a_*\colon \Delta^l\rightarrow\Delta^k$ be the linear extension of $a$ regarding it as the correspondence between ordered vertices. Now

$$\Gamma(a_*)=\{(a_*(t), t),\ t\in\Delta^l\}\subset\Delta^k\times\Delta^l$$

is the graph of $a_*$. The second projection induces the homeomorphism $p\colon \Gamma(a_*)\approx\Delta^l$, which we regard as the canonical one.

We define a functor

$$a^*\colon \mathscr{E}_G(X, \mathscr{F})_k \longrightarrow \mathscr{E}_G(X, \mathscr{F})_l$$

as follows. For each object $(V, X\times\Delta^k)$ of $\mathscr{E}_G(X, \mathscr{F})_k$ with the parametrization $\pi\colon V\rightarrow\Delta^k$, we consider $(\pi\times 1)^{-1}\Gamma(a_*)\subset V\times\Delta^l$. $(\pi\times 1)^{-1}\Gamma(a_*)\cap X\times\Delta^k\times\Delta^l=X\times\Gamma(a_*)$, which we replace by $X\times\Delta^l$ through the canonical homeomorphism $1\times p\colon X\times\Gamma(a_*)\approx X\times\Delta^l$. For each open $G$-cell $b^n=G/H\times\mathrm{Int}\,E^n\times\Delta^k$ of $V$, $(\pi\times 1)^{-1}\Gamma(a_*)\cap b^n\times\Delta^l=G/H\times\mathrm{Int}\,E^n\times\Gamma(a_*)$, which we replace by $\tilde{b}^n=G/H\times\mathrm{Int}\,E^n\times\Delta^l$ by the canonical homeomorphism $1\times p$ and regard it as an open $G$-cell of a new $l$-parametrized relative $G$-$CW$-complex $(\tilde{V}, X\times\Delta^l)$; thereby the attaching $G$-map of $\tilde{b}^n, \tilde{f}\colon G/H\times S^{n-1}\times\Delta^l\rightarrow\tilde{V}^{n-1}$ is given by $\tilde{f}(x, t)=(1\times p)\circ(f(x, a_*(t)), t)$ for $f$ the attaching $G$-map of $b^n$. Clearly $X\times\Delta^l$ is an $l$-parametrized $G$-deformation retract of $\tilde{V}$, and hence $(\tilde{V}, X\times\Delta^l)$ is an object of $\mathscr{E}_G(X, \mathscr{F})_l$. The assignment $a^*\colon (V, X\times\Delta^k)\rightarrow(\tilde{V}, X\times\Delta^l)$ preserves also parametrized formal $G$-expansions, and we obtain the desired functor $a^*$.

It is also easy to see that $a^*$ is a functor between symmetric monoidal categories, and

$$b^*\circ a^*=(a\circ b)^*\colon \mathscr{E}_G(X, \mathscr{F})_k\longrightarrow\mathscr{E}_G(X, \mathscr{F})_m$$

for order-preserving maps $b\colon [m]\rightarrow[l]$ and $a\colon [l]\rightarrow[k]$.

Thus

$$\mathscr{E}_G(X, \mathscr{F})_{.} = \{\mathscr{E}_G(X, \mathscr{F})_k,\ k \geq 0\}$$

is a simplicial symmetric monoidal category.

Let $f\colon X \longrightarrow Y$ be a $G$-map. Define

$$f_{\#}(V, X\times\Delta^k) = (V \cup_{f\times\mathrm{id}} Y\times\Delta^k,\ Y\times\Delta^k)$$

for each object $(V, X\times\Delta^k)$ of $\mathscr{E}_G(X, \mathscr{F})_k$. Clearly $f_{\#}(V, X\times\Delta^k)$ is an object of $\mathscr{E}_G(X, \mathscr{F})_k$ and $f_{\#}$ preserves morphisms. Thus we get a (covariant) functor

$$f_{\#}\colon \mathscr{E}_G(X, \mathscr{F})_k \longrightarrow \mathscr{E}_G(Y, \mathscr{F})_k$$

for each $k \geq 0$. $f_{\#}$ commutes also with simplicial structural morphisms $a^*$, $a\colon [k]\to[l]$, and preserves sums. $(g\circ f)_{\#} = g_{\#}\circ f_{\#}$ for $G$-maps $f\colon X\to Y$ and $g\colon Y\to Z$. Thus we get a functor

$$\mathscr{E}_G(\ ,\mathscr{F})_{.}\colon G\textbf{-Top} \longrightarrow \textbf{Simp-SM-Cat},$$

where **Siml-SM-Cat** denotes the category of simplicial symmetric monoidal categories. Passing to classifying spaces we get a functor

$$B\mathscr{E}_G(\ ,\mathscr{F})_k\colon G\textbf{-Top} \longrightarrow \textbf{H-Spaces},$$

the category of $H$-spaces, with homotopy commutative $H$-spaces as values for each $k \geq 0$, and also a functor

$$|B\mathscr{E}_G(\ ,\mathscr{F})_{.}|\colon G\textbf{-Top} \longrightarrow \textbf{H-Spaces}$$

by taking realizations by simplicial spaces. Further, taking $\pi_0$ of the above functor for $k=0$ we get a functor

$$\mathrm{Wh}_G(\ ,\mathscr{F})\colon G\textbf{-Top} \longrightarrow \textbf{Ab}$$

by our earlier remark and definition.

Next we look at $\pi_0|B\mathscr{E}_G(X, \mathscr{F})_{.}|$. By definition vertices of $|B\mathscr{E}_G(X, \mathscr{F})_{.}|$ are given by objects of $\mathscr{E}_G(X, \mathscr{F})_0$. Thus the canonical map

$$\kappa\colon B\mathscr{E}_G(X, \mathscr{F})_0 \longrightarrow |B\mathscr{E}_G(X, \mathscr{F})_{.}|$$

induces an epimorphism of $\pi_0$. An edge path in $|B\mathscr{E}_G(X, \mathscr{F})_{.}|$ connecting $(V_1, X)$ to $(V_2, X)$ is given by a finite sequence of arrows, inverted arrows in $\mathscr{E}_G(X, \mathscr{F})_0$ and objects of $\mathscr{E}_G(X, \mathscr{F})_1$. An object $(W, X\times\Delta^1)$ of $\mathscr{E}_G(X, \mathscr{F})_1$ connects $\partial_0^*(W, X\times\Delta^1)$ to $\partial_1^*(W, X\times\Delta^1)$, where $\partial_i\colon [0]\to[1]$ are the standard face maps. Now look at $(\tilde{W}, X)$, $\tilde{W} = W/\sim$, where the relation is given by $(x, t)\sim x$ for $x\in X$ and $t\in\Delta^1$. $(\tilde{W}, X)$ contains

