

A Stratification Theoretical Method of Construction of Holomorphic Vector Bundles

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In this paper, we propose an explicit method of construction of holomorphic vector bundles over a complex variety. In the construction, our guiding model is the universal quotient bundle over Grassmann variety. The content of this paper is rather provisional and experimental, but may be used as a general method for treatments of bundles.

Introduction

1. Letting \bar{X} be a normal complex variety, the purpose of this paper is to construct holomorphic vector bundles over \bar{X} , by the following two steps:

(I) To find a bundle E_X over $X = \bar{X} - \{\text{a codimension two subvariety of } \bar{X}\}$, which is endowed with a suitable 'stratification theoretical representation', and

(II) to investigate structure of the direct image $E_X = i_* E_X$, i being the injection: $X \rightarrow \bar{X}$, by giving a similar representation to the one in (I). (See Section 0 for more details of (I) and (II).)

Encouraging facts for our proposed approach are: (i) If \bar{X} is a quasi projective or a Stein variety, then each bundle over \bar{X} is obtained in the manner (I), (II), and (ii) the procedure (I), (II) may be regarded as a generalization of classical methods in treatments of bundles over a Riemann surface [Bir], [Weil] and [Tj] (cf. § 0).

2. The content of this paper is briefly as follows: In Section 1 we give some explicit coherent sheaf theoretical expressions of the bundle E_X as in (I). In Section 2, we introduce the notion of 'type (G)' for such a bundle and give some basic properties of the bundle. A bundle E_X of type (G) is, in our context, an abstraction of a bundle obtained as the pull back of the universal quotient bundle over a Grassmann variety. The main results of this paper are given to such a bundle E_X and are as follows:

(i) An explicit determination of the local structure of its direct image E_X (Theorems 3.1 ~ 3.4), (ii) that of $\Gamma(E_X)$ and $\Gamma(\text{End } E_X)$ (Theorems 4.1 ~ 4.5) and (iii) a type of residue formula for the characteristic classes of E_X (Theorem 5.1), which is based on the Čech theoretical treatment of the classes due to Atiyah ([At]). The basic tools for getting (i) (iii) are:

(a) *Some subvarieties of \bar{X} and a coherent sheaf* (cf. § 2.1), where the former is gotten by taking Schubert subvarieties of Grassmann variety as our model and the latter is gotten by applying the arguments in Section 1.

Actually, investigations of the above two data and uses of them in the proof of (i) ~ (iii) are, the author feels, the central part of this paper.

As an application of (i), (ii) we give a criterion in order that the direct image sheaf E_X is locally free and simple (§ 4). This criterion gives a general method to get simple bundles over a smooth variety of dimension ≥ 2 (Theorem 4.6), and is our main results for the original purpose of the construction of bundles.

3. Very many important results have been known for constructions of vector bundles. (See, for example, the surveys in [Har-1] and [Schn], [Ok-Schn-Sp]). In connection with our proposed approach, we like to say that many important results on the constructions seem to be concentrated to the projective space ([Har-1] and [Schn]). On the other hand, in [Mar], Maruyama developed a general algebraic argument on elementary transformation, and got, among others, a basic result which says that there are a 'lot of' simple bundles over a smooth projective variety of any characteristic and of dimension ≥ 2 . (For more precise formulation, see [Mar]. See also a recent work of Sumihiro ([Sum]) generalizing [Mar].) From a general character of the result of Maruyama, it seems, to the author, to be suitable to take his result as a starting point for the constructions in general situation. Now, our result mentioned above may be an analytic analogue of the above result of Maruyama, and will be also a starting point for further considerations.

Remark. In getting 'starting data' for the constructions of bundles, there are some similarities between the view point in the theory of the elementary transformation ([Mar] and [Sum]) and ours. Some relations between them are discussed in Section 2.2 (cf. Remark 2.6).

As was stated in the beginning of this paper, the content should be regarded as provisional, and some speculations arise concerning how to push the present results further. We list them in the form of Question in the course of the arguments. Good parts of them concern singularity problems arising naturally in the construction of bundles. We hope that they may be interesting for readers.

Remark. This paper is a continuation of our previous works on stratification theory and cohomology with growth and division ([Sa 1-4]). We hope to write a survey paper for this and previous papers elsewhere.

Remark. The author does not present the proof of Theorem 5.1 and Lemma 2.5 in this paper (chiefly because of its length). The proof will be given elsewhere in a near future. The other parts of the present paper is read independently from the above two results.

A fact on morphisms to a Grassmann variety ([Hir] and [Ka]) (cf. (2.6.4)) was informed by Kazama, by answering to my question. The author expresses his thank to Professor Kazama for his kindness.

Notation and Terminologies

Here we summarize notation and terminologies, which are used throughout this paper: First we make the convention:

- (1) $\left\{ \begin{array}{l} \text{complex variety} = \text{complex reduced space} \\ \text{bundle} = \text{holomorphic vector bundle} \end{array} \right.$

For a complex variety \bar{X} , we use the symbol $\mathcal{O}_{\bar{X}}$ for its structure sheaf without mentioning it. Letting $\mathfrak{F}_{\bar{X}}$ be a coherent sheaf over \bar{X} and U an open set of \bar{X} , we use the following notation:

(2) $GL_r(U, \mathcal{O}_{\bar{X}}) := GL_r(\Gamma(U, \mathcal{O}_{\bar{X}|U}))$, and $M_r(U, \mathfrak{F}_{\bar{X}}) = M_r(\Gamma(U, \mathfrak{F}_{\bar{X}|U}))$

where, for an abelian group A and a commutative ring B , we mean:

(3) $\left\{ \begin{array}{l} M_r(A) = \text{the abelian group of } r \times r\text{-matrices with coefficients in } A \\ GL_r(B) = \text{group of } r \times r\text{-matrices with coefficients in } B, \text{ whose} \\ \text{determinant is a unit in } B. \end{array} \right.$

By a Cartier (resp. set theoretical Cartier) divisor of \bar{X} , we mean a codimension one subvariety \bar{Y} of \bar{X} such that, for each point $p \in \bar{Y}$, there is an open set U of p (in \bar{X}) with which we have:

(4) $\left\{ \begin{array}{l} \text{There is an element } f \in \Gamma(U, \mathcal{O}_{\bar{X}}) \text{ which generates the ideal of } \bar{Y} \cap U \\ \text{(resp. whose reduce divisor } (f)_{0, \text{red}} \text{ coincides with } (U \cap \bar{Y}). \end{array} \right.$

As usual \mathbb{C} and \mathbb{Z} are the field of complex numbers and the ring of integers. Also we mean by \mathbb{Z}_+ the set of positive integers and we set $\mathbb{Z}_{+,0} = \mathbb{Z}_+ \cup \{0\}$.

§ 0. Preliminaries

Here we add some remarks to the line (I), (II) in Introduction.

1. First of all, letting \bar{X} be a complex variety and $E_{\bar{X}}$ a bundle^{*)} over \bar{X} , what we have in mind by ‘stratification theoretical treatment’ of the bundle $E_{\bar{X}}$ is:

(*-1) To stratify \bar{X} , to attach an open neighborhood to each stratum and to form a frame of $E_{\bar{X}}$ over each neighborhood, and

(*-2) to use the stratification, the neighborhoods and the frames for investigations of the bundle.

Thus our treatment is based on the Čech cohomology theory, to which methods of stratification theory are applied. Now the following definition is used throughout this paper.

Definition 0.1.1. In this paper we mean, by a *prebundle* over \bar{X} , a pair $D_1=(\bar{X}^2, E_{\bar{X}})$ consisting of a codimension two subvariety \bar{X}^2 and a bundle $E_{\bar{X}}$ over $X:=\bar{X}-\bar{X}^2$.

Definition 0.1.2. (1) By an *s-representation* of D_1 , we mean a datum $D_2=(\bar{X}^1, N, e^0, e^1)$ as follows:

$$(0.1) \quad \begin{cases} \bar{X}^1 = (\text{reduced}) \text{ divisor of } \bar{X}, \text{ which contains } \bar{X}^2, \\ N_1 = \text{open neighborhood of } X^1 := \bar{X}^1 - \bar{X}^2 \text{ in } X, \\ e^i = (e_1^i, \dots, e_r^i), r = \text{rank of } E_{\bar{X}}, \text{ is a frame of } E_{X^1 N_i} \ (i=0, 1), \\ \text{where we set } N_0 = \bar{X} - \bar{X}^1. \end{cases}$$

(2) By an *s-prebundle* over \bar{X} , we mean a pair $D=(D_1, D_2)$ as above. When there is no fear of confusions, we call $E_{\bar{X}}$ also a *prebundle* (or an *s-pre bundle*) over \bar{X} .

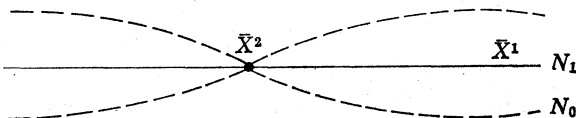


Figure I.

The datum D_2 is stratification theoretical, because writing N_0 also as X^0 , we have a stratification $\mathcal{S}'=(X^0, X^1)$ of X and the frames $e^i:=(e^0, e^1)$ ($i=0, 1$) are attached to the system of neighborhoods $\mathcal{N}':=(N_0, N_1)$ of \mathcal{S}' . Now we sharpen the line (I), (II) in Introduction as follows:

(III) To find an *s-pre bundle* $E_{\bar{X}}$ over \bar{X} , and investigate its direct image E_X by the stratification theoretical method.

In connection with (III), let $h_{10} \in GL(N_1 \cap N_0, \mathcal{O}_{\bar{X}})$ be the transition

^{*)} Bundle=holomorphic vector bundle (cf. Introduction).

matrix for the frames $(e^0, e^1): e^0 = e^1 h_{10}$. Then our basic technique for the investigations of E_X will be (cf. § 1):

(IV) Analysis of growth properties of h_{10} with respect to the first and second boundaries \bar{X}^1 and \bar{X}^2 of $X^0 = N_0$.

Remark 0.1. In Definition 0.1, the neighborhood N_1 of $X^1 = \bar{X}^1 - \bar{X}^2$ is not determined uniquely by (\bar{X}^1, \bar{X}^2) . As a type of such a neighborhood, what we have in mind is a ‘tubular neighborhood’ of X^1 in the stratification theory (cf. [Th] and [Mat]. See also [Sa-1].) Zariski open neighborhoods of X^1 are also important. In this paper, we check what a type of neighborhoods are used, according to arguments in question.

2. Here we check that our approach to bundle theory along the line (I), (II) have generalities. For this letting E_X be a locally free sheaf over \bar{X} , we make:

Definition 0.2. We say that E_X is of type (e), if there is an s -pre bundle $D = (D_1, D_2)$ as in Definition 0.1 such that E_X is the direct image sheaf of the prebundle appearing in D .

Lemma 0.1. Assume that \bar{X} is normal and is a quasi projective variety. Then an algebraic bundle E_X over \bar{X} is of type (e).

Proof. First take a codimension one subvariety \bar{X}^1 of \bar{X} so that $E_{X^1|X^0}$, $X^0 = \bar{X} - \bar{X}^1$, is a product bundle, and we fix a frame e^0 of it. Next take a codimension two subvariety \bar{X}^2 of \bar{X} which is also contained in \bar{X}^1 . Then one can assume that the restriction of E_X to $X^1 := \bar{X}^1 - \bar{X}^2$ is a product bundle, and take a frame e^1 of it. Assume that X^1 is an affine variety, and take an open neighborhood N_1 of X^1 in $X := \bar{X} - \bar{X}^2$. Then we have elements $e^1 \subset \Gamma(N_1, E_X)$ whose restriction to X^1 coincides with e^1 . Assuming that N_1 is small enough, we can assume that e^1 is a frame of $E_{X^1|N_1}$. Now we set $D_1 = (\bar{X}^2, E_X := E_{X^1|X})$ and $D_2 = (\bar{X}^1, N_1, e^0, e^1)$. Because \bar{X} is normal, we see that the direct image $i_* E_X$ of E_X , $i = \text{injection: } X \rightarrow \bar{X}$, coincides with E_X , and we finish the proof of this lemma. q.e.d.

Assume that \bar{X} is a normal Stein variety, and take a bundle E_X over \bar{X} . Then, by the similar reasoning to the proof of Lemma 0.1, we see that E_X is of type (e).

3. Now assume that $\dim \bar{X} = 1$ and \bar{X} is compact. Then we see easily that ‘to give an s -pre bundle E_X over \bar{X} ’ is equivalent to give data as follows:

$$(**-1) \quad \begin{cases} \text{finite points } p_1, \dots, p_u \text{ on } \bar{X}, \text{ neighborhoods } N_\alpha \text{ of } p_\alpha \text{ and} \\ \text{matrices } h_\alpha \in GL(N_\alpha - p_\alpha, \mathcal{O}_{\bar{X}}) (1 \leq \alpha \leq u). \end{cases}$$

Repeating the arguments below Definition 0.1, we set $N_1 = \bigcup_a N_a$ and $N_0 = X - \bigcup_a P_a$. Then the bundle E_X is defined by the element:

$$(**-2) \quad h := \{h_a\}_a \in GL(N_1 \cap N_0, \mathcal{O}_X).$$

Though we do not enter into details of (**-1,2), we like to point out that the procedure (**-1, 2) has many similarities to classical treatments of bundles over a Riemann surface ([Bir], [Weil] and [Tj]). In particular, the notion of ‘matrix divisor’ in [Weil] and [Tj] is essentially equivalent to our notion of s -prebundle in the present situation. This observation is a starting point of the present paper (cf. Introduction).

§ 1. Explicit expressions of E_X

In the remainder of this paper, we fix a normal complex variety \bar{X} . Also we fix an s -pre bundle $D = (D_1, D_2)$ over \bar{X} , with $D_1 = (\bar{X}^2, E_X)$ and $D_2 = (\bar{X}^1, N_1, e^0, e^1)$, once for all, where

$$(*) \quad \left\{ \begin{array}{l} \text{the subvarieties: } \bar{X}^1 \supset \bar{X}^2, \text{ the bundle } E_X \text{ over } X := \bar{X} - \bar{X}^2, \text{ the} \\ \text{neighborhoods } N_i \text{ and the frames } e^i = (e^i_1, \dots, e^i_r) \text{ of } E_{X|N_i}, r = \text{rank} \\ \text{of } E_X \text{ (} i=0, 1) \end{array} \right.$$

has the similar meaning to (0.1).

In Section 1 we assume that there is an element $y \in \Gamma(\mathcal{O}_{\bar{X}})$ such that

$$(1.0) \quad \bar{X}^1 = (y)_{0, \text{red}} \text{ in } \bar{X}.$$

1. First we check that if the transition matrix h_{10} for $(e^0, e^1) \in GL_r(N_1 \cap N_0, \mathcal{O}_X): e^0 = e^1 h_{10}$ in $N_1 \cap N_0$, has a suitable growth property with respect to \bar{X}^1 , then E is imbedded into \mathcal{O}_X^r .

Proposition 1.1. *Assume that $h_{01} := h_{10}^{-1}$ admits the following expression:*

$$(1.1) \quad h_{01} = y^{-a} h'_{01}, \text{ with elements } a \in \mathbf{Z}_{+0} \text{ and } h'_{01} \in M_r(N_1, \mathcal{O}_X). \text{ Then } E_X \text{ is imbedded into } \mathcal{O}_X^r.$$

Proof. Define an \mathcal{O}_X -homomorphism $\tau: E_X \rightarrow \mathcal{O}_X^r$ by the following:

$$(1.2) \quad \left\{ \begin{array}{l} \tau_{N_1}: E_X|_{N_1} \ni e^1 \cdot \zeta^1 \longrightarrow \mathcal{O}_{N_1}^r \ni h'_{01} \zeta^1 \\ \tau_{N_0}: E_X|_{N_0} \ni e^0 \cdot \zeta^0 \longrightarrow \mathcal{O}_{N_0}^r \ni y^a \cdot \zeta^0 \end{array} \right.$$

where $\zeta^i = (\zeta^i_1, \dots, \zeta^i_r)$ is an element of $\mathcal{O}_{N_i}^r$ and $e^i \zeta^i = \sum_{j=1}^r \zeta^i_j \cdot e^i_j$. Note that (1.1) implies $\tau_{N_1} = \tau_{N_0}$ in $N_1 \cap N_0$. Because \bar{X} is normal, we see that τ is injective. q.e.d.

Denote by L_X and L'_X the determinant bundles of E_X and $E'_X := \tau(E_X)$. Then we have the following commutative diagram:

$$(1.3) \quad \begin{array}{ccc} E_X^{\otimes r} & \xrightarrow{\tau^{\otimes r}} & E_X'^{\otimes r} \\ \wedge^r \downarrow & & \downarrow \wedge^r \\ L_X & \xrightarrow{(\wedge^r \tau)} & L'_X \end{array}$$

where \wedge^r denotes the r -th exterior product morphism. Note that, by taking $f^i := \wedge^r e^i$ to be a frame of $L_{X|N_i}$ ($i=0, 1$), the isomorphism $(\wedge^r \tau)$ is explicitly as follows:

$$(1.4) \quad \begin{cases} (\wedge^r \tau)|_{N_1}: L_X|_{N_1} \ni f^1 \cdot \zeta^1 \longrightarrow \mathcal{D}_{N_1} \ni (\det h'_{01}) \cdot \zeta^1, \\ (\wedge^r \tau)|_{N_0}: L_X|_{N_0} \ni f^0 \cdot \zeta^0 \longrightarrow \mathcal{D}_{N_0} \ni y^{ar} \cdot \zeta^0, \end{cases}$$

where ζ^i is an element of \mathcal{D}_{N_i} ($i=0, 1$).

For an element $\varphi \in \Gamma^r(E_X)$ we mean by the divisor of φ the one of $\wedge^r \varphi \in \Gamma(L_X)$. Letting D_φ and $D_{\varphi'}$ be the divisors of φ and $\varphi' = \tau(\varphi) \in \Gamma^r(E'_X)$, we have the following from (1.4):

(1.5) $D_{\varphi'} = D_\varphi + D_0$, where D_0 is defined as follows: $D_0|_{N_1}$ = locus of $\det h'_{01}$ and $D_0|_{N_0}$ = that of 1.

Note that $D_{\varphi'}$ is the divisor of $\wedge^r \varphi' \in \Gamma(X, \mathcal{D}_X)$, and treatments of it are easier than those of D_φ in general. Divisors like D_φ will play basic roles in our arguments henceforth (cf. § 2).

2. Next we check that a suitable growth property of h_{10} with respect to \bar{X}^2 , in addition to (1.2), will insure a more explicit expression of E_X . For this letting x be an element of $\Gamma(\mathcal{D}_X)$ which does not vanish on N_1 , we assume the following for the matrix $h'_{10} := h'_{01}{}^{-1}$ (cf. (1.1)).

(1.6) $h'_{10} = y^{-b} \cdot (x^{-c} h' + y^b \cdot h'')$, where b and c are elements of \mathbf{Z}_{+0} , and h' and h'' are respectively elements of $M_r(\mathcal{D}_X)$ and $M_r(N_1, \mathcal{D}_X)$. (We may say that (1.6) claims that the main part of the matrix h_{10} is meromorphic with respect to y and x .) Now let μ and ω denote the \mathcal{D}_X -morphism: $\mathcal{D}_X^r \ni \zeta \rightarrow \mathcal{D}_X^r \ni h' \zeta$ and the quotient morphism: $\mathcal{D}_X^r \rightarrow \mathcal{D}_X^r / y^b \mathcal{D}_X^r$. Then we have:

Lemma 1.1. $E'_X (= \tau(E_X) \subset \mathcal{D}_X^r)$ is the kernel of the \mathcal{D}_X -homomorphism $\omega \cdot \mu: \mathcal{D}_X^r \rightarrow \mathcal{D}_X^r / y^b \cdot \mathcal{D}_X^r$. (We write ω, μ also for their restrictions to X .)

Proof. Take a point $p \in X^1$ and an element $\zeta \in \mathcal{D}_{X,p}^r$. Then we easily have the following equivalence:

$$(a) \quad \zeta \in E'_{X,p} \iff h'_{10} \cdot \zeta \in \mathcal{D}_{X,p}^r \iff h' \cdot \zeta \in y^b \cdot \mathcal{D}_{X,p}^r,$$

and we have $E'_{X,p} = \text{kernel of } (\omega \cdot \mu)_p$. On the other hand, for a point $p \in N_0 (= \bar{X} - \bar{X}^1)$, we obviously have: $E_{X,p} = \text{kernel of } \omega \cdot \mu_p = \mathcal{D}_{X,p}^r$. q.e.d.

Lemma 1.2. *The direct image sheaf $E_X (= i_* E_X, i$ being the injection: $X \rightarrow \bar{X})$ is coherent.*

Proof. It suffices to check the coherency of $E'_X = i_* E'_X$. But, the normality of \bar{X} implies that E'_X coincides with the kernel of $\omega \cdot \mu: \mathcal{D}_{\bar{X}}^r \rightarrow \mathcal{D}_{\bar{X}}^r / y^b \mathcal{D}_{\bar{X}}^r$. q.e.d.

Remark 1.1. The above explicit form of E'_X :

(*) $E'_X = \text{the kernel of the } \mathcal{D}_{\bar{X}}\text{-homomorphism } \omega \cdot \mu: \mathcal{D}_{\bar{X}}^r \rightarrow \mathcal{D}_{\bar{X}}^r / y^b \mathcal{D}_{\bar{X}}^r$ will be used frequently in later arguments (cf. § 2 and § 3). Moreover, remark that the isomorphism $\tau: E_X \xrightarrow{\sim} E'_X$ is extended to the isomorphism: $E_{\bar{X}} \xrightarrow{\sim} E'_{\bar{X}} (\subset \mathcal{D}_{\bar{X}}^r)$. In later arguments we use τ also for its extension.

3. Coherency conditions. Let $\bar{X}^{/2}$ be a codimension two subvariety of \bar{X} and $F_{X'}$, a locally free sheaf over $X' := \bar{X} - \bar{X}^{/2}$. Recall that a basic condition of Serre ([Se]) for the coherency of the $F_{\bar{X}} := i_* F_{X'}$, i being the injection $X' : \hookrightarrow \bar{X}$, is as follows:

(L.A) For each $p \in \bar{X}^{/2}$ there is an open neighborhood U of p in \bar{X} such that $\Gamma(X' \cap U, \mathfrak{F}_X)$ generates $F_{X',q}$ for each $q \in U \cap X'$.

Next we say that $F_{X'}$ satisfies condition (L.G) if, for each $p \in \bar{X}^{/2}$ there are an open neighborhood U of p in \bar{X} and a codimension two subvariety $\bar{X}^{/2}$ of U containing $\bar{X}^{/2} \cap U$ with which the following holds:

(L.G) There are an s -representation $D_2 = (\bar{X}^1, N_1, e^0, e^1)$ (cf. (0.1)) of $D_1 := (\bar{X}^{/2}, F_{X'}|_{U - \bar{X}^{/2}})$ and an element $y \in \Gamma(U, \mathcal{D}_{\bar{X}})$ with which the following holds:

(1.7) $\bar{X}^1 := (y'')_{0, \text{red}}$ and the transition matrix h_{10} for the frames e^0, e^1 of F_X over $U - \bar{X}^1, N_1$ (= open neighborhood of $X^1 := \bar{X}^1 - \bar{X}^{/2}$ in $U - \bar{X}^{/2}$) admits the expression of the form (1.1) and (1.6).

