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The Versality Theorem for *RL*-Morphisms of Foliation Unfoldings

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In [7], we proved a versality theorem for unfoldings of codim 1 foliation germs, which generalizes the versality theorem with respect to right morphisms in the unfolding theory of function germs. The purpose of this paper is to prove a similar theorem with respect to *RL*-morphisms. These morphisms generalize right-left morphisms in the function case and play an important role in the determinacy problem of foliation germs (see [11]). We also note that the definition naturally involves integrating factors of the given foliation germ (see Definitions (1.1) and (1.2) and (1.3) Remark).

In Section 1, we recall terminologies and describe the set of *RL*-isomorphism classes of first order unfoldings of a foliation germ. We prove, in Section 2, the versality theorem ((2.1) Theorem), which says that an infinitesimally *RL*-versal unfolding of a codim 1 foliation germ F is *RL*-versal. Let $\mathcal{F}$ be an infinitesimally *RL*-versal unfolding of F and let $\mathcal{F}'$ be an arbitrarily given unfolding of F. The proof consists of, as in [7], (I) construction of an *RL*-morphism from $\mathcal{F}'$ to $\mathcal{F}$ as a formal power series in the parameters of $\mathcal{F}'$ and (II) proof of the existence of a convergent solution. Basically the infinitesimal *RL*-versality is sufficient for (I), although the procedure is rather involved. For (II), we need some side condition ((*) in (2.1) Theorem, see also (2.2) Remark), which is satisfied in most cases. We compare the series obtained in (I) with convergent series similar to the one used in Kodaira-Spencer [6]. For this, we use the privileged neighborhood theorem of Malgrange [3] as well as the unfolding theory of integrating factors developed in the appendix of this paper. In Section 3, we explain how our theorem is related to the versality theorem for unfoldings of function germs with respect to right-left morphisms (cf. Wassermann [13]). We consider the "meromorphic" case in Section 4. Namely, for a foliation F generated by a germ ω of the form $\omega = gdf - fdg$, where f and g are holomorphic function germs, we determine the set of integrating factors ((4.1) Lemma) and apply (2.1) Theorem to obtain an *RL*-universal unfolding of F explicitly ((4.6)

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Theorem). In the appendix we deal with the unfolding (extension) problem of integrating factors. Although the extendability of integrating factors is proved in Cerveau-Moussu [2] (see also [1]) by a different approach, here we employed the power series method, since detailed estimates are needed for the proof of (2.1) Theorem.

§ 1. Preliminaries

Let $\mathcal{O}_n$ denote the ring of germs of holomorphic functions at the origin 0 in $C^n=\{(x_1, \cdots, x_n)\}$ and let Ω_n and Θ_n denote, respectively, the $\mathcal{O}_n$-modules of germs of holomorphic 1-forms and of holomorphic vector fields at 0 in C^n. A codim 1 foliation germ at 0 in C^n is a rank 1 free sub-$\mathcal{O}_n$-module $F=(\omega)$ of Ω_n with a generator ω satisfying the integrability condition $d\omega\wedge\omega=0$. The germ at 0 of the analytic set $\{x\,|\,\omega(x)=0\}$ is denoted by $S(\omega)$ or $S(F)$ and is called the singular set of F. We always assume that codim $S(F)\geq 2$ (cf. [7] 1, [9] (5.1) Lemma, [10] (1.1) Lemma).

An unfolding of $F=(\omega)$ is a codim 1 foliation germ $\mathcal{F}=(\tilde{\omega})$ at 0 in $C^n\times C^m$, for some m, with a generator $\tilde{\omega}$ such that $\iota^*\tilde{\omega}=\omega$, where ι denotes the embedding of $C^n=\{x\}$ into $C^n\times C^m=\{(x, t)\}$ given by $\iota(x)=(x, 0)$. We call C^m the parameter space of $\mathcal{F}$. We recall the following ([11] (2.1) Definition, see also [10] (1.2) Definition, [9] (2.5) Definition).

(1.1) **Definition.** Let $\mathcal{F}=(\tilde{\omega})$ and $\mathcal{F}'=(\theta)$ be two unfoldings of F with parameter spaces C^m and $C^l=\{(s_1, \cdots, s_l)\}$, respectively.

(I) A *morphism* from $\mathcal{F}'$ to $\mathcal{F}$ is a triple (Φ, ψ, u) such that

(a) Φ and ψ are holomorphic map germs making the diagram

$$\begin{array}{ccc} (C^n\times C^l, 0) & \xrightarrow{\Phi} & (C^n\times C^m, 0) \\ \downarrow & & \downarrow \\ (C^l, 0) & \xrightarrow{\psi} & (C^m, 0) \end{array}$$

commutative, where the vertical maps are the projections. u is a unit in $\mathcal{O}_{n+l}$.

(b) $\Phi(x, 0)=(x, 0)$ and $u(x, 0)=1$.

(c) $u\theta=\Phi^*\tilde{\omega}$.

(II) An *RL-morphism* from $\mathcal{F}'$ to $\mathcal{F}$ is a quadruple (Φ, ψ, u, α), where Φ, ψ and u are germs satisfying (a) and (b) in (I) and $\alpha=(\alpha_1, \cdots, \alpha_l)$ is a germ in $\mathcal{O}_{n+l}^l$. Instead of (c), we require

(c)′ $u\theta=\Phi^*\tilde{\omega}+\sum_{k=1}^l \alpha_k ds_k$.

Morphisms and *RL*-morphisms turn out to generalize (strict) right morphisms and right-left morphisms, respectively, in the unfolding theory of function germs ([10] (3.11) Remark, [12]).

A first order unfolding of $F=(\omega)$ is a rank 1 free sub-$\mathcal{O}_{n+1}$-module $\mathcal{F}^{(1)}=(\tilde{\omega})$ of Ω_{n+1} with a generator $\tilde{\omega}$ such that $\iota^*\tilde{\omega}=\omega$, where ι denotes the embedding of $\boldsymbol{C}^n=\{x\}$ into $\boldsymbol{C}^n\times\boldsymbol{C}=\{(x, t)\}$ given by $\iota(x)=(x, 0)$, and that $d\tilde{\omega}\wedge\tilde{\omega}\equiv 0 \bmod t^2, tdt$ (integrable to the first order). If we write

$$\tilde{\omega}\equiv\omega+\omega^{(1)}t+hdt \qquad \bmod t^2,\ tdt$$

with $\omega^{(1)}$ in Ω_n and h in $\mathcal{O}_n$, then the first order integrability is equivalent to

$$d\omega\wedge\omega^{(1)}+d\omega^{(1)}\wedge\omega=0$$

and

$$hd\omega+(\omega^{(1)}-dh)\wedge\omega=0.$$

It is not difficult to show that the second equation above implies the first. Also, h is determined uniquely by $\mathcal{F}^{(1)}$ ([9] (4.9) Lemma, [11] 2). Hence if we set

$$I(\omega)=\{h \in \mathcal{O}_n \mid hd\omega=\eta\wedge\omega \text{ for some } \eta \text{ in } \Omega_n\},$$

then each first order unfolding of $F=(\omega)$ determines an element in $I(\omega)$ and vice versa. We also set

$$J(\omega)=\{h \in \mathcal{O}_n \mid h=\langle X, \omega\rangle \text{ for some } X \text{ in } \Theta_n\},$$
$$K(\omega)=\{\alpha \in \mathcal{O}_n \mid \alpha d\omega=d\alpha\wedge\omega\}$$

and

$$\Omega(\omega)=\{\theta \in \Omega_n \mid \theta\wedge\omega=dh\wedge\omega-hd\omega \text{ for some } h \text{ in } \mathcal{O}_n\},$$

where $\langle\, ,\rangle$ denotes the natural pairing of a vector field and a 1-form. Note that $I(\omega)$ and $J(\omega)$ are ideals in $\mathcal{O}_n$, $K(\omega)$ is a sub-$\boldsymbol{C}$-vector space of $I(\omega)$ and $\Omega(\omega)$ is a sub-$\boldsymbol{C}$-vector space of Ω_n. It is shown that $J(\omega)\subset I(\omega)$ ([11] (2.8) Corollary).

(1.2) **Definition.** Let $\mathcal{F}^{(1)}=(\tilde{\omega})$ and $\mathcal{F}'^{(1)}=(\theta)$ be two first order unfoldings of $F=(\omega)$.

(I) $\mathcal{F}^{(1)}$ and $\mathcal{F}'^{(1)}$ are isomorphic (cf. [9] (4.10) Definition) if there is a pair (Φ, u) such that

(a) Φ is a germ of biholomorphic map making the diagram

$$\begin{array}{ccc} (\boldsymbol{C}^n\times\boldsymbol{C}, 0) & \xrightarrow{\ \Phi\ } & (\boldsymbol{C}^n\times\boldsymbol{C}, 0) \\ & \searrow \qquad \swarrow & \\ & (\boldsymbol{C}, 0) & \end{array}$$

commutative, where the other maps are the projections. u is a unit in $\mathcal{O}_{n+1}$.

(b) $\Phi(x, 0)=(x, 0)$ and $u(x, 0)=1$.

(c) $u\theta \equiv \Phi^*\tilde{\omega}$ mod t^2, tdt.

(II) $\mathscr{F}^{(1)}$ and $\mathscr{F}'^{(1)}$ are RL-isomorphic if there is a triple (Φ, u, α), where Φ and u are germs satisfying (a) and (b) in (I) and α is a germ in $\mathcal{O}_{n+1}$. Instead of (c), we require

(c)′ $u\theta \equiv \Phi^*\tilde{\omega}+\alpha dt$ mod t^2, tdt.

(1.3) **Remark.** In (II) of the above, we may assume that α is in $\mathcal{O}_n$. Moreover, the first order integrability of $\Phi^*\tilde{\omega}$ and $\Phi^*\tilde{\omega}+\alpha dt$ implies that α is in $K(\omega)$.

(1.4) **Proposition.** (I) *The set of isomorphism classes of first order unfoldings of $F=(\omega)$ is naturally identified with $I(\omega)/J(\omega)$.*

(II) *The set of RL-isomorphism classes of first order unfoldings of $F=(\omega)$ is naturally identified with $I(\omega)/J(\omega)+K(\omega)$.*

Proof. (I) is proved in [9] § 6 (see also [7] 1).

(II) Let $\mathscr{F}^{(1)}=(\tilde{\omega})$ and $\mathscr{F}'^{(1)}=(\theta)$ be two first order unfoldings of F. We write

$$\tilde{\omega} \equiv \omega+\omega^{(1)}t+hdt$$

and

$$\theta \equiv \omega+\theta^{(1)}t+edt \qquad \text{mod } t^2,\ tdt,$$

with $\omega^{(1)}$ and $\theta^{(1)}$ in Ω_n and h and e in $\mathcal{O}_n$. Suppose $\mathscr{F}^{(1)}$ and $\mathscr{F}'^{(1)}$ are RL-isomorphic and let (Φ, u, α) be as in (1.2) Definition (II). We assume that α is in $K(\omega)$ ((1.3) Remark). By (a) in (1.2) (I), we may write $\Phi(x, t)=(\phi(x, t), t)$ for some holomorphic map germ $\phi\colon (\boldsymbol{C}^n\times \boldsymbol{C}, 0)\to(\boldsymbol{C}^n, 0)$. By (b), we may also write

$$\phi(x, t)\equiv x+\phi^{(1)}(x)t \quad \text{and} \quad u(x, t)=1+u^{(1)}(x)t \quad \text{mod } t^2,$$

with $\phi^{(1)}$ in $\mathcal{O}_n^n$ and $u^{(1)}$ in $\mathcal{O}_n$. Then we have

$$\Phi^*\tilde{\omega}\equiv\omega+(\omega^{(1)}+L_X\omega)t+(h+\langle X, \omega\rangle)dt \qquad \text{mod } t^2,\ tdt,$$

where $L_X\omega$ denotes the Lie derivative of ω with respect to the vector field $X=\sum_{i=1}^n \phi_i^{(1)}(\partial/\partial x_i)$. Thus, from (1.2) (II) (c)′, we get

$$e=h+\langle X, \omega\rangle+\alpha,$$

which shows that h and e determine the same element in $I(\omega)/J(\omega)+K(\omega)$.

Conversely, suppose $e=h+\langle X, \omega\rangle+\alpha$ for some $X=\sum_{i=1}^{n}\xi_i(\partial/\partial x_i)$ in Θ_n and α in $K(\omega)$. Then by [11] (2.6) Lemma, there exists v in $\mathcal{O}_n$ such that

$$\theta^{(1)}+v\omega=\omega^{(1)}+L_X\omega.$$

Hence if we set

$$\Phi(x, t)=(\phi(x, t), t), \qquad \phi(x, t)=x+\xi(x)$$

and

$$u(x, t)=1+v(x)t,$$

then (Φ, u, α) gives an RL-isomorphism between $\mathcal{F}^{(1)}$ and $\mathcal{F}'^{(1)}$.