$\partial_0^*(W, X\times\Delta^1)$ and $\partial_1^*(W, X\times\Delta^1)$ as $G$-subcomplexes respectively. By the usual skeleton-wise and $G$-cell-wise arguments (in which each pair $(\partial_1^* b_\lambda^n, b_\lambda^n)$ or $(\partial_0^* b_\lambda^n, b_\lambda^n)$ of $G$-cells of $(\tilde{W}, X)$ forms the elementary $G$-expansion at each stage for each 1-parametrized $G$-cell $b_\lambda^n$ of $(W, X\times\Delta^1)$), we get

$$\partial_0^*(W, X\times\Delta^1)\nearrow_{\mathscr{F}}(\tilde{W}, X), \qquad \partial_1^*(W, X\times\Delta^1)\nearrow_{\mathscr{F}}(\tilde{W}, X).$$

Thus $\partial_0^*(W, X\times\Delta^1)$ is connected to $\partial_1^*(W, X\times\Delta^1)$ by an edge path in $B\mathscr{E}_G(X, \mathscr{F})_0$, which shows that

$$\kappa_*\colon \pi_0 B\mathscr{E}_G(X, \mathscr{F})_0 \cong \pi_0|B\mathscr{E}_G(X, \mathscr{F})_.|$$

is an isomorphism of abelian monoids. Clearly $\kappa_*$ is natural with respect to $X$, and we obtain

**Theorem 3.1.** *There holds an isomorphism*

$$\kappa_*\colon \mathrm{Wh}_G(X, \mathscr{F}) \cong \pi_0|B\mathscr{E}_G(X, \mathscr{F})_.|$$

*which is a natural isomorphism of functors on* $G$-**Top**.

In particular we see that $\pi_0|B\mathscr{E}_G(X, \mathscr{F})_.|$ is also an abelian group. By the above theorem we identify $\pi_0|B\mathscr{E}_G(X, \mathscr{F})_.|$ with $\mathrm{Wh}_G(X, \mathscr{F})$ through $\kappa_*$.

Now there holds

**Theorem 3.2.** *The functor*

$$|B\mathscr{E}_G(\ , \mathscr{F})_.|\colon G\text{-}\mathbf{Top} \longrightarrow \mathbf{H\text{-}Spaces}$$

*preserves homotopies in the sense that*

$$|B\mathscr{E}_G(f, \mathscr{F})_.| \simeq |B\mathscr{E}_G(g, \mathscr{F})_.|, \quad \textit{homotopic},$$

*if two G-map f, g: X→Y are G-homotopic.*

For the proof we refer to [2]. From this theorem it follows

**Corollary 3.3** (Illman). *The functor*

$$\mathrm{Wh}_G(\ , \mathscr{F})\colon G\text{-}\mathbf{Top} \longrightarrow \mathbf{Ab}$$

*is a G-homotopy functor.*

Cf., [7], (II.2.1.).

## § 4. Homotopy decomposition of $\mathscr{E}_G(X)$.

Let $\{\mathscr{C}_\lambda, \lambda \in \Lambda\}$ be a collection of symmetric monoidal (small) categories. We define the direct sum $\coprod_{\lambda\in\Lambda}\mathscr{C}_\lambda$ in the following way.

When $\Lambda$ is a finite set, we put

$$\coprod_{\lambda\in\Lambda}\mathscr{C}_\lambda=\prod_{\lambda\in\Lambda}\mathscr{C}_\lambda$$

which is also a symmetric monoidal category by coordinate-wise sum.

In general case, let $\{\Lambda_\alpha\subset\Lambda, \alpha \in \Gamma\}$ be the set of all finite subset of $\Lambda$. When $\Lambda_\alpha\subset\Lambda_\beta$, $\alpha$, $\beta \in \Gamma$, we define and embedding

$$i_{\alpha,\beta}\colon \coprod_{\lambda\in\Lambda_\alpha}\mathscr{C}_\lambda\subset\coprod_{\lambda\in\Lambda_\beta}\mathscr{C}_\lambda$$

of symmetric monoidal categories by identifying each object and morphism of the former coordinate-wisely with the same one for $\lambda \in \Lambda_\alpha$ and with constant (0-object and identity morphism) for $\lambda \in \Lambda_\beta-\Lambda_\alpha$. Now we define

$$\coprod_{\lambda\in\Lambda}\mathscr{C}_\lambda=\operatorname*{colim}_{\alpha\in\Gamma}\coprod_{\lambda\in\Lambda_\alpha}\mathscr{C}_\lambda$$

with respect to the above embeddings as a colimit of a directed system. The obtained category is also symmetric monoidal.

An alternative description is that

$$\coprod_{\lambda\in\Lambda}\mathscr{C}_\lambda\subset\prod_{\lambda\in\Lambda}\mathscr{C}_\lambda,$$

the full subcategory of which each object has 0-objects in all coordinates except a finite number.

The same definition works also for simplicial symmetric monoidal categories.

Let $\mathscr{F}$ be a family of subgroups of $G$. We observe the simplicial symmetric monoidal category $\coprod_{(H)\subset\mathscr{F}}\mathscr{E}_G(X, (H))$. When $\mathscr{F}$ is *finite* (i.e., it contains only finite many conjugacy classes), then

$$|B\coprod_{(H)\subset\mathscr{F}}\mathscr{E}_G(X, (H))_\cdot|\cong\prod_{(H)\subset\mathscr{F}}|B\mathscr{E}_G(X, (H))_\cdot|$$

and

$$\pi_i|B\coprod_{(H)\subset\mathscr{F}}\mathscr{E}_G(X, (H))_\cdot|\cong\coprod_{(H)\subset\mathscr{F}}\pi_i|B\mathscr{E}_G(X, (H))_\cdot|$$

for all $i\geq 0$ with respect to any base point. In the general case, as a directed colimit we get also an isomorphism

$$\pi_i|B\coprod_{(H)\subset\mathscr{F}}\mathscr{E}_G(X,(H))_\bullet|\cong\coprod_{(H)\subset\mathscr{F}}\pi_i|B\mathscr{E}_G(X,(H))_\bullet|$$

for all $i\geq 0$ with respect to any base point, which is obviously natural with respect to $X$. In the particular case of $i=0$ we get a natural isomorphism

$$\Psi\colon\pi_0|B\coprod_{(H)\subset\mathscr{F}}\mathscr{E}_G(\ ,(H))_\bullet|\cong\coprod_{(H)\subset\mathscr{F}}\mathrm{Wh}_G(\ ,(H))$$

of functors on $G$-**Top**.

