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Regular System of Weights and Associated Singularities

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Introduction

Let $(a, b, c; h)$ be a system of four positive integers, called the weights, such that $h > \max(a, b, c)$. We call $(a, b, c; h)$ regular, if the following rational function in T becomes a polynomial function. ((1.2) Definition)

$$\frac{(T^h - T^a)(T^h - T^b)(T^h - T^c)}{(T^a - 1)(T^b - 1)(T^c - 1)}.$$

(Notably, then the function becomes a polynomial with *positive integral* coefficients. ((1.3) Theorems 1 and 1*))

The purpose of the present paper is the study of such *regular systems of weights* in certain good cases, since it gives a systematic viewpoint for certain class of discontinuous groups and associated surface singularities, which are studied by several authors such as Arnold, Brieskorn, Dolgachev, Looijenga, Milnor, Orlik, Pinkham, Saito, Sherbak, Slodowy, Wagreich aud Wahl in connection with $\boldsymbol{C}^*$-action.

Namely we put $\varepsilon := a + b + c - h$ and classify regular systems of weights for $\varepsilon = 1$ (>0), $\varepsilon = 0$ and $\varepsilon = -1$ ((2.1) Theorem 2 and Tables 1, 2 and 3). Then each regular system for these three cases corresponds to a certain discrete subgroup of the groups of the motions of $\boldsymbol{P}^1(\boldsymbol{C})$, $\boldsymbol{C}$, and $\boldsymbol{H}$ (the upper half plane) respectively (cf. (3.4) Note). The correspondence is established through surface singularities. Namely let $(a, b, c; h)$ be a regular system of weights. Then the hypersurface $X_0 := \{(x, y, z) \in \boldsymbol{C}^3 : f(x, y, z) = 0\}$ defined by a weighted homogeneous polynomial $f(x, y, z) = \sum a_{ijk} x^i y^j z^k$ (here the sum is over indexes (i, j, k) with $ai + bj + ck = h$ with generic coefficients), has singularity only at the origin. ((3.2) Theorem 3). For the cases $\varepsilon = 1, 0$ or -1, the 2-manifold $X_0 - \{0\}$ is a quotient variety by the free action of the corresponding discrete group on the canonical, trivial or anti-canonical $\boldsymbol{C}^*$-bundle over $\boldsymbol{P}$, $\boldsymbol{C}$ or $\boldsymbol{H}$ respectively. (Some of such description is very old, going back to Schwarz's

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triangle groups. [57])

Precisely, for the case $\varepsilon=1$ (resp. 0), our list of regular systems of weights corresponds to that of binary polyhedral groups (resp. certain Heisenberg groups). The corresponding X_0 are rational double points (resp. simple elliptic singularities). (These cases are classical. Cf. for instance [8] [47]).

For the case $\varepsilon=-1$, our list corresponds one to one to the table of some Fuchsian groups of the first kind, which are listed by Wagreich and Dolgachev, as the rings of automorphic forms with three generators. ([64, Table 1]). We shall prove the correspondence intrinsically without using the table of classifications (cf. (3.5) Theorems 4, 5, (5.5) and (5.6) Theorem 7). This also reproves the result of Wagreich ([64, Theorems (3.1), (4.6)]). Our proof uses the solution of Fenchel's conjecture [11] [18] [44].

Finally, we give a formula for the volume of the fundamental domain F of the discrete groups in terms of weights (cf. (5.6.3)):

$$\mathrm{vol}\,(F)/\pi=h/abc.$$

The proof of the formula uses the self intersection number of the Weil divisor at infinity in a compactification of the Milnor fiber of the singular surface X_0 (§ 6). The minimal model of the compactification is a $K3$ surface for $\varepsilon=-1$. (The idea of the compactification is due to H. Pinkham [42]. For the case $\varepsilon=1$, see [54].)

Each paragraph of the paper has its own introduction. For convenience of the reader and for completeness, sometimes we recalled well known facts with references. The readers are suggested to skip the proofs (1.5)–(1.11) of Theorems 1 and 1* in Section 1, the proofs (2.3)–(2.5) of the classification Theorem 2 in Section 2, the proof of (3.2) Theorem 1) and 2) Section 3 and (4.3)–(4.5) at first reading to avoid getting into details.

As explained in Section 4, we aim to study *the period mapping* for the singularities, which appear in this paper. Thus the present paper is the first attempt to the aim. For the purpose we want to describe the set of vanishing cycles for X_0 in terms of *root systems*. For $\varepsilon>0$, it is done by classical root systems A_l, D_l or E_l [5]. For $\varepsilon=0$, it is done by extended affine root systems $E_l^{(1,1)}$ [53]. The root systems for the case $\varepsilon=-1$ with $\mu_0=0$ will be studied in a forthcoming note [55].

It might be worthwhile to mention a classification work of critical points by V.I. Arnold [3], since most of the singular points which appear in the present paper have appeared already in the tables of V.I. Arnold [loc. cit.]. He took the viewpoint of modality, whereas we took the viewpoints of the regular systems of weights as above, which is much more special. The reason for this restriction comes from the fact that we wanted to single out the singularities in connection with group theoretic veiwpoint.

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Contents

§ 1. Regular system of weights and its exponents

This paragraph is devoted to a purely arithmetic study of regular systems of weights (see Definition (1.2)). The main result is *the positivity of coefficients* a_k *of the characteristic function* $\chi(T)$ (see (1.3) Theorem 1, Theorem 1*), which enables us to introduce exponents (see (1.4) Definition) for every regular system of weights.

(1.1) Let us start with a numerical datum,

$$(a, b, c; h) \in N^3 \times N \tag{1.1.1}$$

such that

$$h > \max(a, b, c). \tag{1.1.2}$$

We shall call such $(a, b, c; h)$ *a system of weights.* It will be called *primitive**, if it satisfies

$$\gcd(a, b, c, h) = 1. \tag{1.1.3}$$

(1.2) To a system of weights, we associate a rational function $\chi(T)$ in a variable T,

$$\chi(T) := T^{-h}\frac{(T^h - T^a)(T^h - T^b)(T^h - T^c)}{(T^a - 1)(T^b - 1)(T^c - 1)}, \tag{1.2.1}$$

*) In a forthcoming paper "On the existence of exponents prime to the Coxeter number, RIMS 529, 1985", we have changed our terminology "primitive" to "reduced".

which we shall call the *characteristic function* for the system of weights (cf. (3.2) Lemma 2).

By definition $\chi(T)$ satisfies the relation,

$$\chi(T^{-1})=T^{-h}\chi(T). \tag{1.2.2}$$

(1.2) **Definition.** *A system of weights $(a, b, c; h)$ is regular, if its characteristic function $\chi(T)$ is regular on $\boldsymbol{C}-\{0\}$. (i.e. $\chi(T)$ does not have any pole on $|T|=1$).*

(1.3) Our interest in this paper is the study of regular systems of weights. The first result in the direction is the following.

Theorem 1. *Let $(a, b, c; h)$ be a regular system of weights. Put*

$$\mu:=\frac{(h-a)(h-b)(h-c)}{abc}. \tag{1.3.1}$$

Then μ is a positive integer. There exist μ integers $m_1, \cdots, m_\mu$ s.t.

$$\chi(T)=\sum_{i=1}^{\mu} T^{m_i}, \tag{1.3.2}$$

$$m_1=\varepsilon<m_2\leqq\cdots\leqq m_{\mu-1}<m_\mu=h-\varepsilon, \tag{1.3.3}$$

$$m_i+m_{\mu+1-i}=h \quad \textit{for } i=1, \cdots, \mu. \tag{1.3.4}$$

Here we define the index ε by,

$$\varepsilon:=a+b+c-h. \tag{1.3.5}$$

The above Theorem 1 is an obvious reformulation of the following Theorem 1*.