Lemma 1.3. *The following three conditions are equivalent.*

(1.8) (a) $F_{\bar{X}}$ is coherent (b) $F_{X'}$ satisfies (L.A) and (c) $F_{X'}$ satisfies (L.G).

The equivalence of (a) and (b) is in ([Se]). The condition (c) is given in terms of the growth properties of the matrix h_{10} , and is concordant to our stratification theoretical approach to bundle theory. Here we give a simple proof of Lemma 1.3, by emphasizing the role of the growth properties.

Proof. The implication: (a) \Leftrightarrow (b) is obvious and (c) \Leftrightarrow (a) follows from Lemma 1.2. We check (b) \Leftrightarrow (c) as follows. For a point $p \in \bar{X}^{/2}$,

take an open neighborhood U of p in \bar{X} and sections $e^0 \in \Gamma^r(U - \bar{X}''^2, F_{\bar{X}'})$, $r = \text{rank of } F_{\bar{X}'}$, such that $\bigwedge^r e^0$ does not vanish identically on $U - \bar{X}''^2$. Take an element $y \in \Gamma(U, \mathcal{O}_{\bar{X}})$ which vanishes on \bar{X}'_2 and the extension of $(\bigwedge^r e^0)_{0, \text{red}}$ to U , and we set $\bar{X}^1 = (y)_{0, \text{red}}$. Next take an element $e^1 \in \Gamma^r(U - \bar{X}''^2, F_{\bar{X}'})$ such that $\bigwedge^r e^1$ does not vanish identically on any irreducible component of \bar{X}^1 . Take a codimension two subvariety \bar{X}''^2 of \bar{X} satisfying the following: $\bar{X}^1 \supset \bar{X}''^2 \supset \bar{X}''^2 \cup (\bar{X}^1 \cap \bar{X}''^1)$, where \bar{X}''^1 is the extension of $(\bigwedge^r e^1)_{0, \text{red}}$ to U . Then, by a simple observation, we have the following relation in $U - \bar{X}''^2$:

$$(1.9) \quad y^a \cdot e^1 = e^0 h'_{01}, \quad \text{with } a \in \mathbb{Z}_{+0} \quad \text{and } h'_{01} \in M_r(U, \mathcal{O}_{\bar{X}}).$$

Thus setting $D_1 = (\bar{X}''^2, F_{\bar{X}'|_{\bar{X} - \bar{X}''^2}})$ and $D_2 = (\bar{X}^1, N_1, e^0, e^1)$, with a suitable open neighborhood N_1 of $X^1 = \bar{X}^1 - \bar{X}''^2$, we get an s -pre bundle $D = (D_1, D_2)$, where the transition matrix h_{10} for (e^0, e^1) clearly satisfies (1.1) and (1.6). q.e.d.

§ 2. Bundles of type (G)

The most important property of the s -pre bundle in the title is the existence of (rank of the prebundle + 1)-sections of the prebundle satisfying suitable conditions (Definition 2.1). This section is divided into three parts according to the nature of the arguments.

§ 2.1. Key definitions

1. First we make:

Definition 2.1. We say that the s -pre bundle $D = (D_1, D_2)$, where $D_1 = (\bar{X}^2, E_X)$ and $D_2 = (\bar{X}^1, N_1, e^0, e^1)$ (cf. the beginning of § 1), is of type (G), if there are sections $e = (e_1, \dots, e_{r+1})$ of E_X , $r = \text{rank of } E_X$, with which the following hold:

(2.1.1) The frames e^i ($i = 0, 1$) are of the form: $e^0 = (e_1, \dots, e_{r-1}, e_r)$ and $e^1 = (e_1, \dots, e_{r-1}, e_{r+1})$.

(2.1.2) $(\bigwedge^r e^0)_0 (\subset X := \bar{X} - \bar{X}^2)$ is a reduced divisor and its closure in \bar{X} coincides with \bar{X}^1 .

(2.1.3) $(\bigwedge^r e^1)_{0, \text{red}} (\subset X)$ is a (reduced) divisor, and letting \bar{X}''^1 be its closure in \bar{X} , the codimension two subvariety \bar{X}^2 is of the form: $\bar{X}^2 = \bar{X}^1 \cap \bar{X}''^1$.

Remark 2.1. Note that $N'_1 := (\bar{X} - \bar{X}''^1) (= (X - X''^1))$ and $X''^1 := \bar{X}''^1 - \bar{X}^2$ satisfy: (1) $N'_1 \supset X''^1 := \bar{X}^1 - \bar{X}^2$ and (2) e^1 is a frame of E_X over N'_1 . Note that they are the conditions imposed on the open set N_1 of X , which appears in D_2 (cf. (0.1), § 0). In order to fix our idea, unless we say otherwise, we assume the following for an s -pre bundle D of type (G):

(2.1.4-1) The neighborhood N_1 of X^1 is taken to be $N'_1 (= (X - X''))$. In connection with this, we remark the following immediate consequence of (2.1.1-3):

$$(2.1.4-2) \quad X^1 \cap X'' = \emptyset, \text{ and } N_1 \supset X^1, N_0 \supset X''.$$

Remark 2.2. One can check easily that the most important property of an s -pre bundle of type (G): 'it admits $(r+1)$ -sections as in Definition 2.1', characterizes also such a bundle in the sense that the following equivalence holds:

(*) To give an s -pre bundle over \bar{X} of type (G) \Leftrightarrow To give an s -pre bundle over \bar{X} , which admits $(r+1)$ -sections e of the prebundle such that the frames e^0 and e^1 , formed in the manner as in (2.1.1), satisfy (2.1.2, 3), where r is the rank of the prebundle.

Next, for convenience of later arguments, we add the following to Definition 2.1:

Definition 2.2. D is said to be of type (W.G.), if (2.1. 1, 2, 4) and the following weaker form of (2.1.3) holds.

(2.1.3)' $\bar{X}^2 = \bar{X}^1 \cap \bar{X}'^1$, where \bar{X}'^1 is a divisor of \bar{X} such that $X'' := \bar{X}'^1 - \bar{X}^2$ is a set theoretical Cartier divisor and $(\bigwedge^r e^1)_{0, \text{red}} \subset X''$. (Note that \bar{X}'^1 is not, in general, determined uniquely by e^1 . When we are concerned with an s -prebundle of type (W.G.), we fix a divisor \bar{X}'^1 as above and the open set N_1 defined as in (2.1.4-2).)

2. Now, letting $D = (D_1, D_2)$ be the s -pre bundle of type (W.G) as in Definition 2.2, we have:

Proposition 2.1. The following holds for D .

(2.1.5) $X^1 = \bar{X}^1 - \bar{X}^2$ and $X'' = \bar{X}'^1 - \bar{X}^2$ are, respectively, Cartier and set theoretical Cartier divisors of X .

(2.1.6) The transition matrix h_{10} for (e^0, e^1) : $e^0 = e^1 h_{10}$ in $N_1 \cap N_0$, is explicitly as follows:

$$(2.1.7) \quad h_{10} = \begin{bmatrix} I_{r-1} & f_1 \\ \vdots & \vdots \\ 0 & f_r \end{bmatrix},$$

where f_1, \dots, f_r are meromorphic functions over X with the pole X'' .

(2.1.8) $(f_r)_0$ is reduced and coincides with X^1 (in $N_1 = \bar{X} - \bar{X}'^1$).

(2.1.9) $1/f_r$ and f_j/f_r ($1 \leq j \leq r-1$) are holomorphic in N_0 ($\supset X''$) (cf. (2.2.4-2)).

Remark 2.3. Note that (2.1.8) gives a defining equation of X^1 in N_1 . In later arguments, we discuss such an equation of \bar{X}^1 in \bar{X} (cf. Lemma 2.6 and Proposition 3.1. Also see Remark 2.5.)

Proof of Proposition 2.1. First (2.1.5) is a direct consequence of (1.1.2) and (2.1.3)'. Next we check (2.1.6, 9) as follows: Remark that e^1 is a frame of E_X over N_1 and e^0 is an element of $\Gamma^r(E_X)$. Then, from (2.1.1), we obviously have:

(a) h_{10} is of the form (2.1.7), by understanding that f_1, \dots, f_r are holomorphic functions on N_1 .

On the otherhand, it is checked easily that $h_{01} = h_{10}^{-1}$ is of the form:

$$(2.1.10) \quad h_{01} = \begin{bmatrix} I_{r-1} & -f_j/f_r \\ 0 & 1/f_r \end{bmatrix} \quad (1 \leq j \leq r-1).$$

But e^0 is a frame of E_X over N_0 , e^1 is an element of $\Gamma^r(E_X)$ and $e^1 = e^0 h_{01}$ in N_0 . Thus the coefficients of h_{01} are holomorphic in N_0 and we have (2.1.9). Next, from (a) and (2.1.9) (cf. also (2.1.4-2)), we see easily that f_1, \dots, f_r are meromorphic functions over X with the pole X^1 and we have (2.1.6). Finally, from (2.1.7), we have:

(b) $(\wedge^r e^0) = f_r (\wedge^r e^1)$ in N_1 .

From this we have (2.1.8), and we finish the proof of this proposition.

q.e.d.

Remark 2.4. Assume that \bar{X} is smooth. Then, in (2.1.6), we have the unique extensions \bar{f}_j of f_j to \bar{X} . Thus, in that place, one can assume that f_1, \dots, f_j are meromorphic functions over \bar{X} with the pole \bar{X}^1 .

3. Basic tools. The arguments here are divided into three parts, according to their nature.

3.0. Letting $D = (D_1, D_2)$ be the s -prebundle of type (W.G) as in Definition 2.2, we assume that there are elements s'_j ($1 \leq j \leq r+1$) and $s_{r+1}, s''_{r+1} \in \Gamma(\mathcal{O}_{\bar{X}})$ such that (2.2.0-1, 2, 3) soon below hold:

(2.2.0-1) $f_j = s'_j/s'_r$ ($1 \leq j \leq r-1$) and $f_r = s'_{r+1}/s'_r$, where f_j are the coefficients of the matrix h_{10} (cf. (2.1.7)).

(2.2.0-2) $\bar{X}^1 = (s'_r)_{0, \text{red}}$,

(2.2.0-3) $(s_{r+1})_0$ is reduced and coincides with \bar{X}^1 . Moreover, $s'_{r+1} = s_{r+1} \cdot s''_{r+1}$ and $(s''_{r+1})_{0, \text{red}} \subset \bar{X}^1$.

Define matrices h_{01} and h'_{01} by $h_{01} = h_{10}^{-1}$ and $h_{01} = s_{r+1}^{-1} \cdot h'_{01}$. Also we set $h'_{10} = h'_{01}^{-1}$. Then, by a simple computation, we have:

(2.2.1) $h'_{01} = \begin{bmatrix} s_{r+1} I_{r-1} & g' \\ 0 & g' \end{bmatrix}$, where $g' = (g'_j)_{j=1}^r$ with $g'_j = -s'_j/s''_{r+1}$ ($1 \leq j \leq r-1$) and $g'_r = s'_r/s''_{r+1}$.

(2.2.2) $h'_{10} = (s_{r+1} \cdot s'_r)^{-1} \cdot h'$, where $h' = \begin{bmatrix} s'_r I_{r-1} & g \\ 0 & g \end{bmatrix}$, and the vector

$g = (g_j)_{j=1}^r$ is as follows: $g_j = s'_j$ ($1 \leq j \leq r-1$) and $g_r = s'_{r+1}$. (Remark that (2.1.9) and (2.2.0) imply:

$$(2.2.1)' \quad g'_j \ (1 \leq j \leq r) \text{ are elements of } \Gamma(\mathcal{D}_X).$$

Remark 2.5. If \bar{X} is smooth, then (2.2.0-1, 2, 3) are legitimate in the local situation. Namely, for a point $p \in \bar{X}$, take a suitable open neighborhood U of p in \bar{X} . Then, by restricting the data \bar{X}^2, E_X, \dots , to U , we check easily that f_j and \bar{X}^1, \bar{X}'' admit the expression given in (2.2.0-1, 2, 3) in U . In the global case, we see easily that a similar expression to (2.2.0-1, 2, 3) holds by understanding that s'_1, \dots are sections of a suitable line bundle over \bar{X} .

3.1. Our first tool is the imbedding of E_X into \mathcal{D}_X^r as in (1.2) (cf. also Remark 1.1) and some resulting explicit expressions of E_X .

Proposition 2.2. (1) *The imbedding $\tau: E_X \hookrightarrow E_X'' (= \tau(E_X) \subset \mathcal{D}_X^r)$ (as in (1.2)) is as follows:*

$$(2.2.3-1) \quad \begin{cases} \tau|_{N_1}: E_X|_{N_1} \ni e^1 \zeta^1 \longrightarrow \mathcal{D}_{N_1}^r \ni \zeta_r^1 \cdot g' + s_{r+1} \cdot \begin{bmatrix} \zeta''^1 \\ 0 \end{bmatrix}, \\ \tau|_{N_0}: E_X|_{N_0} \ni e^0 \zeta^0 \longrightarrow \mathcal{D}_{N_0}^r \ni s_{r+1} \cdot \zeta^0. \end{cases}$$

(2) *We have:*

(2.2.3-2) $\tau(e_j) = s_{r+1} u_j$ ($1 \leq j \leq r$) and $\tau(e_{r+1}) = g'$, where u_j = the j -th unit vector of \mathcal{D}_X^r (i.e., the i -th component of $u_j = \delta_{ij}$ ($1 \leq i \leq r$)) and e_1, \dots, e_{r+1} are as in (2.1.1).

(3) $E_X' (= \tau(E_X))$ is the kernel of $\omega: \mathcal{D}_X^r \rightarrow \mathcal{D}_{X_1}$, where ω is the quotient morphism: $\mathcal{D}_X^r \rightarrow \mathcal{D}_{X_1}$ and the \mathcal{D}_X -homomorphism μ is given by:

$$(2.2.4) \quad \mu: \mathcal{D}_X^r \ni \zeta \longrightarrow \mathcal{D}_X^r \ni \zeta_r g + s_r \cdot \begin{bmatrix} \zeta' \\ 0 \end{bmatrix}.$$

(In (2.2.3-1) and (2.2.4), ζ^i and ζ are the elements of $\mathcal{D}_{N_i}^r$ ($i=0, 1$) and \mathcal{D}_X^r . Moreover, ζ_r^1 and ζ_r are the r -th components of ζ^1 and ζ , and ζ''^1, ζ' are the subvectors of ζ^1, ζ consisting of the first $(r-1)$ -components.)

Proof. (1) and (3) are just a rewritten form of Propositions 1.1 and 1.2 in the present situation, while (2) follows easily from (1). q.e.d.

Now, we give a simple but quite useful expression of E_X' . For this let h'' be the submatrix of h' consisting of the first $(r-1)$ -rows of it, and we set $\tilde{h}'' = \omega(h'')$ with the quotient morphism $\omega: \mathcal{D}_X \rightarrow \mathcal{D}_1 := \mathcal{D}_{X_1}$.

$$(2.2.5-1) \quad \tilde{h}'' = [\tilde{s}'_r I_{r-1}, \tilde{g}], \text{ where } \tilde{s}'_r = \omega(s'_r) \text{ and } \tilde{g} = (\tilde{g}_j)_{j=1}^{r-1} \text{ with } \tilde{g}_j = \omega(g_j).$$

Let χ denote the \mathcal{D}_1 -homomorphism: $\mathcal{D}_1^r \ni \tilde{\zeta} \rightarrow \mathcal{D}_1^{r-1} \ni \tilde{h}'' \cdot \tilde{\zeta}$, and we define

an \mathfrak{D}_1 -module $\mathfrak{F}_{X^1} (\subset \mathfrak{D}_1^r)$ to be the kernel of χ . Thus an element $\tilde{\zeta} = (\tilde{\zeta}_j)_{j=1}^r \in \mathfrak{D}_1^r$ is in \mathfrak{F}_{X^1} , if and only if

$$(2.2.5-2) \quad \tilde{\zeta}_r \cdot \tilde{g} + \tilde{s}'_r \cdot \tilde{\zeta}' = 0, \quad \text{with } \tilde{\zeta}' = (\tilde{\zeta}_j)_{j=1}^{r-1}.$$

Lemma 2.1. *We have the following exact sequence:*

$$(2.2.6) \quad 0 \longrightarrow s_{r+1} \mathfrak{D}_{\bar{X}}^r \xrightarrow{i} E'_X \xrightarrow{\omega} \mathfrak{F}_{X^1} \longrightarrow 0,$$

where ω is the restriction of $\omega: \mathfrak{D}_{\bar{X}}^r \rightarrow \mathfrak{D}_1^r$ to $E'_X (\subset \mathfrak{D}_{\bar{X}}^r)$ and i is the injection of $s_{r+1} \mathfrak{D}_{\bar{X}}^r$ into $\mathfrak{D}_{\bar{X}}^r$.

Proof. First remark that (2.2.6) is reduced to $s_{r+1} \mathfrak{D}_{\bar{X}}^r \xrightarrow{i} E'_X$ in N_0 . Next take a point $p \in \bar{X}^1$ and an element $\zeta \in \mathfrak{D}_{\bar{X},p}^r$. Then we easily have:

$$(a) \quad \zeta \in E'_{\bar{X},p} \iff h' \cdot \zeta \in s_{r+1} \mathfrak{D}_{\bar{X}}^r \iff \tilde{h}'' \omega(\zeta) = 0$$

Thus we have: $\mathfrak{F}_{X^1,p} = \omega(E'_{\bar{X},p})$. Because \mathfrak{F}_{X^1} is the submodule of \mathfrak{D}_1^r , we obviously have: kernel of $\chi = s_{r+1} \mathfrak{D}_{\bar{X}}^r$. q.e.d.

By Lemma 2.1, investigations of E'_X are reduced to those of \mathfrak{F}_{X^1} . The sheaf \mathfrak{F}_{X^1} and the exact sequence (2.2.6) will play basic roles in later arguments (cf. § 3.2).

3.2. Next, from the sections $e = (e_1, \dots, e_{r+1}) \in \Gamma^{r+1}(E_X)$ (cf. Definition 2.1), we form some subvarieties of \bar{X} . First we set:

(2.2.7-1) $\bar{X}_j^1 :=$ the closure of $(\bigwedge^r e^j)_{0,\text{red}} (\subset X)$ in \bar{X} , where we set: $e^j = (e_1, \dots, \check{e}_{r+1-j}, \dots, e_{r+1})$ (e_{r+1-j} is omitted) ($0 \leq j \leq r$).

(Thus we have:

(2.2.7-2) $\bar{X}_0^1 = \bar{X}^1$ and $\bar{X}_1^1 \subset \bar{X}^{r1}$. Moreover, if D is of type (G), then $\bar{X}_1^1 = \bar{X}^{r1}$ (cf. (2.1.2, 3) and (2.1.3)).

Also we have:

Proposition 2.3. $\bar{X}_j^1 = (g'_{r+1-j})_{0,\text{red}}$ ($1 \leq j \leq r$) and $\bar{X}_0^1 = (s_{r+1})_0$.

Proof. First assume that $2 \leq j \leq r$. Then, by a simple computation, we have:

$$(2.2.8-1) \quad \begin{cases} \bigwedge^r e^j = g'_{r+1-j} (\bigwedge^r e^0) & \text{in } N_0 \\ \bigwedge^r e^j = f_{r+1-j} (\bigwedge^r e^1) & \text{in } N_1 \end{cases}$$

From the explicit form of f_{r+1-j} and g'_{r+1-j} (cf. (2.2.0, 1)) we get this proposition for $2 \leq j \leq r$, once we see that s_r and s'_{r+1} are units in N_1 and N_1 . On the other hand, from (2.2.0, 1), we have:

$$(2.2.8-2) \quad \bigwedge^r e^0 = f_r (\bigwedge^r e^1) \text{ in } N_1 \text{ and } \bigwedge^r e^1 = (g'_r / s_{r+1}) (\bigwedge^r e^0) \text{ in } N_0.$$

Thus remarking that s'_r and s_{r+1} are units in N_1 and N_0 , we see that

$$(2.2.8-3) \quad (\wedge^r e^0)_0 = (s_{r+1})_0 \quad \text{and} \quad (\wedge^r e^1)_0 = (g'_r)_0 \quad \text{in } X.$$

(For the case of $j=0$, we also use (3) in (2.2.0.) q.e.d.

Now, using the varieties \bar{X}^1_j ($0 \leqq j \leqq r$), we form the following subvarieties of \bar{X} :

(2.2.9-1) $\bar{Y}' = (\bigcap_{j=0}^r \bar{X}^1_j)$, and \bar{Y} = the union of the irreducible components of \bar{Y}' that are not contained in \bar{X}^1_{sing} .

(2.2.9-2) $\bar{Z} = \bar{Y} \cap \bar{X}^1_{\text{sing}}$.

Then we obviously have:

$$(2.2.9-3) \quad \bar{X}^2 (= \bar{X}^1 \cap \bar{X}'^1) \supset \bar{Y} \supset \bar{Z}.$$

Moreover, we define a closed subvariety \bar{Y}_j in the following manner:

(2.2.9-4) \bar{Y}_j = the union of the irreducible components of $(\bar{X}^1 \cap \bar{X}^1_j)$ that are not contained in $(\bar{Y} \cup \bar{X}^1_{\text{sing}})$ ($1 \leqq j \leqq r$).

Thus we have:

$$(2.2.9-5) \quad (\bar{X}^1 \cap \bar{X}^1_j) = \bar{Y} \cup \bar{Y}_j \cup (\bar{X}^1_{\text{sing}} \cap \bar{X}^1_j).$$

$$(\bar{X} \supset \bar{X}^1 \supset \bar{X}^2) \supset \bar{Y} \supset \bar{Z}$$

$$\cup \\ \bar{Y}_j \quad (1 \leqq j \leqq r)$$

Figure II.

These varieties admit a clear interpretation from a view point of Schubert calculus, when the s -pre bundle D is obtained from the universal quotient bundle over a Grassmann variety (§ 2.3 and Appendix I).

§ 2.2. A remark on finding an s -pre bundle of type (G)

1. First, let us start with a datum $U = (\bar{X}^1, \bar{X}'^1, f)$ as follows:

(2.3.1) \bar{X}^1 and \bar{X}'^1 are reduced divisors of \bar{X} such that $\bar{X}^2 = \bar{X}^1 \cap \bar{X}'^1$ is of codimension two in \bar{X} and $X^1 := \bar{X}^1 - \bar{X}^2$ as well as $X'^1 := \bar{X}'^1 - \bar{X}^2$ are Cartier and set theoretical Cartier divisors of X .

(2.3.2) $f = (f_1, \dots, f_r)$ consists of meromorphic functions f_j ($1 \leqq j \leqq r$) over X with the pole X'^1 , and $(f_r)_0$ is reduced and coincides with X^1 in $N_1 = \bar{X} - \bar{X}'^1$.

Then we set:

$$(*) \quad D_{U1} = (\bar{X}^2, E_X) \quad \text{and} \quad D_{U2} = (\bar{X}^1, N_1, e^0, e^1),$$

where the bundle E_X over $X := \bar{X} - \bar{X}^2$ is characterized by (1) $E_X|_{N_i}$ has a

frame e^i ($i=0, 1$), with $N_0 = \bar{X} - \bar{X}^1$, and (2) the frame relation between e^0 and e^1 is: $e^0 = e^1 h_{10}$ in $N_1 \cap N_0$, with the matrix $h_{10} = \begin{bmatrix} I_{r-1} & f_1 \\ 0 & f_r \end{bmatrix}$.