Now we define a functor

$$\Gamma_X\colon\coprod_{(H)\subset\mathscr{F}}\mathscr{E}_G(X,(H))_k\rightarrow\mathscr{E}_G(X,\mathscr{F})_k$$

for each $k\geq 0$ by $\Gamma_X(x)=(V_1\cup_{X\times\Delta^k}V_2\cup\cdots\cup_{X\times\Delta^k}V_n,\ X\times\Delta^k)$ for each object $x$ of the former, where $(V_1, X\times\Delta^k),\cdots,(V_n, X\times\Delta^k)$ are the all non-zero coordinates of $x$. Again, $\Gamma_X$ is natural with respect to $X$ and commutes with simplicial structural maps, so we get a natural transformation

$$\Gamma\colon\coprod_{(H)\subset\mathscr{F}}\mathscr{E}_G(\ ,(H))_\bullet\rightarrow\mathscr{E}_G(\ ,\mathscr{F})_\bullet$$

of **Simp-SM-Cat**-valued functors on $G$-**Top**.

There holds

**Theorem 4.1.** *The functor*

$$\Gamma_X\colon\coprod_{(H)\subset\mathscr{F}}\mathscr{E}_G(X,(H))_\bullet\rightarrow\mathscr{E}_G(X,\mathscr{F})_\bullet$$

*is a homotopy equivalence of simplicial categories which is natural with respect to $X$.*

The proof will be referred to [2].

**Corollary 4.2.** (Hauschild). *There hold isomorphisms* $\pi_0|B\Gamma|\cdot\Psi^{-1}$:

$$\mathrm{Wh}_G(\ ,\mathscr{F})\cong\coprod_{(H)\subset\mathscr{F}}\mathrm{Wh}_G(\ ,(H)),$$
$$\mathrm{Wh}_G\cong\coprod_{(H)}\mathrm{Wh}_G(\ ,(H))$$

*of functors on $G$-**Top**, where, in the second isomorphism, $(H)$ runs over all distinct conjugacy classes of closed subgroups of $G$.*

Cf., [6], (III.2).

## § 5. A reduction of $\mathscr{E}_G(X, (H))$.

Hereafter we assume that $G$ is *compact*. Let $H$ be a closed subgroup of $G$ and $X$ a $G$-space. $X^H$ is closed in $X$. Since $G$ is compact, $X^{(H)} = G \cdot X^H$ (by definition) is also closed in $X$. Hence $X^{(H)}$ lies in $G$-**Top**. Let $i_{(H)}^X : X^{(H)} \subset X$ be the inclusion, and observe the simplicial functor

$$i_{(H)\#}^X : \mathscr{E}_G(X^{(H)}, (H))_. \to \mathscr{E}_G(X, (H))_. .$$

The assignment $j_{(H)}^X : (V, X \times \Delta^k) \to (V^{(H)}, X^{(H)} \times \Delta^k)$ gives rise to a simplicial functor

$$j_{(H)}^X : \mathscr{E}_G(X, (H))_. \to \mathscr{E}_G(X^{(H)}, (H))_.$$

and clearly $i_{(H)\#}^X$ and $j_{(H)}^X$ are inverse to each other, i.e., $i_{(H)\#}^X$ (or $j_{(H)}^X$) is an equivalence of simplicial symmetric monoidal categories. It is also clear that $j_{(H)}^X$ is natural with respect to $X$, i.e., we obtain a natural equivalence

$$j_{(H)} : \mathscr{E}_G(\ , (H))_. \cong \text{“} X \mapsto \mathscr{E}_G(X^{(H)}, (H))_. \text{”}$$

of simplicial functors.

Next, take the $H$-fixed point set $(V^H, X^H \times \Delta^k)$ of each object $(V, X^{(H)} \times \Delta^k)$ of $\mathscr{E}_G(X^{(H)}, (H))_k$, $k \geq 0$. $(V^H, X^H \times \Delta^k)$ is a $k$-parametrized finite relative $WH$-$CW$-complex (with the same number of $WH$-cells as that of $G$-cells of $(V, X^{(H)} \times \Delta^k)$), where $WH = NH/H$, and the inclusion $X^H \times \Delta^k \subset V^H$ is *a* $WH$-homotopy equivalence. Since $(G/H)^H = NH/H = WH$, each $k$-parametrized $WH$-cell of $(V^H, X^H \times \Delta^k)$ is a free $WH$-cell. The assignment: “$(V, X^{(H)} \times \Delta^k) \mapsto (V^H, X^H \times \Delta^k)$” commutes also with simplicial structural maps and hence gives rise to a simplicial functor

$$\Phi_H^X : \mathscr{E}_G(X^{(H)}, (H))_. \to \mathscr{E}_{WH}(X^H, \{1\})_. .$$

This functor also have an inverse given by

$$(W, X^H \times \Delta^k) \mapsto \pi_\#(G \times_{NH} W, (G \times_{NH} X^H) \times \Delta^k),$$

where $\pi : G \times_{NH} X^H \to X^{(H)}$ is the canonical $G$-map defined by $\pi(g, x) = gx$. $\Phi_H^X$ is also natural with respect to $X$, whence we get an equivalence

$$\Phi_H : \text{“} X \to \mathscr{E}_G(X^{(H)}, (H))_. \text{”} \cong \text{“} X \to \mathscr{E}_{NH}(X^H, \{1\})_. \text{”}$$

of simplicial functors. Composing $j_{(H)}$ with $\Phi_H$ we obtain

**Theorem 5.1.** *There holds a natural equivalence*

$$\Phi_H \circ j_{(H)} : \mathscr{E}_G(\ , (H))_. \cong \text{“} X \mapsto \mathscr{E}_{WH}(X^H, \{1\})_. \text{”}$$

*of* **Simp-SM-Cat**-*valued functors on* $G$-**Top**.

**Corollary 5.2.** (Hauschild). *There hold natural isomorphisms*

$$\mathrm{Wh}_G(X, (H)) \cong \mathrm{Wh}_{WH}(X^H, \{1\}).$$

Cf., [6], (III.3).

When we change the representative $H$ of $(H)$, i.e., $H \sim K$, we obtain an isomorphism

$$\Phi_K \circ \Phi_H^{-1}: \text{“}X \mapsto \mathscr{E}_{WH}(X^H, \{1\})_{\cdot}\text{”} \cong \text{“}X \mapsto \mathscr{E}_{WK}(X^K, \{1\})_{\cdot}\text{”}$$

of functors. The concrete isomorphism

$$\Phi_K^X \circ (\Phi_H^X)^{-1}: \mathscr{E}_{WH}(X^H, \{1\})_{\cdot} \cong \mathscr{E}_{WK}(X^K, \{1\})_{\cdot}$$

of simplicial categories may be given by an action of $g \in G$ such that $K = gHg^{-1}$.