Theorem 1*. *Let $(a, b, c; h)$ be a regular system of weights. Let μ be defined as* (1.3.1).

The characteristic function $\chi(T)$ has the following additive expression:

$$\chi(T)=\sum_{m=\varepsilon}^{h-\varepsilon} a_m T^m, \tag{1.3.5}$$

where a_m $(m \in Z)$ are nonnegative integers, s.t.

$$\sum_{m=\varepsilon}^{h-\varepsilon} a_m=\mu, \tag{1.3.6}$$

$$a_m=a_{h-m} \quad \textit{for } m=\varepsilon, \varepsilon+1, \cdots, h-\varepsilon, \tag{1.3.7}$$

$$a_\varepsilon=a_{h-\varepsilon}=1. \tag{1.3.8}$$

The relationship among $m_1, \cdots, m_\mu$ *and* $a_\varepsilon, \cdots, a_{h-\varepsilon}$ *is*

(1.3.9) $$a_m = \#\{i\colon 1 \leqq i \leqq \mu,\ m_i = m\},$$

(1.3.10) $$\{m_1, \cdots, m_\mu\} = \bigcup_{m=\varepsilon}^{h-\varepsilon} \{\underbrace{m, \cdots, m}_{a_m \text{ copies}}\}.$$

(1.4) **Definition.** *Let* $(a, b, c; h)$ *be a regular system of weights.*

1. *We shall call* μ *of* (1.3.1) *the rank of* $(a, b, c; h)$.

2. *We shall call the integers* $m_1, \cdots, m_\mu$ *of* (1.3.2) *the exponents of* $(a, b, c; h)$.

3. *We shall call the integer* a_m *of* (1.3.5), *the multiplicity of an exponent* m *of* $(a, b, c; h)$.

The remaining (1.5)–(1.11) of this paragraph is devoted to the proof of the above Theorem 1*.

(1.5) First let us introduce a notation.

Definition. *For a set of positive integers* $c_1, \cdots, c_r$ *and for* $k \in \boldsymbol{Z}$, *put*

(1.5.1) $$\begin{aligned} &N(k; c_1, \cdots, c_r) \\ &= \text{The coefficient of } T^k \text{ in the Taylor expansion of } 1 \Big/ \prod_{i=1}^{r} (1 - T^{c_i}) \\ &= \#\Big\{(p_1, \cdots, p_r) \in (\boldsymbol{Z}_+)^r\colon \sum_{i=1}^{r} p_i c_i = k\Big\}. \end{aligned}$$

where $\boldsymbol{Z}_+ := \{p \in \boldsymbol{Z}; p \geqq 0\}$.

By definition $N(0; c_1, \cdots, c_r) = 1$ and $N(k; c_1, \cdots, c_r) = 0$ for $k < 0$.

In Section 1–Section 3, we shall always assume that $r = 3$ and $c_1 = a$, $c_2 = b$ and $c_3 = c$ for some system of weights. Therefore if the weights in reference is clear from context, we shall simply denote $N(k)$ instead of $N(k; a, b, c)$.

(1.6) The following Assertions are basic.

Assertion. *Let* $(a, b, c; h)$ *be a regular system of weights.*

1. *The weight* a *divides* $h-a$, $h-b$, *or* $h-c$.

2. *There exists a positive integer* u *s.t. either* $h = (u+1)a$, $h = ua + b$ *or* $h = ua + c$.

Statements similar to 1 *and* 2 *obtained by permutations of* a, b *and* c *hold.*

3. $N(h-a) > 0$, $N(h-b) > 0$, *and* $N(h-c) > 0$,

4. $\gcd(a, b) \mid h$, $\gcd(b, c) \mid h$ *and* $\gcd(c, a) \mid h$.

Proof. **1.** The denominator of (1.2.1) is divisible by ϕ_a (=: the irreducible cyclotomic polynomial primitive a-th roots of unity). Hence the numerator $(T^{h-a}-1)(T^{h-b}-1)(T^{h-c}-1)$ is also divisible by ϕ_a, so that either $a|(h-a)$, $a|(h-b)$ or $a|(h-c)$ holds.

2. Due to the above **1,** either $(h-a)/a$, $(h-b)/a$ or $(h-c)/a$ is an integer, say u. If u were not positive, $h=\max(a, b, c)$. This is a contradiction to the definition of the weights (cf. (1.1.2)).

3. Due to **2**, $h-a$ is equal to either ua, $(u-1)a+b$ or $(u-1)a+c$. This implies $N(h-a)>0$.

4. Suppose $g:=\gcd(a, b)>1$. Then the denominator of $\chi(T)$ in (1.2.1) is divisible by ϕ_g^2. Therefore at least two of $h-a$, $h-b$, $h-c$ is divisible by g. This implies either $g|(h-a)$ or $g|(h-b)$. Therefore $g|h$.

q.e.d.

Note. The above **1–4** together give enough conditions for a system of weights to be regular.

Corollary. *If $(a, b, c; h)$ is a primitive regular system of weights, then we have,*

$$\gcd(a, b, c)=1. \tag{1.6.1}$$

Proof. Due to Assertion 4, $\gcd(a, b, c)|h$. q.e.d.

(1.7) The characteristic function $\chi(T)$ has an expression,

$$T^{\varepsilon}(1-T^{h-a})(1-T^{h-b})(1-T^{h-c})\sum_{p,q,r=0}^{\infty} T^{pa+qb+rc} \tag{1.7.1}$$

on the unit disc $|T|<1$, so that χ has a power series development at $T=0$,

$$\sum_{k\in \boldsymbol{Z}} a_k T^k \quad (a_k\in \boldsymbol{Z} \text{ for } k\in \boldsymbol{Z}), \tag{1.7.2}$$

such that $a_k=0$ for $k<\varepsilon$ and $a_\varepsilon=1$.

In general the series (1.7.2) converges only on $0<|T|<1$.

(1.8) *If the weights $(a, b, c; h)$ for the characteristic function $\chi(T)$ is regular, then the series* (1.7.2) *is a finite sum,*

$$\chi(T)=\sum_{k=\varepsilon}^{h-\varepsilon} a_k T^k, \tag{1.8.1}$$

with the duality,

$$a_k=a_{h-k} \qquad \textit{for } k\in \boldsymbol{Z}. \tag{1.8.2}$$

Proof. Since $\chi(T)$ is a regular function on $\boldsymbol{C}-\{0\}$, the series (1.7.2) converges on $0<|T|<\infty$. Thus one may apply the relation (1.2.2) to the series. Comparing the coefficients, one obtains the duality (1.8.2). q.e.d.

(1.9) In the following, we want to show the positivity $a_k \geqq 0$ $(k \in \boldsymbol{Z})$ of the coefficients of (1.8.1). For the purpose, let us give an expression for them.

Formula.