(1.5) **Remark.** Recall that for any ideal I in $\mathcal{O}_n$, we have an isomorphism of vector spaces

$$I(\omega)/I\cdot J(\omega)+K(\omega)\simeq\Omega(\omega)/L_I(\omega)+\mathcal{O}_n\omega,$$

where $L_I(\omega)=\{L_X\omega \mid X\in I\cdot\Theta_n\}$ ([11] (2.11) Proposition). Hence the set of RL-isomorphism classes of first order unfoldings of $F=(\omega)$ is also given by $\Omega(\omega)/L_{\mathcal{O}_n}(\omega)+\mathcal{O}_n\omega$.

§ 2. The versality theorem

Let $F=(\omega)$ be a codim 1 foliation germ at 0 in C^n and let $\mathcal{F}=(\tilde{\omega})$ be an unfolding of F with parameter space $C^m=\{(t_1, \cdots, t_m)\}$. We write

$$\tilde{\omega}=\omega+\sum_{j=1}^{m}\omega^{(1_j)}t_j+\sum_{j=1}^{m}h_j dt_j+\text{terms of order}\geq 2 \text{ in } t,$$

with $\omega^{(1_j)}$ in Ω_n and h_j in $\mathcal{O}_n$, $1\leq j\leq m$. We say that $\mathcal{F}$ is infinitesimally RL-versal if the classes $[h_1], \cdots, [h_m]$ span the C-vector space $I(\omega)/J(\omega)+K(\omega)$, or equivalently, $[\omega^{(1_1)}], \cdots, [\omega^{(1_m)}]$ span $\Omega(\omega)/L_{\mathcal{O}_n}(\omega)+\mathcal{O}_n\omega$ ((1.5) Remark). Also, we say that $\mathcal{F}$ is RL-versal if for any unfolding $\mathcal{F}'$ of F, there is an RL-morphism from $\mathcal{F}'$ to $\mathcal{F}$.

In this section, we prove the following

(2.1) **Theorem.** *Let $F=(\omega)$ be a codim 1 foliation germ at 0 in C^n with*

$$(*) \qquad \dim K(\omega)<+\infty.$$

Then any infinitesimally RL-versal unfolding of F is RL-versal.

(2.2) **Remark.** Let $\mathscr{F}$ be an infinitesimally *RL*-versal unfolding of F and let $\mathscr{F}'$ be an arbitrarily given unfolding of F. The theorem is proved by first constructing an *RL*-morphism from $\mathscr{F}'$ to $\mathscr{F}$ as a formal power series in the parameters of $\mathscr{F}'$ and then showing the existence of a convergent solution. The condition (*) above is used in the convergence part. Thus if we do not impose (*), it is shown that any infinitesimally *RL*-versal unfolding of F is "formally" *RL*-versal. Actually, in what follows, we prove a stronger statement: Let $F=(\omega)$ be a codim 1 foliation germ with

(**) $\quad \dim K(\omega)^l/J(\omega)\cap K(\omega)<+\infty$ for some natural number l.

Then, if $\mathscr{F}$ is an infinitesimally *RL*-versal unfolding of F, for any unfolding $\mathscr{F}'$ of F whose parameter space has dimension $\leq l$, there is an *RL*-morphism from $\mathscr{F}'$ to $\mathscr{F}$. In the above, $K(\omega)^l$ denotes the direct sum of l copies of the $\boldsymbol{C}$-vector space $K(\omega)$. It is identified with a subspace of the direct sum $I(\omega)^l$ in a natural manner. We also identify $I(\omega)$ with a subspace of $I(\omega)^l$ by the "diagonal" embedding $h\rightarrow(h, \cdots, h)$.

We note that (*) is satisfied in most cases.

Proof of (2.1) *Theorem.* Let $\mathscr{F}=(\tilde{\omega})$ be an infinitesimally *RL*-versal unfolding of F and let $\boldsymbol{C}^m=\{(t_1, \cdots, t_m)\}$ be the parameter space of $\mathscr{F}$. We write

$$\tilde{\omega}=\sum_{i=1}^{n}\tilde{f}_i(x, t)dx_i+\sum_{j=1}^{m}\tilde{h}_j(x, t)dt_j$$

with $\tilde{f}_i$ and $\tilde{h}_j$ in $\mathcal{O}_{n+m}$. If we set $\omega_t=\sum_{i=1}^{n}\tilde{f}_i(x, t)dx_i$, then the integrability condition $d\tilde{\omega}\wedge\tilde{\omega}=0$ is equivalent to the following four identities;

$$d_x\omega_t\wedge\omega_t=0, \tag{2.3}$$

$$\tilde{h}_jd_x\omega_t+\left(\frac{\partial}{\partial t_j}\omega_t-d_x\tilde{h}_j\right)\wedge\omega_t=0, \qquad 1\leq j\leq m, \tag{2.4}$$

$$\tilde{h}_j\left(\frac{\partial}{\partial t_k}\omega_t-d_x\tilde{h}_k\right)-\tilde{h}_k\left(\frac{\partial}{\partial t_j}\omega_t-d_x\tilde{h}_j\right)-h_{jk}\omega_t=0, \quad 1\leq j, k\leq m, \tag{2.5}$$

and

$$\tilde{h}_ih_{jk}+\tilde{h}_jh_{ki}+\tilde{h}_kh_{ij}=0, \qquad 1\leq i, j, k\leq m, \tag{2.6}$$

where d_x denotes the exterior derivation with respect to $x=(x_1, \cdots, x_n)$,

$(\partial/\partial t_j)\omega_t=\sum_{i=1}^n(\partial\tilde{f}_i/\partial t_j)(x, t)dx_i$ and $h_{jk}=\partial\tilde{h}_j/\partial t_k-\partial\tilde{h}_k/\partial t_j$. Also, we set $f_i(x)=\tilde{f}_i(x, 0)$, $1\leq i\leq n$, $h_j(x)=\tilde{h}_j(x, 0)$ and $\omega^{(1_j)}=\sum_{i=1}^n(\partial\tilde{f}_i/\partial t_j)(x, 0)dx_i$, $1\leq j\leq m$. Then the condition that the unfolding $\mathscr{F}$ of F is infinitesimally RL-versal is given by

(2.7) the classes $[h_1], \cdots, [h_m]$ span $I(\omega)/J(\omega)+K(\omega)$,

or equivalently

(2.8) the classes $[\omega^{(1_1)}], \cdots, [\omega^{(1_m)}]$ span $\Omega(\omega)/L_{\mathcal{O}_n}(\omega)+\mathcal{O}_n\omega$.

Let $\mathscr{F}'=(\theta)$ be an arbitrarily given unfolding of F. Letting $C^l=\{(s_1, \cdots, s_l)\}$ be the parameter space of $\mathscr{F}'$, we write

$$\theta=\sum_{i=1}^n g_i(x, s)dx_i+\sum_{k=1}^l \tilde{e}_k(x, s)ds_k$$

with g_i and $\tilde{e}_k$ in $\mathcal{O}_{n+l}$, and set $\theta_s=\sum_{i=1}^n g_i(x, s)dx_i$. If we let $e_k(x)=\tilde{e}_k(x, 0)$, the integrability of θ implies that

(2.9) the germs $e_1, \cdots, e_l$ are in $I(\omega)$.

We shall prove the existence of an RL-morphism (Φ, ψ, u, α) from $\mathscr{F}'$ to $\mathscr{F}$ ((1.1) Definition). Note that by the condition (a) in (1.1), Φ must be of the form $\Phi(x, s)=(\phi(x, s), \psi(s))$ for some holomorphic map germ $\phi: (C^n\times C^l, 0)\to(C^n, 0)$ and the condition $\Phi(x, 0)=(x, 0)$ is equivalent to $\phi(x, 0)=x$ and $\psi(0)=0$.

Part I. First we show the existence of (Φ, ψ, u, α) as formal power series in s.

We express ϕ, ψ, u and α as power series in $s=(s_1, \cdots, s_l)$;

$$\phi(x, s)=\sum_{|\nu|\geq 0}\phi^{(\nu)}s^\nu, \qquad \psi(s)=\sum_{|\nu|\geq 0}c^{(\nu)}s^\nu,$$
$$u(x, s)=\sum_{|\nu|\geq 0}u^{(\nu)}(x)s^\nu \quad\text{and}\quad \alpha(x, s)=\sum_{|\nu|\geq 0}\alpha^{(\nu)}(x)s^\nu$$

with $\phi^{(\nu)}$ in $\mathcal{O}_n^n$, $c^{(\nu)}$ in C, $u^{(\nu)}$ in $\mathcal{O}_n$ and $\alpha^{(\nu)}$ in $\mathcal{O}_n^l$. In the above ν denotes an l-tuple $(\nu_1, \cdots, \nu_l)$ of non-negative integers, $|\nu|=\nu_1+\cdots+\nu_l$ and $s^\nu=s_1^{\nu_1}\cdots s_l^{\nu_l}$ as usual. In general, for a series $\sigma=\sum_{|\nu|\geq 0}\sigma^{(\nu)}s^\nu$ in s with $\sigma^{(\nu)}\in\mathcal{O}_n^r$ for some r, we set

$$[\sigma]_\nu=\sigma^{(\nu)}s^\nu,\ [\sigma]_p=\sum_{|\nu|=p}[\sigma]_\nu \quad\text{and}\quad \sigma^{|p}=\sum_{|\nu|=0}^{p}[\sigma]_\nu$$

for $p\geqq 0$. If we set $\Phi^{|p}=(\phi^{|p}, \psi^{|p})$, we have

$$(\Phi^{|p})^*\tilde{\omega}=\Theta(p)+\sum_{k=1}^{l} E(p)_k ds_k,$$

where

$$\Theta(p)=\sum_{i=1}^{n}\tilde{f}_i(\phi^{|p}, \psi^{|p})d_x\phi_i^{|p} \tag{2.10}$$

and

$$E(p)_k=\sum_{i=1}^{n}\tilde{f}_i(\phi^{|p}, \psi^{|p})\frac{\partial\phi_i^{|p}}{\partial s_k}+\sum_{j=1}^{m}\tilde{h}_j(\phi^{|p}, \psi^{|p})\frac{\partial\psi_j^{|p}}{\partial s_k}. \tag{2.11}$$

The quadruple (Φ, ψ, u, α) is an RL-morphism from $\mathscr{F}'$ to $\mathscr{F}$ if and only if we have

$$\phi^{(0)}(x)=x, \quad c^{(0)}=0 \quad \text{and} \quad u^{(0)}(x)=1, \tag{2.12}$$

where 0 in ( ) denotes the l-tuple $(0, \cdots, 0)$, and we have the congruences

$$u^{|p}\theta_s \underset{p}{\equiv} \Theta(p) \tag*{(2.13)$_p$}$$

for $p\geq 0$, and the congruences

$$u^{|p-1}\tilde{e}_k \underset{p-1}{\equiv} E(p)_k+\alpha_k^{|p-1}, \qquad 1\leq k\leq l, \tag*{(2.14)$_p$}$$

for $p\geq 1$, where $\equiv_p$ denotes the equality mod s^ν, $|\nu|=p+1$. We also look for auxiliary germs $\alpha^{(\lambda,\mu)}=(\alpha_1^{(\lambda,\mu)}, \cdots, \alpha_l^{(\lambda,\mu)})$ in $\mathcal{O}_n^l$ such that

$$\alpha_k^{(\lambda,0)}\in K(\omega), \quad 1\leq k\leq l, \quad 0\leq|\lambda|\leq p-1, \tag*{(2.15)$_p$}$$

$$\text{if } |\lambda|=q\leq p-1, \qquad \sum_{|\mu|=0}^{p-q}\alpha_k^{(\lambda,\mu)}s^\mu \tag*{(2.16)$_p$}$$

is a $(p-q)$-th order unfolding of $\alpha_k^{(\lambda,0)}$ subject to $(\Phi^{|p})^*\tilde{\omega}$ (see (A.1) Definition) and

$$\alpha^{(\nu)}=\sum_{\lambda+\mu=\nu}\alpha^{(\lambda,\mu)}, \qquad 0\leq|\nu|\leq p-1, \tag*{(2.17)$_p$}$$

for all $p\geq 1$.

We set $\phi^{(0)}(x)=x$, $c^{(0)}=0$ and $u^{(0)}(x)=1$ hereafter.