Further reductions of $\mathrm{Wh}_G(X, (H))$ will be discussed along the line of [1], [8], appart from that of [6].

## § 6. Decompositions of $\mathscr{E}_{WH}(X^H, \{1\})_{\cdot}$

Now we restrict our discussions to the full subcategory $l$-$G$-0-**Top** of $G$-**Top** consisting of locally $G$-0-connected $G$-spaces. For each object $X$ of $l$-$G$-0-**Top** $X^H$ is locally path-connected and decomposes into disjoint union of path-components. Take $WH$-orbits of path-components of $X^H$, then we get a decomposition

$$X^H = \coprod_\alpha WH \cdot X^H{}_\alpha$$

as a topological sum of $WH$-subspaces, where $X^H{}_\alpha$'s are path-components of $X^H$ [7], [8]. We call each summand $WH \cdot X^H{}_\alpha$ a *$WH$-component* of $X^H$ and $X^H{}_\alpha$ a *representative path-component* of the $WH$-component $WH \cdot X^H{}_\alpha$.

Let $(V, X^H \times \Delta^k)$ be an object of $\mathscr{E}_{WH}(X^H, \{1\})_k$, $k \geq 0$. Since the inclusion $X^H \times \Delta^k \subset V$ is a $WH$-homotopy equivalence and $V$ is obviously locally path-connected, the inclusion gives a bijection of path-components and hence of $WH$-components. Let $V_\alpha$ be the $WH$-component of $V$ containing $WH \cdot X^H{}_\alpha$ and put $\tilde{V}_\alpha = V_\alpha \cup X^H \times \Delta^k$. Then we get a sum decomposition

$$(V, X^H \times \Delta^k) = \coprod_\alpha (\tilde{V}_\alpha, X^H \times \Delta^k),$$

which is eventually finite (all summands are zero objects except a finite number) as $V - X^H \times \Delta^k$ contains only a finitely many $WH$-cells. Thus, denoting by $\mathscr{E}_k^\alpha$ the subcategory of $\mathscr{E}_{WH}(X^H, \{1\})_k$ consisting of objects $(W, X^H \times \Delta^k)$ such that $W \cap (X^H \times \Delta^k - WH \cdot X^H{}_\alpha \times \Delta^k) = X^k \times \Delta^k - WH \cdot X^H{}_\alpha \times \Delta^k$ for each index $\alpha$, we get the direct sum decomposition

$$\mathscr{E}_{WH}(X^H, \{1\})_k \cong \coprod_\alpha \mathscr{E}_k^\alpha$$

of symmetric monoidal categories. The inclusion $i_\alpha$: $WH \cdot X^H{}_\alpha \subset X^H$ induces a functor

$$i_{\alpha\#}\colon \mathscr{E}_{WH}(WH \cdot X^H{}_\alpha, \{1\})_k \to \mathscr{E}_k^\alpha \quad (\subset \mathscr{E}_{WH}(X^H, \{1\})_k)$$

for each $\alpha$, which is easily seen to be an equivalence. Thus we get an equivalence

$$\coprod_\alpha i_{\alpha\#}\colon \coprod_\alpha \mathscr{E}_{WH}(WH \cdot X^H{}_\alpha, \{1\})_k \cong \mathscr{E}_{WH}(X^H, \{1\})_k$$

of symmetric monoidal categories, which is clearly compatible with simplicial structural maps. Hence we obtain

**Theorem 6.1.** *There holds a direct sum decomposition*

$$\coprod_\alpha i_{\alpha\#}\colon \coprod_\alpha \mathscr{E}_{WH}(WH \cdot X^H{}_\alpha, \{1\}).\cong \mathscr{E}_{WH}(X^H, \{1\}).$$

*of simplicial symmetric monoidal category for each object $X$ of $l$-$G$-0-***Top**, *which is natural with respect to $X$ in the following sense: let $f$: $X \to Y$ be a $G$-map of $l$-$G$-0-***Top** *such that $f^H$: $X^H \to Y^H$ gives a bijection of path-components (and of $WH$-components); decompose $X^H = \coprod_\alpha WH \cdot X^H{}_\alpha$ and $Y^H = \coprod_\alpha WH \cdot Y^H{}_\alpha$ into $WH$-components with the same indices $\alpha$ so that $f^H(WH \cdot X^H{}_\alpha) \subset WH \cdot Y^H{}_\alpha$ for each $\alpha$; let $\bar{f}^H{}_\alpha = f^H \mid WH \cdot X^H{}_\alpha$: $WH \cdot X^H{}_\alpha \to WH \cdot Y^H{}_\alpha$ for each $\alpha$, then the following diagram*

$$\begin{array}{ccc} \coprod_\alpha \mathscr{E}_{WH}(WH \cdot X^H{}_\alpha, \{1\}). & \xrightarrow{\coprod_\alpha i_{\alpha\#}} & \mathscr{E}_{WH}(X^H, \{1\}). \\ \downarrow \coprod_\alpha \bar{f}^H{}_{\alpha\#} & & \downarrow f^H{}_\# \\ \coprod_\alpha \mathscr{E}_{WH}(WH \cdot Y^H{}_\alpha, \{1\}). & \xrightarrow[\coprod_\alpha i_{\alpha\#}]{} & \mathscr{E}_{WH}(Y^H, \{1\}). \end{array}$$

*is commutative.*

The naturality in the above theorem is also obvious from definitions.

**Corollary 6.2.** *There holds a direct sum decomposition*

$$\mathrm{Wh}_{WH}(X^H, \{1\}) \cong \coprod_\alpha \mathrm{Wh}_{WH}(WH \cdot X^H{}_\alpha, \{1\})$$

*for each object X such that* $X^H=\coprod_\alpha WH\cdot X^H{}_\alpha$ *of l-G-0-***Top***, which is natural with respect to X in the sense as in the above theorem, i.e., let f: X →Y be a G-map of l-G-0-***Top** *such that* $f^H: X^H\to Y^H$ *gives a bijection of path-components; using the above notations of the above theorem there holds the commutative diagram*

$$\begin{array}{ccc} \mathrm{Wh}_{WH}(X^H,\{1\}) \cong \coprod_\alpha \mathrm{Wh}_{WH}(WH\cdot X^H{}_\alpha,\{1\}) \\ \downarrow f^H{}_* \qquad\qquad\qquad \downarrow \coprod_\alpha \bar{f}^H{}_{\alpha *} \\ \mathrm{Wh}_{WH}(Y^H,\{1\}) \cong \coprod_\alpha \mathrm{Wh}_{WH}(WH\cdot Y^H{}_\alpha,\{1\}). \end{array}$$