(1.9.1) $$a_k = N(k-\varepsilon) - N(k-b-c) - N(k-c-a) - N(k-a-b) \qquad \textit{for } k < h + \inf(a, b, c)$$

Proof. Put $d := \inf(a, b, c)$. Noting the following inequalties,

$$\begin{aligned} \varepsilon+(h-a)+(h-b) &= h+c \geqq h+d \\ \varepsilon+(h-b)+(h-c) &= h+a \geqq h+d \\ \varepsilon+(h-c)+(h-a) &= h+b \geqq h+d \\ \varepsilon+(h-a)+(h-b)+(h-c) &= 2h \geqq h+d, \end{aligned}$$

and the equalities,

$$\varepsilon+h-a=b+c, \quad \varepsilon+h-b=c+a, \quad \varepsilon+h-c=a+b,$$

one obtains the following congruence relation,

$$T^{\varepsilon}(1-T^{h-a})(1-T^{h-b})(1-T^{h-c}) \equiv T^{\varepsilon} - T^{b+c} - T^{c+a} - T^{a+b} \bmod T^{h+d}.$$

Therefore,

$$\chi(T) \equiv (T^{\varepsilon} - T^{b+c} - T^{c+a} - T^{a+b}) \sum_{p,q,r=0}^{\infty} T^{pa+qb+rc} \bmod T^{h+d}.$$

This implies the formula (1.9.1). q.e.d.

(1.10) The following is the main step for the proof of Theorem (1.3).

Assertion. *For a regular system of weights, the coefficients of the additive expression* (1.8.1) *of the characteristic function are non-negative, i.e.*

$$a_k \geqq 0 \qquad \textit{for } k \in \boldsymbol{Z}.$$

Proof. Due to the duality (1.8.2), it is enough to show the assertion only for $k \leqq h/2$.

Let us define elements A, B and C of $\boldsymbol{Z}_+^3$ by,

$A:=$ either $(u, 0, 0)$, $(u, 1, 0)$ or $(u, 0, 1)$ according as either $h-a=ua$, $h-a=ua+b$ or $h-a=ua+c$.
$B:=$ either $(0, v, 0)$, $(0, v, 1)$ or $(1, v, 0)$ according as either $h-b=vb$, $h-b=vb+c$ or $h-b\ vb+a$.
$C:=$ either $(0, 0, w)$, $(1, 0, w)$ or $(0, 1, w)$ according as either $h-c=wc$, $h-c=wc+a$ or $h-c=wc+b$.

These A, B and C are well defined due to (1.6) Assertion 2.

Let us define subsets M_A, M_B and M_C of Z_+^3 by,

$$M_A:=\{A+(p, q, r): (p, q, r) \in \boldsymbol{Z}_+^3 \text{ s.t. } pa+qb+rc=k-b-c\},$$
$$M_B:=\{B+(p, q, r): (p, q, r) \in \boldsymbol{Z}_+^3 \text{ s.t. } pa+qb+rc=k-c-a\},$$
$$M_C:=\{C+(p, q, r): (p, q, r) \in \boldsymbol{Z}_+^3 \text{ s.t. } pa+qb+rc=k-a-b\}.$$

Here in the definition, the symbol $+$ for two elements of $\boldsymbol{Z}_+^3$ means the addition as elements of $\boldsymbol{Z}^3$.

By definition, we have obviously,

$$\# M_A=N(k-b-c), \quad \# M_B=N(k-c-a) \quad \text{and} \quad M_C=N(k-a-b).$$

On the other hand, M_A, M_B and M_C are subsets of

$$M:=\{(p, q, r) \in \boldsymbol{Z}_+^3: pa+qb+rc=k-\varepsilon\}.$$
$$(\because\ (h-a)+(k-b-c)=k+h-a-b-c=k-\varepsilon, \text{ etc.})$$

Therefore if we have shown that M_A, M_B and M_C are disjoint in M, then the inequality $0 \leqq \# M-\# M_A-\# M_B-\# M_C=N(k-\varepsilon)-N(k-b-c)-N(k-c-a)-N(k-a-b)$ implies the positivity $a_k \geqq 0$ due to (1.9.1).

In the following i)–iv), we shall show that if M_A, M_B and M_C are not disjoint, then after a suitable change of A, B, C to A', B', C' the newly defined $M_{A'}$, $M_{B'}$ and $M_{C'}$ become disjoint.

i) If *A is of the form* $(u, 0, 0)$, *then* M_A *is disjoint from* M_B *and from* M_C.

Proof. Suppose $^\exists(p, q, r) \in M_A \cap M_B$. On one hand, $(p, q, r) \in M_A$ implies $p \geqq u=(h-a)/a$. On the other hand, $(p, q, r) \in M_B$ implies $p \leqq 1+(k-c-a)/a$. Therefore $(h-a)/a \leqq 1+(k-c-a)/a$ and hence $h-a+c \leqq k$. Combining this with $k \leqq h/2$, one gets $h/2 \leqq a-c$. Then $k-a-c \leqq h/2-a-c \leqq -2c<0$ so that $N(k-a-c)=0$ and $M_B=\phi$. This is a contradiction to $M_A \cap M_B \neq \phi$. The same argument shows $M_A \cap M_C \neq \phi$. q.e.d.

ii) If *A is of the form* $(u, 0, 1)$, *then* $M_A \cap M_B = \phi$.

Proof. Suppose $\exists(p, q, r) \in M_A \cap M_B$. Then $(p, q, r) \in M_A$ implies $p \geqq u = (h-a-c)/a$ and $(p, q, r) \in M_B$ implies $p \leqq 1 + (k-c-a)/a$. Therefore $(h-a-c)/a \leqq 1 + (k-c-a)/a$ and hence $h \leqq k+a$. Noting $k \leqq h/2$, one computes $k-a-c \leqq h/2-a-c \leqq h-k-a-c \leqq -c < 0$. Thus $N(k-a-c)=0$ and $M_B = \phi$. This contradicts $M_A \cap M_B \neq \phi$. q.e.d.

iii) *Let* $A=(u, 1, 0)$ *and* $B=(1, v, 0)$. *Then one of the following three cases occurs*

a) $M_A \cap M_B = \phi$,

b) $u=1$, $a=vb$ *and* $h=(2v+1)b$,

c) $v=1$, $b=ua$ *and* $h=(2u+1)a$.

Proof. The assumption on A and B implies,

$$h = ua + a + b = vb + a + b.$$

Therefore $ua = vb$ and hence $u=0$ iff $v=0$. If $u=v=0$, then $h=a+b$. Without loss of generality let us assume $a \geqq b$. Then

$$k-c-a \leqq \frac{h}{2} - c - a = \frac{b-a}{2} - c < 0$$

so that $N(k-c-a)=0$ and $M_B=\phi$. Thus $M_A \cap M_B = \phi$.

Assume $u \geqq 1$ and $v \geqq 1$. Suppose $M_A \cap M_B \neq \phi$. Take (p_1, q_1, r_1) and (p_2, q_2, r_2) or Z_+^3 such that

$$*) \qquad A + (p_1, q_1, r_1) = B + (p_2, q_2, r_2) \in M_A \cap M_B.$$

Written componentwise, this means,

$$**) \qquad p_1 + u = p_2 + 1,\ q_1 + 1 = v + q_2,\ r_1 = r_2.$$

First let us show $p_1 = q_2 = 0$.