Proposition 2.4. $D_U := (D_{U_1}, D_{U_2})$ is an s -pre bundle and satisfies the following:

(2.3.3) The components e_j^0 ($1 \leq j \leq r$) of e^0 are sections of E_X and the first $(r-1)$ -components of e^0 and e^1 coincide, and

(2.3.4) the last element e_r^1 of e^1 is a meromorphic section of E_X over X with the pole X^1 .

Proof. This is straightforward from (2.3.1, 2). q.e.d.

Note that, in general, e_r^1 is not a (holomorphic) section, and D is, in general, not of type (W.G). But it is easy to see that

(2.3.5) e_r^1 is a (holomorphic) section of E_X , if and only if f satisfies (2.1.9), and we clearly have:

(2.3.6) D_U is of type (W.G), if and only if f satisfies (2.1.9). Now, let \mathcal{U} be the collection of all data $U = (\bar{X}^1, \bar{X}^1, f)$ as in (2.3.1, 2), which also satisfy (2.1.9). Then, by Proposition 2.4, D_U is an s -pre bundle of type (W.G). Moreover, we have:

Proposition 2.5. (1) The map:

(2.4.1) $\mathcal{U} \ni U = (\bar{X}^1, \bar{X}^1, f) \rightarrow \{s\text{-pre bundles of type (W.G)}\} \ni D_U$ is surjective, where D_U is defined in the manner (*).

(2) D_U is of type (G), if and only if

(2.4.2) $(\bigwedge^r e^1)_{0, \text{red}}$ coincides with X^1 .

Proof. (1) follows easily from Proposition 2.1, while (2) is a direct consequence of (2.1.3). q.e.d.

Thus, in order to find s -pre bundles of type (W.G) and (G), it suffices to find a more naive datum U .

2. Now, a simplest (but a most important) method in getting a datum U just above may be as follows: Start with a line bundle L_X over \bar{X} and sections $s = (s_1, \dots, s_{r+1}) \in \Gamma^{r+1}(\bar{X}, L_X)$ satisfying the following:

(2.5.1) $(s_{r+1})_0$ is reduced and $(s_{r+1})_0 \cap (s_r)_{0, \text{red}}$ is of codimension two in \bar{X} .

Then setting

(2.5.2) $\bar{X}^1 = (s_{r+1})_0$, $\bar{X}^1 = (s_r)_{0, \text{red}}$ and $f_j = s_j/s_r$ ($1 \leq j \leq r-1$), $f_r = s_{r+1}/s_r$,

one checks easily that the datum $U(s)=(\bar{X}^1, \bar{X}^n, f=(f_j)_{j=1}^r)$ satisfies (2.3.1, 2). Also it is easily seen that the s -pre bundle $D_s=D_{U(s)}$ is of type (G). Some discussions for such an s -pre bundle D , will be given later (cf. § 4).

Remark 2.6. In the construction of algebraic vector bundle in [Mar] (cf. Introduction), Maruyama starts with a smooth divisor D of a smooth quasi projective variety \bar{X} (of any characteristic) of dimension ≥ 2 . Then he takes sections $s_1, \dots, s_r \in \Gamma^r(L_D)$, L_D being a line bundle of D , such that $\bigcap_{j=1}^r (s_j)_{0, \text{red}} = \phi$ (cf. Principles 2.5 and 2.6, [Mar]. See also [Sum] where the smoothness condition for D and (s_1, \dots, s_r) is dropped to certain degree). Assume that \bar{X} is defined over \mathbb{C} . Then, letting $L_{\bar{X}}$ and $s=(s_1, \dots, s_{r+1}) \subset \Gamma(\bar{X}, L_{\bar{X}})$ be as in (2.5.1, 2), we have the divisor $D(=(s_{r+1})_{0, \text{red}})$ and the sections s_1, \dots, s_r of L_D , where $L_D := \mathcal{O}_D \otimes L_{\bar{X}}$ and s_1, \dots are the restrictions of s_1, \dots to D . Thus we have similar datum $(D, (s_1, \dots, s_r))$ to the one in the theory of Maruyama (though we do not assume the corresponding conditions for D and (s_1, \dots, s_r)). Conversely, starting with a datum $(D, (s_1, \dots, s_r))$ as in [Mar], take a line bundle M_D over D suitably so that $N_D = L_D \otimes M_D$ has an extension to \bar{X} . Then it looks like that one can get our datum $(L_X, (s_1, \dots, s_{r+1}))$ by regarding s_1, \dots, s_r as sections of N_D and extending them to X . (We like to discuss relations more precisely in another place.) From what are mentioned just above it seems to be better that our theory is giving an another treatment of the results of Maruyama, when the conditions on the smoothness of the divisor and on the disjoints of the loci of the sections are satisfied. How to treat bundles without the above conditions may be an open problem (from either view point of the elementary transformation ([Mar] and [Sum]) or of Čech-stratification method in this paper). In spite of the above similarities between the starting data for the constructions of bundles, we point out that our view point and techniques differ largely from the ones in [Mar].

§ 2.3. Relations to Grassmannian geometry

1. Let F be a vector space of dimension n over \mathbb{C} , and let V be the Grassmann variety of d -dimensional subspaces of F , where $1 < d < n - 1$. Then letting $F_{\bar{V}}$ denote the product bundle $\bar{V} \times F$, we have the exact sequence of the universal bundles:

$$(2.6.0) \quad 0 \rightarrow G_{\bar{V}} \rightarrow F_{\bar{V}} \xrightarrow{\omega} E_{\bar{V}} \rightarrow 0, \text{ where } G_{\bar{V}} \text{ and } E_{\bar{V}} \text{ are the sub and quotient universal bundles over } \bar{V}. \text{ (Recall that } G_{\bar{V}} \text{ is defined to be: } G_{\bar{V}, p} = \text{tautological subspace } G_p(\subset F) \text{ of } p \in \bar{V}.) \text{ Take a basis } e'_1, \dots, e'_n \text{ of } F, \text{ and we write } e'^1, \dots, \text{ also for the corresponding sections of } F_{\bar{V}}.$$

Setting $e_j = \omega(e'_j)$ ($1 \leq j \leq n$), we form the following subvarieties of \bar{V} :

(2.6.1) $\bar{V}^1 = (\bigwedge^r e^0)_0$ and $\bar{V}^1 = (\bigwedge^r e^1)_0$, and $\bar{V}^2 = \bar{V}^1 \cap \bar{V}^1$, where we set: $e^0 = (e_1, \dots, e_{r-1}, e_r)$ and $e^1 = (e_1, \dots, e_{r-1}, e_{r+1})$.

Lemma 2.2. *The s-pre bundle $D = (D_1, D_2)$, where $D_1 = (\bar{V}^2, E_{\bar{V}}|_{\bar{V}})$ and $D_2 = (\bar{V}^1, N_1, e^0, e^1)$, $V = \bar{V} - \bar{V}^2$ and $N_1 = \bar{V} - \bar{V}^1$, is of type G.*

The proof is given in Appendix I, where we summarize some explicit computations for $E_{\bar{V}}$ which are obtained from Schubert calculus or elementary direct computations.

Next, returning to our original variety \bar{X} , we will give a corresponding fact to Lemma 0.1 for the s-pre bundle of type G. For this we say that a bundle $F_{\bar{X}}$ over \bar{X} is of type G, if there is an s-pre bundle $D = (D_1, D_2)$ of type G such that $F_{\bar{X}}$ is the direct image of the pre bundle which appears in D .

Lemma 2.3. *Assume that \bar{X} is normal and a quasi projective variety and that $E_{\bar{X}}$ is an algebraic bundle over \bar{X} . Then letting L be the line bundle corresponding to the hyperplane cut, we have:*

$$(2.6.2) \quad E_{\bar{X}} \otimes L^m \quad (m \gg 0) \text{ is of type (G).}$$

Proof. It is well known that $E_{\bar{X}} \otimes L^m$ is the pull back of the universal quotient bundle of a Grassmann variety (cf., for example, [F]). Then from the generic position argument in [KL-2], we have this lemma. q.e.d.

When \bar{X} is a Stein variety and $F_{\bar{X}}$ is a bundle over \bar{X} , the following stronger form of Theorem A of H. Cartan holds ([Hir] and [Ka]).

(2.6.3) There are finitely many sections $s = (s_1, \dots, s_i) \subset \Gamma(F_{\bar{X}})$ which generate $F_{\bar{X},p}$ for each $p \in \bar{X}$.

Thus the bundle $F_{\bar{X}}$ is induced from the universal quotient bundle of a Grassmann variety. Lemma 2.3 and (2.6.4) are supporting facts for our introduction of the notion of type (G). In connection with (2.6.4) we make:

Question 2.1.1. Is any bundle over a Stein variety of type (G)?

Question 2.1.2. (1) Is any bundle over a Stein variety induced from the universal bundle over a Grassmann variety? An affirmative answer to Question 2.1.1 seems to follow from 2.1.2 and a corresponding fact to the generic position argument in [KI-2].

Remark 2.7. Lemma 2.2 is a starting point for our introduction of

the notion of type (G). As a matter of fact, our idea in introducing the notion of type (G) may be given as follows:

(**) *To form bundles over a complex variety, which have similar properties to the universal bundle over a Grassmann variety, by direct computations (assuming only the facts for line bundles).*

4. Here we check that the 'notion of type (G)' appears also in analysis of singularities of coherent sheaves. Namely, start with a coherent sheaf F_X over X and a subvariety Y of X of codimension ≥ 2 , which satisfies the following:

(2.7.0) $F_X := F_X|_X$, $X = \bar{X} - Y$, is locally free and F_X coincides with the direct image sheaf of F_X .

Then what we want to do is:

(*) *To attach a suitable s -pre bundle of type (G) to F_X and to use it for analysis of properties (like local freeness) of F_X .*

For this we first recall the following basic fact concerning the singularities of coherent sheaves:

Lemma 2.4 (Scheja [Sc] and Siu-Trautmann [S-T]). *Assume that \bar{X} is smooth. Then the singular set $S(F_X) := \{q \in \bar{X}; F_{X,q} \text{ is not } \mathcal{O}_{\bar{X},q}\text{-free}\}$ is of codimension ≥ 3 in \bar{X} .*

Next take integers $(1 \leq) r < t$. For each index $I = (i_1, \dots, i_r): 1 \leq i_1 < \dots < i_r \leq t$, take an element $f_I \in \Gamma(\mathcal{O}_{\bar{X}})$. Then for an $t \times r$ -matrix $A \in M_{tr}(\mathbb{C})$, we form an element $f_A \in \Gamma(\mathcal{O}_{\bar{X}})$ by

(2.7.1) $f_A = \sum_I \det A^I \cdot f_I$, where A^I is the submatrix of A consisting of $I = (i_1, \dots, i_r)$ -rows.

We denote by D_A the divisor of f_A . Also setting $f := (f_I)_I$, let B_f denote the base locus of f :

(2.7.2) $B_f = \bigcap_I (f_I)_{0, \text{red}}$.

Take a matrix $A_0 \in M_s(\mathbb{C})$ and a point $p \in \bar{X}$. Then choosing a suitable open neighborhood U of p (in \bar{X}) and \bar{V} of A_0 (in $M_{tr}(\mathbb{C})$), we have the following

Lemma 2.5. *Take a proper subvariety \bar{W} of \bar{V} . Then, for each $A \in \bar{V} - \bar{W}$, we have:*

(2.7.3) $(D_{A, \text{sing}} - (B_f \cap \bar{X}_{\text{sing}}))|_U$ is of codimension ≥ 4 in \bar{X} .

This may be an analogue of the theorem of Bertini (on the moving singularities of the divisors in a linear system) to our 'Grassmannian system' of divisors $D_A; A \in M_{tr}(\mathbb{C})$. The proof is given by reducing it to the original Bertini's theorem, by a certain induction argument. The

proof requires some pages, and is given in an another place (see also an algebraic analogue of Lemma 2.5 in [Sa-5]).

Now, take a point $p \in \bar{Y}$ and an open neighborhood U of p in \bar{X} , and we choose sections $e = (e_1, \dots, e_{r+1}) \subset \Gamma(U, E_{\bar{X}})$ suitably. Then, setting $e^0 = (e_1, \dots, e_{r-1}, e_r)$ and $e^1 = (e_1, \dots, e_{r-1}, e_{r+1})$, we form subvarieties of U as follows:

(2.8.1) \bar{X}^1 and \bar{X}^{r1} are the closures of $(\wedge^r e^0)$ and $(\wedge^r e^1)$ ($\subset U \cap X$) in U , and $\bar{X}^2 = \bar{X}^1 \cap \bar{X}^{r1}$, $N_1 = \bar{X} - \bar{X}^{r1}$.

Lemma 2.6. *Assume that \bar{X} is smooth. Then the s -pre bundle $D = (D_1, D_2)$, where*

$$(2.8.2) \quad D_1 = (\bar{X}^2, F_{\bar{X}|_X}), \quad X = \bar{X} - \bar{X}^2, \quad \text{and} \quad D_2 = (\bar{X}^1, N_1, e^0, e^1)$$

is of type (G). Moreover, we have:

$$(2.8.3) \quad \bar{X}^1 \text{ is irreducible and } \text{codim}_X \bar{X}^1_{\text{sing}} \geq 3.$$

Proof. Take sections $s_1, \dots, s_t \in \Gamma(U, F_{\bar{X}})$, which generates $F_{\bar{X}}$ over U . For a general matrix $A \in M_{tr}(\mathbf{C})$, we set: $e_A = s \cdot A$ and $D_A =$ the closure of $(\wedge^r e_A)_0$ in \bar{X} . Then Lemmas 2.4 and 2.5 imply: $\text{codim } \bar{X}^1_{\text{sing}} \geq 3$. This also implies that \bar{X}_1 is reduced and irreducible. Moreover, taking a suitable section $e_{r+1} \in \Gamma(U, F_{\bar{X}})$, we see easily that (e_A, e_{r+1}) satisfies (2.1.1 ~ 3), and we have this lemma. q.e.d.

For our s -pre bundles (in particular, those of type (G)), a quite basic problem is to discuss properties of the varieties \bar{X}^1, \bar{X}^2 (and $\bar{Y}, \bar{Y}_j, \dots$ as in (2.2.9)). Lemma 2.6 concerns the singularity of the divisor \bar{X}^1 , and we are led to make the following

Question 2.2. Letting sections $e = (e_1, \dots, e_{r+1})$ have the similar meaning to the one in Lemma 2.6, discuss the nature of the singularity of the divisor \bar{X}^1 . Also discuss the similar things for the varieties \bar{X}^2, \bar{Y} and \bar{Y}_j ($1 \leq j \leq r$) (cf. (2.2.9)). The above question seems to have relations to the theory of Le-Teisser-Navaro ([Le-Te] and [Nav]) on treatments of singularities of coherent sheaves (and underlying theories of Nash modifications).

§ 3. Local structures of the direct image $E_{\bar{X}}$

In Section 3 we assume:

(*) \bar{X} is smooth, and the s -pre bundle $D = (D_1, D_2)$ is of type (W.G).

We use freely the notations for D , which were introduced in the beginning of Section 1 and in Section 2.1.

§ 3.1. Plans for investigations

1. The arguments in Section 3 will be done, by using the $\mathcal{O}_{\bar{X}^1}$ -module $\mathfrak{F}_{\bar{X}^1}$ and the varieties \bar{Y} , \bar{Z} and \bar{Y}_j ($1 \leq j \leq r$) (cf. (2.2.5, 9) and Figure II, § 2.1). Our arguments will be basically divided into the following three steps:

(*-1) *Those for $(\bar{X} - \bar{Y})$, $(\bar{Y} - \bar{Z})$ and \bar{Z} .*

The first step is done by an entirely elementary argument. The main subject in the second step is:

(*-2) *To construct explicitly frames of $F_{\bar{X}^1}$ and $E_{\bar{X}}$, by using structures of $\mathfrak{F}_{\bar{X}^1}$ and the variety \bar{Y} (Theorems 3.1 and 3.2). Now the third step of the investigation of $\mathfrak{F}_{\bar{X}^1}$ and $E_{\bar{X}}$ over \bar{Z} concerns the singular locus of the divisor \bar{X}^1 (cf. (2.2.9)), and the arguments become substantially harder than the ones in the first two steps. Our main idea in the third step is then stated as follows:*

(*-3) *To make a full use of the explicit form of the frames in (*-2) for the investigations of $E_{\bar{X}}$ over \bar{Z} .*

The main result in the third step describes the germ $F_{\bar{X}^1, p}$ ($p \in \bar{Z}$) explicitly in terms of the ideals of \bar{Y}_j (cf. Theorem 3.3).

2. Here, using the assumption of the smoothness of \bar{X} , we sharpen the arguments in (2.2.0)~(2.2.9). First, recall that the explicit form of the transition matrix h_{10} for the frames e^0, e^1 is as follows (cf. (2.1.6)):

$$(3.0) \quad h_{10} = \begin{bmatrix} & f_1 \\ I_{r-1} & \vdots \\ & f_r \\ 0 & f_r \end{bmatrix}, \text{ with meromorphic functions } f_1, \dots, f_r \text{ over } \bar{X}$$

with the pole \bar{X}^1 ($:=$ the closure of $(\wedge^r e^1)_{0, \text{red}} (\subset X)$ in \bar{X}).

For the divisor \bar{X}^1 of \bar{X} , we assume:

(3.1.1) \bar{X}^1 has the finite irreducible components (and we write the irreducible decomposition of \bar{X}^1 as: $\bar{X}^1 = \bar{X}_1^1 \cup \dots \cup \bar{X}_u^1$.)

(3.1.2) There are elements $s_{r,1}, \dots, s_{r,u} \in \Gamma(\mathcal{O}_{\bar{X}})$, which generate the ideals of $\bar{X}_1^1, \dots, \bar{X}_u^1$.

Moreover, we assume the existence of data as follows:

(3.1.3-1) s_1, \dots, s_{r-1} and $s_{r+1} \in \Gamma(\mathcal{O}_{\bar{X}})$ such that none of them vanishes identically on \bar{X}_t^1 ($1 \leq t \leq u$), and

(3.1.3-2) $m(j) = (m(j, 1), \dots, m(j, u)) \in \mathbb{Z}^u$ ($1 \leq j \leq r$), with which f_1, \dots, f_r are expressed in the following form:

(3.1.4) $f_j = s_j / s_r^{m(j)}$ ($1 \leq j \leq r-1$) and $f_r = s_{r+1} / s_r^{m(r)}$, where we set $s_r^{m(j)} = s_{r,1}^{m(j,1)} \dots s_{r,u}^{m(j,u)}$ ($1 \leq j \leq r$).

Remark 3.1.1. Because \bar{X} is smooth, (3.1.1)~(3.1.4) are valid in the

local situation (cf. Remark 2.5) and are legitimate for the investigations of the local structure of E_X .

Remark 3.1.2. For $t = 1, \dots, u$, we set:

$$(3.1.5) \quad m_t = \max(1, \max_{j=1}^r m(j, t)), \text{ and set:}$$

$$(3.1.6) \quad s'_r = \prod_{t=1}^u s_{r,t}^{m_t}, \quad s'_j = \prod_{t=1}^u s_{r,t}^{m_t - m(j,t)} \quad (1 \leq j \leq r-1) \quad \text{and} \\ s'_{r+1} = \left(\prod_{t=1}^u s_{r,t}^{m_t - m(r,t)} \right) \cdot s_{r+1}.$$

Then the elements s'_1, \dots, s'_{r+1} satisfy (2.2.0). The arguments below follows easily from (2.2.1 ~ 9).

3. First we set:

$$(3.2.1) \quad n(j) := m(r) - m(j) \in \mathbb{Z}^u \quad (1 \leq j \leq r-1),$$

and define vectors as follows:

$$(3.2.2) \quad g = (g_j)_{j=1}^r, \text{ with } g_j = -s_j s_r^{n(j)} \quad (1 \leq j \leq r-1) \text{ and } g_r = s_r^{m(r)}.$$

Then the matrices $h_{01} := h_{10}^{-1}$, $h'_{01} := s_{r+1} h_{01}$ and $h'_{10} = h'_{01}{}^{-1}$ are explicitly as follows:

$$(3.2.3) \quad h'_{01} = \begin{bmatrix} s_{r+1} I_{r-1} & g \\ 0 & \end{bmatrix}, \text{ and } h'_{10} = (s_{r+1} s_r^{m(r)})^{-1} h', \text{ with} \\ h' = \begin{bmatrix} g_r I_{r-1} & -g_j \\ 0 & s_{r+1} \end{bmatrix} \quad (1 \leq j \leq r-1).$$

Proposition 3.1. We have the following.

(3.3.1) $(\bigwedge^r e^0)_0 = (s_{r+1})_0$ in X (and so $(s_{r+1})_0$ is reduced and coincides with \bar{X}^1 (in \bar{X})).

(3.3.2) $(\bigwedge^r e^1)_0 = (g_r)_0 (= (s_r^{m(r)})_0)$ (in X), and so $m(r) \in \mathbb{Z}_{+0}^u$.

(3.3.3) The elements $n(j)$ are in \mathbb{Z}_{+0}^u ($1 \leq j \leq r-1$).

Proof. The first two facts follow directly from (2.2.8), and the last follows from the explicit form of h_{10} (cf. (3.2.3)) and (2.1.9). q.e.d.

The varieties $\bar{X}^1_j, \bar{Y}' \dots$ ars as follows:

$$(3.3.4) \quad \bar{Y}' = \bigcap_{j=1}^r (g_j)_{0,\text{red}} \text{ on } \bar{X}^1, \text{ and } \bar{X}^1_j = (g_{r+1-j})_{0,\text{red}} \quad (1 \leq j \leq r).$$

For completeness we rewrite Figure II, Section 2.1:

$$(3.3.4-2) \quad \bar{X}^1_j = (g_{r+1-j})_{0,\text{red}} \quad (2 \leq j \leq r), \quad \bar{X}^1_0 = \bar{X}^1_1 = \bar{X}^1$$

$$(3.3.5) \quad \bar{Y}' = \bigcap_{j=1}^r (g_j)_{0,\text{red}} \text{ on } \bar{X}^1.$$

For completeness we rewrite Figure II, Section 2.1.

$$(3.3.6) \quad (\bar{X} \supset \bar{X}^1 \supset \bar{X}^2) \supset \bar{Y} \supset \bar{Z} \quad (:= \bar{Y} \cap \bar{X}_{\text{sing}}^1) \\ \cup \\ \bar{Y}_j \quad (1 \leq j \leq r)$$

4. Thirdly, letting the injection $\tau: E_X \hookrightarrow E'_X (= \tau(E_X) \subset \mathcal{O}_X^r)$ be as in (2.2.3), we have:

(3.4.1) $\tau(e_j) = s_{r+1} u_j$ ($1 \leq j \leq r$) and $\tau(e_{r+1}) = g$, where u_j is the j -th unit vector of \mathcal{O}_X^r (cf. (2.2.3)).

Let h'' be the submatrix of h' consisting of its first $(r-1)$ -rows, and we set $\tilde{h}'' = \omega(h'')$ with the quotient morphism $\omega: \mathcal{O}_X \rightarrow \mathcal{O}_{X^1}$.

$$(3.4.2) \quad \tilde{h}'' = [\tilde{g}_r \cdot I_{r-1}, (-\tilde{g}_j)] \quad (1 \leq j \leq r-1), \text{ with } \tilde{g}_j = \omega(g_j).$$

Then the basic \mathcal{O}_{X^1} -module \mathfrak{F}_{X^1} is defined as follows:

(3.4.3) An element $\tilde{\zeta} = (\tilde{\zeta}_j)_{j=1}^r \in \mathcal{O}_{X^1}^r$ is in $\mathfrak{F}_{X^1, p}$, if and only if it satisfies: $\tilde{g}_r \cdot \tilde{\zeta}_j = \tilde{g}_j \cdot \tilde{\zeta}_r$ ($1 \leq j \leq r-1$).