## § 7. A reduction of $\mathscr{E}_{WH}(WH\cdot X^H{}_\alpha, \{1\})$.

In this section we continue to discuss in *l*-*G*-0-**Top**. Let $X$ be an object $G$-space and $WH\cdot X^H{}_\alpha$ a $WH$-component of $X^H$. Put

$$W_\alpha H=\{w\in WH,\ w\cdot X^H{}_\alpha\subset X^H{}_\alpha\},$$

which is a closed subgroup of $WH$. $X^H{}_\alpha$ is a $W_\alpha H$-space and we can express

$$WH\cdot X^H{}_\alpha=WH\times_{W_\alpha H}X^H{}_\alpha.$$

Let $(WH)_0$ be the 1-component of $WH$. Clearly $W_\alpha H\supset(WH)_0$ and hence $[WH: W_\alpha H]<\infty$ since $WH$ is compact.

Let $(V, WH\cdot X^H{}_\alpha\times\Delta^k)$ be an object of $\mathscr{E}_{WH}(WH\cdot X^H{}_\alpha, \{1\})_k$, $k\geq 0$. The inclusion $i$: $WH\cdot X^H{}_\alpha\times\Delta^k\subset V$ is a $WH$-homotopy equivalence and gives a bijection of path-components. Let $V_\alpha$ be the path-component of $V$ containing $X^H{}_\alpha\times\Delta^k$. Then $V_\alpha$ is a $W_\alpha H$-space,

$$V=WH\cdot V_\alpha=WH\times_{W_\alpha H}V_\alpha$$

and the inclusion $X^H{}_\alpha\times\Delta^k\subset V_\alpha$ is a $W_\alpha H$-homotopy equivalence. Let $b^n$ be a parametrized (free) $WH$-cell of $(V, WH\cdot X^H{}_\alpha\times\Delta^k)$. Then $V_\alpha\cap b^n$ is a parametrized free $W_\alpha H$-cell and we can express $V_\alpha\cap b^n=W_\alpha H\times e^n\times\Delta^k$. Thus $(V_\alpha, X^H{}_\alpha\times\Delta^k)$ is an object of $\mathscr{E}_{W_\alpha H}(X^H{}_\alpha, \{1\})_k$. The assignment $\rho$: $(V, WH\cdot X^H{}_\alpha\times\Delta^k)\mapsto(V_\alpha, X^H{}_\alpha\times\Delta^k)$ preserves parametrized free $WH$-expansions to free $W_\alpha H$-expansions and we obtain a functor

$$\rho\colon \mathscr{E}_{WH}(WH\cdot X^H{}_\alpha, \{1\})_k\to\mathscr{E}_{W_\alpha H}(X^H{}_\alpha, \{1\})_k$$

for each $k\geq 0$.

Clearly $\rho$ commutes with simplicial structural maps and the assignment "$(V_\alpha, X^H{}_\alpha\times\Delta^k)\mapsto(WH\times_{W_\alpha H}V_\alpha, WH\times_{W_\alpha H}X^H{}_\alpha\times\Delta^k)$" gives rise to an inverse functor to $\rho$. Hence we obtain

**Theorem 7.1.** *There holds a natural equivalence*

$$\rho\colon \mathscr{E}_{WH}(WH\cdot X^H{}_\alpha, \{1\})_\bullet \cong \mathscr{E}_{W_\alpha H}(X^H{}_\alpha, \{1\})_\bullet$$

*of simplicial symmetric monoidal categories for each object $X$ of $l$-$G$-0-***Top**, *which is natural with respect to $X$ in the following sense: let $f\colon X\to Y$ be a $G$-map in $l$-$G$-0-***Top** *such that $f^H\colon X^H\to Y^H$ gives a bijection of path-components; let $WH\cdot X^H{}_\alpha$ and $WH\cdot Y^H{}_\alpha$ (with the same indices) be $WH$-components of $X^H$ and $Y^H$ such that $f^H(WH\cdot X^H{}_\alpha)\subset WH\cdot Y^H{}_\alpha$ and choose representative components $X^H{}_\alpha$ and $Y^H{}_\alpha$ so that $f^H(X^H{}_\alpha)\subset Y^H{}_\alpha$; put $f^H{}_\alpha = f^H | X^H{}_\alpha\colon X^H{}_\alpha \to Y^H{}_\alpha$ and $\bar f^H{}_\alpha = f^H | WH\cdot X^H{}_\alpha\colon WH\cdot X^H{}_\alpha \to WH\cdot Y^H{}_\alpha$, then the following diagram*

$$\begin{array}{ccc}
\mathscr{E}_{WH}(WH\cdot X^H{}_\alpha, \{1\})_\bullet & \xrightarrow[\rho]{} & \mathscr{E}_{W_\alpha H}(X^H{}_\alpha, \{1\})_\bullet \\
\downarrow \bar f^H{}_{\alpha\#} & & \downarrow f^H{}_{\alpha\#} \\
\mathscr{E}_{WH}(WH\cdot Y^H{}_\alpha, \{1\})_\bullet & \xrightarrow[\rho]{} & \mathscr{E}_{W_\alpha H}(Y^H{}_\alpha, \{1\})_\bullet
\end{array}$$

*is commutative.*

As to the naturality in the above theorem, we remark that $W_\alpha H$ for $Y^H{}_\alpha$ is the smae as that for $X^H{}_\alpha$ under the situation in the above theorem. The rests of the proof of naturality is obvious by definitions.

**Corollary 7.2.** *There holds an isomorphism*

$$\rho_*\colon \mathrm{Wh}_{WH}(WH\cdot X^H{}_\alpha, \{1\}) \cong \mathrm{Wh}_{W_\alpha H}(X^H{}_\alpha, \{1\})$$

*for each object $X$ of $l$-$G$-0-***Top**, *which is natural with respect to $X$ in the same sense as in the above theorem, i.e., let $f\colon X\to Y$ be a $G$-map in $l$-$G$-0-***Top** *such that $f^H\colon X^H\to Y^H$ gives a bijection of path-components; using the notations of the above theorem there holds the commutative diagram*

$$\begin{array}{ccc}
\mathrm{Wh}_{WH}(WH\cdot X^H{}_\alpha, \{1\}) & \underset{\rho_*}{\cong} & \mathrm{Wh}_{W_\alpha H}(X^H{}_\alpha, \{1\}) \\
\downarrow \bar f^H{}_{\alpha *} & & \downarrow f^H{}_{\alpha *} \\
\mathrm{Wh}_{WH}(WH\cdot Y^H{}_\alpha, \{1\}) & \underset{\rho_*}{\cong} & \mathrm{Wh}_{W_\alpha H}(Y^H{}_\alpha, \{1\}).
\end{array}$$