Since $(h-a-b)/a = u = p_2 + 1 - p_1 = (k-c-a-q_2 b - r_2 c)/a + 1 - p_1$, one gets

$$***) \qquad h-k = -(1+r_2)c + (1-p_1)a + (1-q_2)b < (1-p_1)a + (1-q_2)b.$$

Therefore if $p_1 \geqq 1$ then $h-k<b$ and if $q_2 \geqq 1$ then $h-k<a$. Since $k \leq h/2$, this implies $k<b$ or $k<a$, so that in the first case $M_A=\phi$ and in the second case $M_B=\phi$. In any case $M_A \cap M_B = \phi$.

Now using $p_1 = q_2 = 0$, one computes easily, $k-b-c = p_1 a + q_1 b + r_1 c = (v-1)b + r_1 c$ so that one gets

$$k=vb+(r_1+1)c.$$

Then recalling $h=vb+a+b$ and $vb=ua$, one gets

$$\frac{h}{2}-k=\frac{vb+a+b}{2}-vb-(r_1+1)c$$
$$=\frac{b}{2u}(v+u-uv)(r_1+1)c.$$

Since $h/2-k\geqq 0$ and $c>0$, one obtains the inequality,

$$v+u-uv>0,$$

where $u, v\in \boldsymbol{Z}$ and $u\geqq 1$, $v\geqq 1$. Hence either $u=1$ or $v=1$. In both cases one sees directly the equalities of iii). q.e.d.

iv) Suppose that M_A, M_B and M_C are not disjoint. By a permutation of the role of a, b and c, suppose $M_A\cap M_B\neq\phi$. Then i) and ii) imply $A=(u, 1, 0)$ and $B=(1, v, 0)$. Then applying ii) again $M_A\cap M_C=\phi$ and $M_B\cap M_C=\phi$. Applying iii) either $a|h$ or $b|h$ occurs. Then we replace $A=(u, 1, 0)$ by $A'=(2u, 0, 0)$ or $B=(1, v, 0)$ by $B'=(0, 2v, 0)$. In any case, using i) and ii) one may check easily that $M_{A'}$, $M_{B'}$ and $M_{C'}$ are disjoint.

These complete the proof of (1.10) Assertion. q.e.d.

(1.11) Since all the coefficients a_k ($k\in \boldsymbol{Z}$) are non-negative, and only finite number of them are positive, let us define,

(1.11.1) $$\mu:=\sum_{k\in\boldsymbol{Z}} a_k.$$

Then μ is counted from the weights as follows.

(1.11.2) $$\mu=\frac{(h-a)(h-b)(h-c)}{abc}$$

By the additive expression (1.8.1) for χ, we have $\mu=\chi(1)$. On the other hand the rational expression (1.2.1) for χ implies,

$$\chi(1)=\frac{(T^{h-a-1}+\cdots+1)(T^{h-b-1}+\cdots+1)(T^{h-c-1}+\cdots+1)}{(T^{a-1}+\cdots+1)(T^{b-1}+\cdots+1)(T^{c-1}+\cdots+1)}\Bigg|_{T=1}$$
$$=\frac{(h-a)(h-b)(h-c)}{abc}.$$

Thus (1.5)–(1.11) altogether prove (1.3) Theorem 1*.

§ 2. Classification of regular systems of weights for $\varepsilon=0, \pm 1$

In the present paragraph, we classify primitive regular systems of weights for $\varepsilon>0$, $\varepsilon=0$ and $\varepsilon=-1$ (cf. (2.2) Tables 1, 2 and 3).

(2.1) **Theorem 2.** *For $\varepsilon>0$, there are two infinite sequences A_l $(l\geq 1)$, and D_l $(l\geq 4)$ of primitive regular systems of weights and three exceptional cases E_6, E_7 and E_8 (see* (2.2) *Table* 1).

For $\varepsilon=0$, there are three exceptional cases $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ of primitive regular systems of weights (see (2,2) *Table* 2).

For $\varepsilon=-1$, there are $14+8+6+3$ cases of exceptional primitive regular systems of weights (see (2.2) *Table* 3).

The proofs of the classification are given in (2.3), (2.4) and (2.5).

Note **1.** For each regular system of weights above, we associate a notation for later use as follows.

For any regular system of weights, we associate some surface singular point in (3.2), (3.3). Therefore if the singular point has already a "standard" notation, we use it also for the regular system.

For some other regular systems $(a, b, c; h)$ in the case $\varepsilon=-1$, we use tentatively R_{abc} as the notation for the system of weights.

Note **2.** We identify two systems of weights, when they differ only by a permutation of the first three weights a, b and c. Thus in Tables 1, 2 and 3, we have assumed $a\leq b\leq c$ except in the case of type A_l.

(2.2) **Primitive regular systems of weights**

Table 1 (*Case* $\varepsilon>0$)

Notation	μ	a	b	c	h	**exponents**	h/abc
A_l $(l\geq 1)$	l	a	b	c	h	$1c, 2c, 3c, \cdots, lc$	$(l+1)/ab$
	Here $h:=a+b$ s.t. $c\mid h$ and $l:=h/c-1$. (cf. Note 1)						
D_l $(l\geq 4)$	l	2	$l-2$	$l-1$	$2(l-1)$	$1, 3, 5, \cdots, l-1, \cdots, 2l-3$ ($l-1$ twice for l even)	$1/(l-2)$
E_6	6	3	4	6	12	1, 3, 5, 7, 8, 11	1/6
E_7	7	4	6	9	18	1, 5, 7, 9, 11, 13, 17	1/12
E_8	8	6	10	15	30	1, 7, 11, 13, 17, 19, 23, 29	1/30

Table 2 (*Case* $\varepsilon=0$)

Notation	μ	a	b	c	h	**exponents**	h/abc
$\tilde{E}_6$	8	1	1	1	3	0, 1, 1, 1, 2, 2, 2, 3	3
$\tilde{E}_7$	9	1	1	2	4	0, 1, 1, 2, 2, 2, 3, 3, 4	2
$\tilde{E}_8$	10	1	2	3	6	0, 1, 2, 2, 3, 3, 4, 4, 5, 6	1

Table 3 (*Case* $\varepsilon = -1$)