(2.18) **Proposition.** *There exist $\phi^{(\nu)}$ and $c^{(\nu)}$ for ν with $|\nu|\leq 1$, $u^{(0)}$, $\alpha^{(0)}$ and $\alpha^{(0,\mu)}$ for μ with $|\mu|\leq 1$ such that* (2.13)$_0$, (2.14)$_1$, (2.15)$_1$, (2.16)$_1$ *and* (2.17)$_1$ *hold.*

Proof. Since $\phi^{(0)}(x)=x$ and $c^{(0)}=0$, we have $\Theta(0)=\omega$. Also, since $\theta_0=\omega$ and $u^{(0)}=1$, we have $(2.13)_0$. The congruence $(2.14)_1$ is equivalent to

$$e_k=\sum_{i=1}^{n}\phi_i^{(1_k)}f_i+\sum_{j=1}^{m}c_j^{(1_k)}h_j+\alpha_k^{(0)}, \qquad 1\leq k\leq l,$$

where 1_k denotes the l-tuple with 1 in the k-th component and 0 in the others. By (2.7) and (2.9), there exist c_{kj} in $\boldsymbol{C}$, $X_k=\sum_{i=1}^{n}\xi_{ki}(\partial/\partial x_i)$ in Θ_n and β_k in $K(\omega)$ such that

$$e_k=\sum_{j=1}^{m}c_{kj}h_j+\langle X_k, \omega\rangle+\beta_k.$$

Thus if we set $\phi_i^{(1_k)}=\xi_{ki}$, $c_j^{(1_k)}=c_{kj}$ and $\alpha_k^{(0)}=\beta_k$, $(2.14)_1$ is satisfied. Let $\alpha_k^{(0,0)}=\alpha_k^{(0)}$ and let $\sum_{|\mu|=0}^{1}\alpha_k^{(0,\mu)}s^\mu$ be the first order unfolding of $\alpha_k^{(0,0)}$ subject to $(\Phi^{|1})^*\tilde{\omega}$ (see (A.2) Theorem). Then we have $(2.15)_1$, $(2.16)_1$ and $(2.17)_1$.

(2.19) **Proposition.** *For any $\phi^{(\nu)}$ and $c^{(\nu)}$ for ν with $|\nu|\leq p$, $u^{(\nu)}$ and $\alpha^{(\nu)}$ for ν with $|\nu|\leq p-1$ and $\alpha^{(\lambda,\mu)}$ for λ, μ with $|\lambda+\mu|\leq p$, $|\lambda|\leq p-1$, satisfying $(2.13)_{p-1}$, $(2.14)_p$, $(2.15)_p$, $(2.16)_p$ and $(2.17)_p$, there exist $\phi^{(\nu)}$ and $c^{(\nu)}$ for ν with $|\nu|=p+1$, $u^{(\nu)}$ and $\alpha^{(\nu)}$ for ν with $|\nu|=p$ and $\alpha^{(\lambda,\mu)}$ for λ, μ with $|\lambda|=p$, $|\mu|=0$ or $|\lambda+\mu|=p+1$, $|\mu|\neq 0$, satisfying $(2.13)_p$, $(2.14)_{p+1}$, $(2.15)_{p+1}$, $(2.16)_{p+1}$ and $(2.17)_{p+1}$.*

Proof. Given $\phi^{(\nu)}$, $c^{(\nu)}$, $u^{(\nu)}$, $\alpha^{(\nu)}$ and $\alpha^{(\lambda,\mu)}$ as in the assumption above. Then we have $\Phi^{|p}$, $u^{|p-1}$ and $\alpha^{|p-1}$. Hence we also have $\Theta(p)$ and $E(p)_k$ (see (2.10), (2.11)).

First we show that we have the congruence

$$\alpha_k^{|p-1}d_x\Theta(p)\underset{p-1}{\equiv}d\alpha_k^{|p-1}\wedge\Theta(p). \tag{2.20}$$

In fact, we compute

$$\begin{aligned}&\alpha_k^{|p-1}d_x\Theta(p)-d\alpha_k^{|p-1}\wedge\Theta(p)\\&=\sum_{|\nu|=0}^{p-1}\sum_{\lambda+\mu=\nu}(\alpha_k^{(\lambda,\mu)}d_x\Theta(p)-d\alpha_k^{(\lambda,\mu)}\wedge\Theta(p))s^\nu\\&=\sum_{q=0}^{p-1}\sum_{|\lambda|=q}\Big(\sum_{|\mu|=0}^{p-q-1}(\alpha_k^{(\lambda,\mu)}d_x\Theta(p)-d\alpha_k^{(\lambda,\mu)}\wedge\Theta(p))s^\mu\Big)s^\lambda.\end{aligned}$$

By $(2.16)_p$ (cf. $(\mathrm{A}.11)_{p-q-1}$),

$$\sum_{|\mu|=0}^{p-q-1}(\alpha_k^{(\lambda,\mu)}d_x\Theta(p)-d\alpha_k^{(\lambda,\mu)}\wedge\Theta(p))s^\mu\underset{p-q-1}{\equiv}0.$$

Hence we get (2.20).

Second, $(2.13)_p$ reads

$$[u]_p\omega+u^{|p-1}\theta_s\underset{p}{\equiv}\Theta(p).$$

Since codim $S(\omega)\geq 2$, to prove the existence of $[u]_p=\sum_{|\nu|=p}u^{(\nu)}s^\nu$ satisfying the above, it suffices to show that

$$(2.21)\qquad [\Theta(p)-u^{|p-1}\theta_s]_p\wedge\omega=0.$$

From the integrability of $(\Phi^{|p})^*\tilde{\omega}$ and $u^{|p-1}\theta$, we get (cf. (2.4))

$$(2.22)_p\qquad E(p)_k d_x\Theta(p)+\left(\frac{\partial}{\partial s_k}\Theta(p)-d_xE(p)_k\right)\wedge\Theta(p)=0$$

and

$$(2.23)_{p-1}\quad (u^{|p-1}\tilde{e}_k)d_x(u^{|p-1}\theta_s)+\left(\frac{\partial}{\partial s_k}(u^{|p-1}\theta_s)-d_x(u^{|p-1}\tilde{e}_k)\right)\wedge(u^{|p-1}\theta_s)=0.$$

Substracting $(2.23)_{p-1}$ from $(2.22)_p$ and using $(2.13)_{p-1}$ and $\Theta(p-1)\equiv_{p-1}\Theta(p)$, we get

$$(E(p)_k-u^{|p-1}\tilde{e}_k)d_x\Theta(p)$$
$$+\left(\frac{\partial}{\partial s_k}(\Theta(p)-u^{|p-1}\theta_s)-d_x(E(p)_k-u^{|p-1}\tilde{e}_k)\right)\wedge\Theta(p)\underset{p-1}{\equiv}0.$$

Then by $(2.14)_p$ and (2.20), we have

$$\frac{\partial}{\partial s_k}(\Theta(p)-u^{|p-1}\theta_s)\wedge\Theta(p)\underset{p-1}{\equiv}0.$$

Hence by $(2.13)_{p-1}$,

$$\frac{\partial}{\partial s_k}[\Theta(p)-u^{|p-1}\theta_s]_p\wedge\omega=0,\qquad 1\leq k\leq l.$$

Therefore we get (2.21).

Third, $(2.14)_{p+1}$ reads

$$(2.24)\qquad u^{|p}\tilde{e}_k\underset{p}{\equiv}E(p)_k+\alpha_k^{|p-1}+\sum_{i=1}^{n}\frac{\partial[\phi_i]_{p+1}}{\partial s_k}\cdot f_i$$
$$+\sum_{j=1}^{m}\frac{\partial[\psi_j]_{p+1}}{\partial s_k}\cdot h_j+[\alpha_k]_p.$$

We look for $[\phi_i]_{p+1}=\sum_{|\nu|=p+1}\phi_i^{(\nu)}s^\nu$, $[\psi_j]_{p+1}=\sum_{|\nu|=p+1}c^{(\nu)}s^\nu$ and $[\alpha_k]_p=\sum_{|\nu|=p}\alpha_k^{(\nu)}s^\nu$ satisfying (2.24). From the integrability of $u^{|p}\theta$, we get

$(2.23)_p$. Subtracting $(2.22)_p$ from $(2.23)_p$ and using $(2.13)_p$, we have

$$(u^{|p}\tilde{e}_k - E(p)_k)d_x\Theta(p) + \left(\frac{\partial}{\partial s_k}(u^{|p}\theta_s - \Theta(p)) - d_x(u^{|p}\tilde{e}_k - E(p)_k)\right)\wedge\Theta(p) \underset{p}{\equiv} 0.$$

Then, by $(2.13)_p$ and $(2.14)_p$, we get

$$(2.25)\quad [u^{|p}\tilde{e}_k - E(p)_k]_p d\omega + \left(\frac{\partial}{\partial s_k}[u^{|p}\theta_s - \Theta(p)]_{p+1} - d_x[u^{|p}\tilde{e}_k - E(p)_k]_p\right)\wedge\omega$$
$$+ [\alpha_k^{|p-1} d_x\Theta(p)]_p - [d_x\alpha_k^{|p-1}\wedge\Theta(p)]_p = 0.$$

Now we compute

$$(2.26)\qquad [\alpha_k^{|p-1} d_x\Theta(p) - d_x\alpha_k^{|p-1}\wedge\Theta(p)]_p$$
$$= \left[\sum_{|\nu|=0}^{p-1}\sum_{\lambda+\mu=\nu}(\alpha_k^{(\lambda,\mu)}d_x\Theta(p) - d\alpha_k^{(\lambda,\mu)}\wedge\Theta(p))s^\nu\right]_p$$
$$= \sum_{q=0}^{p-1}\sum_{|\lambda|=q}\left[\sum_{|\mu|=0}^{p-q-1}(\alpha_k^{(\lambda,\mu)}d_x\Theta(p) - d\alpha_k^{(\lambda,\mu)}\wedge\Theta(p))s^\mu\right]_{p-q} s^\lambda.$$

On the other hand, by $(2.16)_p$ (cf. $(\mathrm{A}.11)_{p-q}$),

$$\sum_{|\mu|=0}^{p-q}(\alpha_k^{(\lambda,\mu)}d_x\Theta(p) - d\alpha_k^{(\lambda,\mu)}\wedge\Theta(p))s^\mu \underset{p-q}{\equiv} 0.$$

Hence

$$\left[\sum_{|\mu|=0}^{p-q-1}(\alpha_k^{(\lambda,\mu)}d_x\Theta(p) - d\alpha_k^{(\lambda,\mu)}\wedge\Theta(p))s^\mu\right]_{p-q}$$
$$= -\sum_{|\mu|=p-q}(\alpha_k^{(\lambda,\mu)}d\omega - d\alpha_k^{(\lambda,\mu)}\wedge\omega)s^\mu.$$

Substituting this in (2.26), we obtain

$$[\alpha_k^{|p-1} d_x\Theta(p) - d_x\alpha_k^{|p-1}\wedge\Theta(p)]_p$$
$$= -\sum_{|\nu|=p}\sum_{\substack{\lambda+\mu=\nu\\ |\mu|>0}}(\alpha_k^{(\lambda,\mu)}d\omega - d\alpha_k^{(\lambda,\mu)}\wedge\omega)s^\nu.$$

Hence if we set

$$\delta_k = \sum_{|\nu|=p}\sum_{\substack{\lambda+\mu=\nu\\ |\mu|>0}}\alpha_k^{(\lambda,\mu)}s^\nu,$$

from (2.25), we get

$$(2.27)\qquad [u^{|p}\tilde{e}_k - E(p)_k - \delta_k]_p d\omega$$
$$+ \left(\frac{\partial}{\partial s_k}[u^{|p}\theta_s - \Theta(p)]_{p+1} - d_x[u^{|p}\tilde{e}_k - E(p)_k - \delta_k]_p\right)\wedge\omega = 0.$$

This shows that the coefficients of $[u^{|p}\theta_s-\Theta(p)]_{p+1}$ are all in $\Omega(\omega)$. Hence we may write

$$[u^{|p}\theta_s-\Theta(p)]_{p+1}=\sum_{|\nu|=p+1}(\sum_{j=1}^{m}c_{\nu j}\omega^{(1j)}+L_{X_\nu}\omega+g_\nu\omega)s^\nu$$

for some $c_{\nu j}$ in $\boldsymbol{C}$, X_ν in Θ_n and g_ν in $\mathcal{O}_n$. Substituting this in (2.27), we see that

$$(2.28)\qquad [u^{|p}\tilde{e}_k-E(p)_k-\delta_k]_p =\frac{\partial}{\partial s_k}\left(\sum_{|\nu|=p+1}\left(\sum_{j=1}^{m}c_{\nu j}h_j+\langle X_\nu,\omega\rangle\right)s^\nu\right)+\sum_{|\lambda|=p}\beta_{\lambda k}s^\lambda$$

for some $\beta_{\lambda k}$ in $K(\omega)$. Writing $X_\nu=\sum_{i=1}^n\xi_{\nu i}(\partial/\partial x_i)$, we set $\phi_i^{(\nu)}=\xi_{\nu i}$ and $c_j^{(\nu)}=c_{\nu j}$ for ν with $|\nu|=p+1$ and $\alpha_k^{(\lambda,0)}=\beta_{\lambda k}$ for λ with $|\lambda|=p$. We also set $\alpha_k^{(\nu)}=\sum_{\lambda+\mu=\nu}\alpha_k^{(\lambda,\mu)}$ for ν with $|\nu|=p$. Then $[\phi_i]_{p+1}=\sum_{|\nu|=p+1}\phi_i^{(\nu)}s^\nu$, $[\psi_j]_{p+1}=\sum_{|\nu|=p+1}c_j^{(\nu)}s^\nu$ and $[\alpha_k]_p=\sum_{|\nu|=p}\alpha_k^{(\nu)}s^\nu=\delta_k+\sum_{|\lambda|=p}\alpha_k^{(\lambda,0)}s^\lambda$ satisfy (2.24). We also have $(2.15)_{p+1}$ and $(2.17)_{p+1}$.