When we change the choice of representative components of a $WH$-component $WH\cdot X_\alpha$, i.e., $WH\cdot X_\alpha = WH\cdot X_\beta$, we obtain isomorphisms

$$\mathscr{E}_{WH}(WH\cdot X_\alpha, \{1\})_\bullet \underset{\rho_\alpha}{\cong} \mathscr{E}_{W_\alpha H}(X_\alpha, \{1\})_\bullet$$

$$\mathscr{E}_{WH}(WH\cdot X_\beta, \{1\})_\bullet \underset{\rho_\beta}{\cong} \mathscr{E}_{W_\beta H}(X_\beta, \{1\})_\bullet$$

of simplicial categories. Hence an isomorphism

$$\rho_\beta \circ \rho_\alpha^{-1}\colon \mathscr{E}_{W_\alpha H}(X_\alpha, \{1\}). \cong \mathscr{E}_{W_\beta H}(X_\beta, \{1\}).$$

of simplicial categories, which may be given directly by a choice $w \in WH$ such that $w \cdot X^H{}_\alpha \subset X^H{}_\beta$ (then $W_\beta H = w \cdot W_\alpha H \cdot w^{-1}$) and by an action of $w$. Cf. [8], Section 8.

## § 8. Passing to universal covering spaces

Here we restrict our discussions to the full subcategory $l$-$G$-1-**Top** of $l$-$G$-0-**Top** consisting of locally $G$-1-connected $G$-spaces. Let $X^H{}_\alpha$ be a path-component of $X^H$. As $X^H{}_\alpha$ is semi-locally 1-connected we define its universal covering space $\tilde{X}^H{}_\alpha$ as the subspace of the fundamental groupoid of $X^H{}_\alpha$ with initial point $x_0 \in X^H{}_\alpha$. $\pi_1 = \pi_1(X^H{}_\alpha, x_0)$ operates on $\tilde{X}^H{}_\alpha$ as the covering transformation group. Let $WH \cdot X^H{}_\alpha$ be the $WH$-component of $X^H$ with $X^H{}_\alpha$ as a representative path-component and $W_\alpha H$ be the closed subgroup of $WH$ defined in the precceding section, then $X^H{}_\alpha$ is a $W_\alpha H$-space. By [8], Section 5 (also [1] for finite group case) we have a Lie group $\Gamma_\alpha$ satisfying the short exact sequence

$$1 \longrightarrow \pi_1 \longrightarrow \Gamma_\alpha \xrightarrow{q} W_\alpha H \longrightarrow 1$$

and $\tilde{X}^H{}_\alpha$ is a $\Gamma_\alpha$-space such that $\Gamma_\alpha$-actions contain $\pi_1$-actions as covering transformations and the covering projection $p\colon \tilde{X}^H{}_\alpha \to X^H{}_\alpha$ is $q$-equivariant. Illman [8], Section 5, defined such actions of $\Gamma_\alpha$ on $\tilde{X}^H{}_\alpha$ in two ways and proved that the two actions coincide.

Let $(W, \tilde{X}^H{}_\alpha \times \Delta^k)$ be an object of $\mathscr{E}_{\Gamma_\alpha}(\tilde{X}^H{}_\alpha, \{1\})_k$, $k \geq 0$. The quotient map $q_k\colon (W, \tilde{X}^H{}_\alpha \times \Delta^k) \mapsto (\pi_1 \backslash W, X^H{}_\alpha \times \Delta^k) \in \operatorname{obj} \mathscr{E}_{W_\alpha H}(X^H{}_\alpha, \{1\})_k$ with respect to $\pi_1$-actions preserves free $\Gamma_\alpha$-expansions to free $W_\alpha H$-expansions and hence we obtain a functor

$$q_k\colon \mathscr{E}_{\Gamma_\alpha}(\tilde{X}^H{}_\alpha, \{1\})_k \longrightarrow \mathscr{E}_{W_\alpha H}(X^H{}_\alpha, \{1\})_k$$

for each $k \geq 0$. Clearly $q_k$ commutes with simplicial structural maps and we get the simplicial functor

$$q. = \{q_k, k \geq 0\}; \mathscr{E}_{\Gamma_\alpha}(\tilde{X}^H{}_\alpha, \{1\}). \longrightarrow \mathscr{E}_{W_\alpha H}(X^H{}_\alpha, \{1\}).\,.$$

Choose a point $t_0 \in \Delta^k$ and $\widetilde{X^H{}_\alpha \times \Delta^k}$ be the universal covering space of $X^H{}_\alpha \times \Delta^k$ with $(x_0, t_0)$ as the base point. The canonical projections $\mathrm{pr}_1\colon X^H{}_\alpha \times \Delta^k \to X^H{}_\alpha$ and $\mathrm{pr}_2\colon X^H{}_\alpha \times \Delta^k \to \Delta^k$ induces the $\Gamma_\alpha$-homeomorphism

$$h_k\colon \widetilde{X^H{}_\alpha \times \Delta^k} \approx \tilde{X}^H{}_\alpha \times \Delta^k.$$

(Cf., the later naturality argument below Theorem 8.1.) For each object $(V, X^H{}_\alpha \times \Delta^k)$ of $\mathscr{E}_{W_\alpha H}(X^H{}_\alpha, \{1\})_k$ let $(\tilde{V}, \widetilde{X^H{}_\alpha \times \Delta^k})$ be the pair of universal covering spaces with respect to the base point $(x_0, t_0)$. Observe that $h_{k\#}(\tilde{V}, \widetilde{X^H{}_\alpha \times \Delta^k})$ is an object of $\mathscr{E}_{\Gamma_\alpha}(\tilde{X}^H{}_\alpha, \{1\})_k$ and the assignment $\lambda_k$; $(V, X^H{}_\alpha \times \Delta^k) \mapsto h_{k\#}(\tilde{V}, \widetilde{X^H{}_\alpha \times \Delta^k})$ preserves free $W_\alpha H$-expansions to free $\Gamma_\alpha$-expansions. Thus we obtain a functor

$$\lambda_k \colon \mathscr{E}_{W_\alpha H}(X^H{}_\alpha, \{1\})_k \longrightarrow \mathscr{E}_{\Gamma_\alpha}(\tilde{X}^H{}_\alpha, \{1\})_k$$

for each $k \geq 0$. We see easily that the pair $(q_k, \lambda_k)$ gives an equivalence between symmetric monoidal categories. ($\lambda_k$'s may not commutes with simplicial structural maps.) Thus

$$Bq_k \colon B\mathscr{E}_{\Gamma_\alpha}(\tilde{X}^H{}_\alpha, \{1\})_k \longrightarrow B\mathscr{E}_{W_\alpha H}(X^H{}_\alpha, \{1\})_k$$

is a homotopy equivalence for each $k \geq 0$, and we obtain the simplicial map

$$Bq_{\bullet} = \{Bq_k, k \geq 0\} \colon B\mathscr{E}_{\Gamma_\alpha}(\tilde{X}^H{}_\alpha, \{1\})_{\bullet} \longrightarrow B\mathscr{E}_{W_\alpha H}(X^H{}_\alpha, \{1\})_{\bullet}$$

of good simplicial spaces which is degreewise homotopy equivalence. Thus, by the "realization lemma" [24], Appendix $A$, we see that $|Bq_{\bullet}|$ is a homotopy equivalence and obtain