Notation	μ	a	b	c	h	exponents	h/abc
E_{12}	12	6	14	21	42	-1, 5, 11, 13, 17, 19, 23, 25, 29, 31, 37, 43	1/42
E_{13}	13	4	10	15	30	-1, 3, 7, 9, 11, 13, 15, 17, 19, 21, 23, 27, 31	1/20
E_{14}	14	3	8	12	24	-1, 2, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 22, 25	1/12
Z_{11}	11	6	8	15	30	-1, 5, 7, 11, 13, 15, 17, 19, 23, 25, 31	1/24
Z_{12}	12	4	6	11	22	-1, 3, 5, 7, 9, 11, 11, 13, 15, 19, 23	1/12
Z_{13}	13	3	5	9	18	-1, 2, 4, 5, 7, 8, 9, 10, 11, 13, 14, 16, 19	2/15
W_{12}	12	4	5	10	20	-1, 3, 4, 7, 8, 9, 11, 12, 13, 16, 17, 21	1/10
W_{13}	13	3	4	8	16	-1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 17	1/6
Q_{10}	10	6	8	9	24	-1, 5, 7, 8, 11, 13, 16, 17, 19, 25	1/18
Q_{11}	11	4	6	7	18	-1, 3, 5, 6, 7, 9, 11, 12, 13, 15, 19	3/28
Q_{12}	12	3	5	6	15	-1, 2, 4, 5, 5, 7, 8, 10, 10, 11, 13, 16	1/6
S_{11}	11	4	5	6	16	-1, 3, 4, 5, 7, 8, 9, 11, 12, 13, 17	2/15
S_{12}	12	3	4	5	13	-1, 2, 3, 4, 5, 6, 7, 8, 9, 10 11, 14	13/60
U_{12}	12	3	4	4	12	-1, 2, 3, 3, 5, 6, 6, 7, 9, 9, 10, 13	1/4
$R_{269}(J_{3,0})$	16	2	6	9	18	-1, 1, 3, 5, 5, 7, 7, 9, 9, 11, 11, 13, 13, 15, 17, 19	1/6
$R_{247}(Z_{1,0})$	15	2	4	7	14	-1, 1, 3, 3, 5, 5, 7, 7, 7, 9, 9, 11, 11, 13, 15	1/4
$R_{245}(Q_{2,0})$	14	2	4	5	12	-1, 1, 3, 3, 4, 5, 5, 7, 7, 8, 9, 9, 11, 13	3/10
$R_{236}(W_{1,0})$	15	2	3	6	12	-1, 1, 2, 3, 4, 5, 5, 6, 7, 7, 8, 9, 10, 11, 13	1/3
$R_{234}(S_{1,0})$	14	2	3	4	10	-1, 1, 2, 3, 3, 4, 5, 5, 6, 7, 7, 8, 9, 11	5/12
$R_{233}(U_{1,0})$	14	2	3	3	9	-1, 1, 2, 2, 3, 4, 4, 5, 5, 6, 7, 7, 8, 10	1/2
$R_{223}(V_{1,0})$	15	2	2	3	8	-1, 1, 1, 2, 3, 3, 3, 4, 5, 5, 5, 6, 7, 7, 9	2/3

$R_{225}(N_{16})$	16	2	2	5	10	-1, 1, 1, 3, 3, 3, 5, 5, 5, 5, 3, 3, 3, 9, 9, 11	1/2
R_{146}	22	1	4	6	12	-1, 0, 1, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, 10, 11, 12, 13	1/2
R_{135}	21	1	3	5	10	-1, 0, 1, 2, 2, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7, 7, 8, 8, 9, 10, 11	2/3
R_{134}	20	1	3	4	9	-1, 0, 1, 2, 2, 3, 3, 3, 4, 4, 5, 5, 6, 6, 6, 7, 7, 8, 9, 10	3/4
R_{124}	21	1	2	4	8	-1, 0, 1, 1, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 7, 7, 8, 9	1
R_{123}	20	1	2	3	7	-1, 0, 1, 1, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 7, 8	7/6
R_{122}	20	1	2	2	6	-1, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 7	3/2
R_{113}	25	1	1	3	6	-1, 0, 0, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 6, 6, 7	2
R_{112}	24	1	1	2	5	-1, 0, 0, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 6	5/2
R_{111}	27	1	1	1	4	-1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, 4, 5	4

Note **1.** The case A_l need special treatment, since l alone does not determine the weights $(a, b, c; h)$ uniquely. In this case we understand $c=1$ as the condition for the system to be primitive.

Under this convention we have always the equality,

$$\varepsilon=1$$

for any primitive regular system of weights with $\varepsilon>0$.

Note **2.** For the case D_l, l even, the exponent $l-1$ is counted twice. (i.e. $a_{l-1}=2$ (l:even), $=1$ (l: odd).)

Note **5.** The division of Table 3 into $14+8+6+3$ types are done according to the following rule.

The first group of 14 types: = Regular systems of weights with $a_0=a_1=0$.

The second group of 8 types: = Regular systems of weights with $a_0=0$, $a_1>0$.

The third group of 6 types: = Regular systems of weights with $a_0=1$, $a_1>0$

The fourth group of 3 types: = Regular systems of weights with $a_0 \geq 2$, $a_1 \geq 0$

Note **6.** We notice that Table 3 of $(a, b, c; h)$ for $\varepsilon = -1$ coincides with the table obtained by P. Wagreich [64, Table 1] for the systems of degrees $(q_0, q_1, q_2; d)$ of three generators and a relation polynomial for certain ring of automorphic forms. This fact will be proven in (3.4) Theorems 3 and 4 and in (5.5). (Compare [loc. cit, Theorem 3.1, Theorem 4.6].)

Proof of (2.1) *Theorem* 2. Without loss of generality, we assume $a \geq b \geq c$.

(2.3) *Case* $\varepsilon > 0$

Due to (1.6) Assertion 2, either i) $h = ua + b$, ii) $h = ua + c$ or iii) $h = (u+1)a$ for an integer $u > 0$. On the other hand, the assumption $\varepsilon = a + b + c - h > 0$ implies $h < 3a$, $h < 2a + b$ and $h < 2a + c$. Therefore,

i) $h = a + b$ and $c = \varepsilon$. Hence $\chi = T^{\varepsilon}(T^{h-c} - 1)/(T^c - 1)$ so that $c \mid (h - c)$ and $\chi = T^c(T^{h-2c} + T^{h-3c} + \cdots + 1)$. This is the case of type A_l for $l = (a+b)/c - 1$.

ii) $h = a + c$ and $b = \varepsilon$. This is also the case of type A_l for $l = (a+c)/b - 1$.

iii) $h = 2a$ and $a = b + c - \varepsilon$. Hence $\chi = T^{\varepsilon}(T^{h-b} - 1)(T^{h-c} - 1)/(T^b - 1)(T^c - 1)$ so that either iii)$_1$ $b \mid (h - c)$ or iii)$_2$ $b \mid (h - b)$.

iii)$_1$ Since $h - c = 2a - c = 2b + c - 2\varepsilon < 3b$, we have $h - c = vb$ for $v = 1$ or 2. If $v = 1$, then $2a = h = b + c$ so that $a = b = c$. This is the case of type A_1.

Assume $v = 2$ so that $h = c + 2b$ and $c = 2\varepsilon$. Then

$$\chi = T^{\varepsilon} \frac{(T^{b+2\varepsilon} - 1)(T^b + 1)}{(T^{2\varepsilon} - 1)} = T^{\varepsilon + b}(T^b + 1) + T^{\varepsilon} \frac{T^{2b} - 1}{T^{2\varepsilon} - 1},$$

so that $2\varepsilon \mid 2b$. This implies 2ε divides $c + 2b = h = 2a$ and hence $\gcd(a, b, c; h) = \varepsilon$.

Thus if the system of weights is primitive, then $\varepsilon = 1$ and $c = 2$, $b = l - 2$, $a = l - 1$ for some $l \geq 4$ and

$$\chi = T(T^l - 1)(T^{l-2} + 1)/(T^2 - 1) = T(T^{2l-4} + T^{2l-6} + \cdots + 1 + T^{l-2}).$$

This is the case of type D_l.

iii)$_2$ Since $h = 2b + 2c - 2\varepsilon < 4b$, we have $h = vb$ for $v = 2$ or 3. If $v = 2$, then $2a = h = 2b$ so that $a = b$ and $h = a + b$. This case reduces to i).