Finally, for λ with $|\lambda|=q\le p$, we let $\sum_{|\mu|=0}^{p-q+1}\beta_{\lambda k}^{(\mu)}s^\mu$ be the $(p-q+1)$-st order unfolding of $\beta_{\lambda k}=\alpha_k^{(\lambda,0)}$ subject to $(\Phi^{|p+1})^*\tilde{\omega}$. Then, by (A.2) Theorem, $\beta_{\lambda k}^{(\mu)}=\alpha_k^{(\lambda,\mu)}$ if $|\lambda+\mu|\le p$. We set $\alpha_k^{(\lambda,\mu)}=\beta_{\lambda k}^{(\mu)}$ for λ,μ with $|\lambda+\mu|=p+1$, $|\mu|\neq 0$. Then we have $(2.16)_{p+1}$.

Part II. Now we prove the existence of convergent solutions. For an n-tuple $\rho=(\rho_1,\cdots,\rho_n)$ of positive real numbers, we set $P(\rho)=\{x\in\boldsymbol{C}^n\,|\,|x_i|\le\rho_i,\ 1\le i\le n\}$. For a germ f in $\mathcal{O}_n$, we write $f(x)=\sum_{|\alpha|\ge 0}a_\alpha x^\alpha$, where $\alpha=(\alpha_1,\cdots,\alpha_n)$ is an n-tuple of non-negative integers, $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $|\alpha|=\alpha_1+\cdots+\alpha_n$, and set $|f|_\rho=\sum_{|\alpha|\ge0}|a_\alpha|\rho^\alpha$. If $f=(f_1,\cdots,f_r)$ is a germ in $\mathcal{O}_n^r$ for some r, we set $|f|_\rho=\sum_{i=1}^r|f_i|_\rho$. The $\mathcal{O}_n$-modules Ω_n and Θ_n are both naturally identified with $\mathcal{O}_n^n$.

We fix a basis $[\gamma_1],\cdots,[\gamma_N]$ of the $\boldsymbol{C}$-vector space $K(\omega)^l/J(\omega)\cap K(\omega)$ ($\gamma_i\in K(\omega)^l\subset\mathcal{O}_n^l$). Also we choose open neighborhoods U, V and W of the origins in $\boldsymbol{C}^n$, $\boldsymbol{C}^m$ and $\boldsymbol{C}^l$, respectively, so that the germs $\omega,\gamma_1,\cdots,\gamma_N$ have representatives on U and the germs $\tilde{\omega}$ and θ have representatives on $U\times V$ and $U\times W$, respectively. Thus the germs f_i, $1\le i\le n$, h_j, $1\le j\le m$, and e_k, $1\le k\le l$, have representatives on U.

Consider the $\mathcal{O}_n$-homomorphisms

$$\lambda:\ \Theta_n\longrightarrow\mathcal{O}_n,\qquad \lambda(X)=\langle X,\omega\rangle$$

and

$$\mu:\ \mathcal{O}_n\longrightarrow\Omega_n,\qquad \mu(g)=g\omega.$$

Then by Malgrange [3] Théorème (1.1), there exists ρ such that $P(\rho)\subset U$

and that the homomorphisms λ and μ have fissions simultaneously adapted to ρ, i.e., we have

(2.29) **Lemma.** *There exist* $\rho=(\rho_1, \cdots, \rho_n)$, $\rho_i>0$, *and a positive constant K such that* $P(\rho)\subset U$ *and that*

(a) *every germ e in* $J(\omega)(=\operatorname{Im}\lambda)$ *can be written as* $e=\langle X, \omega\rangle$ *for X in* Θ_n *with*

$$|X|_{a\rho}\leq K|e|_{a\rho} \qquad \textit{for } \frac{1}{2}\leq a\leq 1,$$

(b) *every germ* η *in* $\mathcal{O}_n\omega(=\operatorname{Im}\mu)$ *can be written as* $\eta=g\omega$ *for g in* $\mathcal{O}_n$ *with*

$$|g|_{a\rho}\leq K|\eta|_{a\rho} \qquad \textit{for } \frac{1}{2}\leq a\leq 1.$$

We choose ρ with the properties in (2.29) Lemma and fix it once for all. We also set

$$K_0=\max\{|f|_\rho, |h|_\rho, |e|_\rho, |\gamma_i|_\rho, 1\leq i\leq N\},$$

where $f=(f_1, \cdots, f_n)$, $h=(h_1, \cdots, h_m)$ and $e=(e_1, \cdots, e_l)$.

Let $\sigma=\sum\sigma^{(\nu)}s^\nu$ be a series with $\sigma^{(\nu)}$ in $\mathcal{O}_n^r$ for some r and let $\sum a^{(\nu)}s^\nu$ be a series with $a^{(\nu)}$ positive real numbers. We say that $\sum a^{(\nu)}s^\nu$ dominates σ in $P(\rho)$ and write

$$\sum\sigma^{(\nu)}s^\nu\ll\sum a^{(\nu)}s^\nu \qquad \text{in } P(\rho)$$

if $|\sigma^{(\nu)}|_\rho\leq a^{(\nu)}$ for all ν. Consider the series (cf. [6] p. 291, [5] p. 50. Note that ours is modified so that it fits to our purpose)

$$A(s)=\frac{1}{32c^{1/2}}\sum_{p\geq 1}\frac{c^p}{p^3}(s_1+\cdots+s_l)^p,$$

where c is a positive constant to be determined later. We let $A'(s)$ be the series obtained from $A(s)$ by differentiation with respect to the variable $s_1+\cdots+s_l$;

$$A'(s)=\frac{1}{32c^{1/2}}\sum_{p\geq 0}\frac{c^{p+1}}{(p+1)^2}(s_1+\cdots+s_l)^p.$$

As in [6] (19) or [5] Lemma 3.6, we can prove that

$$(2.30) \qquad A(s)^r\ll\left(\frac{1}{c^{1/2}}\right)^{r-1}A(s) \qquad \text{for } r\geq 1.$$

From this we get

$$A'(s)A(s) \ll \frac{1}{2c^{1/2}} A'(s). \tag{2.31}$$

Similarly we can show that

$$A'(s)^r \ll \left(\frac{c^{1/2}}{2}\right)^{r-1} A'(s) \qquad \text{for } r \geq 1. \tag{2.32}$$

We set $A_i(s) = c^{-i/4}A(s)$, $1 \leq i \leq 4$, and prove that there exist ϕ, ψ, u and α satisfying (2.12), $(2.13)_p$ and $(2.14)_p$ for all p such that, if we choose c sufficiently large, then the following estimates hold for all $p \geq 0$;

$$(2.33)_p \qquad u^{|p} \ll 32 A_2'\left(\frac{s}{1-a}\right),$$

$$(2.34)_p \qquad \phi_i^{|p} - x_i \ll (1-a)A\left(\frac{s}{1-a}\right), \qquad 1 \leq i \leq n,$$

$$(2.35)_p \qquad \psi_j^{|p} \ll (1-a)A\left(\frac{s}{1-a}\right), \qquad 1 \leq j \leq m,$$

and

$$(2.36)_p \qquad \alpha_k^{|p} \ll A'\left(\frac{s}{1-a}\right), \quad 1 \leq k \leq l, \quad \text{in } P(a\rho) \quad \text{for } \frac{1}{2} \leq a < 1.$$

We write $\gamma_i = (\beta_{i1}, \cdots, \beta_{il})$, $1 \leq i \leq N$, with β_{ik} in $K(\omega)$ and denote by $\tilde{\beta}_{ik}(p)$ the p-th order unfolding of β_{ik} subject to $(\Phi^{|p})^*\tilde{\omega}$ ((A.2) Theorem). This time we look for auxiliary elements $a^{(\nu)} = (a_1^{(\nu)}, \cdots, a_N^{(\nu)})$ in $\boldsymbol{C}^N$ for all ν with $|\nu| \geq 1$ such that

$(2.37)_p$ if we define $\alpha^{(\lambda,\mu)} = (\alpha_l^{(\lambda,\mu)}, \cdots, \alpha_l^{(\lambda,\mu)})$ in $\mathcal{O}_n^l$ for λ, μ with $|\lambda + \mu| \leq p$, $|\lambda| \leq p-1$ by

$$(\lambda_k + 1) \sum_{i=1}^{N} a_i^{(\lambda+1_k)} \tilde{\beta}_{ik}(p) \underset{p-q}{\equiv} \sum_{|\mu|=0}^{p-q} \alpha_k^{(\lambda,\mu)} s^\mu,$$

then $(2.15)_p$, $(2.16)_p$ and $(2.17)_p$ are satisfied and that we have the estimate

$$(2.38)_p \qquad \sum_{|\nu|=1}^{p} a_i^{(\nu)} s^\nu \ll (1-a)A_1\left(\frac{s}{1-a}\right), \qquad 1 \leq i \leq N,$$

for all $p \geq 1$.

Let $\nu=(\nu_1, \cdots, \nu_l)$ be an l-tuple of non-negative integers and set $L(\nu)=\{k \mid 1\leq k\leq l, \nu_k\neq 0\}$. Also, let $l(\nu)$ be the cardinality of $L(\nu)$ and define

$$\pi^{(\nu)}: K(\omega)^l \longrightarrow K(\omega)^{l(\nu)}$$

by $\pi^{(\nu)}(\beta_1, \cdots, \beta_l)=(\cdots, \beta_k, \cdots)$, $k \in L(\nu)$. This induces a linear map

$$K(\omega)^l/J(\omega)\cap K(\omega) \longrightarrow K(\omega)^{l(\nu)}/J(\omega)\cap K(\omega),$$

which we also denote by $\pi^{(\nu)}$. Note that $[\pi^{(\nu)}(\gamma_1)], \cdots, [\pi^{(\nu)}(\gamma_N)]$ span the vector space $K(\omega)^{l(\nu)}/J(\omega)\cap K(\omega)$.

(2.39) **Lemma.** *There exists a constant K_1 such that for any l-tuple ν and for any σ in $I(\omega)+K(\omega)^{l(\nu)}$ we have*

$$\sigma=\sum_{j=1}^{m} c_j h_j+\langle X, \omega\rangle+\sum_{i=1}^{N} a_i \pi^{(\nu)}(\gamma_i)$$

for X in Θ_n and constants c_j, $1\leq j\leq m$, and a_i, $1\leq i\leq N$, with

$$|c_j|\leq K_1|\sigma|_{a\rho}, \quad |X|_{a\rho}\leq K_1|\sigma|_{a\rho} \quad \textit{and} \quad |a_i|\leq K_1|\sigma|_{a\rho} \quad \textit{for} \quad \frac{1}{2}\leq a\leq 1.$$

Proof. First we show the existence of a constant K' such that for any l-tuple ν and for any σ in $J(\omega)+K(\omega)^{l(\nu)}$, there exist constants $a_1, \cdots, a_N$ such that $[\sigma]=\sum_{i=1}^{N} a_i[\pi^{(\nu)}(\gamma_i)]$ in $J(\omega)+K(\omega)^{l(\nu)}/J(\omega)\simeq K(\omega)^{l(\nu)}/J(\omega)\cap K(\omega)$ and that

$$|a_i|\leq K'|\sigma|_{\rho/2}.$$

This is done by modifying the arguments in [6] Lemma 1 (see also [7] (3.18) Lemma). Thus for an l-tuple ν and σ in $J(\omega)+K(\omega)^{l(\nu)}$, we set

$$\iota(\sigma)=\inf\{\max|a_i| \mid [\sigma]=\sum_{i=1}^{N} a_i[\pi^{(\nu)}(\gamma_i)]\}$$

and show the existence of a constant K' such that for any $\sigma\neq 0$,

$$\iota(\sigma)<K'|\sigma|_{\rho/2}.$$

Suppose that this is false. Then for any natural number p, there exist an l-tuple $\nu(p)$ and an element $\sigma(p)$ in $J(\omega)+K(\omega)^{l(\nu(p))}$ such that $\iota(\sigma(p))=1$ and $|\sigma(p)|_{\rho/2}<1/p$. For any l-tuple ν, $L(\nu)$ is a subset of the finite set $\{1, \cdots, l\}$. Hence $L(\nu)$ are the same for infinitely many p. Thus we may assume that $\sigma(p)$ are all in $J(\omega)+K(\omega)^{l'}$ for some $l'\leq l$. We denote the projection $K(\omega)^l\to K(\omega)^{l'}$ by π. By definition of $\iota(\sigma(p))$, there exist constants

$a_i(p)$, $1\leq i\leq N$, such that $[\sigma(p)]=\sum_{i=1}^{N}a_i(p)[\pi(\gamma_i)]$ and that $|a_i(p)|<2$. Since, for each i, the sequence $\{a_i(p)\}$ is bounded, replacing $\{\sigma(p)\}$ by a suitable subsequence, if necessary, we may assume that $a_i=\lim_{p\to\infty}a_i(p)$ exists. Since $\sum_{i=1}^{N}a_i[\pi(\gamma_i)]=[0]$, if we set $a_i'=a_i(p)-a_i$, we have, for a sufficiently large p,

$$\sum_{i=1}^{N}a_i'[\pi(\gamma_i)]=[\sigma(p)] \quad \text{and} \quad |a_i'|<\frac{1}{2}.$$

This contradicts with $\iota(\sigma(p))=1$.