Moreover, the exact sequence (2.2.5) takes the following form:

$$(3.4.4) \quad 0 \longrightarrow s_{r+1} \mathcal{O}_X^r \longrightarrow E'_X \longrightarrow \mathfrak{F}_{X^1} \longrightarrow 0.$$

§ 3.2. Frame constructions

In Section 3.2 we write \mathcal{O}_{X^1} and \mathfrak{F}_{X^1} as \mathcal{O}_1 and \mathfrak{F}_1 .

5. First we check that E'_X and \mathfrak{F}_1 have very simple properties outside \bar{Y}' (cf. (3.3.5)). For this setting $N_j = \bar{X} - \bar{X}_j^1$ ($0 \leq j \leq r$), we obviously have:

$$(3.5.1) \quad \bar{X} - \bar{Y}' = \bigcup_{j=0}^r N_j.$$

Lemma 3.1.1. E_X is locally free over $(\bar{X} - \bar{Y}')$, and we have:

$$(3.5.2) \quad e^j \text{ is a frame of } E_X|_{N_j} \quad (0 \leq j \leq r) \text{ (cf. (3.3.4)).}$$

Proof. Recalling the definition of \bar{X}_j^1 (cf. (3.3.4)), it is clear that e^j is a frame of E_X over $(N_j - \bar{X}^0)$. This implies also that, for a point $p \in (N_j \cap \bar{X}^0)$, $E_{X,p}$ is $\mathcal{O}_{X,p}$ -free and e^j is a base of it. Thus we have this lemma. q.e.d.

Next, take a point $p \in \bar{X}^1 - \bar{Y}'$. Then for $\tilde{g} = (\tilde{g}_j)_{j=1}^r (= \omega\tau(e_{r+1}))$ (cf. also (3.4.2)), we have:

Lemma 3.1.2. Assume that each \tilde{g}_j ($1 \leq j \leq r$) is not a zero divisor in $\mathcal{O}_{1,p}$. Then the $\mathcal{O}_{1,p}$ -module $\mathfrak{F}_{1,p}$ is spanned by \tilde{g} .

Proof. Take an element $\tilde{\zeta} \in \mathfrak{F}_{1,p} (\subset \mathcal{O}_{1,p}^r)$. Then, from the explicit form of F_1 (cf. (3.4.3)), we see easily the following:

(3.5.3.1) If one of the j -th component $\tilde{\zeta}_j$ of $\tilde{\zeta} = 0$ in $\mathcal{O}_{1,p}$ ($1 \leq j \leq r$), then $\tilde{\zeta} = 0$ in $\mathfrak{F}_{1,p}$.

From this we see easily the following:

(3.5.3.2) If there is an element $\tilde{\zeta} \in \mathfrak{F}_{1,p}$ such that one of its component, say $\tilde{\zeta}_j$, does not vanish at p , then the $\mathfrak{D}_{1,p}$ -module $\mathfrak{F}_{1,p}$ is generated by $\tilde{\zeta}$.

On the other hand, from (3.3.5), we see that one of \tilde{g}_j ($1 \leq j \leq r$) does not vanish at p , and we have this lemma. q.e.d.

Remark 3.2. We take e^j to be a standard frame of $E_X|_{N_j}$ ($0 \leq j \leq r$). Also, if $\mathfrak{D}_{1,p}$ does not contain a zero divisor for each $p \in \bar{X}^1 - \bar{Y}'$, we take \tilde{g} to be a frame of the invertible sheaf \mathfrak{F}_1 over $\bar{X}^1 - \bar{Y}'$. Assume that

(3.5.3.3) \tilde{g}_j does not vanish identically on \bar{X}^1 ($1 \leq j \leq r$).

Then, if \bar{X}^1 is normal, we have:

(3.6.0)' \tilde{g}_j ($1 \leq j \leq r$) is not a zero divisor in $\mathfrak{D}_{1,p}$

and \tilde{g} is a frame of \mathfrak{F}_1 over $(\bar{X}^1 - \bar{Y}')$. In the remainder of Section 3, we assume:

(3.6.0) the generic condition (3.5.3.3) for \tilde{g}_j ($1 \leq j \leq r$) holds.

6. Here we examine the local structure of \mathfrak{F}_1 and E'_X over $\bar{Y} - \bar{Z}$. For this, in the remainder of n.6, we fix a point $p \in \bar{Y} - \bar{Z} (\subset \bar{X}^1_{\text{reg}})$ and we write the irreducible decomposition of \tilde{g}_j ($1 \leq j \leq r$) as follows:

(3.6.1-0) $\tilde{g}_r = \tilde{i}_1^{a(1)} \cdots \tilde{i}_v^{a(v)}$ and $\tilde{g}_j = \tilde{i}_1^{b(j,1)} \cdots \tilde{i}_v^{b(j,v)} \tilde{g}''_j$ ($1 \leq j \leq r-1$), where $\tilde{i}_1, \dots, \tilde{i}_v$ vanishes at p and irreducible in $\mathfrak{D}_{1,p}$. Moreover, the elements $a(1), \dots, a(k)$ and $b(j, 1), \dots, b(j, v)$ are elements of \mathbf{Z}_+ and \mathbf{Z}_{+0} respectively, and $\tilde{g}''_j \in \mathfrak{D}_{1,p}$ is not divided by \tilde{i}_i ($1 \leq i \leq v$).

Next, for purpose of explicit computations here, define a subset I_j of $\{1, \dots, v\}$ ($1 \leq j \leq r-1$) by:

(3.6.1-1) $I_j = \{i \in \{1, \dots, v\} : b(j, i) < a(j)\}$,

and we set:

(3.6.1-2) $I = \bigcup_{j=1}^{r-1} I_j (\subset \{1, \dots, v\})$ (or, alternatively, $I = \{i \in \{1, \dots, v\} : b(j, i) < a(j)$ for an element $j \in \{1, \dots, r\}\}$), and

(3.6.1.3) $b_i = \min_{j=1}^{r-1} b(j, i)$ (for $i \in I$).

Then the following lemma determines explicitly the $\mathfrak{D}_{1,p}$ -module $\mathfrak{F}_{1,p}$.

Lemma 3.2.1. Define an element $\tilde{\eta} = (\tilde{\eta}_j)_{j=1}^r \in \mathfrak{D}_{1,p}^r$ by

$$(3.6.2) \quad \tilde{\eta}_r = \prod_{i \in I} \tilde{i}_i^{a(i) - b(i)} \quad \text{and} \quad \tilde{\eta}_j = \left(\prod_{i \in I} \tilde{i}_i^{b(j,i) - b(i)} \right) \cdot \left(\prod_{i \notin I} \tilde{i}_i^{b(j,i) - a(i)} \right) \cdot \tilde{g}''_j \quad (1 \leq j \leq r-1).$$

(When $I = \emptyset$ we understand that $\tilde{\eta}_r$ and the first factor of $\tilde{\eta}_j = 1$ ($1 \leq j \leq r-1$).

The element $\tilde{\eta}$ is in $\mathfrak{F}_{1,p}$, and generates the $\mathfrak{D}_{1,p}$ -module $\mathfrak{F}_{1,p}$.

Proof. Recall that an element $\tilde{\zeta} = (\tilde{\zeta}_j)_{j=1}^r \in \mathfrak{D}_{1,p}^r$ is in $\mathfrak{F}_{1,p}$, if and only if

$$(a-1) \quad \tilde{g}_r \tilde{\zeta}_j - \tilde{g}_j \tilde{\zeta}_r = 0 \quad (1 \leq j \leq r-1) \quad (\text{cf. (3.4.3)}).$$

But, by a simple computation, we have:

$$(a-2) \quad \tilde{g}_j \cdot \tilde{\eta}_r = \tilde{g}_r \cdot \tilde{\eta}_j = (\prod_{i \in I} \tilde{i}_i^{a(i)-b(i)+b(j,t)}) \cdot (\prod_{i \in I} \tilde{i}_i^{b(j,t)}) \cdot \tilde{g}_j'',$$

and we have: $\tilde{\eta} \in \mathfrak{F}_{1,p}$. On the other hand, if $\tilde{\zeta} = (\tilde{\zeta}_j)_{j=1}^r$ is in $\mathfrak{F}_{1,p}$ then, from (a-1) and the definition of I (cf. (3.6.1-2)), we have easily: $\tilde{\zeta}_r \equiv 0 \pmod{\tilde{i}^{a(i)-b(i)}} (i \in I)$ and $\tilde{\zeta}_r = \tilde{\alpha} \cdot \tilde{\eta}_r$ with an element $\tilde{\alpha} \in \mathfrak{D}_{1,p}$. By (3.5.3.1) and (3.6.0), we have $\tilde{\zeta} = \tilde{\alpha} \cdot \tilde{\eta}$, and we are done. q.e.d.

Take an element $\eta \in E'_{X,p}$ such that $\omega(\eta) = \tilde{\eta}$. Then, from Lemma 3.2.1 and (3.3.5), (3.4.1), we have:

Lemma 3.2.2. *The $\mathfrak{D}_{X,p}$ -module $E'_{X,p}$ is generated by $s_{r+1} \cdot u_j = (\tau(e^j)) (1 \leq j \leq r)$ and η , where u_j is the j -th unit vector (cf. (2.2.3-2)).*

From this lemma and (3.3.4), (3.6.2) we have:

$$(3.7) \quad \tilde{g} (= \omega\tau(e_{r+1})) = (\prod_{i \in I} \tilde{i}_i^{b(i)}) (\prod_{i \in I} \tilde{i}_i^{a(i)}) \cdot \tilde{\eta}.$$

Now, we give a condition for the $\mathfrak{D}_{X,p}$ -freeness of $E_{X,p}$ in the following form:

Theorem 3.1.1. *$E_{X,p}$ is $\mathfrak{D}_{X,p}$ -free if and only if there is an element $j \in \{1, \dots, r\}$ such that*

$$(3.8) \quad \tilde{g}_k \equiv 0 \pmod{\tilde{g}_j} \quad \text{in } \mathfrak{D}_{1,p} \text{ for all } k = 1, \dots, r.$$

Next define a vector $\tilde{\eta}(j) (1 \leq j \leq r)$ by

$$(3.9.1) \quad \tilde{\eta}(j) = (1/\tilde{g}_j)\tilde{g}.$$

Theorem 3.1.2. *Assume that (3.8) holds for an element $j \in \{1, \dots, r\}$.*

(1) *We have:*

$$(3.9.2) \quad \tilde{g}'' \text{ does not vanish at } p, \text{ and } \tilde{\eta}(j) = (1/\tilde{g}''_j)\tilde{\eta}, \text{ if } j \neq r, \tilde{\eta}(r) = \tilde{\eta}, \text{ if } j = r.$$

(2) *We can take $\tilde{\eta}(j)$ to be an $\mathfrak{D}_{1,p}$ -basis of $\mathfrak{F}_{1,p}$.*

Remark 3.3. The base $\tilde{\eta}(j)$ as above is given in a global form in comparison to the one $\tilde{\eta}$ in Lemma 3.1.2.

Proof of Theorems 3.1.1. and 3.1.2. (i) It is obvious that (2) in Theorem 3.1.2 follows from Lemma 3.1.2, Theorem 3.1.1 and (1), Theorem 3.1.2.

(ii) We prove Theorem 3.1.1 and (1), Theorem 3.1.2 as follows:

(ii-1) First, letting $\tilde{\eta} \in E'_{X,p}$ be as in Lemma 3.2.1, we define the germ $\bar{X}_{j,\tilde{\eta},p}^1$ of a divisor at p as follows:

(a-1) $\bar{X}_{j,\tilde{\eta},p}^1$ = the closure of $(e_{1 \wedge \dots \wedge \check{e}_j \wedge \dots \wedge e_r \wedge \tau^{-1}(\tilde{\eta}))_0 (\subset X_p)$ in \bar{X}_p , where X_p, \dots are the germs of X, \dots at p .

Then, from Lemma 3.2.2, we easily have the following equivalence:

$$(a-2) \quad E_{X,p} \text{ is } \mathfrak{D}_{X,p}\text{-free} \Leftrightarrow \bigcap_{j=1}^r \bar{X}_{j,\eta,p}^1 = \phi.$$

But, from (1.5), we have:

$$(a-3) \quad \bar{X}_{j,\eta,p}^1 = (\det(\tau(e_1), \dots, \check{\tau}(e_j), \dots, \tau(e_r), \eta)/s_{r+1}^{r-1})_{0,\text{red}}.$$

On the other hand, by (3.5.2), we see that the right hand side of (a-3) = η_j (= the j -th component of η), and we rewrite (a-2) as follows:

$$(a-4) \quad E_{X,p} \text{ is } \mathfrak{D}_{X,p}\text{-free} \Leftrightarrow \eta_j \text{ does not vanish at } p \text{ for an element } j \in \{1, \dots, r\}.$$

(ii-2) Next we analyze the right hand side of (a-4) as follows: First assume that $I = \phi$. Then, from a simple observation, we have:

$$(b-1) \quad \eta_r (= 1) \text{ does not vanish at } p, \text{ and } \tilde{g}_j/\tilde{g}_r (= \tilde{\eta}_j) \in \mathfrak{D}_{X,p} \quad (1 \leq j \leq r).$$

Thus we have the following for $j = 1, \dots, r-1$ (cf. also (3.2.3)):

$$(b-2) \quad \eta_j \text{ does not vanish at } p \Leftrightarrow \tilde{g}_r \equiv 0 \pmod{\tilde{g}_j} \Leftrightarrow \tilde{g}_k \equiv 0 \pmod{\tilde{g}_j} \quad (1 \leq k \leq r).$$

Next assume that $I \neq \phi$. Then, by (3.6.2), we obviously have:

$$(b-3) \quad \tilde{\eta}_r \text{ vanishes at } p.$$

On the other hand, for an element $j \in \{1, \dots, r-1\}$, we also have the following from (3.8):

$$(b-4) \quad \eta_j \text{ does not vanish at } p \Leftrightarrow (1) \tilde{g}_j'' \text{ does not vanish at } p \text{ and } (2) b(j, i) = b(i); i \in I \text{ and } b(j, i) = a(i); i \notin I.$$

On the other hand, we obviously have the following from (3.5.1):

$$(b-5-1) \quad \tilde{g}_r \equiv 0 \pmod{\tilde{g}_j} \Leftrightarrow (1) a_i \geq b(j, i) \text{ for any } i \in \{1, \dots, r\} \text{ and } (2) \tilde{g}_j'' \text{ does not vanish at } p,$$

$$(b-5-2) \quad \tilde{g}_k \equiv 0 \pmod{\tilde{g}_j} \quad (1 \leq k \neq j \leq r-1) \Leftrightarrow b(k, i) \geq b(j, i) \text{ for any } i \in \{1, \dots, r\} \text{ and } (2) \tilde{g}_k'' \equiv 0 \pmod{\tilde{g}_j'}.$$

Combining (b-4) with (b-5-1, 2), we clearly have (cf. also (3.5.4)):

$$(b-6) \quad \text{right hand side of (b-4)} \Leftrightarrow \tilde{g}_k \equiv 0 \pmod{\tilde{g}_j} \quad (1 \leq k \leq r).$$

We summarize (b-1)–(b-6) in the following manner:

$$(b-7-1) \quad \tilde{\eta}_r \text{ does not vanish at } p \Leftrightarrow \tilde{g}_k \equiv 0 \pmod{\tilde{g}_j} \quad (1 \leq k \leq r) \Leftrightarrow I = \phi.$$

$$(b-7-2) \quad \tilde{\eta}_j \quad (1 \leq j \leq r-1) \text{ does not vanish at } p \Leftrightarrow \tilde{g}_k \equiv 0 \pmod{\tilde{g}_j} \quad (1 \leq k \leq r). \quad (\text{In (b-7-2), the both cases: } I = \phi \text{ and } I \neq \phi \text{ can occur.})$$

Clearly, we have Theorem 3.1.1 from (a-4) and (b-7). Moreover, we see easily that if (b-7-1) holds, then we have:

$$(c-1) \quad \tilde{\eta}_r (= \tilde{g}_r/\tilde{g}_r) = 1, \tilde{\eta}_j = (\prod_{i \in I} \tilde{t}_i^{b(j,i) - a(i)}) \cdot \tilde{g}_j'' (= \tilde{g}_j/\tilde{g}_r) \quad (1 \leq j \leq r-1), \text{ and, if (b-7-2) holds, then we have the following from (b-4) and (3.5.3):}$$

$$(c-2) \quad \tilde{\eta}_k = (\tilde{g}_k/\tilde{g}_j) \cdot \tilde{g}_k'' \quad (1 \leq k \leq r).$$

From (c-1, 2) we have (1) in Theorem 3.1.2.

q.e.d.

From Theorem 3.1.1 we easily have:

Corollary 3.1. *If $E_{X,p}$ is $\mathfrak{D}_{X,p}$ -free, then the germ \bar{Y}_p of \bar{Y} at p is that of a divisor.*

This will show that the local freeness of E_X gives a very strong condition for the s -pre bundle D .

7. Here, we summarize some consequences of the arguments hitherto in Section 3.2: First define the following subvarieties, which are supplementary to the ones in (2.2.9) and (3.3.6):

(3.10.1) $\bar{Z}' = \bar{Y}' \cap \bar{X}_{\text{sing}}^1$ and $\bar{Y}'' =$ the union of the irreducible components of \bar{Y}' that are contained in \bar{X}_{sing}^1 (cf. (2.2.9-1, 2)).

Thus we have:

(3.10.2) $\bar{Z}' = \bar{Z} \cup \bar{Y}''$ (cf. (2.2.9-2)).

Theorem 3.1.3. *The direct image E_X is locally free over $\bar{X} - \bar{Z}'$, if and only if one of the following holds:*

(3.10.3) $\bar{Y} = \phi$, or $\bar{Y} (\neq \phi)$ is a divisor of \bar{X}^1 and (3.8) holds for each $p \in \bar{Y} - \bar{Z}$.

Proof. This is clear from Lemma 3.1.2 and Theorem 3.1.1. q.e.d.

Remark 3.3. Theorem 3.1.3 and Lemma 3.1 determine completely the local structure of E_X over $\bar{X} - \bar{Z}' (\supset \bar{X}_{\text{reg}}^1)$. A more global version of Theorem 3.1.3 will be given in Theorem 3.4.1.

Corresponding to our basic diagram (3.3.6), we make:

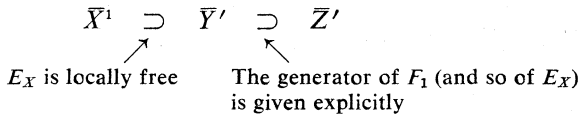


Figure III.

8. Now we investigate the structure of E_X for a point $p \in \bar{Z}'$. This differs according to whether

(3.11.0) E_X is locally free over $\bar{Y} - \bar{Z} (\supset \bar{X}_{\text{reg}}^1)$

or not. The latter case is more complicated than the former. In the remainder of Section 3.2, we assume that (3.11.0) holds. This assumption is legitimate for the discussions of the local freeness of E_X . For the investigation of $E_{X,p}$ ($p \in \bar{Z}$), we first do the following:

(*) To write $Y := \bar{Y} - \bar{Z}$ as a union of locally closed varieties of \bar{Y} and to attach a frame of \mathfrak{F}_1 to each locally closed variety.

Remark 3.4. This is a little weaker form than the one required in (*-1, 2), Section 0 in the point that \bar{Y} is not stratified. But it may give hints for stratifying \bar{Y} .

Now, remark that (3.11.0) and Theorem 3.1.3 imply:

(3.11.1) $\bar{Y} = \phi$, or \bar{Y} is a divisor of \bar{X}^1 and (3.8) holds.

In the second case, without losing generality for investigations of $E_{X,p}(p \in \bar{Z})$, we assume the following:

(3.11.2) \bar{Y} has the finite irreducible components (and we write the irreducible decomposition of \bar{Y} as: $\bar{Y} = \bigcup_{\beta=1}^s \bar{Y}_\beta$).

Setting $Y_\beta = \bar{Y}_\beta - \bar{Z}$, we define a multiplicity $m(j, \beta)$ of \tilde{g}_j along Y_β as follows:

(3.11.3) $\tilde{g}_j \equiv 0 \pmod{\mathfrak{S}_{\beta,p}^{m(j,\beta)}}$ but $\not\equiv 0 \pmod{\mathfrak{S}_{\beta,p}^{m(j,\beta)+1}}$ for each $p \in Y_\beta$, where $\mathfrak{S}_{\beta,p}$ is the stalk at p of the ideal $\mathfrak{S}_\beta (\subset \mathcal{O}_1)$ of \bar{Y}_β .

(The divisor $Y_\beta (\subset X_{\text{reg}}^1)$ is connected, and one checks easily that $m(j, \beta)$ is independent of the point $p \in Y_\beta$.)

For a subset $\beta = (\beta(1), \dots, \beta(t))$ of $(1, \dots, s)$, we set:

$$(3.11.4) \quad \bar{Y}_\beta = \bigcap_{u=1}^t \bar{Y}_{\beta(u)} \text{ and } Y_\beta = \bar{Y}_\beta - \bar{X}_{\text{sing}}^1.$$

Proposition 3.4.1. *We have the disjoint union:*

$$(3.12.1) \quad Y = \bigsqcup_{\beta} Y_\beta, \text{ where } Y_\beta \text{ satisfies: } Y_\beta - \left(\bigcup_{\tau \neq \beta} Y_\tau \right) \neq \emptyset.$$

Proof. Take a point $p \in Y$, and we define a subset $\beta(p)$ of $\{1, \dots, s\}$ by: $\beta(p) = \{\beta; Y_\beta \ni p\}$. Then it is clear that $\beta(p)$ satisfies the condition in (3.12.1) and that Y is the finite disjoint union of such $Y_{\beta(p)}; p \in Y$.

q.e.d.

Next, for an index $\beta = (\beta(1), \dots, \beta(t))$ satisfying the condition in (3.12.1), we set:

$$(3.12.2-1) \quad m(j, \beta) = (m(j, \beta(1)), \dots, m(j, \beta(t))) \quad (1 \leq j \leq r) \text{ and } m(\beta) = (m(\beta(1)), \dots, m(\beta(t))) \text{ with } m(\beta(1)) = \min_{j=1}^r m(j, \beta(1)), \dots, \text{ and}$$

$$(3.12.2-2) \quad Y_{\beta,j} = \bar{Y}_\beta - \bar{Y}_j. \text{ (For the variety } \bar{Y}_j, \text{ see (3.3.5).)}$$

Proposition 3.4.2. (1) $Y_\beta = \bigcup_j Y_{\beta,j}$, where $j \in \{1, \dots, r\}$ must satisfy:

$$(3.12.3) \quad Y_{\beta,j} \neq \emptyset \text{ and } m(j, \beta) = m(\beta).$$

(2) The element $\tilde{\eta}(j)$ (cf. (3.9.1)) is an $\mathcal{O}_{1,p}$ -basis of $\mathfrak{F}_{1,p}$ for each $p \in Y_{\beta,j}$.

Proof. First take a point $p \in Y$. Then from the assumption (3.11.0) and Theorem 3.1.1, we see that there is an element j such that $Y_j \ni p$ and $m(j, \beta) = m(\beta)$. From this we have (1). Also we get (2) easily from (2), Theorem 3.1.2.

q.e.d.