Assume $v = 3$, so that $h = 3b$ and $b = 2c - 2\varepsilon$. Then

$$\chi=T^{\varepsilon}\frac{(T^{2c-2\varepsilon}+1)(T^{5c-6\varepsilon}-1)}{T^{c}-1}\equiv -T^{-7\varepsilon}\frac{(T^{2\varepsilon}+1)(T^{6\varepsilon}-1)}{T^{c}-1}\quad \text{mod } Z[T]_{T}.$$

(Here $Z[T]_T$ means the localization of $Z[T]$ by T.) Therefore either iii)$_{2,1}$ $c|4\varepsilon$ or iii)$_{2,2}$ $c|6\varepsilon$. On the other hand by the assumption $b\geq c$, we have $b-c=2c-2\varepsilon-c=c-2\varepsilon\geq 0$, that is,

*) $c\geq 2\varepsilon$.

iii)$_{2,1}$ Put $c=p\varepsilon/q$ where $p|4$ and $q|\varepsilon$.

*) implies $p\varepsilon/q\geq 2\varepsilon$ so that $4\geq p\geq 2q$ and hence $q=1$ or $q=2$. Then, either

$c=2\varepsilon$, $b=2\varepsilon$, $a=3\varepsilon$, $h=6\varepsilon$ and $\chi=T^{\varepsilon}(T^{2\varepsilon}+1)^2$, or

$c=4\varepsilon$, $b=6\varepsilon$, $a=9\varepsilon$, $h=18\varepsilon$ and

$$\chi=T^{\varepsilon}(T^{4\varepsilon}-T^{2\varepsilon}+1)(T^{12\varepsilon}+T^{10\varepsilon}+\cdots+1).$$

iii)$_{2,2}$ Put $c=p\varepsilon/q$ where $p|6$ and $q|\varepsilon$.

*) implies $p\varepsilon/q\geq 2\varepsilon$ so that $6\geq p\geq 2q$ and hence $q=1$, 2 or 3. Then, either

$c=3\varepsilon$, $b=4\varepsilon$, $a=6\varepsilon$, $h=12\varepsilon$ and $\chi=T^{\varepsilon}(T^{4\varepsilon}+1)(T^{6\varepsilon}+T^{3\varepsilon}+1)$, or

$c=6\varepsilon$, $b=10\varepsilon$, $a=15\varepsilon$, $h=30\varepsilon$ and $\chi=T^{\varepsilon}(T^{10\varepsilon}+1)(T^{18\varepsilon}+T^{12\varepsilon}+T^{6\varepsilon}+1)$.

This completes the proof for Table 1.

(2.4) *Case* $\varepsilon=0$

Since $h=a+b+c$, due to (1.6) Assertion either $a|(b+c)$, $a|b$ or $a|c$. Therefore $a\geq b\geq c\geq 1$ implies either i) $a=b$ or ii) $a=b+c$.

i) $a=b$. Then $\chi=(T^{b+c}-1)^2(T^b+1)/(T^b-1)(T^c-1)$ so that $b|c$. Therefore $a=b=c$ and $\chi=(T^c+1)^3$.

ii) $a=b+c$. Then $\chi=(T^{2c+b}-1)(T^{2b+c}-1)/(T^b-1)(T^c-1)$ so that $b|2c$ or $b|c$. Therefore $b\geq c$ implies either $b=c$ or $b=2c$. Thus either

$a=2c$, $b=c$ and $\chi=(T^{2c}+T^c+1)^2$, or

$a=3c$, $b=2c$ and $\chi=(T^{2c}+1)(T^{4c}+T^{3c}+T^{2c}+T^c+1)$.

This completes the proof for Table 2.

(2.5) *Case* $\varepsilon=-1$

Since $h=a+b+c+1$, due to (1.6) Assertion either $a|(b+c+1)$, $a|(b+1)$ or $a|(c+1)$. Therefore $a\geq b\geq c$ implies either i) $a=b+c+1$, ii) $a=(b+c+1)/2$, iii) $a=b+1$, iv) $a=c+1$ or v) $a=b=c=1$.

i) $a=b+c+1$. Then $\chi(T)=T^{-1}(T^{b+2c+2}-1)(T^{2b+c+2}-1)/(T^b-1)\times(T^c-1)$ so that either $b|(2c+2)$ or $b|(c+2)$. Thus $b\geq c$ implies either i)$_1$ $b=2c+2$, i)$_2$ $b=c+1$, i)$_3$ $b=c+2$ or i)$_4$ $b=c=2$.

i)$_1$ $b=2c+2$. Then $\chi(T)=T^{-1}(T^{2c+2}+1)(T^{5c+6}-1)/(T^c-1)$ so that either $c|4$ or $c|6$. Thus we have one of the following:

$c=1,\ b=4,\ \ a=6,\ \ \chi=T^{-1}(T^4+1)(T^{10}+T^9+\cdots+1)$;
$c=2,\ b=6,\ \ a=9,\ \ \chi=T^{-1}(T^6+1)(T^{14}+T^{12}+\cdots+1)$;
$c=3,\ b=8,\ \ a=12,\ \chi=T^{-1}(T^8+1)(T^{18}+T^{15}+\cdots+1)$;
$c=4,\ b=10,\ a=15,\ \chi=T^{-1}(T^8-T^6+\cdots+1)(T^{24}+T^{22}+\cdots+1)$;
$c=5,\ b=14,\ a=21,\ \chi=T^{-1}(T^{14}+1)(T^{30}+T^{24}+\cdots+1)$.

i)$_2$ $b=c+1$. Then $\chi(T)=T^{-1}(T^{2c+2}+T^{c+1}+1)(T^{3c+4}-1)/(T^c-1)$ so that $c|3$ or $c|4$. Thus we have one of the following:

$c=1,\ b=2,\ a=4,\ \ \chi=T^{-1}(T^4+T^2+1)(T^6+T^5+\cdots+1)$;
$c=2,\ b=3,\ a=6,\ \ \chi=T^{-1}(T^6+T^3+1)(T^8+T^6+\cdots+1)$;
$c=3,\ b=4,\ a=8,\ \ \chi=T^{-1}(T^6-T^5+T^3-T+1)(T^{12}+T^{11}+\cdots+1)$;
$c=4,\ b=5,\ a=10,\ \chi=T^{-1}(T^{10}+T^5+1)(T^{12}+T^8+T^4+1)$.

i)$_3$ $b=c+2$. Then $\chi(T)=T^{-1}(T^{3c+4}-1)(T^{2c+4}+T^{c+2}+1)/(T^c-1)$ so that either $c|4$ or $c|6$. Thus we have one of the following:

$c=1,\ b=3,\ a=5,\ \ \chi=T^{-1}(T^6+T^5+\cdots+1)(T^6+T^3+1)$;
$c=2,\ b=4,\ a=7,\ \ \chi=T^{-1}(T^8+T^6+\cdots+1)(T^8+T^4+1)$;
$c=3,\ b=5,\ a=9,$
$$\chi=T^{-1}(T^{12}+T^{11}+\cdots+1)(T^8-T^7+T^5-T^4+T^3-T+1);$$
$c=4,\ b=6,\ a=11,\ \chi=T^{-1}(T^{12}+T^8+T^4+1)(T^{12}+T^6+1)$;
$c=6,\ b=8,\ a=15,\ \chi=T^{-1}(T^{20}+T^{18}+\cdots 1)$
$$\times(T^{14}-T^{13}+T^{11}-T^{10}+T^8-T^7+T^6-T^4+T^3-T+1).$$

i)$_4$ $c=2,\ b=2,\ a=5,\ \chi=T^{-1}(T^6+T^4+T^2+1)^2$.