Next, noting that we have the natural isomorphisms

$$\begin{aligned} I(\omega)+K(\omega)^{l(\nu)}/J(\omega)+K(\omega)^{l(\nu)} &\simeq I(\omega)/I(\omega)\cap(J(\omega)+K(\omega)^{l(\nu)}) \\ &\simeq I(\omega)/J(\omega)+K(\omega), \end{aligned}$$

it is shown by similar arguments as above that there exists a constant K_1' such that for any l-tuple ν and σ in $I(\omega)+K(\omega)^{l(\nu)}$, there exist constants c_j, $1\leq j\leq m$, such that $[\sigma]=\sum_{j=1}^{m}c_j[h_j]$ in $I(\omega)+K(\omega)^{l(\nu)}/J(\omega)+K(\omega)^{l(\nu)}$ and that

$$|c_j|\leq K_1'|\sigma|_{\rho/2}.$$

Then, since $\sigma-\sum_{j=1}^{m}c_jh_j$ is in $J(\omega)+K(\omega)^{l(\nu)}$, by the above, there exist constants a_i, $1\leq i\leq N$, such that $[\sigma-\sum_{j=1}^{m}c_jh_j]=\sum_{i=1}^{N}a_i[\pi^{(\nu)}(\gamma_i)]$ in $J(\omega)+K(\omega)^{l(\nu)}/J(\omega)$ and that

$$|a_i|\leq K'|\sigma-\sum_{j=1}^{m}c_jh_j|_{\rho/2}.$$

Noting that

$$\left|\sigma-\sum_{j=1}^{m}c_jh_j\right|_{\rho/2}\leq\left(1+\sum_{j=1}^{m}|h_j|_{\rho/2}\right)|\sigma|_{\rho/2}\leq(1+K_0)|\sigma|_{\rho/2},$$

we have the estimate for a_i.

Finally, using (2.29) Lemma (a), we have the estimate for X.

The following is proved similarly as in [7] p. 42.

(2.40) **Lemma.** (a) *We have*

$$A\left(\frac{s}{1-a}\right)\ll\frac{2}{c}\frac{1}{1-a}A'\left(\frac{s}{1-a}\right) \qquad \textit{for } \frac{1}{2}\leq a<1.$$

(b) *If, for a series* $\sum\sigma^{(\nu)}s^{\nu}$,

$$\sum \sigma^{(\nu)} s^{\nu} \ll \frac{1}{b-a} A\left(\frac{s}{1-b}\right) \quad \textit{in } P(a\rho) \quad \textit{for } \frac{1}{2} \leq a < b < 1,$$

then

$$\sum \sigma^{(\nu)} s^{\nu} \ll \frac{8e}{c} \frac{1}{1-a} A'\left(\frac{s}{1-a}\right) \quad \textit{in } P(a\rho) \quad \textit{for } \frac{1}{2} \leq a < 1.$$

The following two propositions will finish the proof of the theorem.

(2.41) **Proposition.** *If we choose c in the series $A(s)$ sufficiently large, then there exist $\phi^{(\nu)}$, $c^{(\nu)}$ and $a^{(\nu)}$ for ν with $|\nu| \leq 1$, $u^{(0)}$ and $\alpha^{(0)}$ such that* $(2.13)_0$, $(2.14)_1$, $(2.33)_0$, $(2.34)_1$, $(2.35)_1$, $(2.36)_0$, $(2.37)_1$ *and* $(2.38)_1$ *hold.*

Proof. We recall the proof of (2.18) Proposition. Thus if we set $\phi^{(0)}(x) = x$, $c^{(0)} = 0$ and $u^{(0)} = 1$, then we have $(2.13)_0$. Also, since

$$32 A_2'\left(\frac{s}{1-a}\right) = 1 + \text{terms of order} \geq 1 \text{ in } s,$$

we have $(2.33)_0$. Next, since e_k is in $I(\omega)$, which is identified with $I(\omega) + K(\omega)^{l(1k)}$, (2.39) Lemma asserts the existence of X_k in Θ_n and constants c_{kj} and a_{ki} such that

$$e_k = \sum_{j=1}^{m} c_{kj} h_j + \langle X_k, \omega \rangle + \sum_{i=1}^{N} a_{ki} \pi^{(1k)}(\gamma_i)$$

and that

$$|X_k|_{a\rho} \leq K_1 |e_k|_{a\rho} \qquad \text{for } \frac{1}{2} \leq a \leq 1.$$

Note that this shows that $|X_k|_\rho$ is bounded. Writing $X_k = \sum_{i=1}^{n} \xi_{ki}(\partial/\partial x_i)$, we set $\phi_i^{(1k)} = \xi_{ki}$, $c_j^{(1k)} = c_{kj}$ and $a_i^{(1k)} = a_{ki}$. We also define $\alpha^{(0,\mu)}$ for μ with $|\mu| \leq 1$ by the identity in $(2.37)_1$ and set $\alpha^{(0)} = \alpha^{(0,0)}$. Then, since $\pi^{(1k)}(\gamma_i) = \beta_{ik}$, we have $(2.14)_1$ as well as $(2.37)_1$. Now, since $|X_k|_\rho$ and $|\alpha^{(0)}|_\rho$ are bounded and

$$(1-a) A\left(\frac{s}{1-a}\right) = \frac{c^{1/2}}{32} (s_1 + \cdots + s_l + \text{terms of order } \geq 2 \text{ in } s)$$

and

$$A'\left(\frac{s}{1-a}\right) = \frac{c^{1/2}}{32} (1 + \text{terms of order } \geq 1 \text{ in } s),$$

we have the estimates $(2.34)_1$, $(2.35)_1$, $(2.36)_1$ and $(2.38)_1$, if c is sufficiently large.

(2.42) **Proposition.** *If we choose c sufficiently large, then for any* $\phi^{(\nu)}$, $c^{(\nu)}$ *and* $a^{(\nu)}$ *for* ν *with* $|\nu|\leq p$, $u^{(\nu)}$ *and* $\alpha^{(\nu)}$ *for* ν *with* $|\nu|\leq p-1$ *satisfying* $(2.13)_{p-1}$, $(2.14)_p$, $(2.33)_{p-1}$, $(2.34)_p$, $(2.35)_p$, $(2.36)_{p-1}$, $(2.37)_p$ *and* $(2.38)_p$, *there exist* $\phi^{(\nu)}$, $c^{(\nu)}$ *and* $a^{(\nu)}$ *for* ν *with* $|\nu|=p+1$, $u^{(\nu)}$ *and* $\alpha^{(\nu)}$ *for* ν *with* $|\nu|=p$ *satisfying* $(2.13)_p$, $(2.14)_{p+1}$, $(2.33)_p$, $(2.34)_{p+1}$, $(2.35)_{p+1}$, $(2.36)_p$, $(2.37)_{p+1}$ *and* $(2.38)_{p+1}$.

Proof. Given $\phi^{(\nu)}$, $c^{(\nu)}$, $a^{(\nu)}$, $u^{(\nu)}$ and $\alpha^{(\nu)}$ as in the assumption above. Then we have $\Phi^{|p}$, $u^{|p-1}$ and $\alpha^{|p-1}$ as well as $\alpha^{(\lambda,\mu)}$ for λ, μ with $|\lambda+\mu|\leq p$, $|\mu|\neq 0$. Hence we also have $\Theta(p)$ and $E(p)_k$ (see (2.10), (2.11)). As is shown in (2.19) Proposition (see (2.21)), we have

$$[\Theta(p)-u^{|p-1}\theta_s]_p\wedge\omega=0.$$

In order to estimate $\Theta(p)$, we consider the power series expansion of $\tilde{f}_i(x+y,t)$ in $(y,t)=(y_1,\cdots,y_n,t_1,\cdots,t_m)$. Note that, since $\tilde{f}_i(x,t)$ is holomorphic in $U\times V$, the coefficient functions (in x) of the power series (in (y,t)) are holomorphic in an open set containing $P(\rho)$. Hence by taking suitable constants b_0 and c_0, we may assume that

$$\tilde{f}_i(x+y,t)-f_i(x)\ll(1-a)A_0\left(\frac{y}{1-a},\frac{t}{1-a}\right)\qquad\text{in } P(\rho),$$

where $A_0(y,t)$ is the series given by

$$A_0(y,t)=\frac{b_0}{c_0}\sum_{p\geq 1}c_0^p(y_1+\cdots+y_n+t_1+\cdots+t_m)^p.$$

As in [7] (3.14), we have

$$\tilde{f}_i(\phi^{|p},\psi^{|p})-f_i(x)\ll K_2(1-a)A\left(\frac{s}{1-a}\right)\tag{2.43}$$

for some constant K_2. On the other hand, using [3] Lemma (2.4), we have, from $(2.34)_p$,

$$\frac{\partial\phi_i^{|p}}{\partial x_j}\ll 1+\frac{c_1(1-b)}{b-a}A\left(\frac{s}{1-b}\right)\qquad\text{in } P(a\rho),\quad \frac{1}{2}\leq a<b<1,$$

for some constant c_1. Using

$$A\left(\frac{s}{1-a}\right)A\left(\frac{s}{1-b}\right)\ll A\left(\frac{s}{1-b}\right)^2\ll\frac{1}{c^{1/2}}A\left(\frac{s}{1-b}\right),$$

and $1-b<1-a\leq 1/2$, we get

$$\Big[\sum_{i=1}^{n}\tilde{f}_i(\phi^{|p},\ \psi^{|p})\frac{\partial\phi_i^{|p}}{\partial x_j}\Big]_p$$
$$\ll n(1-a)\Big\{K_2A\Big(\frac{s}{1-a}\Big)+\Big(K_0+\frac{K_2}{2c^{1/2}}\Big)\frac{c_1}{b-a}A\Big(\frac{s}{1-b}\Big)\Big\}.$$

Thus by (2.40) Lemma, we get

$$(2.44)\qquad \Big[\sum_{i=1}^{n}\tilde{f}_i(\phi^{|p},\ \psi^{|p})\frac{\partial\phi_i^{|p}}{\partial x_j}\Big]_p\ll n\Big(\frac{2K_2+8ec_1K_0}{c}+\frac{4ec_1K_2}{c^{3/2}}\Big)A'\Big(\frac{s}{1-a}\Big)$$
$$=n\Big(\frac{2K_2+8ec_1K_0}{c^{1/2}}+\frac{4ec_1K_2}{c}\Big)A_2'\Big(\frac{s}{1-a}\Big).$$

Now, we may assume that

$$\theta_s-\omega\ll A\Big(\frac{s}{1-a}\Big)\qquad \text{in } P(a\rho),\quad \frac{1}{2}\leq a<1.$$

Hence, using $(2.33)_{p-1}$ and (2.31), we get

$$[u^{|p-1}\theta_s]_p=[u^{|p-1}(\theta_s-\omega)]_p\ll 32A_2'\Big(\frac{s}{1-a}\Big)A\Big(\frac{s}{1-a}\Big)$$
$$\ll\frac{16}{c^{1/2}}A_2'\Big(\frac{s}{1-a}\Big).$$

Hence, by (2.29) Lemma (b), for each ν with $|\nu|=p$, there exists $u^{(\nu)}$ such that

$$[u]_p\omega+u^{|p-1}\theta_s\underset{p}{\equiv}\Theta(p),\qquad [u]_p=\sum_{|\nu|=p}u^{(\nu)}(x)s^\nu,$$

and that

$$[u]_p\ll\Big(\frac{K_3}{c^{1/2}}+\frac{K_4}{c}\Big)A_2'\Big(\frac{s}{1-a}\Big)$$

for some constants K_3 and K_4. Therefore, if c is sufficiently large, we have the estimate $(2.33)_p$ as well as the congruence $(2.13)_p$.