Now we summarize Propositions 3.4.1 and 3.4.2 as follows:

Theorem 3.2. *Writing Y as the union of their locally closed varieties in the form:*

(3.13.1) $Y = \bigcup_{\beta, j} Y_{\beta, j}$, where the indices (β, j) must satisfy the conditions in (3.12.1) and (3.12.4), we see that

(3.13.2) $\tilde{\eta}(j)$ is a frame of \mathfrak{F}_1 over $Y_{\beta, j}$.

Here we add a remark to Theorem 3.2, which says that $\tilde{\eta}(j)$ is a frame of \mathfrak{F}_1 in a Zariski open neighborhood of $Y_{\beta, j}$ in \bar{X}_{reg}^1 .

Remark 3.5. Take a subset $\beta = (\beta(1), \dots, \beta(t))$ of $(1, \dots, s)$ such that $Y_\beta \neq \emptyset$ and we assume:

(3.14.1) $m(j, \beta) > m(\beta)$. (Namely, for an element $u \in \{1, \dots, t\}$, we have: $m(j, \beta(u)) > m(\beta(u))$.)

Then, for a subset γ of $\{1, \dots, s\}$ satisfying $\gamma \supset \beta$, we obviously have:

$$(3.14.2) \quad m(j, \gamma) > m(\gamma).$$

Now, for an element $j \in \{1, \dots, r\}$, we define a closed subvariety \bar{W}_j of \bar{X}^1 as follows:

$$(3.14.3) \quad \bar{W}_j = \bar{Y}_j \cup (\bigcup_{\gamma} \bar{Y}_\gamma), \text{ where } Y_\gamma (\neq \emptyset) \text{ satisfies (3.14.1).}$$

From the explicit form of $\tilde{\eta}(j)$ ($= (1/\tilde{g}'_j)\tilde{\eta}$ or $= \tilde{\eta}$, according as $j \neq$ or $= r$) (cf. Theorem 3.1.2 and (3.14.1, 2)), we easily see:

$$(3.14.4) \quad \tilde{\eta}(j) \text{ generates } \mathfrak{F}_{1, p} \text{ for each } p \in \bar{X}_{\text{reg}}^1 - \bar{W}_j.$$

Moreover, from (3.13.2), we have the following for a locally closed subvariety $\bar{Y}_{\beta, j}$ in Theorem 3.2:

$$(3.14.5) \quad \text{The frame } \tilde{\eta}(j) \text{ of } \mathfrak{F}_1 \text{ over } Y_{\beta, j} \text{ is actually defined on } (\bar{X}_{\text{reg}}^1 - \bar{W}_j) (\supset Y_{\beta, j}).$$

Example. The simplest form of the expression (3.14.5) is obtained in the case where the following holds:

(3.15.1) \bar{Y} is irreducible and the multiplicity of \tilde{g}_j along $Y (= \bar{Y} - \bar{Z})$ is independent of $j = 1, \dots, r$.

Actually, we see easily that (3.14.3) then takes the following form:

$$(3.15.2) \quad Y = \bigcup_{j=1}^r (\bar{Y} - \bar{Y}_j),$$

and (3.14.4) is read as follows:

$$(3.15.3) \quad \tilde{\eta}(j) \text{ generates } \mathfrak{F}_{1, p} \text{ for each } p \in (\bar{X}_{\text{reg}}^1 - \bar{Y}_j).$$

Thus, in this case, the frame construction in Theorem 3.2 takes a simple form. Also note that, if the s -pre bundle D is obtained from the universal bundle over a Grassmann variety (Lemma 2.2), we have (cf. Appendix I):

(3.15.4) The condition (3.15.1) holds with a stronger form that the multiplicity in question = 1.

In connection with this, we make:

Question 3.1. Confirm conditions for the s -pre bundles D , to which (3.15.4) holds.

8. Now, we will examine the structure of $\mathfrak{F}_{1,p}$ ($p \in \bar{X}_{\text{sing}}^1$). First we check that if $\bar{Y} = \phi$ (cf. (3.11.1)), then the structure is very simple.

Lemma 3.3. *If \bar{X}^1 is normal and $\bar{Y} = \phi$, then we have:*

(3.16.1) $\mathfrak{F}_{1,p}$ is generated by \tilde{g} over $\mathfrak{D}_{1,p}$ for any $p \in \bar{X}^1$, where we regard \tilde{g} ($= \omega\tau(e_{r+1})$) as the element of $\Gamma(\bar{X}^1, \mathfrak{F}_1)$.

Proof. By Lemma 3.1.2, the element \tilde{g} generates \mathfrak{F}_1 over \bar{X}_{reg}^1 . On the other hand, because \mathfrak{F}_1 is an \mathfrak{D}_1 -submodule of \mathfrak{D}_1^r (cf. (3.4.3)), the normality of \bar{X}^1 implies:

(a) $\mathfrak{F}_1 \simeq i_* i^* \mathfrak{F}_1$, with the injection $i: \bar{X}_{\text{reg}}^1 \hookrightarrow \bar{X}^1$, and we have this lemma. q.e.d.

Next assume that $\bar{Y} \neq \phi$. In this case our arguments will be done under the following strong condition.

(3.16.2) The condition (3.15.1) holds.

Now, for an element $j \in \{1, \dots, r\}$, we define an $\mathfrak{D}_{1,p}$ -submodule $\mathfrak{U}_{p,j}$ of $\mathfrak{D}_{1,p}$ as follows:

$$\mathfrak{U}_{p,j} = \{ \alpha \in \mathfrak{D}_{1,p}; \alpha \tilde{\eta}(j) \in \mathfrak{D}_{1,p}^r \}.$$

Theorem 3.3.1. *Assume that \bar{X}^1 is normal. Then we have an isomorphism of $\mathfrak{D}_{1,p}$ -module for a point $p \in \bar{Z}$:*

(3.16.3)
$$\mathfrak{U}_p \simeq \mathfrak{F}_{1,p}.$$

Proof. Take a small open neighborhood U of p in \bar{X}^1 . Then, from the explicit form of \mathfrak{F}_1 (cf. (3.4.3)) and from that $\tilde{\eta}(j)$ is a frame of \mathfrak{F}_1 over $\bar{X}_{\text{reg}}^1 - \bar{Y}_j$ (cf. (3.14.4)), we have:

(a) $\Gamma(U \cap \bar{X}_{\text{reg}}^1, \mathfrak{F}_1) = \{ \alpha \tilde{\eta}(j); \alpha \text{ is an element of } \Gamma((U \cap \bar{X}_{\text{reg}}^1) - \bar{Y}_j, \mathfrak{D}_1) \text{ such that } \alpha \tilde{\eta}(j) \text{ is extended to } \Gamma(U \cap \bar{X}_{\text{reg}}^1, \mathfrak{D}_1^r) \}.$

But the j -th component of $\tilde{\eta}(j) = 1$ (cf. (3.9.1)), and we have:

(b) the right hand side of (a) = $\{ \alpha \tilde{\eta}(j); \alpha \text{ is an element of } \Gamma(U \cap \bar{X}_{\text{reg}}^1, \mathfrak{D}_1) \text{ and } \alpha \tilde{\eta}(j) \text{ is extended to } \Gamma(U \cap \bar{X}_{\text{reg}}^1, \mathfrak{D}_1^r) \}.$

Thus, from the similar reasoning to the proof of Theorem 3.1, we have this theorem. q.e.d.

The above theorem may insure:

(**) *the structure of $\mathfrak{F}_{1,p}$ is 'determined' by the boundary behavior of $\tilde{\eta}(j)$ along the divisor $(\bar{Y}_j \cap \bar{X}_{\text{reg}}^1)$.*

We give some consequences of Theorem 3.3.1, in which (**) appears explicitly. For this, we assume:

(3.17.0) \bar{Y}_j consists of the finite irreducible components and we write the irreducible decomposition of \bar{Y}_j as $\bar{Y}_j = \bigcup_{\alpha} \bar{Y}_{j,\alpha}$.

(This does not lose generalities in local investigations of \mathfrak{F}_1 and $E_{\bar{X}}$.)

Now, for each $\bar{Y}_{j,\alpha}$, we let $m(j; \alpha) (\in \mathbb{Z}_+)$ be the multiplicity of \tilde{g}_j along $Y_{j,\alpha} := \bar{Y}_{j,\alpha} \cap X_{\text{reg}}^1$, which is defined in the similar manner to (3.11.3). We then define an \mathfrak{D}_1 -submodule \mathfrak{B}_j of \mathfrak{D}_1 to be the one which is associated to the following pre sheaf:

$$(3.17.1) \quad U \rightarrow \mathfrak{B}_j(U) = \{ \alpha \in \Gamma(U, \mathfrak{D}_1); \alpha \text{ is in } \Gamma(U \cap \bar{X}_{\text{reg}}^1, \mathfrak{S}_{j,\alpha}^{m(j;\alpha)}) \},$$

where $\mathfrak{S}_{j,\alpha}$ denotes the ideal sheaf of $Y_{j,\alpha} (\subset \mathfrak{D}_1)$.

Theorem 3.3.2. *Assume that \bar{X}^1 is normal. Then we have the following isomorphism of $\mathfrak{D}_{1,p}$ -module for a point $p \in \bar{Z}$:*

$$(3.17.2) \quad \mathfrak{B}_{j,p} \simeq \mathfrak{F}_{1,p}.$$

(a) an element $\alpha \in \mathfrak{D}_{1,p}$ is in $\mathfrak{B}_{j,p} \Leftrightarrow (\tilde{g}_k / \tilde{g}_j) \alpha \in \mathfrak{D}_{1,p} (1 \leq k \leq r)$.

But the definition of the divisor \bar{Y}_j of \bar{X}^1 (cf. (2.2.9-5)) implies the following for each irreducible component $\bar{Y}_{j,\alpha}$:

(b) There is an element $k \in \{1, \dots, r\}$ such that $\tilde{g}_k \equiv 0 \pmod{Y_{j,\alpha}}$.

Then we see easily that the present theorem is nothing more than the rewritten form of Theorem 3.3.1, by using (a) and (b). q.e.d.

Theorem 3.3.3. *Assume that \bar{X} is normal and that the multiplicity $m(j, \alpha) = 1$ for each $\bar{Y}_{j,\alpha}$. Then we have:*

$$(3.17.3) \quad \mathfrak{F}_{1,p} \xrightarrow{\sim} \mathfrak{S}_{j,p} \quad (:= \text{ideal of } \bar{Y}_j) (\subset \mathfrak{D}_1); \quad p \in \bar{Z}.$$

This follows directly from Theorem 3.3.2. See also Appendix, where corresponding facts to (3.17.3) are given for the universal bundle on a Grassmann variety, by using Schubert calculus.

A natural question may be:

Question 3.2.1. Give conditions for the local freeness of E_X . In particular, give the conditions, when E_X arises from geometric situations like foliations ([Ba-Bo] and [Suwa]) and monodromy representations ([De]).

When E_X is not locally free, it is desirable to analyze the singularities of E_X . As in the end of Section 2, we make a question in this direction:

Question 3.2.2. Discuss possible relations between the treatments of the singularities of coherent sheaves by Le-Teisser-Navarro ([Le-Te] and [Nav]) and ours as in Section 3 hitherto.

Finally, the arguments in the hardest part (Theorems 3.2 and 3.3) in Section 3.2 were done under the local freeness assumption of E_X over $Y-Z$ (cf. (3.11.0)). The following question then naturally arises:

Question 3.2.3. Discuss the local structure of E_X also in the case where we do not assume (3.11.0).

Remark 3.8. Quite recently, Sumihiro ([Sum]) gives some treatments of singularities which arise in the elementary transformation.

§4. Structure of $\Gamma(E_X)$ and $\Gamma(\mathcal{E}_{nd} E_X)$

Here we start with a line bundle L_X over \bar{X} and sections $s = (s_1, \dots, s_{r+1}) \subset \Gamma(L_X)$ satisfying (2.5.1), and we assume that our s -pre bundle $D (= D_s)$ is obtained in the manner of (2.5.2) (cf. also (2.3.1, 2)). Thus D is of type (G) and we have:

$$(4.0) \quad \bar{X}^1 = (s_{r+1})_0, \bar{X}^{r1} = (s_r)_{0, \text{red}} \text{ and } f_j = s_j/s_r \ (1 \leq j \leq r-1) \text{ and } f_r = s_{r+1}/s_r. \text{ (For } X^1, X^{r1} \text{ and } f_1, \dots, f_r, \text{ see (2.5.2).)}$$

By (3.3.1), we see that $L_X := L_{\bar{X}|X}$, with $X = \bar{X} - (\bar{X}^1 \cap \bar{X}^{r1})$ coincides with the determinant bundle of the s -pre bundle E_X . Our arguments here will be given by reducing the structure in the title to that of certain subspaces of $\Gamma(L_X)$. In doing it, the varieties formed from s as in (2.2.1 ~ 9) will be basic.

Part A. Structure of $\Gamma(E_X)$

1.1. First, for the frames e^i of $E_X|_{N_i}$ ($i=0, 1$), we take $\ell^i := \bigwedge^r e^i$ to be a frame of $L_X|_{N_i}$ ($i=0, 1$), and we form a frame $\tilde{\ell}^i = (\tilde{\ell}^i(1), \dots, \tilde{\ell}^i(r))$ of $L_X^r|_{N_i}$ ($= L_X|_{N_i} \bigoplus^r \dots \bigoplus^r L_X|_{N_i}$) as follows:

$$(4.1.1-1) \quad \text{The } k\text{-th component of } \tilde{\ell}^i(j) = 0 \ (k \neq j) \text{ and } \ell^i \ (k = j).$$

Then letting ω denote the quotient morphism: $L_X \rightarrow L_{X^1} := L_X \otimes \mathcal{D}_{X^1}$, we have the following (cf. also Proposition 1.1 and Lemma 1.1):

Proposition 4.1. (1) We have an injection $\tau: E_X \rightarrow L_X^r$ as follows:

$$(4.1.1-2) \quad \begin{cases} \tau|_{N_0}: E_X|_{N_0} \ni e^0 \zeta^0 \longrightarrow L_X^r|_{N_0} \ni \tilde{\ell}^0 \cdot \zeta^0 \\ \tau|_{N_1}: E_X|_{N_1} \ni e^1 \zeta^1 \longrightarrow L_X^r|_{N_1} \ni \tilde{\ell}^1 \cdot (f_r h_{01} \zeta^1) \end{cases}$$

where ζ^i is an element of $\mathcal{D}_{N_i}^r$ ($i=0, 1$).

(2) Define an \mathcal{D}_X -submodule E'_X of L_X^r as follows:

$$(4.1.1-3) \quad E'_X|_{N_0} = L_X^r|_{N_0} \text{ and } E'_X|_{N_1} = \text{kernel of } (\omega \cdot \mu): L_X^r|_{N_1} \rightarrow L_1^r,$$

where μ is the \mathcal{D}_X -morphism: $L_X^r|_{N_1} \ni \zeta \rightarrow L_X^r|_{N_1} \ni \begin{bmatrix} I_{r-1} & f_j \\ 0 & f_r \end{bmatrix} \cdot \zeta \ (1 \leq j \leq r-1)$

and we set $L_1 = L_{X^1}$.

Then, we have:

$$(4.1.1-4) \quad \tau: E_X \xrightarrow{\sim} E'_X.$$

Proof. (1) is checked by a simple observation. To see (2), remarking that

$$(4.1.1-5) \quad f_r h_{01} = \begin{bmatrix} f_r \cdot I_{r-1} & (-f_j) \\ 0 & 1 \end{bmatrix} \quad (1 \leq j \leq r-1)$$

we easily have the following for $\zeta'^1 (= (\zeta'_j{}^1)_{j=1}^r) := (f_r h_{01}) \cdot \zeta^1$ (in (4.1.1-2)).

$$(4.1.1-6) \quad \zeta'_j{}^1 = \zeta_j^1 \cdot f_r - \zeta_r^1 \cdot f_j \quad (1 \leq j \leq r-1) \quad \text{and} \quad \zeta'_r{}^1 = \zeta_r^1.$$

From this we easily have (2). q.e.d.

1.2. Next, we form a C -morphism (=homomorphism of C -modules) $\chi: \Gamma(E_X) \rightarrow \Gamma(X^1, L_1)$ by the following commutative diagram:

$$(4.1.2-1) \quad \begin{array}{ccc} \Gamma(E_X) & \xrightarrow{\tau} & \Gamma(L_X^r) \\ \chi \searrow & & \swarrow \omega p \\ & & \Gamma(X^1, L_1) \end{array} \quad \text{with } L_1 = L_{X^1}.$$

with the projection $p: L_X^r \ni \zeta = (\zeta_j)_{j=1}^r \rightarrow$ its last factor $L_X \ni \zeta_r$. The C -morphism χ is useful for analysis of $\Gamma(E_X)$. In order to examine χ , we define a C -subspace Γ' of $\Gamma(X^1, L_1)$ by

$$(4.1.2-2) \quad \Gamma' = \{ \tilde{s} \in \Gamma(X^1, L_1); \tilde{s} \text{ and } \tilde{f}_j \tilde{s} \ (1 \leq j \leq r-1) \text{ are in } \omega \Gamma(L_X^r) \},$$

with $\tilde{f}_j = \omega(f_j)$. (Here we use ω also for the projection: $\mathfrak{D}_X \rightarrow \mathfrak{D}_1$ ($:= \mathfrak{D}_{X^1}$)).

Also letting $\mathfrak{S}_1 (\subset \mathfrak{D}_X)$ denote the ideal of X^1 , we define a C -subspace of $\Gamma(E_X)$ as follows:

$$(4.1.2-3) \quad \Gamma_c(E_X) = \tau^{-1} \Gamma(X, \mathfrak{S}_1 L_X^r).$$

Lemma 4.1. *We have the following exact sequence:*

$$(4.1.3) \quad 0 \longrightarrow \Gamma_c(E_X) \longrightarrow \Gamma(E_X) \xrightarrow{\chi} \Gamma' \longrightarrow 0.$$

Proof. The desired fact for χ follows easily from (4.1.1-4).

A little more precisely, take an element $\zeta = (\zeta_j)_{j=1}^r \in \Gamma^r(L_X)$. Then, by (4.1.1-3), we have:

(a) ζ is in the image of $\chi \Leftrightarrow \zeta_j + f_j \cdot \zeta_r \equiv 0 \pmod{f_r} \ (1 \leq j \leq r-1)$. From this and (4.1.3-5), we have: $\chi \Gamma(E_X) = \Gamma'$. Also, from (4.1.1-2) and (4.1.3-5), we have: (kernel of χ) = $\Gamma_c(E_X)$. q.e.d.

In the remainder of Part A we examine Γ' and $\Gamma_c(E_X)$.

2. Here we determine the structure of $\Gamma_c(E_X)$ as follows:

Lemma 4.2. Assume that

(4.2.0) $(C \simeq) \Gamma(X, C_X) \simeq \Gamma(\mathcal{O}_X)$, where C_X is the constant sheaf with the stalk C .

Then we have:

$$(4.2.1) \quad \dim_C \Gamma_c(E_X) = r \text{ and } \Gamma_c(E_X) \text{ is spanned by } (e_1, \dots, e_r), \text{ and} \\ r + 1 \leq \dim_C \Gamma(E_X) (= \dim_C \Gamma' + r).$$

Proof. Letting $\mathcal{O}_X[X^1]$ be the sheaf over X of meromorphic functions over X with the pole X^1 , we define an obvious C -isomorphism:

$$(4.2.2-1) \quad \theta: \Gamma(L_X) \xrightarrow{\sim} \Gamma(X, \mathcal{O}_X[X^1]),$$

where θ is characterized by: $\theta(\ell^0) = 1$ (= constant function over X with the value 1).

Then, for the injection: $\mathcal{O}_X \hookrightarrow \mathcal{O}_X[X^1]$, we have the commutative diagram:

$$(4.2.2-2) \quad \begin{array}{ccc} \Gamma(L_X) & \xrightarrow{\theta} & \Gamma(X, \mathcal{O}_X[X^1]) \\ \uparrow & & \uparrow \\ \Gamma(X, \mathfrak{S}_1 L_X) & \xrightarrow{\theta} & \Gamma(\mathcal{O}_X) \end{array}$$

The first assertion in (4.2.1) follows from (4.2.2-2) and (4.2.0). The second is insured by checking $\tau(e_{r+1}) \neq 0$. q.e.d.

3. Here, in order to determine Γ' , we define a C -subspace Γ'' of $\Gamma(\bar{X}^1, \bar{L}_1)$ as follows:

(4.3.1-1) $\Gamma'' = \{ \tilde{s} \in \Gamma(\bar{X}^1, \bar{L}_1); \tilde{f}_j \cdot \tilde{s}|_{X^1} (\in \Gamma(X^1, L_1)) \text{ are also in } \Gamma(\bar{X}^1, \bar{L}_1), \text{ with } \bar{L}_1 := L_X \otimes \mathcal{O}_{\bar{X}^1} \}$.

The relation of Γ' to Γ'' is:

(4.3.1-2) $\Gamma' = \{ \tilde{s} \in \Gamma''; \tilde{s} \text{ and } \tilde{f}_j \cdot \tilde{s} (1 \leq j \leq r-1) \text{ are in } \omega\Gamma(L_X), \text{ where we identify } \tilde{f}_j \cdot \tilde{s} \text{ with its extension to } \Gamma(\bar{X}^1, \bar{L}_1) \}$.

The key point for Γ'' is that it is defined by \bar{L}_1 and $\tilde{f}_j (1 \leq j \leq r-1)$ and concerns only the structure of the divisor \bar{X}^1 .

Now, letting the varieties $\bar{Y}_j (1 \leq j \leq r)$ be the ones formed in the manner (2.2.9), we describe Γ'' explicitly in terms of them, under the following conditions:

(4.3.2-1) $\bar{Y} = \phi$, or is a divisor of \bar{X}^1 .

In the second case we also assume:

(4.3.2-2) $\tilde{s}_k \equiv 0 \pmod{\tilde{s}_r}$ along Y_β for each $Y_\beta (1 \leq k \leq r)$, where $Y_\beta = \bar{Y}_\beta \cap \bar{X}^1_{\text{reg}}$ with an irreducible component \bar{Y}_β of \bar{Y} . Also $\tilde{s}_i = \omega(s_i) \in \Gamma(\bar{X}^1, \bar{L}_1)$.

Remark 4.1. By Theorem 3.1.1 and Corollary 3.1, the condition

(4.3.2-1) holds if $E_{\bar{X}}$ is locally free. (4.3.2-2) is a sharpened form of (4.3.2-1) and may be mild. (For example, if \bar{X}'^1 intersects simply with \bar{X}^1 along each \bar{Y}_β , then (4.3.2-2) holds.)