ii) $a=(b+c+1)/2$. Then the inequality $a\geq b\geq c$ implies that $a=b=c+1$ and hence $\chi(T)=T^{-1}(T^{c+1}+1)^2(T^{2c+3}-1)/(T^c-1)$. Therefore $c|2$ or $c|3$. Thus we have one of the following:

$c=1,\ b=2,\ a=2,\ \chi=T^{-1}(T^2+1)^2(T^4+T^3+\cdots+1)$;
$c=2,\ b=3,\ a=3,\ \chi=T^{-1}(T^3+1)(T^2-T+1)(T^6+T^5+\cdots+1)$;
$c=3,\ b=4,\ a=4,\ \chi=T^{-1}(T^4+1)^2(T^6+T^3+1)$.

iii) $a=b+1$. Then $\chi(T)=T^{-1}(T^{b+c+1}-1)(T^{b+c+2}-1)(T^{b+1}-1)/(T^b-1)(T^c-1)$. Thus $b\geq c$ implies either iii)$_1$ $b=c+2$, iii)$_2$ $b=c+1$, iii)$_3$ $b=2$ or iii)$_4$ $b=c=1$.

iii)$_1$ $b=c+2$. Then $\chi(T)=T^{-1}(T^{2c+3}-1)(T^{c+2}+1)(T^{c+3}+1)/(T^c-1)$ so that either $c|4$ or $c|6$. Thus we have one of the following:

$c=1,\ c=3,\ a=4,\ \chi=T^{-1}(T^4+T^3+\cdots+1)(T^3+1)(T^4+1)$;
$c=2,\ c=4,\ a=5,$
$$\chi=T^{-1}(T^6+T^5+\cdots+1)(T^4+1)(T^4-T^3+T^2-T+1);$$
$c=3,\ c=5,\ a=6,\ \chi=T^{-1}(T^6+T^3+1)(T^5+1)(T^6+1)$;
$c=4,\ c=6,\ a=7,\ \chi=T^{-1}(T^{10}+T^9+\cdots+1)(T^4-T^2+1)$
$$\times(T^6-T^5+\cdots+1);$$
$c=6,\ c=8,\ a=9,\ \chi=T^{-1}(T^{12}+T^9+\cdots+1)(T^8+1)(T^6-T^3+1)$.

iii)$_2$ $b=c+1$. Then $\chi(T)=T^{-1}(T^{c+1}+1)(T^{2c+3}-1)(T^{c+2}+1)/(T^c-1)$ so that either $c|3$ or $c|4$. Thus we have the following:

$c=1,\ b=2,\ a=3,\ \chi=T^{-1}(T^2+1)(T^3+1)(T^4+T^3+\cdots+1)$;
$c=2,\ b=3,\ a=4,\ \chi=T^{-1}(T^2+T+1)(T^6+T^5+\cdots+1)(T^4+1)$;
$c=3,\ b=4,\ a=5,\ \chi=T^{-1}(T^4+1)(T^5+1)(T^6+T^3+1)$;
$c=4,\ b=5,\ a=6,\ \chi=T^{-1}(T^4-T^3+\cdots+1)(T^{10}+T^9+\cdots+1)$
(T^4-T^2+1).

iii)$_3$ $b=2$. Then $\chi(T)=T^{-1}(T^3+1)(T^{c+4}-1)(T^{c+3}-1)/(T^2-1)\times(T^c-1)$ so that either $c|3$, $c|4$ or $c|6$ and $c\leq b=2$. Thus either

$c=1,\ b=2,\ a=3$, and $\chi=T^{-1}(T^3+1)(T^4+T^3+\cdots+1)(T^2+1)$, or
$c=2,\ b=2,\ a=3$, and
$$\chi=T^{-1}(T^2-T+1)(T^4+T^2+1)(T^4+T^3+\cdots+1).$$

iii)$_4$ $c=1,\ b=1,\ a=2$ and $\chi=T^{-1}(T^2+T+1)(T^3+T^2+T+1)\times(T^2+1)$.

iv) $a=c+1$. Since $a\geq b\geq c$, then either iv)$_1$ $a=b=c+1$ or iv)$_2$ $a=b+1,\ b=c$. Thus iv)$_1$ is reduced to ii) and iv)$_2$ is reduced to iii).

v) $a=b=c=1$ and $\chi=T^{-1}(T^2+T+1)^3$.

This completes the proof for Table 3.

(2.6) Before closing this paragraph, let us give a numerical equality, which will explain a dimensional relation among certain moduli spaces (cf. (3.6) Assertion).

Assertion. *Let $(a, b, c; h)$ be a regular system of weights which is not of type A_l. Then we have,*

(2.6.1) $$a_{-\varepsilon}=a_{h+\varepsilon}=N(h)-N(a)-N(b)-N(c)$$

where $a_{-\varepsilon}$ is the multiplicity of exponents equal to $-\varepsilon$ and $N(k)$ are defined in (1.5).

Proof. Due to the duality (1.3.7), we have only to show,

$$a_{h+\varepsilon}=N(h)-N(a)-N(b)-N(c).$$

Due to the classification Table 1, an inequality $h+\varepsilon<h+\inf(a, b, c)$ holds except in the case of type A_l. Then the formula (1.9.1) obviously implies the Assertion. q.e.d.

Notation. We shall denote the number $a_{-\varepsilon}=a_{h+\varepsilon}$ by m_0. (cf. (3.6) Assertion 1 and (4.1.2))

§ 3. Quotient singularities

For any regular system of weights $(a, b, c; h)$, there exists weighted homogeneous polynomials $f(x, y, z)$ of degree h with $\deg x=a$, $\deg y=b$ and $\deg z=c$ such that the hypersurface $X_0:=\{(x, y, z) \in \boldsymbol{C}^3: f(x, y, z)=0\}$ has singular points only at the origin. ((3.2) Theorem)

According as $\varepsilon=1, 0$ or -1, the 2-manifold $X_0-\{0\}$ is a quotient variety by the free action of the discrete subgroups of the motion groups of $\boldsymbol{P}^1(\boldsymbol{C})$, $\boldsymbol{C}$ or $\boldsymbol{H}:=\{z \in \boldsymbol{C}: \operatorname{Im}(z)>0\}$ on their canonical, trivial or anti-canonical $\boldsymbol{C}^*$-bundles. ((3.4) Note.)

(3.1) Let $(a, b, c; h)$ be a system of weights. The polynomial ring $R:=\boldsymbol{C}[x, y, z]$ in three variables x, y and z has a natural graded ring structure,

(3.1.1)
$$R=\bigoplus_{n=0}^{\infty} R_n$$

by weights $\deg x=a$, $\deg y=b$ and $\deg z=c$. Here R_n is spanned by the monomials $x^p y^q z^r$ s.t. $pa+qb+rc=n$, whose rank is equal to $m(n)$ (recall (1.5)).

We shall denote by,

(3.1.2)
$$R_+:=\bigoplus_{n>0} R_n,$$

the maximal ideal of R generated by x, y and z.