Next, we set

$$\sigma_k=u^{|p}\tilde{e}_k-E(p)_k-\delta_k,\qquad \delta_k=\sum_{|\nu|=p}\ \sum_{\substack{\lambda+\mu=\nu\\|\mu|>0}}\alpha_k^{(\lambda,\mu)}s^\nu,$$

and estimate σ_k. First, from (2.11), we have

$$[E(p)_k]_p$$
$$=\left[\sum_{i=1}^{n}(\tilde{f}_i(\phi^{|p},\ \psi^{|p})-f_i(x))\frac{\partial\phi_i^{|p}}{\partial s_k}+\sum_{j=1}^{m}(\tilde{h}_j(\phi^{|p},\ \psi^{|p})-h_j(x))\frac{\partial\psi_j^{|p}}{\partial s_k}\right]_p.$$

Using the estimate (2.43) and a similar one for $\tilde{h}_j$, we get, from $(2.34)_p$, $(2.35)_p$, (2.31) and $1-a\leq 1/2$,

$$(2.45)\qquad [E(p)_k]_p\ll(n+m)K_2(1-a)A\left(\frac{s}{1-a}\right)A'\left(\frac{s}{1-a}\right)$$
$$\ll\frac{(n+m)K_2}{4c^{1/2}}A'\left(\frac{s}{1-a}\right)\qquad\text{in } P(a\rho),\ \frac{1}{2}\leq a<1.$$

Second, since we may assume that

$$\tilde{e}_k\ll K_0+A\left(\frac{s}{1-a}\right),$$

we get

$$[u^{|p}\tilde{e}_k]_p\ll 32A_2'\left(\frac{s}{1-a}\right)\left\{K_0+A\left(\frac{s}{1-a}\right)\right\}$$
$$\ll\left(\frac{32K_0}{c^{1/2}}+\frac{16}{c}\right)A'\left(\frac{s}{1-a}\right).$$

Third, we have

$$\delta_k=\sum_{|\nu|=p}\sum_{\substack{\lambda+\mu=\nu\\|\mu|>0}}\sum_{i=1}^{N}(\lambda_k+1)a_i^{(\lambda+1_k)}[\tilde{\beta}_{ik}(p)]_\mu s^\lambda.$$

By (2.44) and (2.45), we may apply (A.2) Theorem to get

$$\tilde{\beta}_{ik}(p)\ll K_0+A_3'\left(\frac{s}{1-a}\right).$$

On the other hand, from $(2.38)_p$, we have

$$\sum_{|\lambda|=0}^{p-1}(\lambda_k+1)a_i^{(\lambda+1_k)}s^\lambda\ll A_1'\left(\frac{s}{1-a}\right).$$

Hence we get

$$\delta_k\ll NA_1'\left(\frac{s}{1-a}\right)A_3'\left(\frac{s}{1-a}\right)=\frac{N}{c}A'\left(\frac{s}{1-a}\right)^2\ll\frac{N}{2c^{1/2}}A'\left(\frac{s}{1-a}\right).$$

Therefore, we have

$$[\sigma_k]_p \ll \left(\frac{K_5}{c^{1/2}} + \frac{K_6}{c}\right) A'\left(\frac{s}{1-a}\right) \tag{2.46}$$

for some constants K_5 and K_6.

Now we take ν with $|\nu| = p+1$ and set

$$\sigma^{(\nu)} = \left(\cdots, \frac{1}{\nu_k}\sigma_k^{(\nu-1_k)}, \cdots\right), \qquad k \in L(\nu).$$

Then we have shown in (2.19) Proposition (see (2.28)) that $\sigma^{(\nu)}$ is in $I(\omega) + K(\omega)^{l(\nu)}$. Hence by (2.39) Lemma, there exist X in Θ_n and constants c_j, $1 \leq j \leq m$, and a_i, $1 \leq i \leq N$, such that

$$\sigma^{(\nu)} = \sum_{j=1}^{m} c_j h_j + \langle X, \omega\rangle + \sum_{i=1}^{N} a_i \pi^{(\nu)}(\gamma_i), \quad \text{i.e.,}$$

$$\sigma_k^{(\nu-1_k)} = \nu_k\left(\sum_{j=1}^{m} c_j h_j + \langle X, \omega\rangle + \sum_{i=1}^{N} a_i \beta_{ik}\right)$$

and that

$$|X|_{a\rho},\ |c_j|,\ |a_i| \leq K_1 |\sigma^{(\nu)}|_{a\rho} \qquad \text{for } \frac{1}{2} \leq a \leq 1.$$

Writing $X = \sum_{i=1}^{n} \xi_i(\partial/\partial x_i)$, we set $\phi_i^{(\nu)} = \xi_i$. $c_j^{(\nu)} = c_j$ and $a_i^{(\nu)} = a_i$. Thus we have $\Phi^{|p+1} = (\phi^{|p+1}, \psi^{|p+1})$. Also, for λ with $|\lambda| = p$, we set $\alpha_k^{(\lambda,0)} = (\lambda_k + 1) \sum_{i=1}^{N} a_i^{(\lambda+1_k)} \beta_{ik}$. Furthermore, for ν with $|\nu| = p$, we set $\alpha_k^{(\nu)} = \sum_{\lambda+\mu=\nu} \alpha_k^{(\lambda,\mu)}$ and, for λ, μ with $|\lambda+\mu| = p+1$, $|\mu| > 0$, let $\alpha_k^{(\lambda,\mu)}$ be defined by using the identity in $(2.37)_{p+1}$. Then we have $(2.14)_{p+1}$ as well as $(2.37)_{p+1}$. On the other hand, from (2.46), we have

$$\begin{aligned}\sum_{|\nu|=p+1} \sigma^{(\nu)} s^\nu &\ll l\left(\frac{K_5}{c^{1/2}} + \frac{K_6}{c}\right)(1-a) A\left(\frac{s}{1-a}\right)\\ &= l\left(\frac{K_5}{c^{1/4}} + \frac{K_6}{c^{3/4}}\right)(1-a) A_1\left(\frac{s}{1-a}\right) \qquad \text{in } P(a\rho),\ \frac{1}{2} \leq a < 1.\end{aligned}$$

Hence if c is sufficiently large, we have the estimates $(2.34)_{p+1}$, $(2.35)_{p+1}$ and $(2.38)_{p+1}$. Finally, we have

$$\begin{aligned}\sum_{|\nu|=p} \alpha^{(\nu)} s^\nu &= \sum_{|\nu|=p} \sum_{\lambda+\mu=\nu} \alpha_k^{(\lambda,\mu)} s^\nu = \sum_{|\nu|=p} \sum_{\lambda+\mu=\nu} \sum_{i=1}^{N} (\lambda_k + 1) a_i^{(\lambda+1_k)} [\tilde{\beta}_{ik}(p)]_\mu s^\lambda\\ &\ll N A_1'\left(\frac{s}{1-a}\right)\left(K_0 + A_3'\left(\frac{s}{1-a}\right)\right)\\ &\ll N\left(\frac{K_0}{c^{1/4}} + \frac{1}{2c^{1/2}}\right) A'\left(\frac{s}{1-a}\right) \qquad \text{in } P(a\rho),\ \frac{1}{2} \leq a < 1.\end{aligned}$$

Hence if c is sufficiently large, we have the estimate $(2.36)_p$.

§ 3. The case $\omega = df$

Suppose $\omega = df$ for some f in $\mathcal{O}_n$ with $f(0)=0$ and codim $S(df) \geq 2$. Then $I(\omega) = \mathcal{O}_n$ and $J(\omega) = (\partial f)$ (the ideal generated by $\partial f/\partial x_1, \cdots, \partial f/\partial x_n$). The assumption codim $S(df) \geq 2$ implies that f is not a power, where a germ h in $\mathcal{O}_n$ is said to be a power if $h = h_0^m$, $m > 1$, for some non-unit h_0 in $\mathcal{O}_n$. Hence by the factorization theorem in [4] p. 472, we have $K(\omega) = f^*\mathcal{O}_1 = \{p \circ f \mid p \in \mathcal{O}_1\}$.

If a germ $\tilde{f}$ in $\mathcal{O}_{n+m}$ is an unfolding of f, then $\mathscr{F} = (d\tilde{f})$ is an unfolding of $F = (df)$ with parameter space C^m and conversely, any unfolding of $F = (df)$ has a generator of the form $d\tilde{f}$ with $\tilde{f}$ an unfolding of f ([7] p. 47). We have seen that the unfolding theory for $F = (df)$ with respect to morphisms is equivalent to that for f with respect to (strict) right-morphisms ([10] (3.11) Remark.) It is shown in [12] that the unfolding theory for $F = (df)$ with respect to RL-morphisms is equivalent to that for f with respect to right-left morphisms ([13] Definition 3.2).

For an unfolding $\tilde{f}$ of f with parameter space $C^m = \{(t_1, \cdots, t_m)\}$, we set $h_j(x) = (\partial \tilde{f}/\partial x_j)(x, 0)$, $1 \leq j \leq m$. We say that $\tilde{f}$ is infinitesimally right-left versal if the classes $[h_1], \cdots, [h_m]$ span the vector space $\mathcal{O}_n/(\partial f) + f^*\mathcal{O}_1$. Thus $\tilde{f}$ is an infinitesimally right-left versal unfolding of f if and only if $\mathscr{F} = (d\tilde{f})$ is an infinitesimally RL-versal unfolding of $F = (df)$. If f admits an infinitesimally right-left versal unfolding, then $\mathcal{O}_n/(\partial f) + f^*\mathcal{O}_1$ is finite dimensional. Hence by [13] Corollary 2.17, $\mathcal{O}_n/(\partial f)$ is finite dimensional. Thus $K(\omega)/J(\omega) \cap K(\omega) \simeq J(\omega) + K(\omega)/J(\omega)$, $\omega = df$, is also finite dimensional. Therefore, by (2.1) Theorem (see (2.2) Remark) we obtain the following result, which is a special case of [13] Theorem 3.22.

(3.1) **Proposition.** *Let f be a germ in $\mathcal{O}_n$ with $f(0)=0$ and* codim $S(df) \geq 2$. *If $\tilde{f}$ is an infinitesimally right-left versal unfolding of f, then for any one parameter unfolding g of f, there is a right-left morphism from g to $\tilde{f}$.*

(3.2) **Remark.** Suppose $\mathcal{O}_n/(\partial f) + f^*\mathcal{O}_1$ is finite dimensional and let $[h_1], \cdots, [h_m]$ be one of its basis. Then $\tilde{f} = f + \sum_{j=1}^m h_j t_j$ is an infinitesimally right-left versal unfolding of f.

§ 4. The case $\omega = gdf - fdg$

Let ω be a germ of holomorphic 1-form at 0 in C^n of the form

$$\omega = gdf - fdg$$

for some f and g in $\mathcal{O}_n$ with $f(0) = g(0) = 0$ and codim $S(\omega) \geq 2$. Note that the last assumption implies that f and g are relatively prime and neither one is a power. The ideal $I(\omega)$ coincides with the ideal (f, g) generated by f and g ([10] (2.1) Lemma). We have $J(\omega)=(g\partial f-f\partial g)$ (the ideal generated by $g(\partial f/\partial x_i)-f(\partial g/\partial x_i)$, $1\leq i\leq n$.)

(4.1) **Lemma.** *The* $\boldsymbol{C}$*-vector space* $K(\omega)$ *is three dimensional and we may take* f^2, fg *and* g^2 *as its basis.*

Proof. If $df\wedge dg=0$, then by the factorization theorem in [4] p. 472, we may write $g=p(f)$ for some p in $\mathcal{O}_1$. Hence $\omega=(g-fp'(f))df$. This contradicts the assumption codim $S(\omega)\geq 2$, since $g-fp'(f)$ is not a unit in $\mathcal{O}_n$. Thus we have $df\wedge dg\neq 0$.

Now, if β is a germ in $K(\omega)$, we have $\beta d\omega=d\beta\wedge\omega$. Hence as is shown in the proof of [10] (2.1) Lemma, there exist germs ϕ and ψ in $\mathcal{O}_n$ such that

$$d\beta=\phi df+\psi dg \quad \text{and} \quad 2\beta=\phi f+\psi g.$$

From these, we get

$$d\phi\wedge df+d\psi\wedge dg=0 \tag{4.2}$$

and

$$fd\phi+gd\psi=\phi df+\psi dg. \tag{4.3}$$

Thus we have

$$\phi d\omega=2d\phi\wedge\omega \quad \text{and} \quad \psi d\omega=2d\psi\wedge\omega.$$

Again, using the arguments in the proof of [10] (2.1) Lemma, we may write

$$2d\phi=\phi_1 df+\phi_2 dg, \qquad 2\phi=\phi_1 f+\phi_2 g$$

and

$$2d\psi=\psi_1 df+\psi_2 dg, \qquad 2\psi=\psi_1 f+\psi_2 g$$

for some ϕ_1, ϕ_2, ψ_1 and ψ_2 in $\mathcal{O}_n$. From these, we get

$$fd\phi_1+gd\phi_2=0 \tag{4.4}$$

and

$$fd\psi_1+gd\psi_2=0. \tag{4.5}$$

On the other hand, the identity (4.2) shows that

$$\phi_2=\psi_1.$$

We set $\xi=\phi_2-\phi_2(0)$ $(=\psi_1-\psi_1(0))$ and claim that $\xi=0$. Suppose $\xi\neq0$. Since $d\phi_1\wedge d\xi=0$, if ξ is not a power, by the factorization theorem in [4], we have $\phi_1=p(\xi)$ for some p in $\mathcal{O}_1$. Hence from (4.4), we get

$$p'(\xi)f+g=0.$$

This contradicts the assumption codim $S(\omega)\geq2$. If ξ is a power, we may write $\xi=\xi_0^m$ with $m>1$ and ξ_0 not a power. Since $d\phi_1\wedge d\xi_0=d\psi_2\wedge d\xi_0=0$, again by the factorization theorem, we may write $\phi_1=p(\xi_0)$ and $\psi_2=q(\xi_0)$ for some p and q in $\mathcal{O}_1$. From (4.4) and (4.5), we have

$$p'(\xi_0)f+m\xi_0^{m-1}g=0$$

and

$$m\xi_0^{m-1}f+q'(\xi_0)g=0.$$

Hence we see that ξ_0 is divisible by fg. We write $\xi_0=f^kg^l\eta$ with $k\geq1$, $l\geq1$ and η not divisible by f or g. If we denote by d the order of the power series $p'(\xi_0)$ of ξ_0, then we must have $kd+1=k(m-1)$ and $ld=l(m-1)+1$. Hence

$$k(m-d-1)=1 \quad\text{and}\quad l(m-d-1)=-1,$$

which is a contradiction, since k, l, m and d are integers with $k\geq1$ and $l\geq1$. Therefore, $\xi=0$ and ϕ_2 and ψ_1 reduce to constants. By (4.4) and (4.5), ϕ_1 and ψ_2 must also be constants, which shows that β is a linear combination of f^2, fg and g^2. Conversely, it is not difficult to see that the linear combinations of these germs are all in $K(\omega)$. Q.E.D.