Now, recalling (2.2.9-5), we have:

$$(4.3.3-1) \quad \bar{X}'^1 \cap \bar{X}_{\text{reg}}^1 (= \bar{X}^2 \cap \bar{X}_{\text{reg}}^1) = Y \cap Y_1, \quad \text{where } Y = \bar{Y} \cap \bar{X}_{\text{reg}}^1 \\ \text{and } Y_1 = \bar{Y}_1 \cap \bar{X}_{\text{reg}}^1,$$

and, for each irreducible component $\bar{Y}_{1,\alpha}, \bar{Y}_\beta$ of \bar{Y}_1, \bar{Y} , we set:

$$(4.3.3-2) \quad \begin{cases} m(\bar{s}_j; \alpha) = \text{the order of } \bar{s}_j \text{ along } Y_{1,\alpha} \quad (:= \bar{Y}_{1,\alpha} \cap \bar{X}_{\text{reg}}^1) \\ n(\bar{s}_j, \beta) = \text{the order of } \bar{s}_j \text{ along } Y_\beta \quad (:= \bar{Y}_\beta \cap \bar{X}_{\text{reg}}^1) \end{cases}$$

Then, setting $m(\alpha) = \min_{j=1}^r m(\bar{s}_j, \alpha) \in \mathbf{Z}_{+0}$, we define divisors \bar{Y}_1^*, \bar{Y}^* and \bar{X}^{*2} of \bar{X}^1 by

$$(4.3.3-3) \quad \bar{Y}_1^* = \sum_{\alpha} m(\alpha) \cdot \bar{Y}_{1,\alpha}, \quad \bar{Y}^* = \sum_{\beta} n(\bar{s}_r; \beta) \cdot \bar{Y}_\beta \quad \text{and} \quad \bar{X}^{*2} = \bar{Y}_1^* + \bar{Y}^*.$$

Then letting $\mathcal{O}_i[X^{*2}]$ be the sheaf over \bar{X}_{reg}^1 of meromorphic functions with the pole $X^{*2} := \bar{X}^{*2} \cap \bar{X}_{\text{reg}}^1$, we have:

Theorem 4.1. *Assume that \bar{X}^1 is normal. Then we have:*

$$(4.3.4) \quad \Gamma'' \cong \Gamma(\bar{X}_{\text{reg}}^1, \mathcal{O}_i[X^{*2}]).$$

Proof. (1) First we define a \mathcal{C} -isomorphism:

$$\Gamma(\bar{X}^1, \bar{L}_1) \simeq \Gamma(\bar{X}^1, \mathcal{O}_{\bar{X}^1}[\bar{s}_r])$$

in the similar manner to (4.1.1-3):

(4.3.5-1) $\theta: \Gamma(\bar{X}^1, \bar{L}_1) \simeq \Gamma(\bar{X}^1, \mathcal{O}_i[\bar{s}_r])$, which is characterized by $\theta(\bar{s}_r) = 1$. (Here $\mathcal{O}_i[\bar{s}_r]$ denotes the sheaf over \bar{X}^1 of meromorphic functions with the pole \bar{s}_r .)

Take an element $\bar{s} \in \Gamma(\bar{X}^1, \bar{L}_1)$ and we write $\bar{s} = \bar{s}(1) \cdot \bar{s}_r$ with an element $\bar{s}(1) \in \Gamma(\bar{X}^1, \mathcal{O}_i)$. We let $m(\bar{s}; \alpha)$ and $n(\bar{s}; \beta)$ be the order of the pole of $\bar{s}(1)$ along $Y_{1,\alpha}$ and Y_β . Then remarking that (4.3.2-2) is equivalent to

$$(4.3.5-2) \quad \bar{f}_j \text{ are holomorphic along each } Y_\beta \quad (1 \leq j \leq r-1),$$

we see that \bar{s} is extendable to $\Gamma(\bar{X}^1, \bar{L}_1)$ if and only if:

$$(a) \quad m(\bar{s}; \alpha) \leq m(\bar{s}_r, \alpha) \text{ and } n(\bar{s}, \beta) \leq n(\bar{s}_r, \beta) \text{ for each } Y_{1,\alpha} \text{ and } Y_\beta.$$

On the other hand, by a simple computation, we have:

$$(b) \quad \text{the order of the pole of } \bar{f}_j \cdot \bar{s}(1) \text{ along } Y_{1,\alpha} = m(\bar{s}, \alpha) + m(\bar{s}_r, \alpha) - m(\bar{s}_j, \alpha) \text{ (resp. along } Y_\beta = n(\bar{s}; \beta) + n(\bar{s}_r, \beta) - n(\bar{s}_j, \beta)).$$

Remark that $\bar{f}_j \cdot \bar{s}$ is in $\Gamma(\bar{X}^1, \bar{L}_1)$ if and only if the orders in (b) satisfy the similar inequalities to (a). But (4.3.0-2) implies the inequalities for Y_β . (When $\bar{Y} = \phi$ one can dispense with this argument. Thus we have this theorem. q.e.d.

Corollary 4.1.1. (1) Assume that \bar{X} is compact and \bar{X}^1 is normal. Then we have:

$$(4.3.5) \quad r + 1 \leq \dim_C \Gamma(E_X) (= r + \dim_C \Gamma') \leq r + \dim_C \Gamma(\bar{X}_{\text{reg}}^1, \mathcal{O}_1[X^{*2}]).$$

(2) If $\omega\Gamma(\bar{X}, L_{\bar{X}}) = \Gamma(\bar{X}^1, \bar{L}_1)$, then the second inequality in (4.3.5) is actually the equality.

The following corollary is also useful for investigations of $\Gamma(E_X)$.

Corollary 4.1.2. Assume that

(4.4.0) $\bar{X}^{*2} = \bar{Y}^*$ (or, equivalently, the multiplicity $m(\alpha) = 0$ for each $Y_{1,\alpha}$ (cf. (4.3.2-2))).

Then we have:

(4.4.1) $\Gamma'' \cong \Gamma(\bar{X}_{\text{reg}}^1, \mathcal{O}_1[Y^*])$, with the sheaf $\mathcal{O}_1[Y^*]$ of meromorphic functions over \bar{X}_{reg}^1 with the pole $Y^* (= \bar{Y}^* \cap \bar{X}_{\text{reg}}^1)$. (Note that if E_X is locally free, then Theorem 3.1.3 insures the condition (4.4.0).)

Proof. Obvious from Theorem 4.1. The structure of $\Gamma(E_X)$ is, of course, very basic for the bundle E_X . By taking account into Corollary 4.1, we make:

Question 4.1.1. Evaluate $\dim_C \Gamma(E_X)$ and $\dim_C \Gamma(\bar{X}_{\text{reg}}^1, \mathcal{O}_1[X^{*2}])$ as well as $\dim_C \Gamma(\bar{X}_{\text{reg}}^1, \mathcal{O}_1[Y^*])$.

It looks like that the structure of $\mathcal{O}_1[Y^*]$ seems to be very interesting in connection with theories of special divisors and treatments of zero cycles as in [G-H] (cf. Introduction). Here we only show that the equality:

$$(4.4.2) \quad \dim_C \Gamma(X, E_X) = r + 1$$

is checked in general situations:

Theorem 4.2.1. Assume that \bar{X} is compact and \bar{X}^1 is normal. Moreover, assume that (4.3.2-1,2) and (4.4.0) hold. Then we have the implication:

$$(4.4.3) \quad \dim_C (\bar{X}_{\text{reg}}^1, \mathcal{O}_1[Y^*]) = 1 \Leftrightarrow \dim_C \Gamma(E_X) = r + 1.$$

(Recall that (4.3.2-1) and (4.4.0) hold if E_X is locally free. Also (4.3.2-2) is mild under that condition.)

Proof. Obvious from Corollary 4.1.

The simplest case, where the condition in (4.4.3) is checked, may be given as follows:

Theorem 4.2.2. Assume that \bar{X} is compact and that the following holds:

(4.4.4-1) $\bigcap_{j=1}^{r+1} (s_j)_{0,\text{red}} = \phi$ and $(s_{r+1})_0$ is reduced and normal.

Then we have:

(4.4.4-2) $\dim_C \Gamma(E_X) = r + 1.$

Proof. First remark that (4.4.4-1) implies the first condition of (4.3.2-1) (and it is not necessary to assume (4.3.2-2)). Also (4.4.4-1) implies (4.4.0). Moreover, it implies: $\mathfrak{D}_1[Y^*] \simeq \mathfrak{D}_1$ (over \bar{X}_{reg}^1), and we have the present theorem. q.e.d.

We finish part A, by adding the following to Question 4.1.1.

Question 4.1.2. Discuss structures of $H^q(\bar{X}, E_X)$ ($q \geq 1$), hopefully, by generalizing the arguments in part A.

B. Structure of $\Gamma(\mathcal{E}_{nd} E_X)$

Our argument here will be done similarly to the one in part A.

5. First, take an element $\varphi \in \Gamma(\mathcal{E}_{nd} E_X)$ and we set $\varphi_i = \varphi|_{N_i}$ ($i=0, 1$). Then we have the ‘matrix representation’ of φ_i as follows:

(4.5.1-1) $\varphi_i(e^i) = e^i \cdot A_i,$ with $A_i \in M_r(N_i, \mathfrak{D}_X)$ ($i=0, 1$),

and we obviously have the following relation:

(4.5.1-2) $\left\{ \begin{aligned} \Gamma(\mathcal{E}_{nd} E_X) &= \{A_1 \in M_r(N_1, \mathfrak{D}_X) \text{ such that:} \\ A_0 &:= h_{01} \cdot A_1 \cdot h_{10} \text{ is extended to } M_r(N_0, \mathfrak{D}_X)\}. \end{aligned} \right.$

Now, let $a_{ij}(k)$ denote the (i, j) -component of A_k ($k=0, 1$) and $1 \leq i, j \leq r$. Then, from a simple computation, we see easily that the relation between A_0 and A_1 just above is equivalent to the following:

(4.5.2) $\begin{aligned} a_{rj}(0) &= a_{rj}(1)/f_r \quad (1 \leq j \leq r-1), \quad a_{rr}(0) = a_{rr}(1) + \sum_{j=1}^{r-1} a_{rj}(1)(f_j/f_r) \\ \text{and } a_{ij}(0) &= a_{ij}(1) - (f_i/f_r)a_{rj}(1) \quad (1 \leq i, j \leq r-1), \\ a_{ir}(0) &= \sum_{j=1}^{r-1} (a_{ij}(1) - (f_i/f_r)a_{rj}(1))f_j + (a_{ir}(1) - (f_i/f_r)a_{rr}(1))f_r \end{aligned}$
(1 ≤ i ≤ r-1).

From this, we easily have:

Proposition 4.5. We have a C -morphism as follows:

(4.5.3) $\Gamma(\mathcal{E}_{nd} E_X) \ni \varphi \longrightarrow M_r(X, L_X) \ni B = [b_{ij}] \quad (1 \leq i, j \leq r)$

where the element $b_{ij} \in \Gamma(X, L_X)$ is given by the following manner: Setting $b_{ij}|_{N_k} = \ell^k \cdot b_{ij}(k)$ with $b_{ij}(k) \in \Gamma(N_k, \mathfrak{D}_X)$ ($k=0, 1$), the element $b_{ij}(k)$ is defined to be:

$$(4.5.4-1) \quad b_{ij}(0) = a_{ij}(0) \quad (1 \leq i, j \leq r),$$

$$(4.5.4-2) \quad b_{ij}(1) = f_r \times (\text{right hand side in the equality (4.5.2) for the index } (i, j)).$$

Next we set:

$$(4.5.5-1) \quad \Gamma_e(\mathcal{E}_{nd} E_X) = \tau^{-1}(M_r(X, \mathfrak{S}_1 L_X)) \quad (\text{cf. (4.1.3)}).$$

Then we obviously have the commutative diagram:

$$(4.5.5-2) \quad \begin{array}{ccc} \Gamma(\mathcal{E}_{nd} E_X) & \xrightarrow{\tau} & M_r(X, L_X) \\ \uparrow & & \uparrow \\ \Gamma_e(\mathcal{E}_{nd} E_X) & \xrightarrow{\tau} & M_r(X, \mathfrak{S}_1 L_X) \end{array}$$

Moreover, we define a \mathcal{C} -morphism $\chi: \Gamma(\mathcal{E}_{nd} E_X) \rightarrow \Gamma(X^1, L_1)$ by the following commutative diagram:

$$(4.5.5-3) \quad \begin{array}{ccc} \Gamma(\mathcal{E}_{nd} E_X) & \xrightarrow{\tau} & M_r(X, L_X) \ni B \\ \chi \searrow & & \swarrow \omega p \\ \Gamma^{r-1}(X^1, L_1) & \ni & b = (b_{r1}, \dots, b_{rr-1}) \end{array}$$

where p is the projection: $M_r(X, L_X) \ni B \rightarrow \Gamma^{r-1}(X, L_X) \ni (b_{r1}, \dots, b_{rr-1}) = (r, 1), \dots$, and $(r, r-1)$ -components of B . In order to analyze χ , we define \mathcal{C} -subspaces Γ' and Γ'' of $\Gamma^{r-1}(X^1, L_1)$ by:

$$(4.5.6-1) \quad \left\{ \begin{array}{l} \tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_{r-1}) \in \Gamma^{r-1}(X^1, L_1) \text{ is in } \Gamma', \text{ if and only if:} \\ \tilde{b}_j \ (1 \leq j \leq r-1) \text{ and the following elements of } \Gamma(X^1, L_1) \text{ are} \\ \text{extendable to } \Gamma(X, L_X): \\ \tilde{f}_i \tilde{b}_j \ (1 \leq i, j \leq r-1), \quad \tilde{f}_i (\sum_{j=1}^{r-1} \tilde{f}_j \tilde{b}_j) \ (1 \leq i \leq r-1) \text{ and} \\ \sum_{j=1}^{r-1} \tilde{f}_j \tilde{b}_j \end{array} \right.$$

and by

$$(4.5.6-2) \quad \tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_{r-1}) \in \Gamma^{r-1}(X^1, L_1) \text{ is in } \Gamma'', \text{ if and only if } \tilde{b}_j \ (1 \leq j \leq r-1) \text{ and all the elements formed in the manner in (4.5.6-1) are extended to } \Gamma(\bar{X}^1, \bar{L}_1). \text{ Similarly to (4.1.4), we have:}$$

$$(4.5.6-3) \quad \Gamma'' = \{ \tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_{r-1}) \in \Gamma''; \tilde{b}_j \ (1 \leq j \leq r-1) \text{ and all the elements } \tilde{f}_i \tilde{b}_j, \dots \text{ in (4.4.6-1) are extendable to } \Gamma(\bar{X}, \bar{L}_X) \}.$$

Lemma 4.5.1. (1) We have the following:

$$(4.5.7-1) \quad \Gamma(\mathcal{E}_{nd} E_X) \subset \Gamma' \subset \Gamma'' \text{ and } (\text{kernel of } \chi) = \Gamma_e(\mathcal{E}_{nd} E_X).$$

$$(2) \quad \Gamma(\mathcal{E}_{nd} E_X) \cong \mathcal{C}, \text{ if:}$$

$$(4.5.7-2) \quad \Gamma'' \cong 0 \text{ and } \Gamma_e(\mathcal{E}_{nd} E_X) \cong \mathcal{C}.$$

Proof. This follows easily from (4.5.5-1) ~ (4.5.6-3).

Next we determine the image of χ explicitly. For this take an element $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_{r-1}) \in \Gamma^{r-1}(X^1, L_1)$.

Lemma 4.5.2. *The element \tilde{b} is in the image of χ , if and only if there are elements $b_{ij} \in \Gamma(L_X)$ ($1 \leq i, j \leq r$) such that*

(4.5.8-1) $\omega(b_{r,j}) = \tilde{b}_j$ ($1 \leq j \leq r-1$) and $\omega(b_{i,j}) = \tilde{f}_i \tilde{b}_j, \dots$ for the other indices (i, j) . Moreover, the elements $\tilde{f}_i \tilde{b}_j$ in the right hand side and the element

(4.5.8-2) $\tilde{b}_{ir} - \sum_{j=1}^r \tilde{b}_{ij} \cdot \tilde{f}_j - \tilde{f}_i \cdot \tilde{b}_{rr}$ ($1 \leq i \leq r-1$) are in $\Gamma(X, \mathfrak{S}_1 L_X)$.

Proof. The first condition is the consequence of that $\tilde{b} \in \Gamma'$. In order to see the second, for the elements b_{ij} as in (4.5.8-1), we define elements $a_{ij}(0) \in \Gamma(N_0, \mathfrak{D}_X)$ by (4.5.2). Then we have unique elements $a_{ij}(1) \in \Gamma(N_1, \mathfrak{D}_X)$ by (4.5.2) except for the indices (i, r) ($1 \leq i \leq r-1$). Finally, from the last equation in (4.5.2), we get (4.5.8-2). q.e.d.

6. Here we determine the structures of Γ'' and $\Gamma_c(\mathcal{E}_{nd} E_X)$. As in Part A, we assume here (4.2.0) and (4.3.2-1, 2). First we give a corresponding fact to Theorem 4.1. For this take an element $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_{r-1}) \in \Gamma^{r-1}(X^1, L_1)$ and we write $\tilde{b}_j = \tilde{b}_j(1) \cdot \tilde{s}_r$ with $\tilde{b}_j(1) \in \Gamma(X^1, \mathfrak{D}_1)$. We then define:

(4.6.1) $\left\{ \begin{matrix} m(\tilde{b}_j, \alpha) \\ m(\tilde{b}, \alpha) \end{matrix} \right\} =$ the order of the pole of $\left\{ \begin{matrix} \tilde{b}_j(1) \\ \sum_{j=1}^{r-1} \tilde{b}_j(1) \tilde{f}_j \end{matrix} \right\}$ along $Y_{1,\alpha}$ (cf. (4.3.1)).

Moreover, let $n(\tilde{b}_j, \beta)$ be the order of the pole of $\tilde{b}_j(1)$ along Y_β (cf. (4.3.1)). Then letting the multiplicities $m(\tilde{s}_j, \alpha)$ and $n(\tilde{s}_j, \beta)$ ($1 \leq j \leq r$) be as in (4.3.1), we have:

Theorem 4.3. *An element $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_{r-1}) \in \Gamma^{r-1}(X^1, L_X)$ is in Γ'' if and only if the following holds for each $Y_{1,\alpha}$ and Y_β :*

(4.6.2) $\left\{ \begin{matrix} \max_j n(\tilde{b}_j, \beta) \leq n(s_r, \beta) \\ \max_j m(\tilde{b}_j, \alpha) \leq \min(\min_j m(\tilde{s}_j, \alpha), m(\tilde{s}_r, \alpha)) \quad (1 \leq j \leq r-1) \\ m(\tilde{b}) \leq \min_j m(\tilde{s}_j, \alpha). \end{matrix} \right.$

The proof is similar to that of Theorem 4.1, and is omitted.
 Next, we define a \mathcal{C} -subspace \mathfrak{S} of $\Gamma^{r-1} := \Gamma^{r-1}(\bar{X}_{reg}^1, \mathfrak{D}_1[Y^*])$ as follows:
 (4.6.3) $\mathfrak{S} = \{ \tilde{b}(1) = (\tilde{b}_j(1)) \ (1 \leq j \leq r-1) \in \Gamma^{r-1}; \tilde{f}_i \cdot (\sum_{j=1}^{r-1} \tilde{b}_j(1) \cdot \tilde{f}_j) \text{ is holomorphic along each } Y_{1,\alpha} \}$ (cf. (4.3.1-1)).
 Then, corresponding to Theorem 4.1, we have the following characterization of Γ .

Lemma 4.6.1. *Assume that \bar{X}^1 is normal and that (4.3.2-1, 2) and (4.4.0) hold. Then we have a C -isomorphism as follows:*

$$(4.6.4) \quad \Gamma'' \simeq \mathfrak{S}.$$

Proof. Take an element $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_{r-1}) \in \Gamma^{r-1}(X^1, L_1)$, and we write $\tilde{b}_j = \tilde{b}_j(1)\tilde{s}_r$ over X^1 ($1 \leq j \leq r-1$). Then the condition for $\tilde{f}_i\tilde{b}_j$ in (4.5.6-1) implies: $\tilde{b}_j(1) \in \Gamma(\bar{X}_{\text{reg}}^1, \mathfrak{D}_1[Y^*])$ (cf. the proof of Theorem 4.1). Moreover, the condition for $\sum_{j=1}^{r-1} \tilde{b}_j(1)\tilde{f}_j$ in (4.5.6-1) is equivalent to the one (4.6.2). Thus we see easily that the attachment: $\tilde{b} \rightarrow \tilde{b}(1) = (\tilde{b}_j(1))_{j=1}^{r-1}$ gives the C -isomorphism in this lemma. q.e.d.

In connection with the condition (4.6.3), we make the following condition for an element $\tilde{b}(1) = (\tilde{b}_j(1))$ ($1 \leq j \leq r-1$) $\in \Gamma^{r-1}(\bar{X}_{\text{reg}}^1, \mathfrak{D}_1[Y^*])$:

$$(4.6.5) \quad \text{The condition in (4.6.1) for } \tilde{b}(1) \Rightarrow \tilde{b}_j(1) \text{ are in } \Gamma(\bar{X}_{\text{reg}}^1, C_{\bar{X}^1}) \text{ (} 1 \leq j \leq r-1 \text{)}.$$

The following will be useful as a condition for the vanishing of Γ'' .

Lemma 4.6.2. *Assume that \bar{X} is compact, \bar{X}^1 is normal and that (4.3.2-1, 2) hold. Then if (4.6.5) and*

$$(4.6.6) \quad \tilde{s}_1, \dots, \tilde{s}_r \text{ are linearly independent over } C$$

holds, we have $\Gamma' \cong 0$.

Proof. Remark that (4.6.5) implies $\sum_{j=1}^{r-1} \tilde{b}_j(1)\tilde{f}_j = c$, where the elements $\tilde{b}_j(1)$ and c are in C . Then (4.6.5) implies that $\tilde{b}_j(1) = 0$ ($1 \leq j \leq r-1$), and we have this lemma. q.e.d.

In arguments soon below, we will check that (4.6.4) (and so the vanishing of Γ'') holds in a general situation. Here we give an explicit description of $\Gamma_c(\text{End } E_X)$. For this we define a C -submodule of $M_r(C)$ as follows:

$$(4.6.7) \quad \mathfrak{C} = \{(c_{ij}) \text{ (} 1 \leq i, j \leq r \text{)}\}; \text{ the elements } c_{ij} \in C \text{ satisfy:}$$

$$c_{ir} = \sum_{j=1}^{r-1} c_{ij}\tilde{f}_j - c_{rr}\tilde{f}_i + \tilde{f}_i(\sum_{j=1}^{r-1} c_{rj}\tilde{f}_j) \quad (1 \leq i \leq r-1).$$

Lemma 4.6.3. *Assume that \bar{X} is compact. Then we have:*

$$(4.6.8) \quad \Gamma_c(\mathcal{E}_{nd} E_X) \simeq \mathfrak{C} \text{ (as } C\text{-modules)}.$$

Proof. Take an element $\varphi \in \Gamma(\mathcal{E}_{nd} E_X)$, and let $a_{ij}(0) \in \Gamma(N_1, \mathfrak{D}_X)$ ($1 \leq i, j \leq r$) have the similar meaning to (4.5.2). Then, from (4.5.2), we see that φ is in $\Gamma_c(\mathcal{E}_{nd} E_X)$ if and only if:

$$(a) \quad a_{ij}(0) \text{ are in } C \text{ and satisfy (4.6.7).}$$

Thus we have this lemma. q.e.d.

We check that the condition $\Gamma_c(\text{End } E_X) \cong C$ holds in general.

Theorem 4.4. *Assume that \bar{X} is compact, \bar{X}^1 is normal and (4.3.2-1, 2) as well as (4.4.0) hold. Then if (4.6.6) holds, we have:*

$$(4.6.9) \quad \Gamma_c(E_X) \cong C.$$

Proof. Take an element $(c_{ij}) \in \mathcal{C}$ ($1 \leq i, j \leq r$). Then we have: $\sum_{j=1}^{r-1} c_{rj} \tilde{f}_j \in C$. (Actually, checking the order of the each summand of the expression (4.6.7), we see that (4.4.0) implies that $\sum_{j=1}^{r-1} c_{rj} \tilde{f}_j$ is holomorphic along each $Y_{1,\alpha}$. This and (4.3.3-2) imply that the assertion just above.) Because of (4.6.6), we see that $c_{rj} = 0$ ($1 \leq j \leq r-1$). Then, from (4.6.6), we also see that (4.6.7) implies: $c_{ij} = 0$ ($i \neq j$) and $c_{ii} = c_{rr}$ ($1 \leq i \leq r$), and we have this theorem. q.e.d.