**Theorem 8.1.** *There holds the homotopy equivalence*

$$q_{\bullet} \colon \mathscr{E}_{\Gamma_\alpha}(\tilde{X}^H{}_\alpha, \{1\})_{\bullet} \simeq \mathscr{E}_{W_\alpha H}(X^H{}_\alpha, \{1\})_{\bullet}$$

*of simplicial symmetric monoidal categories for each object $X$ of $l$-$G$-1-***Top**.

As to the naturality with respect to $X$, we observe the following situation. Let $f \colon X \rightarrow Y$ be a $G$-map in $l$-$G$-1-**Top** such that $f^H \colon X^H \rightarrow Y^H$ gives bijection of path-components and induces isomorphisms of fundamental groups for any base point. Let $f^H{}_\alpha = f^H | X^H{}_\alpha \colon X^H{}_\alpha \rightarrow Y^H{}_\alpha$ be the $W_\alpha H$-map of the considered path-components. Take universal covering spaces $\tilde{X}^H{}_\alpha$ and $\tilde{Y}^H{}_\alpha$ with respect to base points $x_0$ and $f(x_0)$. Express the groups $\Gamma_\alpha$ for $\tilde{X}^H{}_\alpha$ and $\tilde{Y}^H{}_\alpha$ by $\Gamma^X_\alpha$ and $\Gamma^Y_\alpha$ respectively. Here we regard $\tilde{Y}^H{}_\alpha$ as a principal $\pi_1$-bundle over $Y^H{}_\alpha$, $\pi_1 = \pi_1(Y^H{}_\alpha, f(x_0))$. Since $f^H{}_{\alpha *} \colon \pi_1(X^H{}_\alpha, x_0) \cong \pi_1(Y^H{}_\alpha, f(x_0))$ we have a canonical identification $\tilde{X}^H{}_\alpha = (f^H{}_\alpha)^* \tilde{Y}^H{}_\alpha \subset X^H{}_\alpha \times \tilde{Y}^H{}_\alpha$. Define $\tilde{\gamma} \cdot (x, \tilde{y}) = (q(\gamma) \cdot x, \gamma \cdot \tilde{y})$ for $\gamma \in \Gamma^Y_\alpha$ and $(x, \tilde{y}) \in \tilde{X}^H{}_\alpha$, then we see that $\tilde{\gamma} \in \mathrm{Homeo}\,(\tilde{X}^H{}_\alpha)$ and its action on $\tilde{X}^H{}_\alpha$ covers the $q(\gamma)$-action on $X^H{}_\alpha$. We may assume that $W_\alpha H$-actions on $X^H{}_\alpha$ and $Y^H{}_\alpha$ are effective. (If not, we may use a device by Illman [8], Section 5, to attach a free elementary $W_\alpha H$-expansion to $X^H{}_\alpha$ and $Y^H{}_\alpha$ respectively so that $f^H{}_\alpha$ extends as a $W_\alpha H$-map). Thus $\bar{\gamma} \in \Gamma^X_\alpha$, and we see easily that the assignment "$\gamma \mapsto \bar{\gamma}$" gives an isomorphism: $\Gamma^Y_\alpha \cong \Gamma^X_\alpha$. We identify the both sides by this isomorphism and denote it simply by $\Gamma_\alpha$. Then we get a commutative diagram

$$\begin{array}{ccc} \tilde{X}^H{}_\alpha & \xrightarrow{\tilde{f}^H{}_\alpha} & \tilde{Y}^H{}_\alpha \\ \downarrow q & & \downarrow q \\ X^H{}_\alpha & \xrightarrow{f^H{}_\alpha} & Y^H{}_\alpha, \end{array}$$

where $\tilde{f}^H{}_\alpha$ is a $\Gamma_\alpha$-map.

**Proposition 8.2.** *Under the above situation the following diagram*

$$\begin{array}{ccc} \mathscr{E}_{\Gamma_\alpha}(\tilde{X}^H{}_\alpha, \{1\}). & \underset{q_\bullet}{\simeq} & \mathscr{E}_{W_\alpha H}(X^H{}_\alpha, \{1\}). \\ \downarrow \tilde{f}^H{}_{\alpha\#} & & \downarrow f^H{}_{\alpha\#} \\ \mathscr{E}_{\Gamma_\alpha}(\tilde{Y}^H{}_\alpha, \{1\}). & \underset{q_\bullet}{\simeq} & \mathscr{E}_{W_\alpha H}(Y^H{}_\alpha, \{1\}). \end{array}$$

*is commutative.*

**Corollary 8.3.** *There holds an isomorphism*

$$q_*: \mathrm{Wh}_{\Gamma_\alpha}(\tilde{X}^H{}_\alpha, \{1\}) \cong \mathrm{Wh}_{W_\alpha H}(X^H{}_\alpha, \{1\}).$$

*Under the situation of the above proposition $q_*$ is natural in the sene that the diagram*

$$\begin{array}{ccc} \mathrm{Wh}_{\Gamma_\alpha}(\tilde{X}^H{}_\alpha, \{1\}) & \underset{q_*}{\cong} & \mathrm{Wh}_{W_\alpha H}(X^H{}_\alpha, \{1\}) \\ \downarrow \tilde{f}^H{}_{\alpha *} & & \downarrow f^H{}_{\alpha *} \\ \mathrm{Wh}_{\Gamma_\alpha}(\tilde{Y}^H{}_\alpha, \{1\}) & \underset{q_*}{\cong} & \mathrm{Wh}_{W_\alpha H}(Y^H{}_\alpha, \{1\}) \end{array}$$