Thus in this case, the condition (*) in (2.1) Theorem is always satisfied. If we denote by $[f^2, fg, g^2]_C$ the $\boldsymbol{C}$-vector subspace of $\mathcal{O}_n$ generated by f^2, fg and g^2, from (2.1) Theorem and (4.1) Lemma, we obtain the following

(4.6) **Theorem.** *Let $F=(\omega)$ be a germ of codim 1 foliation at 0 in $\boldsymbol{C}^n$ generated by a germ ω of the form $\omega=gdf-fdg$ for some f and g in $\mathcal{O}_n$ with $f(0)=g(0)=0$ and* codim $S(\omega)\geq2$. *If the dimension of the $\boldsymbol{C}$-vector space*

$$(f,g)/(g\partial f-f\partial g)+[f^2,fg,g^2]_C$$

is finite, then F admits an RL-versal unfolding. In fact, if $[u_1g-v_1f],\cdots,[u_mg-v_mf]$ $(u_j, v_j\in\mathcal{O}_n)$ is a $\boldsymbol{C}$-basis of the above vector space, then the unfolding $\mathscr{F}=(\tilde{\omega})$ of F with parameter space $\boldsymbol{C}^m=\{(t_1,\cdots,t_m)\}$ generated by

$$\tilde{\omega}=\tilde{g}d\tilde{f}-\tilde{f}d\tilde{g},$$

where $\tilde{f}$ and $\tilde{g}$ are germs in $\mathcal{O}_{n+m}$ given by

$$\tilde{f}=f+\sum_{j=1}^{m} u_j t_j \quad \textit{and} \quad \tilde{g}=g+\sum_{j=1}^{m} v_j t_j$$

is RL-versal.

(4.7) **Remark**. We could say that the unfolding $\mathscr{F}$ in the above is an RL-universal unfolding of F (cf. [10] (2.4) Theorem).

(4.8) **Example.** Let $f=x-y$ and $g=xy$ on $C^2=\{(x,y)\}$. Then $\omega=gdf-fdg=y^2dx-x^2dy$. We have $I(\omega)=(x-y, xy)$, $J(\omega)=(x^2, y^2)$ and $K(\omega)=[(x-y)^2, (x-y)xy, x^2y^2]_C$. We have $\mathcal{O}_2/I(\omega)=C^2$ and we may take $[1]$ and $[x]$ as its basis and $\mathcal{O}_2/J(\omega)+K(\omega)=C^3$ and we may take $[1]$, $[x]$ and $[x-y]$ as its basis. Hence from the exact sequence

$$0\longrightarrow I(\omega)/J(\omega)+K(\omega)\longrightarrow\mathcal{O}_2/J(\omega)+K(\omega)\longrightarrow\mathcal{O}_2/I(\omega)\longrightarrow 0,$$

we see that $I(\omega)/J(\omega)+K(\omega)=C$ and we may take $[x-y]=[0\cdot g-(-1)f]$ as its basis. Therefore, by (4.6) Theorem, if we set $\tilde{f}=f=x-y$ and $\tilde{g}=g-t=xy-t$, then the unfolding $\mathscr{F}=(\tilde{\omega})$ with parameter space $C=\{t\}$ generated by

$$\tilde{\omega}=\tilde{g}d\tilde{f}-\tilde{f}d\tilde{g}=(y^2-t)dx-(x^2-t)dy+(x-y)dt$$

is an RL-universal unfolding of $F=(\omega)$.

We note that $I(\omega)/J(\omega)=C^2$ and we may take $[x-y]=[0\cdot g-(-1)f]$ and $[xy]=[1\cdot g-0\cdot f]$ as its basis. Hence if we set $\tilde{f}'=f+t_2=x-y+t_2$ and $\tilde{g}'=g-t_1=xy-t_1$, then the unfolding $\mathscr{F}'=(\tilde{\omega}')$ with parameter space $C^2=\{(t_1, t_2)\}$ generated by

$$\begin{aligned}\tilde{\omega}'&=\tilde{g}'d\tilde{f}'-\tilde{f}'d\tilde{g}'\\ &=(y^2-t_2y-t_1)dx-(x^2+t_2x-t_1)dy+(x-y+t_2)dt_1+(xy-t_1)dt_2\end{aligned}$$

is a universal unfolding of $F=(\omega)$ (see [10] (2.4) Theorem and [8] (5.11) Example).

Appendix. Unfoldings of integrating factors

We recall that $K(\omega)$ denotes the C-vector space of integrating factors of a germ of integrable 1-form ω at the origin 0 in C^n;

$$K(\omega)=\{\beta\in\mathcal{O}_n \mid \beta d\omega=d\beta\wedge\omega\}.$$

Let Λ denote the graded algebra $\oplus_{r=0}^{n+l}\Omega_{n+l}^r$ $(\Omega_{n+l}^0=\mathcal{O}_{n+l})$ of germs of holomorphic forms at 0 in $C^n\times C^l=\{(x, s)\}$ and for a non-negative integer

p, let Λ_p denote the ideal in Λ generated by s^ν and ds^ν for all l-tuples $\nu=(\nu_1, \cdots, \nu_l)$ of non-negative integers with $|\nu|\geq p$. For σ and τ in Λ, we write

$$\sigma \underset{p}{\equiv} \tau$$

if $\sigma-\tau$ is in Λ_{p+1}.

If f is a germ in $\mathcal{O}_n$, a germ $\tilde{f}$ in $\mathcal{O}_{n+l}$ is said to be an unfolding of f, or to unfold f, if $\tilde{f}(x, 0)=f(x)$.

(A.1) **Definition.** Let $F=(\omega)$ be a codim 1 foliation germ at 0 in $\boldsymbol{C}^n$ and let β be a germ in $K(\omega)$. Also, let $\mathscr{F}=(\tilde{\omega})$ be an unfolding of F with parameter space $\boldsymbol{C}^l$. A p-th order unfolding of β subject to $\mathscr{F}$ is a germ $\tilde{\beta}$ in $\mathcal{O}_{n+l}$ such that $\tilde{\beta}$ is an unfolding of β and that

$$\tilde{\beta}\, d\tilde{\omega} \underset{p}{\equiv} d\tilde{\beta}\wedge\tilde{\omega}.$$

In this appendix we prove the following theorem. Here we use the dominant series in Section 2. We also use the notations there.

(A.2) **Theorem.** *Let $F=(\omega)$ be a codim 1 foliation germ at 0 in $\boldsymbol{C}^n$. If β is a germ in $K(\omega)$, then for any unfolding $\mathscr{F}=(\tilde{\omega})$ of F (with parameter space $\boldsymbol{C}^l=\{s\}$) and for any non-negative integer p, there is a p-th order unfolding $\tilde{\beta}(p)$ of β subject to $\mathscr{F}$. If we write*

$$\tilde{\omega}=\sum_{i=1}^{n}\tilde{f}_i(x, s)dx_i+\sum_{k=1}^{l}\tilde{h}_k(x, s)ds_k$$

and set $\tilde{\omega}^{|p}=\omega_s^{|p}+\sum_{k=1}^{l}\tilde{h}_k^{|p-1}ds_k$, $\omega_s^{|p}=\sum_{i=1}^{n}\tilde{f}_i^{|p}dx_i$, then the germ $\tilde{\beta}(p)$ is determined by $\tilde{\omega}^{|p}$ uniquely modulo Λ_{p+1}.

Moreover, if we choose the constant c in the series $A(s)$ sufficiently large, then the estimates

$$(\mathrm{A.3})_p \qquad \omega_s^{|p}-\omega \ll LA_4'\left(\frac{s}{1-a}\right)$$

and

$$(\mathrm{A.4})_p \qquad \tilde{h}^{|p-1}-h \ll LA_2'\left(\frac{s}{1-a}\right) \quad \text{in } P(a\rho),\ \frac{1}{2}\leq a<1,$$

imply the estimate

$$(\mathrm{A.5})_p \qquad \tilde{\beta}(p)^{|p}-\beta \ll A_3'\left(\frac{s}{1-a}\right) \quad \text{in } P(a\rho),\ \frac{1}{2}\leq a<1,$$

for all $p \geq 1$, *where* $\tilde{h} = (\tilde{h}_1, \cdots, \tilde{h}_l)$, $h(x) = \tilde{h}(x, 0)$ *and* L *is a constant.*

(A.6) **Remark.** In the above, we choose open neighborhoods U and W of the origins in $\boldsymbol{C}^n$ and $\boldsymbol{C}^l$, respectively, so that the germs ω and β have representatives on U and the germ $\tilde{\omega}$ has a representative on $U \times W$. Then we choose an n-tuple $\rho = (\rho_1, \cdots, \rho_n)$ of positive numbers so that $P(\rho) \subset U$ and that ρ has the properties in (2.29) Lemma.

(A.7) **Remark.** The dominant series above are chosen so that the theorem can be directly applied to the proof of (2.1) Theorem. We could use, for instance, the following series instead ([6] p. 291, [5] p. 50, [7], [11]);

$$A^*(s) = \frac{b}{16c} \sum_{p \geq 1} \frac{c^p}{p^2} (s_1 + \cdots + s_l)^p.$$

In this case, we can prove that if we choose b and c sufficiently large, then the estimates

$$\omega_s^{|p} - \omega \ll A^*\left(\frac{s}{1-a}\right)$$

and

$$\tilde{h}^{|p-1} - h \ll A^*\left(\frac{s}{1-a}\right) \qquad \text{in } P(a\rho), \ \frac{1}{2} \leq a < 1,$$

imply the estimate

$$\tilde{\beta}(p)^{|p} - \beta \ll A_\varepsilon^*\left(\frac{s}{1-a}\right) \qquad \text{in } P(a\rho), \ \frac{1}{2} \leq a < 1,$$

for all $p \geq 1$, where ε is any positive number and $A_\varepsilon^*(s) = c^\varepsilon A^*(s)$.

For more applications of the theorem, see [11] (4.31) Lemma.

(A.8) **Corollary** (Cerveau-Moussu [2]). *Let* $F = (\omega)$ *be a* codim 1 *foliation germ at* 0 *in* $\boldsymbol{C}^n$. *If* β *is a germ in* $K(\omega)$, *for any unfolding* $\mathscr{F} = (\tilde{\omega})$ *of* F, *there exists a unique germ* $\tilde{\beta}$ *in* $K(\tilde{\omega})$ *that unfolds* β.

Proof of (A.2) *Theorem.* Let $\mathscr{F} = (\tilde{\omega})$ be an unfolding of F with parameter space $\boldsymbol{C}^l = \{(s_1, \cdots, s_l)\}$. To prove the first half of the theorem, it suffices to show that there exists a formal power series

$$\tilde{\beta} = \sum_{|\nu| \geq 0} \beta^{(\nu)}(x) s^\nu, \qquad \beta^{(\nu)} \in \mathcal{O}_n,$$

in s satisfying

(A.9) $$\beta^{(0)}=\beta$$

and

(A.10)$_p$ $$\beta^{|p}d\tilde{\omega}\underset{p}{\equiv}d\beta^{|p}\wedge\tilde{\omega} \qquad \text{in } \Lambda,$$

for $p\geq 0$, and that $[\beta]_p$ is uniquely determined by $\beta^{|p-1}$ and $\tilde{\omega}^{|p}$ for $p\geq 1$.

The congruence (A.10)$_p$ is equivalent to

(A.11)$_p$ $$\beta^{|p}d_x\omega_s\underset{p}{\equiv}d_x\beta^{|p}\wedge\omega_s,$$

(A.12)$_p$ $$\frac{\partial\beta^{|p}}{\partial s_k}\omega_s\underset{p-1}{\equiv}\beta^{|p-1}\left(\frac{\partial}{\partial s_k}\omega_s-d_x\tilde{h}_k\right)+\tilde{h}_k d_x\beta^{|p-1}, \qquad 1\leq k\leq l,$$

and

(A.13)$_p$ $$\tilde{h}_j\frac{\partial\beta^{|p-1}}{\partial s_k}-\tilde{h}_k\frac{\partial\beta^{|p-1}}{\partial s_j}\underset{p-2}{\equiv}\beta^{|p-2}h_{jk}, \qquad 1\leq j,\ k\leq l,$$

where $h_{jk}=(\partial\tilde{h}_j/\partial s_k)-(\partial\tilde{h}_k/\partial s_j)$. We set $\beta^{(0)}=\beta$ so that (A.9) and (A.11)$_0$ are satisfied. We think of (A.12)$_0$, (A.13)$_0$ and (A.13)$_1$ as void conditions.