In this paper, we give a condition for the simpleness of E_X in the following form:

Theorem 4.5.1. *Assume that*

(4.7.1) \bar{X} is compact, \bar{X}^1 is normal, and (4.3.2-1, 2) as well as (4.4.0) holds.

Then, if (4.6.5) and (4.6.6) hold, we have:

$$(4.7.2) \quad \Gamma(\mathcal{E}_{nd} E_X) \cong C.$$

(As in the case of Theorems 4.2.1 and 4.4, the assumption (4.7.1) does not lose generalities, when E_X is locally free.)

Proof. This is an immediate consequence of Lemma 4.6.2 and Theorem 4.4. q.e.d.

The condition (4.6.6) for the linearly independence of $\tilde{s}_1, \dots, \tilde{s}_r$ is quite mild. The key point is the validity of (4.6.5). Here we see that one can check (4.6.5) in a rather general condition.

Theorem 4.5.2. *Assume that (4.4.4-1) and (4.6.6) hold. (Namely:*

(4.8.0) $\bigcap_{j=1}^{r+1} (s_{j,0})_{\text{red}} = \phi$, $\tilde{s}_1, \dots, \tilde{s}_r$ are linearly independent over C , and $(s_{r+1})_{0,\text{red}}$ is reduced and irreducible.)

Then we have:

$$(4.8.1) \quad \Gamma(\mathcal{E}_{nd} E_X) \cong C.$$

Remark 4.2. By Remark 3.2, the above theorem seems to be valid under the assumption that \bar{X} is normal.

Remark 4.3. In [Mar], Maruyama gave a condition for the simpleness of the bundles (constructed by the method of elementary transformation), by algebro-geometrical method. His result contains group theoretical (or invariant theoretical) studies of the endmorphisms (§ 2 and

§ 3, [Mar]), corresponding facts to which do not exist in the present paper. Our point in the investigations of $\mathcal{E}_{nd} E_X$ (as hitherto in part B) is, as in part A, that the stratification theoretical data (attached to E_X) appears explicitly in our discussions.

7. Here we quickly summarize our arguments for bundles of type (G) (in § 2~§ 4) and discuss some possibilities about generalizations of our arguments. The discussions here will be divided into three parts in a concordant manner to our basic diagram of the varieties in Figure II, Section 2.1.

7.1. First from Theorem 4.5.2 and Theorem 3.1.3, we have:

Theorem 4.6. *Assume that \bar{X} is smooth and compact. Also assume that (4.8.0) holds. Then $E_{\bar{X}}$ is locally free and is simple.*

Remark that, if $\dim \bar{X} \geq 2$ and \bar{X} is a projective variety, then one can find arbitrary many pairs $(L_{\bar{X}}, s = (s_j)_{j=1}^{r+1})$ (consisting of a line bundle $L_{\bar{X}}$ and sections s of $L_{\bar{X}}$ satisfying (4.8.0). (For example, taking an ample bundle $L_{\bar{X}}$ over \bar{X} . Then a pair $(L_{\bar{X}}^m, s)$, where $m \gg 0$ and $s = (s_1, \dots, s_{r+1})$ is a generic element of $\Gamma^{r+1}(L_{\bar{X}}^m)$ ($r \geq \dim \bar{X}$), satisfies (4.8.0).) This also insures:

(*) For a smooth projective variety \bar{X} of dimension ≥ 2 , there are arbitrary many simple bundles (whose rank $\geq \dim \bar{X}$) over \bar{X} . At the present moment, we regard (*) as our analogue of the basic theorem of Maruyama (cf. Introduction), which was given for a smooth projective variety (of any characteristic) of dimension ≥ 2 in an elegant manner.

Remark 4.4. By Remarks 3.2 and 4.2, it seems that Theorem 4.6 is true if \bar{X} is normal.

In the remainder of Section 4, we discuss about how to generalize (and sharpen) Theorem 4.6 and (*).

7.2. Here, assume that, for our s -pre bundle E_X of type (G), the direct image $E_{\bar{X}}$ is locally free. Taking a 'generic' element $e = (e_1, \dots, e_{r+1}) \in \Gamma^{r+1}(E_{\bar{X}})$, we make the following:

(4.9.1) The divisor $(e_{1 \wedge \dots \wedge e_r})_0$ is smooth, but the variety $\bar{Y} := \bigcap_{j=0}^r (e_{1 \wedge \dots \wedge \check{e}_{r+1-j} \wedge \dots \wedge e_{r+1}})_0, \text{red} = \phi$.

Question 4.2.1. Discuss general methods to find a bundle of type (G), which satisfies (4.9.1).

Thirdly, using the similar notation to (4.9.1), we make:

(4.9.2) the divisor \bar{X}^1 has singularities and the variety $\bar{Y} = \phi$.

Question 4.2.2. Discuss some general methods to find a bundle of type (G), which satisfies (4.9.2).

In connection with Questions 4.2.1 and 4.2.2, we like to make the following (rather vague) question.

Question 4.2.3. Give interpretations (from our view points of s -pre bundles of type (G)) of known examples of bundles (as in [Har-2], [Mu-Ho], [Ta] and [Mar]).

Finally, we like to add the following questions, which arise naturally from our arguments hitherto in the main body of the present paper.

Question 4.3.1. Discuss the condition for the simpleness of E_X more closely (and discuss (4.6.3) in more general). (This seems to be much easier than the corresponding one in Question 4.1.1.)

Question 4.3.2. Discuss structures of $H^q(\bar{X}, \mathcal{E}_{nd} E_X)$ ($q \geq 1$), hopefully, by generalizing the arguments in Section 4.

Also recall that the simpleness of a bundle implies the indecomposability of it. But the simpleness fails for the case where \bar{X} is a Stein variety, and we add the following to Question 4.1.1:

Question 4.3.3. Give some criterions for the indecomposability of bundles over a Stein variety.

§ 5. A type of residue formula

In this section we will be concerned with some explicit representations of the characteristic class (in the sense of Atiyah ([At]).

1. First we quickly recall the characteristic class of Atiyah (cf. [At] and [Bo-2]). Let M be a complex manifold and E_M a bundle over it. Moreover, let $\mathcal{N} = \{N_\lambda\}_\lambda$ be an open covering of M such that $E_M|_{N_\lambda}$ has a frame, denoted by e_λ . We denote by $h_{\lambda\mu} \in GL(N_\lambda \cap N_\mu, \mathcal{D}_M)$ the transition matrix for (e_λ, e_μ) . Letting e denote the collection $\{e_\lambda\}_\lambda$, we have an element:

(5.1.1-1) $\theta = \theta(\mathcal{N}, e) \in Z^1(N, \mathcal{E}_{nd} E_M \otimes \Omega_M^1)$, where Ω_M^1 is the sheaf of holomorphic differential one form over M and, for each $N_\lambda \cap N_\mu (\neq \emptyset)$ the component $\theta_{\lambda\mu}$ of θ is as follows:

$$(5.1.1-2) \quad \theta_{\lambda\mu} = dh_{\lambda\mu} \cdot h_{\lambda\mu}^{-1} \in \Gamma(N_\lambda \cap N_\mu, \mathcal{E}_{nd} (E_M) \otimes \Omega_M^1).$$

(For the intrinsic meaning of such an element θ , see [At] and ([Bo-2]). Here we only recall the following two facts:

(*-1) The element $\tilde{\theta} \in H^1(M, \mathcal{E}_{nd} E_M \otimes \Omega_M^1)$, which is determined by θ , is the obstruction for the existence of the holomorphic connection for E_M .

(*-2) Through the Dolbeaut isomorphism, the element $\tilde{\theta}$ corresponds to the curvature form $\kappa (\in H^{1,1}(M, \mathcal{E}_{nd} (E_M)))$ of a suitable c^∞ -differentiable connection of E_M .

Now, let $I'_p: M_r(\mathcal{C}) \rightarrow \mathcal{C}$ be the basic invariant symmetric polynomial of degree p . (Thus I'_p gives the $(r-p)$ -th coefficients of the characteristic polynomial of each element of $M_r(\mathcal{C})$.) Then, letting I_p be the corresponding \mathfrak{D}_M -morphism:

$$(5.1.2) \quad \mathcal{E}_{nd} E_M \overbrace{\otimes \cdots \otimes}^p \mathcal{E}_{nd} E_M \longrightarrow \mathfrak{D}_M,$$

one attaches to an element $\omega^p = \omega^p(N, \mathbf{e}) \in Z^p(N, \Omega_M^p)$ as follows:

$$(5.1.3) \quad \begin{array}{ccc} Z^1(N, \mathcal{E}_{nd} E_M \otimes \Omega_M^1) & \xrightarrow{\cup^p} & Z^p(N, \mathcal{E}_{nd} E_M^{\otimes p} \otimes (\Omega_M^1)^{\otimes p}) \\ \tilde{I}_p \searrow & \uparrow & \swarrow I_p \otimes \wedge^p \\ & Z^p(N, \Omega_M^p) & \end{array}$$

where \cup^p and \wedge^p are the p -th cup and exterior products. Then the characteristic class of Atiyah is defined to be:

(5.1.4) $\tilde{\omega}^p \in H^p(M, \Omega_M^p)$, which is determined by $\omega^p = \tilde{I}_p(\theta)$. In our context, the pair $(\mathcal{N}, \mathbf{e})$ is a basic stratification theoretical datum for investigations of E_M , and our interest is the element $\omega^p \in Z^p(N, \Omega_M^p)$ rather than its class $\tilde{\omega}^p$.

2. Now we return to our original situation: We start with a stratification \mathcal{S} of M and its neighborhood system $\mathcal{N} = \{N_\lambda; S_\lambda \in \mathcal{S}\}$, where N_λ is an open neighborhood of S_λ . We then assume that $E_M|_{N_\lambda}$ is a product bundle and we fix a frame e_λ of it. Setting $e = (e_\lambda)$, we have an element $\omega^p \in Z^p(N, \Omega_M^p)$, which is an invariant of $(\mathcal{S}, \mathcal{N}, \mathbf{e})$. Now we let $NV_{\mathcal{S}}^p$ (= nerve of \mathcal{S}) to be: $\mathcal{U} = (S_p, \dots, S_0); S_j \in \mathcal{S} (0 \leq j \leq p)$ such that $S_p \prec \dots \prec S_0$, where $S_1 \prec S_0, \dots$ means that $S_1 \subset \bar{S}_0 - S_0, \dots$. We then take an element $\mathcal{U} = (S_p, \dots, S_0) \in NV_{\mathcal{S}}^p$ such that

$$(5.2.1) \quad \text{codim}_X S^i = i \quad (0 \leq i \leq p)$$

and we will concentrate our attention to the lowest stratum S_p : We set $\mathcal{N}_{\mathcal{U}} = \{N_j\}_{j=0}^p$, with $N_j = N_{S_j}$. Then letting $i_{\mathcal{U}}$ be the injection: $\mathcal{N}_{\mathcal{U}} \hookrightarrow \mathcal{N}$, we define:

$$(5.2.2) \quad \omega_{\mathcal{U}}^p = i_{\mathcal{U}}^* \omega^p \in Z^p(N_{\mathcal{U}}, \Omega_M^p) \in \Gamma(|\mathcal{N}_{\mathcal{U}}|, \Omega_M^p), \quad \text{where } |\mathcal{N}_{\mathcal{U}}| = \bigcap_{j=0}^p N_j$$

(For the open set $\mathcal{N}_{\mathcal{U}}$, see Figure I, Section 0 for the case of $p=1$. The general case of $p \geq 2$ will be figured, by extending Figure I). Now we introduce a condition for the element $\omega_{\mathcal{U}}^p$, which may be regarded as a 'boundary value' of $\omega_{\mathcal{U}}^p$ around the lowest element S^p of U . For this we set: $\mathcal{N}'_{\mathcal{U}} := \mathcal{N}_{\mathcal{U}} - \{N_{S^p}\}$, and we define a relative cochain complex $C_{S^p}^*(\mathcal{N}_{\mathcal{U}}, \Omega_M^p)$ by the following exact sequence:

$$(5.2.3) \quad 0 \longrightarrow C^*_{Sp}(\mathcal{N}_q, \Omega^p_M) \longrightarrow C^*(N_q, \Omega^p_M) \longrightarrow C^*(N'_q, \Omega^p_M) \longrightarrow 0.$$

Remarking that N_q consists of $(p+1)$ -elements, we have:

$$(5.2.4) \quad \begin{array}{ccc} C^p_{Sp}(N_q, \Omega^p_M) & \longrightarrow & C^p(N_q, \Omega^p_M) \\ \downarrow \wr & & \downarrow \wr \\ Z^p_{Sp}(N_q, \Omega^p_M) & \longrightarrow & Z^p(N_q, \Omega^p_M) \end{array}$$

and we regard ω^p_q as the relative p -cocycle:

$$(5.2.5) \quad \omega^p_q \in Z^p_{Sp}(N_q, \Omega^p_M).$$

Now assume that there are elements $f = (f_1, \dots, f_p) \in \Gamma(N_{Sp}, \mathcal{O}_M)$ such that

$$(5.2.6-1) \quad S^p (\subset N_{Sp}) \text{ is the (set theoretical) locus of } f$$

and

$$(5.2.6-2) \quad f_j \ (1 \leq j \leq p) \text{ does not vanish in } |\mathcal{N}_q| := \bigcap_{i=0}^p N_j.$$

Then setting $d \log f = d \log f_1 \wedge \dots \wedge d \log f_p \in Z^p_{Sp}(N_q, \Omega^p_M)$, we make:

Definition 5.1. We say that ω^p_q satisfies residue condition with respect to f , if one can write:

$$(5.2.7) \quad \omega^p_q = a \cdot d \log f + \delta \omega^{p-1}_q, \text{ with an element } a \in \mathcal{C} \text{ and an element } \omega^{p-1}_q \in C^{q-1}_{Sp}(\mathcal{N}_q, \Omega^p_M).$$

Thus, as we said earlier, the residue condition concerns a boundary behavior of ω^p_q along the stratum S^p . Remark that, from its construction, we may regard that ω^p_q is an invariant of $(\mathcal{S}, \mathcal{N}, \mathbf{e})$, and we may regard:

(*) The residue condition is a basic condition for $(\mathcal{S}, \mathcal{N}, \mathbf{e})$.

Besides the above naive interpretation of the residue condition, we remark that it also insures a topological meaning to ω^p_q . Actually, let Ω^p_M be the (abelian) sheaf of d -closed holomorphic differential forms over M . Then we have the following diagram:

$$(5.3.1) \quad \begin{array}{ccc} H^p(\mathcal{N}_q, \Omega^p_M) \ni a \cdot d \log f & \xrightarrow{\beta} & H^{2p}(\mathcal{N}_q, \Omega^p_M) \\ \downarrow \alpha & & \\ H^p(\mathcal{N}_q, \Omega^p_M) \ni \tilde{\omega}^p_q & & \end{array}$$

where H denotes the symbol of ‘hyper cohomology’. Note that the two cohomology groups in the top line (5.3.1) are of topological nature, while the one in the bottom is of complex analytic nature. Now, letting $\tilde{\omega}^p_q \in H^p(N_q, \Omega^p_M)$ be the element defined by ω^p_q , the residue condition implies:

(5.3.2) $\tilde{\omega}^p_q \in$ image of α (and ω^p_q is endowed with (at least one) topological meaning.

Moreover, if the condition:

$$(5.3.3) \quad \text{kernel of } \alpha \subset \text{kernel of } \beta$$

holds, then one can attach to $\tilde{\omega}_\mathfrak{u}^p$ the unique element of $H_{\mathbb{S}^p}^{2p}(\mathcal{N}_\mathfrak{u}, \Omega_M^p)$, and $\tilde{\omega}_\mathfrak{u}^p$ is endowed with a true topological meaning. (We like to make clear (5.3.3) in another place.) In this connection, we remark that the Atiyah characteristic class is, in general, of complex analytic nature ([At] and [Bo-1, 2]), and the insurance of the topological meaning is given by using some global properties (like Kahlerian property) (cf. [At]). We like to emphasize that our residue condition concerns only the local behaviors of $\omega_\mathfrak{u}^p$ (or, tracing back to the definition, of the pair $(\mathcal{S}, \mathbf{e})$). The above fact would justify our emphasize of the residue condition (5.2.7).

3. Now assume that E_M is of type (G). (In other words, there is an s -pre bundle D of type (G) such that E_M is the direct image of the prebundle appearing in D .) We assume that E_M (or the s -pre bundle D) comes from the (local) geometric situation as in (2.2.0). Namely, we start with elements $s'_j \in \Gamma(\mathfrak{S}_M)$ ($1 \leq j \leq r+1$) satisfying (2.2.0-1, 2, 3). We then assume the following generic condition for the sections s'_j :

$$(5.4.0) \quad \bar{X}^{j+1} := (s'_{r+1}, \dots, s'_{r+1-j}) \quad (0 \leq j \leq d), \text{ with } d = \min(r, \dim M),$$

is of codimension $j+1$. Setting

$$(5.4.1) \quad X^{j+1} = \bar{X}^{j+1} - \bar{X}^{j+2} \quad (0 \leq j \leq r) \quad \text{and} \quad X^0 = M - \bar{X}^1,$$

we have a stratification of $\mathcal{S} := \{X^j\}_{j=0}^d$ of M . Also take a neighborhood N_j of X^j suitably ($0 \leq j \leq d$). (We take $N_0 = X^0$.) Moreover, letting $\mathbf{e} = (e_1, \dots, e_{r+1})$ be the sections of the bundle E_M of type (G) (as in (2.1.1) and (2.3.1~4)), we take $\mathbf{e}^j := (e_1, \dots, e_{r+1-j}, \dots, e_{r+1})$ to be a frame of $E_M|_{N_j}$. Now, for each $p = 1, \dots, d$, we set $\mathcal{U}(p) = (X^0, \dots, X^p)$. Then, setting $N(p) := \bigcap_{k=0}^p N_k$, we have the element

$$(5.4.2) \quad \omega^p \in \Gamma(N(p), \Omega_M^p)$$

as in (5.2.2). Recall that the residue condition for ω^p concerns the main part of the boundary value of ω^p around the subvariety X^p (cf. Definition 5.1).

Theorem 5.1. *For each $p = 1, \dots, d$, the element ω_p satisfies the residue condition with respect to $(s'_{r+1}, \dots, s'_{r+2-p})$.*

The proof takes some pages, and will be given elsewhere.

Remark 5.1. In the main part $a_p \cdot d \log (s'_{r+1} \wedge \dots \wedge s'_{r+2-p})$ of $\omega_\mathfrak{u}^p$ (cf. (5.2.7)), the element $a_p \in \mathcal{C}$ is actually in \mathcal{Z} , and is interpreted as a certain multiplicity of the $(r+1-p)$ -vector: $e'_1 \wedge \dots \wedge e'_{r+1-p}$. (This will be also given elsewhere.)

Question 5.1. To globalize the definition for the residue condition (in Definition 5.1) and the result in Theorem 5.1. If M is a Grassmann variety and E_M is its universal quotient bundle then Definition 5.1 and Theorem 5.1 are checked to be applied only to the generic open part of the Schubert subvariety (representing the Chern class of E_M). Thus a good answer to Question 5.1 must be applied to the singular part of the Schubert varieties. We finish Section 5, by adding the following (naive) question to Question 5.1:

Question 5.2. Generalize Definition 5.1 and Theorem 5.1 for an element $(S_{i(0)}, \dots, S_{i(p)}) \in Nu_{\mathcal{S}}$, where the stratification \mathcal{S} is as in Definition 5.1, where we do not assume: $(i(0), \dots, i(p)) = (0, \dots, p)$. (For Question 5.2, it seems to be necessary to drop the (set theoretical) complete intersection condition from Definition 5.1 and Theorem 5.1.

Appendix. Grassmannian computations

Here we summarize explicit computations for the quotient universal bundle over a Grassmann variety, which correspond to the ones for ‘ s -pre bundles of type G ’ (cf. § 2 and § 3).

Standard facts for Grassmann variety will be found in Bott ([Bo]) Griffiths-Harriths ([G-H]), Hodge-Pedoe ([H-P]), Kleiman ([K1-1, 2]) Kleiman-Lundoff ([KL-Lu]), Laskov ([La]) and Musili ([Mu-1, 2]).

1.1. Let the complex euclidean space F (of dimension n), the Grassmann variety \bar{V} of d -dimensional subspaces of F ($1 < d < n$), the exact sequence of the universal bundles:

$$(I.1.0) \quad 0 \longrightarrow G_{\bar{V}} \longrightarrow F_{\bar{V}} (= F \times \bar{V}) \xrightarrow{\omega} E_{\bar{V}} \longrightarrow 0$$

and the basis $e' = (e'_1, \dots, e'_d)$ of F be as in Section 2.3. As in that place, we write e'_i, \dots , also for the corresponding sections of $F_{\bar{V}}$ and we set $e_i = \omega(e'_i)$ ($1 \leq i \leq d$). A most theoretical approach to the Grassmann variety may be the one from the group theoretical view point (cf. [Bo] and [Mu-1]). Here we do not enter into this view point. Instead, for purpose of explicit computations, we work with the Stiefel variety \bar{W} corresponding to \bar{V} :

$$(I.1.1) \quad \bar{W} = \{f = (f_1, \dots, f_d) \in F^d; f_1 \wedge \dots \wedge f_d \neq 0\}.$$

Recall that \bar{V} is the quotient variety of \bar{W} by the right action of the general linear group $GL_d(\mathbb{C})$:

$$(I.1.2) \quad \pi: \bar{W} \ni f \longrightarrow \bar{V} = \bar{W}/GL_d \ni p,$$

where $p = \pi(f)$ is also characterized by that the tautological subspace $G_{\bar{V}, p}$ for $p(\subset F)$ (= the fiber of $G_{\bar{V}}$ at p) is spanned by f .

For the Stiefel variety \bar{W} , the corresponding exact sequence to (1.1.0) takes the form:

$$(I.1.3) \quad 0 \longrightarrow G_{\bar{W}} \longrightarrow F_{\bar{W}} (= F \times \bar{W}) \longrightarrow E_{\bar{W}} \longrightarrow 0,$$

where the fiber $G_{\bar{W}, f}$ of $G_{\bar{W}}$ at $f \in \bar{W}$ is the subspace of F spanned by f . (Note that $G_{\bar{W}}$ has a tautological frame: $\bar{W} \ni f \rightarrow f$, and $G_{\bar{W}}$ is a product

bundle.) Next identify F^d with $\mathbb{C}^{nd} (= \mathbb{C}^n \times \cdots \times \mathbb{C}^n)$ by the basis e' : $F^d \ni f (= (f_1, \dots, f_d)) = e' \cdot Y_f \rightarrow \mathbb{C}^{nd} \ni Y_f$. For an index $I = (i(1), \dots, i(d))$: $1 \leq i(1) < \dots < i(d) \leq n$, let $P_I(Y)$; $Y \in \mathbb{C}^{nd}$ be the Plücker function for I :

$$(I.1.4) \quad p_I(Y) = \det Y^J, \text{ where } Y^J = J\text{-submatrix of } M_{nd}(\mathbb{C}), \text{ with } J = (n+1-i(d), \dots, n+1-i(1)).$$

$$\left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \begin{array}{l} i(d) \\ i(1) \end{array}$$

Then one can identify \bar{W} with the Zariski open set $(\mathbb{C}^{nd} - (\cap_I (p_I)_0))$ of \mathbb{C}^{nd} .