*is commutative.*

## § 9. The final step of reductions of $\mathrm{Wh}_G(X, (H))$

Let $X$ be an object of $l$-$G$-1-**Top** and $(V, \tilde{X}^H{}_\alpha)$ be an object of $\mathscr{E}_{\Gamma_\alpha}(\tilde{X}^H{}_\alpha, \{1\})_0$ using the notations of the preceding section. Put

$$C_n(V) = H_n(V^n, V^{n-1}; Z), \quad n \geq 0, \quad V^{-1} = \tilde{X}^H{}_\alpha,$$

where $V^n$ is the $n$-skeleton of $(V, \tilde{X}^H{}_\alpha)$. By Illman [8], Proposition 3.1, $C_* = \{C_n(V), n \geq 0\}$ is a bounded acyclic chain complex. Here

$$C_n(V)=H_n(V^n, V^{n-1}; \boldsymbol{Z})=\coprod_i H_n(\Gamma_\alpha\times E_i^n; \Gamma_\alpha\times S_i^{n-1}; \boldsymbol{Z})$$
$$=\coprod_i H_0(\Gamma_\alpha)=\coprod_i \boldsymbol{Z}\cdot\Gamma_\alpha/\Gamma_{\alpha,0},$$

where $\Gamma_{\alpha,0}$ is the 1-component of $\Gamma_\alpha$. Thus $C_n(V)$ is a finitely generated free $(\Gamma_\alpha/\Gamma_{\alpha,0})$-module with preferred basis consisting of path-components of free $n$-$\Gamma_\alpha$-cells. Thus, putting $C_*(V)=\{C_n(V), n\geq 0\}$, $C_*(V)$ represents an element

$$[C_*(V)]\in \mathrm{Wh}(\Gamma_\alpha/\Gamma_{\alpha,0}),$$

where the right hand side is the classical Whitehead group of the discrete group $\Gamma_\alpha/\Gamma_{\alpha,0}$. By [8], Section 10, we know that the assignment "$(V, \tilde{X}^H{}_\alpha)\mapsto[C_*(V)]$" gives rise to well defined homomorphism

$$\Phi\colon \mathrm{Wh}_{\Gamma_\alpha}(\tilde{X}^H{}_\alpha, \{1\})\longrightarrow \mathrm{Wh}(\Gamma_\alpha/\Gamma_{\alpha,0}).$$

**Proposition 9.1.** *Let* $f\colon X\rightarrow Y$ *be a* $G$*-map in* $l$-$G$-1-**Top** *such that* $f^H\colon X^H\rightarrow Y^H$ *gives a bijection of path-components and induces isomorphisms of fundamental groups for any base point. Then, under the situation of Proposition* 8.2 *we get the following commutative diagram*

$$\begin{array}{ccc}
\mathbf{Wh}_{\Gamma_\alpha}(\tilde{X}^H{}_\alpha, \{1\}) & \xrightarrow{\ \Phi_X\ } & \\
\Big\downarrow \tilde{f}^H{}_{\alpha*} & & \mathbf{Wh}(\Gamma_\alpha/\Gamma_{\alpha,0})\ . \\
\mathbf{Wh}_{\Gamma_\alpha}(\tilde{Y}^H{}_\alpha, \{1\}) & \xrightarrow{\ \Phi_Y\ } & 
\end{array}$$

The proof is obvious since $\tilde{f}^H{}_{\alpha*}\colon H_n(V^n, V^{n-1})\cong H_n(f_\#(V^n), f_\#(V^{n-1}))$ is essentially an identity map.

**Theorem 9.2.** $\Phi\colon \mathrm{Wh}_{\Gamma_\alpha}(\tilde{X}^H{}_\alpha, \{1\})\cong \mathrm{Wh}(\Gamma_\alpha/\Gamma_{\alpha,0})$, *an isomorphism.*

The proof of this theorem is essentially the same as that of [8], Section 11. We remark only that, for the proof of $\Phi$ being an epimorphism in our relative version, we may apply Theorem 1.3 for the inclusion $(\tilde{X}^H{}_\alpha, \tilde{X}^H{}_\alpha)\subset(V, \tilde{X}^H{}_\alpha)$.

Now let $f\colon X\rightarrow Y$ be a $G$-map in $l$-$G$-1-**Top** satisfying the conditions of Proposition 9.1. By Proposition 9.1 and Theorem 9.2 we see that

$$\tilde{f}^H{}_{\alpha*}\colon \mathrm{Wh}_{\Gamma_\alpha}(\tilde{X}^H{}_\alpha, \{1\})\cong \mathrm{Wh}_{\Gamma_\alpha}(\tilde{Y}^H{}_\alpha, \{1\})$$

is an isomorphism for each corresponding pair of path-components of $X^H$ and $Y^H$. Then, by Corollary 8.3

$$f^H{}_{\alpha*}\colon \mathrm{Wh}_{W_\alpha H}(X^H{}_\alpha, \{1\})\cong \mathrm{Wh}_{W_\alpha H}(Y^H{}_\alpha, \{1\})$$

is an isomorphism and by Corollary 7.2

$$\bar{f}^{H}{}_{\alpha *}\colon \mathrm{Wh}_{WH}(WH\cdot X^{H}{}_{\alpha}, \{1\}) = \mathrm{Wh}_{WH}(WH\cdot Y^{H}{}_{\alpha}, \{1\})$$

is an isomorphism for each corresponding pair of $WH$-components. Thus, by Corollary 6.2

$$f^{H}{}_{*}\colon \mathrm{Wh}_{WH}(X^{H}, \{1\}) \cong \mathrm{Wh}_{WH}(Y^{H}, \{1\})$$

is an isomorphism. Hence, by Corollary 5.2

$$f_{*}\colon \mathrm{Wh}_{G}(X, (H)) \cong \mathrm{Wh}_{G}(Y, (H))$$

is an isomorphism. Thus we obtain

**Theorem 9.3.** *Let $f\colon X\to Y$ be a $G$-map in $l$-$G$-1-***Top** *and $H$ a closed subgroup of $G$ such that $f^{H}\colon X^{H}\to Y^{H}$ gives a bijection of path-components and induces isomorphisms of fundamental groups for any base point. Then there holds an isomorphism*

$$f_{*}\colon \mathrm{Wh}_{G}(X, (H)) \cong \mathrm{Wh}_{G}(Y, (H)).$$

**Added in Proof.** In course of the proof of Theorem 4.1 (Homotopy decomposition) the author has found a little difficulty if we are based on the present definition of parametrized G-expansion categories, of which the objects are a bit too tight. We need to replace them by ones with somewhat relaxed form. This task is done in [2], whereby $\pi_0$ is *not* changed and also the whole development of sections 4–9 of the present article is still *not* changed. For details, cf., [2].

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