(A.14) **Proposition.** *If we have $\beta^{(\nu)}$, for ν with $|\nu|\leq p$, satisfying* (A.11)$_p$, (A.12)$_p$ *and* (A.13)$_p$, *then, for each ν with $|\nu|=p+1$, there exists unique $\beta^{(\nu)}$ satisfying* (A.11)$_{p+1}$, (A.12)$_{p+1}$ *and* (A.13)$_{p+1}$, *for $p\geq 0$.*

Proof. Given $\beta^{(\nu)}$, $|\nu|\leq p$, as in the assumption above. First we show that (A.13)$_{p+1}$ is satisfied. If we set

$$\varepsilon_{jk}=\tilde{h}_j\frac{\partial\beta^{|p}}{\partial s_k}-\tilde{h}_k\frac{\partial\beta^{|p}}{\partial s_j}-\beta^{|p-1}h_{jk},$$

for our purpose, it suffices to show that

$$\varepsilon_{jk}\omega_s\underset{p-1}{\equiv}0,$$

since ε_{jk} does not have terms of order less than $p-1$ by (A.13)$_p$. Using (A.12)$_p$, we have

$$\varepsilon_{jk}\omega_s\underset{p-1}{\equiv}\beta^{|p-1}\left\{\tilde{h}_j\left(\frac{\partial}{\partial s_k}\omega_s-d_x\tilde{h}_k\right)-\tilde{h}_k\left(\frac{\partial}{\partial s_j}\omega_s-d_x\tilde{h}_j\right)-h_{jk}\omega_s\right\},$$

which is zero by the integrability of $\tilde{\omega}$ (cf. (2.5)).

Second, by (A.12)$_p$, (A.12)$_{p+1}$ is equivalent to

(A.15) $$\frac{\partial[\beta]_{p+1}}{\partial s_k}\omega=[\zeta_k]_p, \qquad 1\leq k\leq l,$$

where

$$\zeta_k=\beta^{|p}\left(\frac{\partial}{\partial s_k}\omega_s-d_x\tilde{h}_k\right)+\tilde{h}_k d_x\beta^{|p}-\frac{\partial\beta^{|p}}{\partial s_k}\omega_s.$$

To prove the existence of $[\beta]_{p+1}=\sum_{|\nu|=p+1}\beta^{(\nu)}s^\nu$ satisfying (A.15), it suffices to show that

$$[\zeta_k]_p\wedge\omega=0,\qquad 1\leq k\leq l, \tag{A.16}$$

and

$$\frac{\partial[\zeta_j]_p}{\partial s_k}=\frac{\partial[\zeta_k]_p}{\partial s_j},\qquad 1\leq j,\ k\leq l. \tag{A.17}$$

Since ζ_k does not have terms of order less than p, if $\tau_k\wedge\omega_s\equiv_p 0$, then we have (A.16). But this easily follows from $(\text{A.11})_p$ and the integrability of $\tilde{\omega}$ (cf. (2.4)). Also, we have (A.17) if

$$\zeta_{jk}\underset{p-1}{\equiv}0, \tag{A.18}$$

where

$$\zeta_{jk}=\frac{\partial\zeta_j}{\partial s_k}-\frac{\partial\zeta_k}{\partial s_j}.$$

Now, using $(\text{A.13})_{p+1}$, we get

$$\frac{1}{2}\zeta_{jk}\underset{p-1}{\equiv}\frac{\partial\beta^{|p}}{\partial s_k}\left(\frac{\partial}{\partial s_j}\omega_s-d_x\tilde{h}_j\right)-\frac{\partial\beta^{|p}}{\partial s_j}\left(\frac{\partial}{\partial s_k}\omega_s-d_x\tilde{h}_k\right)+h_{jk}d_x\beta^{|p}.$$

Then, by $(\text{A.12})_p$ and $(\text{A.13})_{p+1}$, we have

$$\beta^{|p}\zeta_{jk}\underset{p-1}{\equiv}0.$$

Since ζ_{jk} does not have terms of order less than $p-1$, this implies that $\beta\zeta_{jk}\equiv_{p-1}0$. Since the theorem is trivial in the case $\beta=0$, we may assume that $\beta\neq 0$. Hence we get (A.18). Note that $[\beta]_{p+1}$ is determined uniquely by $\beta^{|p}$ and $\tilde{\omega}^{|p+1}$.

Third, we prove that $(\text{A.11})_{p+1}$ is satisfied. If we set

$$\eta=\beta^{|p+1}d_x\omega_s-d_x\beta^{|p+1}\wedge\omega_s,$$

for our purpose, it suffices to show that

$$\frac{\partial}{\partial s_k}\eta\underset{p}{\equiv}0,\qquad 1\leq k\leq l. \tag{A.19}$$

Using the identity obtained by taking d_x of $(A.12)_{p+1}$, we have

$$\frac{1}{2}\frac{\partial}{\partial s_k}\eta \underset{p}{\equiv} \frac{\partial \beta^{|p+1}}{\partial s_k} d_x\omega_s + \left(\frac{\partial}{\partial s_k}\omega_s - d_x\tilde{h}_k\right) d_x\beta^{|p}.$$

Denoting the right hand side of the above by τ_k, we get, using $(A.11)_p$ and $(A.12)_{p+1}$,

$$\beta^{|p}\tau_k \underset{p}{\equiv} \left\{\beta^{|p}\left(\frac{\partial}{\partial s_k}\omega_s - d_x\tilde{h}_k\right) - \frac{\partial \beta^{|p+1}}{\partial s_k}\omega_s\right\} \wedge d\beta^{|p},$$

which is zero by $(A.12)_{p+1}$. By $(A.11)_p$, η does not have terms of order less than $p+1$. Hence τ_k does not have terms of order less than p. Thus we have $\beta\tau_k \equiv_p 0$. Again, we may assume that $\beta \neq 0$. Therefore, we have (A.19).

Now we prove the second half of the theorem. For our purpose, it suffices to show that if we choose c sufficiently large, then the estimates $(A.3)_p$ and $(A.4)_p$ imply the estimate

$$(A.20)_p \qquad \beta^{|p} - \beta \ll A_3'\left(\frac{s}{1-a}\right) \qquad \text{in } P(a\rho),\ \frac{1}{2} \leq a < 1,$$

for all $p \geq 1$.

The following two propositions will finish the proof of the theorem.

(A.21) **Proposition.** *If we choose c sufficiently large, then we have* $(A.20)_1$.

Proof. From (A.15), we have

$$\beta^{(1_k)}\omega = \beta(\omega^{(1_k)} - dh_k) + h_k d\beta.$$

Since the germs on the right hand side have representatives on U (see (A.6) Remark), $|\beta^{(1_k)}|_\rho$ is bounded by (2.29) Lemma (b). On the other hand, we have

$$A_3'\left(\frac{s}{1-a}\right) = \frac{1}{32}\left(\frac{1}{c^{1/4}} + \frac{c^{3/4}}{4}\frac{s_1 + \cdots + s_l}{1-a} + \text{terms of order} \geq 2 \text{ in } s\right).$$

Hence, if c is sufficiently large, we have $(A.20)_1$.

(A.22) **Proposition.** *If we choose c sufficiently large, then the estimates* $(A.3)_{p+1}$, $(A.4)_{p+1}$ *and* $(A.20)_p$ *imply* $(A.20)_{p+1}$ *for* $p \geq 1$.

Proof. We recall that $[\beta]_{p+1}$ is determined from (A.15). Setting $L_0 = \max\{|\omega|_\rho, |h|_\rho, |dh|_\rho, |\beta|_\rho, |d\beta|_\rho\}$, we estimate ζ_k.

First, we have, from $(A.3)_{p+1}$ and $(A.20)_p$,

$$\left[\beta^{|p}\frac{\partial}{\partial s_k}\omega_s\right]_p=\left[\beta^{|p}\frac{\partial}{\partial s_k}\omega_s^{|p+1}\right]_p\ll\left(L_0+A_3'\left(\frac{s}{1-a}\right)\right)\frac{L}{1-a}A_4''\left(\frac{s}{1-a}\right).$$

By (2.32), we have

$$A'\left(\frac{s}{1-a}\right)A''\left(\frac{s}{1-a}\right)\ll\frac{c^{1/2}}{4}A''\left(\frac{s}{1-a}\right).$$

Hence we get

$$\left[\beta^{|p}\frac{\partial}{\partial s_k}\omega_s\right]_p\ll L\left(\frac{L_0}{c^{1/4}}+\frac{1}{4c^{1/2}}\right)\frac{1}{1-a}A_3''\left(\frac{s}{1-a}\right)$$

$$\text{in } P(a\rho),\quad \frac{1}{2}\leq a<1.$$

Second, using [3] Lemma (2.4), we have, from $(A.4)_{p+1}$ and $(A.20)_p$,

$$[\beta^{|p}d_x\tilde{h}_k]_p\ll\left(L_0+A_3'\left(\frac{s}{1-a}\right)\right)\left(L_0+\frac{nLc_1}{b-a}A_2'\left(\frac{s}{1-b}\right)\right)$$

$$\text{in } P(a\rho),\quad \frac{1}{2}\leq a<b<1,$$

for some constant c_1. Noting that $p\geq 1$ and $A'(s/1-a)\ll A'(s/1-b)$, we get, using (2.32),

$$[\beta^{|p}d_x\tilde{h}_k]_p\ll L_0A_3'\left(\frac{s}{1-a}\right)+nc_1L\left(L_0c^{1/4}+\frac{1}{2}\right)\frac{1}{b-a}A_3'\left(\frac{s}{1-b}\right).$$

As in [7] p. 42 we can prove that

$$\text{(A.23)}\qquad A'\left(\frac{s}{1-a}\right)\ll\frac{4}{c}\frac{1}{1-a}A''\left(\frac{s}{1-a}\right)$$

and that

(A.24) if, for a series $\sum\sigma^{(\nu)}s^\nu$,

$$\sum\sigma^{(\nu)}s^\nu\ll\frac{1}{b-a}A'\left(\frac{s}{1-b}\right)\qquad\text{in } P(a\rho)\quad\text{for } \frac{1}{2}\leq a<b<1,$$

then

$$\sum\sigma^{(\nu)}s^\nu\ll\frac{4e}{c}\frac{1}{1-a}A''\left(\frac{s}{1-a}\right)\quad\text{in } P(a\rho)\quad\text{for } \frac{1}{2}\leq a<1.$$

Thus we have

$$[\beta^{|p}d_x\tilde{h}_k]_p \ll \left(\frac{4nec_1L_0L}{c^{3/4}}+\frac{4L_0+2nec_1L}{c}\right)\frac{1}{1-a}A_3''\left(\frac{s}{1-a}\right).$$

Third, we have

$$\begin{aligned}\left[\frac{\partial\beta^{|p}}{\partial s_k}\omega_s\right]_p &= \left[\frac{\partial\beta^{|p}}{\partial s_k}(\omega_s^{|p}-\omega)\right]_p \\ &\ll \frac{1}{1-a}A_3''\left(\frac{s}{1-a}\right)LA_4'\left(\frac{s}{1-a}\right) \\ &\ll \frac{L}{4c^{1/2}}\frac{1}{1-a}A_3''\left(\frac{s}{1-a}\right) \qquad \text{in } P(a\rho),\quad \frac{1}{2}\le a<1.\end{aligned}$$

Fourth, we have

$$[\tilde{h}_kd_x\beta^{|p}]_p \ll \left(L_0+LA_2'\left(\frac{s}{1-a}\right)\right)\left(L_0+\frac{nc_1}{b-a}A_3'\left(\frac{s}{1-b}\right)\right).$$

Since $p\ge 1$, we get, using (2.32),

$$[\tilde{h}_kd_x\beta^{|p}]_p \ll L_0Lc^{1/4}A_3'\left(\frac{s}{1-a}\right)+nc_1\left(L_0+\frac{L}{2}\right)\frac{1}{b-a}A_3'\left(\frac{s}{1-b}\right).$$

Then, by (A.23) and (A.24), we have

$$[\tilde{h}_kd_x\beta^{|p}]_p \ll \left(\frac{4L_0L}{c^{3/4}}+\frac{2nec_1(2L_0+L)}{c}\right)\frac{1}{1-a}A_3''\left(\frac{s}{1-a}\right).$$

From the above, we get

$$[\zeta_k]_p \ll \left(\frac{L_1}{c^{1/4}}+\frac{L_2}{c^{1/2}}+\frac{L_3}{c^{3/4}}+\frac{L_4}{c}\right)\frac{1}{1-a}A_3''\left(\frac{s}{1-a}\right).$$

Hence by (2.29) Lemma (b), we get

$$[\beta]_{p+1} \ll K\left(\frac{L_1}{c^{1/4}}+\frac{L_2}{c^{1/2}}+\frac{L_3}{c^{3/4}}+\frac{L_4}{c}\right)A_3'\left(\frac{s}{1-a}\right).$$

Therefore, if c is sufficiently large, we have the estimate $(A.20)_{p+1}$.

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