1.2. Next let A_i be the subspace of F spanned by (e_1, \dots, e_i) ($1 \leq i \leq n$). Recall that, for an element $I = (i(1) < \dots < i(d))$, the closed Schubert variety \bar{S}_I (for I) is defined to be:

$$(I.1.5-1) \quad \bar{S}_I = \{p \in \bar{V}; \dim(G_{\bar{V}, p} \cap A_{i(j)}) \geq j\} \quad (1 \leq j \leq d),$$

where $G_{\bar{V}, p}$ is the tautological subspace of F for p . In addition to the closed variety \bar{S}_I , let \bar{V}_I^1 be the Plücker divisor for I : $\bar{V}_I^1 = (\wedge^r e^J)_0$, where $J = (j(1) < \dots < j(r))$ is the complement of I in $(1, \dots, n)$ and $e^J = (e_{j(1)}, \dots, e_{j(r)})$. Then, setting $V_I = \bar{V} - \bar{V}_I^1$, the open Schubert variety S_I (for I) is defined to be:

$$(I.1.5-2) \quad S_I = \bar{S}_I \cap V_I.$$

Among very many important facts on the closed and open Schubert varieties (as in the references in the beginning of this appendix), we recall here two facts as follows: First we have:

Theorem I.1. (1) *The Chow group $A(\bar{V})$ of \bar{V} is isomorphic to $\oplus_I \mathbb{Z}[c(\bar{S}_I)]$ as \mathbb{Z} -module, where the element $c(\bar{S}_I) \in A(\bar{V})$ is defined by \bar{S}_I .*

(2) *(Schubert-Bruhat stratification) The following expression gives a stratification of \bar{V} :*

$$(I.1.5-3) \quad \bar{V} = \coprod_I S_I.$$

(For the proof, see [H-P], [K1-1] and [Mu-1].) Next, letting $\sigma_k(A_i)$ denote the special Schubert variety ([K1-1]):

$$(I.1.6) \quad \sigma_k(A_i) = \{p \in \bar{V}; \dim(A_i \cap G_{\bar{V},p}) \geq k\} \quad (1 \leq i \leq r, k \geq 1)$$

(= \bar{S}_I , with $I = (i+1-k, \dots, i, r+k+1, \dots, n)$), we have:

Theorem I.2. (1) $\sigma_1(A_i)$ represents the $(r+1-i)$ -th Chern class $c_{r+1-i}(E_{\bar{V}})$ of $E_{\bar{V}}$, and $\sigma_k(A_i)$ ($k \geq 2$) describe completely the singular locus of $\sigma_1(A_i)$:

$$(I.1.7) \quad \sigma_1(A_i) \supset \sigma_2(A_i) \supset \dots \supset \sigma_k(A_i) \supset \dots,$$

where $\sigma_k(A_i)$ is the singular locus of $\sigma_{k-1}(A_i)$ ($k=2, \dots$).

(2) The Chow ring $A(\bar{V})$ of \bar{V} is generated by the Chern classes $c_i(E_{\bar{V}})$ ($1 \leq i \leq r$). (For the above, see [H-P], [La] and [K1-1].)

Now, remark that the special Schubert varieties just above are defined in terms of the r -sections $e^0 = (e_1, \dots, e_r)$ (and, for notational reason, we write $\sigma_k(A_i)$ also as $\sigma_{i,k}(e^0)$). On the other hand, recall that a most basic property of our 's-pre bundle' is the existence of $(r+1)$ -sections (Definition 2.1) (and that the basic varieties in our frame construction are formed from those sections. (cf. Figure II, § 2.1)). Taking account into this, let us start with the $(r+1)$ -sections $\tilde{e} = (e_1, \dots, e_{r+1}) \subset \Gamma(E_{\bar{V}})$. Then setting $e^j = (e_1, \dots, \check{e}_{r+1-j}, \dots, e_{r+1})$ ($0 \leq j \leq r$), we form a closed variety $\sigma_{i,k}(e^j)$ ($1 \leq j \leq r$) in the similar manner to $\sigma_{i,k}(e^0)$. Also we attach to e a closed subvariety of V , which may be an analogue of $\sigma_{i,k}(e^0)$ for \tilde{e} :

$$(I.1.8-1) \quad \sigma_{r,1}(\tilde{e}) = \bigcap_{j=0}^r \sigma_{r,1}(e^j) \quad (= p \in \bar{V}; \dim(A_{r+1} \cap G_{\bar{V},p}) \geq 2)$$

Recall that, in our arguments in Section 2~Section 4, the corresponding variety \bar{Y} to the above one (cf. (2.2.9)) plays very basic roles. In light of this the fact that

(I.1.8-2) $\sigma_{1,r}(e)$ is the Schubert variety (more precisely, = \bar{S}_I , with $I = r-k+1, \dots, r+1, r+p+2, \dots, n$),

may be worthwhile pointing out (because that the variety like \bar{Y} has a corresponding fact in the Schubert calculus is an encouraging support for our present experimental stage.) Next, in comparison with the main part of this paper, we write here the corresponding diagram to the one in Figure II, Section 2.1 for the universal bundle E_X (cf. also Lemma 2.2):

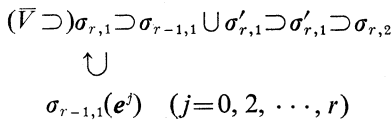


Figure I.

where we write: $\sigma_{r,1} = \sigma_{r,1}(e^0) \cdots$ and $\sigma'_{r,1} = \sigma_{r,1}(\tilde{e})$. (That this corresponds to Figure II, Section 2.1 is checked from the definition of $\sigma'_{r,1}$ as in (I.1.8-1) and (I.1.10-1, 2) soon below:

(I.1.10-1) $\sigma_{r,1} \cap \sigma_{r,1}(e^1) = \sigma_{r-1,1} \cup \sigma'_{r,1}$ and the Schubert varieties in the right hand side are of codimension two in \bar{V} (cf. [Mu-1]).

(I.1.10-2) $\sigma_{r,2}$ = the singular locus of $\sigma_{r,1}$, and $\sigma'_{r,1} = \bigcap_{j=0}^r \sigma_{r,1}(e^j)$ (cf. (I.1.8)).

In connection with the above arguments, we quickly check the validity of Lemma 2.2: From its formulation, it is clear that it suffices to see the following for the check:

(I.1.10-3) The divisors $(\wedge e^j)_0$ is reduced and irreducible ($0 \leq j \leq r$), and $(\wedge e^0)_0 \cap (\wedge e^1)_0$ is of codimension two.

But this is well known from the Schubert calculus (cf. [K1 ~ 1, 2] and [Mu-1, 2]).

2. Next recall that a main result in the present paper is the explicit construction of the frames as in Section 3 (Theorem 3.1 ~ 3.3). Here we will check that the corresponding fact for the universal bundle E_X is obtained in a very clear form from the Schubert calculus.

2.0. First take an open Schubert variety $S_I, I = (i_1 < \cdots < i_d)$. Then we give the explicit form of the standard frames of the universal bundles $G_{\bar{V}}$ and $E_{\bar{V}}$ over the ambient space V_I of S_I (cf. (I.1.5-2)). For this we quickly recall the standard affine structure of V_I ([K1-1] and [Mu-1]): Corresponding to V_I , we set $W_I = \pi^{-1}(V_I) (= \bar{W} - (p_I)_0)$. Then we easily have the following commutative diagram:

$$(I.2.1) \quad \begin{array}{ccc} (\mathbb{C}^{nd} \supset) W_I \ni Y = X(I) \cdot g & \xrightarrow{\sim} & V_I GL_d \ni (X(I), g) \\ \pi \downarrow & & \downarrow \lambda \\ V_I & \xrightarrow{\sim} & \mathbb{C}^{rd} \ni X(I) \end{array}$$

Here λ is the projection, and we identify \mathbb{C}^{rd} with the linear subspace of \mathbb{C}^{nd} defined by the following condition: The $i(d), \dots, i(1)$ -rows are:

$(\overbrace{1, 0, \dots, 0}^d), \dots, (\overbrace{0, \dots, 0, 1}^d)$, and we write the coordinates $X(I)$ of \mathbb{C}^{rd} in the following form:

$$(I.2.2) \quad X(I) = \begin{bmatrix} x(j(r)) \\ 1, 0, \dots, 0 \\ 0, \dots, 1 \\ x(j(1)) \end{bmatrix} \cdots \begin{matrix} i(d) \\ i(1) \end{matrix}$$

(= $n \times d$ -matrix whose $i(1), \dots, i(1)$ -rows are just as above, and we write $j(1)$ -, \dots ,-rows as follows: $x(j(1)) = (x(j(1), d), \dots, x(j(1), 1))$, \dots).

Next, recall that the exact sequence (I.1.0) turns out to be the direct sum over V_r ([K1-1]):

$$(I.2.3) \quad F_{\bar{V}}|_I = G_{\bar{V}}|_I + E_{\bar{V}}|_I, \text{ where } F_{\bar{V}}|_I, \dots \text{ are the restriction of } F_{\bar{V}} \text{ to } V_I, \dots$$

and we take, as standard frames of $E_{\bar{V}}|_I$ and $G_{\bar{V}}|_I$, the following:

$$(I.2.4) \quad e^I := (e_{j(1)}, \dots, e_{j(r)}) \text{ and } f^I := (f_1, \dots, f_d) := e^I \cdot X(I).$$

2.1. Next we give an explicit form of the transition matrix h_{10} for the frames (e^0, e^1) and some resulting sheaves (cf. § 2 and § 3). Our arguments will be done for the open Schubert varieties as follows:

$$(I.3.0) \quad \begin{cases} S^0 := S_{r+1, \dots, n}, & S^1 := S_{r, r+2, \dots, n}, & S^2 := S_{r-1, r+2, \dots, n}, \\ S'^2 := S_{r, r+1, r+3, \dots, n} & \text{and} & S^4 := S_{r-1, r, r+3, \dots, n}. \end{cases}$$

Remark that these open varieties are, respectively, the generic open parts of \bar{V} , $\bar{S}^1 := \sigma_{r,1}(e^0)$, $\bar{S}^2 := \sigma_{r-1,1}(e^0)$, $\bar{S}'^2 := \sigma_{r,1}(\tilde{e})$ and $\bar{S}^4 := \sigma_{r,2}(e^0)$. (For the roles of the Schubert varieties, see Figure I and Theorem I.2. We write V for the ambient affine space (cf. (I.1.5-2)) for each open Schubert variety in (I.3.0). Moreover, for each V , the standard coordinates $x(j, i)$ are the ones in (I.2.4). From the explicit form of the d and $(d+1)$ -th components of the frame f_I (cf. (I.2.4)), we have the following relation for $e_1, \dots, e_{r+2} \subset \Gamma(E_{\bar{V}})$:

$$(I.3.1) \quad \begin{cases} e_{r+1} + \sum_{j=1}^r x(j, 1) \cdot e_j = 0, & \text{for } S^0, \\ x(r+1) \cdot e_{r+1} + e_r + \sum_{j=1}^{r-1} x(j, 1) \cdot e_j = 0, & \text{for } S^1, \\ x(r+1, 1) \cdot e_{r+1} + x(r, 1) \cdot e_r + e_{r-1} + \sum_{j=1}^{r-2} x(j, 1) \cdot e_j = 0, & \text{for } S^2. \end{cases}$$

Moreover, for S'^2 we have:

$$(I.3.2) \quad \begin{cases} x(r+2, 1) \cdot e_{r+2} + e_r + \sum_{j=1}^{r-1} x(j, 1) \cdot e_j = 0, \\ x(r+2, 2) \cdot e_{r+2} + e_{r+1} + \sum_{j=1}^{r-1} x(j, 2) \cdot e_j = 0, \end{cases}$$

and for S^4 we have:

$$(I.3.3) \quad x(r+2, i) \cdot e_{r+2} + x(r+1, i) \cdot e_{r+1} + e_{r-2+i} + \sum_{j=1}^{r-2} x(j, i) \cdot e_j = 0 \quad (i=1, 2).$$

In a concordant manner to the arguments in Section 2 ~ Section 4, we will be here concerned with S^2 , S'^2 and S^4 .

3.1. First, for S^2 , we have:

(I.3.4-1) $S^2 = \text{locus of } x(r+1, 1), x(r, 1)$, and the standard frame for S^2 (cf. (I.2.4)) is $(e_1, \dots, e_{r-2}, e_r, e_{r+1})$. (Moreover, $\bar{S}^1 \cap V = \text{locus of } x(r+1, 1)$.)

Thus the transition matrix h_{10} for (e^0, e^1) is explicitly as follows:

$$(I.3.4-2) \quad h_{10} = \begin{bmatrix} I_{r-1} & (-x(j, 1)/x(r, 1)) \\ 0 & (-x(r+1, 1)/x(r, 1)) \end{bmatrix} \quad (1 \leq j \leq r-1).$$

S^2	S^1
	$x(r+1, 1) \nearrow$
	$x(r, 1)$
	\swarrow

This gives the explicit growth properties of the matrix h_{10} with respect to S^2 .

3.2. Next we will be concerned with S'^2 , which corresponds to our basic variety \bar{Y} in the frame construction (cf. § 3.1). First, from (I.3.3), we have the following relation for (e_1, \dots, e_{r+1}) :

$$(I.3.4-3) \quad -x(r+2, 1) \cdot e_{r+1} + x(r+2, 2) \cdot e_r + \sum_{j=1}^{r-1} s_j e_j = 0, \quad \text{with}$$

$$s_j = \det \begin{bmatrix} x(r+2, 2) & x(r+2, 1) \\ x(j, 1) & x(j, 2) \end{bmatrix}.$$

Also we have:

$$(I.3.4-4) \quad \wedge^r e^0 = \varepsilon_0 x(r+2, 1) (\wedge^r e^1), \quad \wedge^r e^1 = \varepsilon_1 x(r+2, 2) (\wedge^r e^1) \quad \text{and} \\ \wedge^r e^j = \varepsilon_j s_j (\wedge^r e^1), \quad \text{where } \varepsilon_j = 1 \text{ or } -1 \text{ and } e^j = (e_1, \dots, e_{r-1}, e_{r+2}).$$

Remark that, from the explicit form of s_j , we have: $x(r+2, 1) = x(r+2, 2) = 0 \Rightarrow s_j = 0$ ($1 \leq j \leq r-1$), and, from (I.3.2-6) and (I.1.8, 9), we have:

$$(I.3.4-5) \quad S'^2 \text{ is the locus of } (x(r+2, 1), x(r+2, 2)).$$

(Also note that $\bar{S}^1 \cap V$ is the locus of $x(r+2, 1)$.) The explicit form of the matrix h_{10} :

$$(I.3.4-6) \quad h_{10} = \begin{bmatrix} I_{r-1} & (s_j/x(r+2, 2)) \\ 0 & (x(r+2, 1)/x(r+2, 2)) \end{bmatrix} \quad (1 \leq j \leq r-1)$$

also gives the explicit growth property of h_{10} with respect to the codimension two subvariety S'^2 . Moreover, remark that the restriction of h_{10} to $\bar{S}^1 \cap V$ is of the form:

$$(I.3.4-7) \quad h_{10} = \begin{bmatrix} I_{r-1} & x(j, 1) \\ 0 & (x(r+2, 1)/x(r+2, 2)) \end{bmatrix} \quad (1 \leq j \leq r-1)$$

and the coefficients of it are holomorphic over $\bar{S}^1 \cap V$ except the (r, r) -component. The similar fact fails for S^2 (cf. (I.3.4-2)).

Remark. The explicit form of the matrix h_{10} for S^2 and S'^2 and the difference mentioned soon above are used in the explicit residue computations (in the form of Theorem 5.1) for the universal bundle E_X . (This will be given elsewhere.)

Now, define an $(r-1) \times r$ -matrix h' by $h' = x(r+2, 2) \cdot [I_{r-1}, x(j, 1)]$ ($1 \leq j \leq r-1$). Then, corresponding to (2.2.5), we define an \mathcal{O}_1 -module $\tilde{\mathcal{F}}_1$, where \mathcal{O}_1 is the structure sheaf of \bar{S}^1 , to be the kernel of $\chi: \mathcal{O}_1^r \ni \zeta \rightarrow \mathcal{O}_1^{r-1} \ni h'\zeta$. Then, corresponding to the basic fact for the frame construction in Lemma 3.1 and Theorem 3.1, we easily have:

(I.3.4-8) F_1 is an invertible sheaf and has $\tilde{\eta} = (\tilde{\eta}_j)$ ($1 \leq j \leq r$), where $\tilde{\eta}_r = 1, \tilde{\eta}_j = x(j, 1)$ ($1 \leq j \leq r-1$) as its frame.

Also defining an injection $\tau: E_{\mathcal{F}}|_V \rightarrow \mathcal{O}_{\bar{V}}^r$ in the manner in (1.1), by a simple computation, we have:

(I.3.4-9) $\omega\tau(e_{r+2}) = \tilde{\eta}$, where ω is the quotient morphism: $\mathcal{O}_{\bar{V}}^r \rightarrow \mathcal{O}_{\bar{S}^1}^r$.

Remarking that $(e_1, \dots, e_{r-1}, e_{r+2})$ is the standard frame of $E_{\mathcal{F}}|_V$ (cf. (I.2.6)), the two facts just above may be worthwhile pointing out in connection with our frame constructions in Section 3.

Now, in the arguments as above for S^2 and S'^2 , the singularity of the divisor S^1 does not appear. But, in the argument soon below for S^4 , the singularity will enter into.

3.3. First from (I.3.3) we have:

$$(I.3.5-1) \quad s_0 \cdot \begin{bmatrix} e_{r+2} \\ e_{r+1} \end{bmatrix} + \begin{bmatrix} x(r+1, 2), & -x(r+1, 1) \\ -x(r+2, 2), & x(r+2, 1) \end{bmatrix} \cdot \begin{bmatrix} e_{r-1} \\ e_r \end{bmatrix} \\ + \begin{bmatrix} s_{r-2}(1), \dots, s_1(1) \\ s_{r-2}(2), \dots, s_1(2) \end{bmatrix} \cdot \begin{bmatrix} e_{r-2} \\ \vdots \\ e_1 \end{bmatrix} = 0,$$

where we set:

$$s_0 = \det \begin{bmatrix} x(r+1, 1) & x(r+1, 2) \\ x(r+2, 1) & x(r+2, 2) \end{bmatrix} \quad \text{and} \quad s_j(i) = \det \begin{bmatrix} x(j, 1) & x(j, 2) \\ x(r+1, i) & x(r+1, i) \end{bmatrix} \\ (1 \leq j \leq r-2, i = 1, 2).$$

Also from a simple computation we have:

$$(I.3.5-2) \quad \wedge^r e^j = \varepsilon_j \cdot s_{r+1-j}(2) (\wedge^r e^1) \quad (3 \leq j \leq r), \quad \text{and} \quad \wedge^r e^j = \varepsilon_j \cdot s_j (\wedge^r e^1) \\ (0 \leq j \leq 2), \quad \text{with} \quad s_j = x(r+2, j) \quad (j = 1, 2). \quad (\text{Here } \varepsilon_j = 1 \text{ or } -1.)$$

From this we easily have:

$$(I.3.5-3) \quad \bar{S}^1 \cap V = (s_0)_0, \quad \bar{S}'^2 \cap V = (s(1), s(2))_0 \quad \text{and} \quad S^4 = (x(r+i, j))_0 \\ (i, j = 1, 2).$$

Moreover, letting e_j^0 be the $(r-1)$ -sections: $(e_1, \dots, \check{e}_{r+1-j}, \dots, e_r)$ ($1 \leq j \leq r-1$), we have (cf. [Mu-1]):

$$(I.3.5-4) \quad \bar{S}^1 \cap (\wedge^r e^j)_0 = \bar{S}^{r2} \cup (\wedge^{r-1} e_j^0)_0.$$

Now, define the imbedding $\tau: E_{\bar{V}}|_{\bar{V}} \rightarrow \mathfrak{D}_{\bar{V}}^r$ as in (1.1). Then, from (1.3.5-1), we have:

$$(I.3.5-5) \quad \tilde{g}(:= \omega\tau(e_{r+1})) = (\tilde{g}_j) (1 \leq j \leq r), \text{ where } \tilde{g}_j = \omega s_j(2) (1 \leq j \leq r-2), \\ \tilde{g}_{r-1} = \omega x(r+2, 2) \text{ and } \tilde{g}_r = -\omega x(r+2, 1).$$

Moreover we define a meromorphic vector $\tilde{\eta}(j)$ in the manner (3.9.1):

$$(I.3.5-6) \quad \tilde{\eta}(j) = (1/\tilde{g}_j) \cdot \tilde{g}.$$

Then, corresponding to Theorem 3.3, we have:

Theorem I.3. *We have:*

(I.3.5-7) $\alpha \cdot \tilde{\eta}(1)$ is an element of $\mathfrak{F}_1 (\subset \mathfrak{D}_1^r)$ for any $\alpha \in \mathfrak{F}_1$ ($:=$ the ideal of $(\wedge^{r-1} e_j^0) (\subset \mathfrak{D}_1)$, and

(I.3.5-8) $\Phi: \mathfrak{F}_1 \ni \alpha \rightarrow \tilde{\mathfrak{F}}_1 \ni \alpha \tilde{\eta}(1)$ is an \mathfrak{D}_1 -isomorphism.

(Here the \mathfrak{D}_1 -module $\tilde{\mathfrak{F}}_1$ is defined similarly to the one in (I.3.4-8) (cf. also (2.2.5)).

Proof. First, by a simple computation, we have:

$$(a-1) \quad e_1 \wedge \dots \wedge e_{r-1} \\ = (\varepsilon_1 x(r+2, 1) \cdot e_{r+1} + \varepsilon_2 x(r+1, 1) \cdot e_{r+2}) \wedge e_1 \wedge \dots \wedge e_{r-2},$$

where $\varepsilon_i = 1$ or -1 ($i = 1, 2$). and:

$$(a-2) \quad (e_1 \wedge \dots \wedge e_{r-1})_0 = (x(r+2, 1), x(r+1, 1))_0.$$

On the otherhand, we obviously have:

$$(b-1) \quad x(r+2, 1) \cdot \tilde{\eta}(1) = \omega\tau(e_{r+1}), \text{ with the quotient morphism: } \mathfrak{D}_{\bar{V}}^r \mathfrak{D}_{\tilde{S}_1}^r.$$

Also from (I.3.5-1) we easily have:

$$(b-2) \quad x(r+1, 1) \cdot \tilde{\eta}(1) = \omega\tau(e_{r+2}).$$

From (b-1, 2) we have (I.3.5-7). On the otherhand we have (cf. (3.4.4)):

$$(c) \quad 0 \longrightarrow s_0 \mathfrak{D}_{\bar{V}}^r \longrightarrow (E_{\bar{V}}|_{\bar{V}}) \longrightarrow \tilde{\mathfrak{F}}_1 \longrightarrow 0,$$

and $\omega\tau(e_j) = 0$ ($1 \leq j \leq r$) (cf. (2.2.6)). From this and that $(e_1, \dots, e_{r-2}, e_{r+1}, e_{r+2})$ is a frame of $E_{\bar{V}}|_{\bar{V}}$, we have (I.3.5-8). q.e.d.

The proof of Theorem 1.3 would show that the isomorphism: $\mathfrak{S}_1 \cong \mathfrak{F}$ is natural and also would be a supporting fact for our frame constructions in Section 3.)

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