(3.2) **Theorem 3.** *For a system of weights $(a, b, c; h)$ the following three statements are equivalent.*

i) *The system of weights is regular.*

ii) *There exists a polynomial $f(x, y, z) \in R_h$ of degree h, with the following property:*

(3.2.1) *The partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ form a regular sequence in the ring R.*

iii) *The set $U:=\{f \in R_h: f$ satisfies (3.2.1)$\}$ is a non-void Zariski open subset of R_h.*

Note [34]. *For a polynomial f satisfying* (3.2.1), *put*

(3.2.2)
$$A_f:=R\Big/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)R.$$

Then A_f is a graded module of finite rank μ over $\boldsymbol{C}$, whose Poincarè polynomial (*cf.* [5, Ch. v, § 5]) *is $T^{-\varepsilon}\chi(T)$.*

Proof. The equivalence of ii) and iii) is an immediate consequence of a more general assertion: the condition for three homogeneous elements f_1, f_2 and f_3 of degrees u, v and w in R_+ to become a regular sequence is an open condition in $R_u \times R_v \times R_w$ for $u, v, w \in N$.

Let us show that i) implies ii).

Due to (1.6) Assertion 2, the space R_h contains a monomial of the form x^{u+1}, $x^u y$ or $x^u z$ for some $u \in N$. (Also R_h contains y^{v+1}, $y^v z$ or $y^v x$ for some $v \in N$ and z^{w+1}, $z^w x$ or $z^w y$ for some $w \in N$.)

A direct calculation shows that some linear combinations of these monomials such as $x^{u+1}+y^{v+1}+z^{w+1}$, $x^{u+1}+y^{v+1}+z^w y$, $x^{u+1}+y^v x+z^w y$, $x^{u+1}+y^v x+z^w y$, $x^u z+y^v x+z^w y$ and linear combinations which are obtained from the above by a permutation of the role of x, y and z, satisfy already the condition (3.2.1) so that there is nothing to prove.

On the other hand, the other types of linear combinations such as $x^u z+y^v z+z^{w+1}$, $x^u z+y^v z+z^w y$ or $x^u z+y^v z+z^w x$ does not satisfy (3.2.1), since they are divisible by z.

This happens, when

$$*)\qquad a\,|\,(h-c) \quad \text{and} \quad b\,|\,(h-c).$$

Since $\gcd(a, b)\,|\,h$ ((1.6) Assertion 4), the above *) implies $\gcd(a, b)\,|\,c$ so that $\gcd(a, b)=\gcd(a, b, c, h)$. Since we may reduce the problem to the primitive system of weights, we shall assume from now on that $\gcd(a, b)=1$. Then *) implies that $\exists d \in N$ s.t.

$$h=c+dab.$$

Furthermore $\gcd(a, b)=1$ implies that $\exists p, q \in Z$ s.t.

$$h=pa+qb.$$

By modifying $h=(p-kb)a+(q+ka)b$ for $k \in Z$, if necessary we may assume that $p \geq 0$, $q \geq 0$ ($\because h>dab\geq ab$.). Thus R_h contains a monomial $x^p y^q$.

Due to (1.6) Assertion 2, we consider the following three cases.

Case **1.** $h=(w+1)c$. Put $f=(x^{db}+y^{da})z+x^p y^q+z^{w+1}$.

A direct calculation shows that $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$ is not a regular sequence of R_+, iff

$$**)\qquad c^c(dab)^{dab}=(-1)^h(pa)^{pa}(qb)^{qb}.$$

Since $c+dab=h=pa+qb$ and since the function $\varphi(t):=t^t(h-t)^{h-t}$ ($t \in (0, h)$) is convex, the equality **) implies that $h=$even and either $c=pa$ or $c=qb$. This implies either a divides $c+dab=h$ or b divides $c+dab=h$.

Then by replacing the monomial $x^u z$ or $y^y z$ by the new monomial $x^{h/a}$ or $y^{h/b}$ in f, we obtain a new polynomial satisfying (3.2.1).

Case **2.** $h=a+wc$. Put $f=(x^{db}+y^{da})z+x^p y^q+z^w x$.

One may assume $w>1$, since otherwise $db=1$ and we may replace the monomial $y^v z$ by y^h in f.

A direct calculation shows that $(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$ is not a regular sequence of R_+, iff

$$***)\qquad (-1)^{dw} d^d (d(w-1))^{d(w-1)} = \left(\frac{(w-1)q}{a}\right)^{((w-1)q)/a} \left(\frac{wp-p+1)}{b}\right)^{(wp-p+1)/b}$$

Since

$$d+d(w-1)=dw=\frac{(w-1)q}{a}+\frac{wp-p+1}{b}$$

and since the function $\varphi(t):=t^t\cdot(dw-t)^{dw-t}$ $(t\in(0, dw))$ is convex, the equality ***) implies that $dw=$even and either $d(w-1)=((w-1)q)/a$ or $d(w-1)=(wp-p+1)/b$. If $d(w-1)=((w-1)q)/a$, then $ad=q$ so that a divides $pa+qb=h$. Then by adding new monomial $x^{h/a}$ instead of $x^u z$ to f, we obtain a polynomial satisfying (3.2.1).

If $d(w-1)=(wp-p+1)/b$, then $(bd-p)(w-1)=1$, so that $w=2$ and $bd=p+1$. Using these relations $c=h-a-(w-1)c=(h-c)-a=abd-a=ap$. Therefore a divides $c+abd=h$. Then by replacing the monomial $x^u z$ by $x^{h/a}$ in f, we obtain a polynomial satisfying (3.2.1).

Case **3.** $h=b+wc$.

We can reduce this case to Case 2 by changing the role of x and y.

This completes the proof of the implication i)$\Rightarrow$ii).

To show that ii) (or iii)) implies i), it is enough to show the Note. ($\because$ If $T^{-\varepsilon}\chi(T)$ is a polynomial, $\chi(T)$ can have poles only at $T=0$.)

Proof of Note. Obviously the Poincaré series for R is $1/(1-T^a)(1-T^b)(1-T^c)$. Since the ideal $(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$ is generated by a regular sequence which are homogeneous of degrees $h-a$, $h-b$ and $h-c$, a generality on Poincaré series (cf. [5]) implies that the Poincaré polynomial for A_f is $(T^{h-a}-1)(T^{h-b}-1)(T^{h-c}-1)/(T^a-1)(T^b-1)(T^c-1)=T^{-\varepsilon}\chi(T)$. q.e.d.

Note **1.** Geometrically, the property (3.2.1) is equivalent to the fact that the affine variety,

$$(3.2.4)\qquad X_0:=\{(x, y, z)\in \boldsymbol{C}^3: f(x, y, z)=0\}$$

has an isolated singular point at the origin 0, where μ is the so called Milnor number. ([32] [34])

The exponents for a hypersurface singular point are introduced in [51, (5.3)] (cf. [44, § 1]). Then the exponents $\alpha_1, \cdots, \alpha_\mu$ for the singular point X_0 and the exponents $m_1, \cdots, m_\mu$ for the regular system $(a, b, c; h)$ defined in this paper are related by

$$\alpha_i - 1 = m_i/h \quad (i = 1, \cdots, \mu)$$

(cf. [44, (3.7.1)]).

The eigenvalues of the Milnor monodromy [32] of the singular point is given by $\exp(2\pi\sqrt{-1}\, m_i/h)$ $(i = 1, \cdots, \mu)$.

Note **2.** Put

$$\Omega_f := \Omega^3_{C^3}/df \wedge \Omega^2_{C^3},$$

where $\boldsymbol{C}^3 := \operatorname{Spec} R$. Then Ω_f is a graded vector space of rank μ whose Poincaré polynomial is $T^h \chi(T)$.

There is a nondegenerate residue pairing, (cf. [69, III § 9])

$$J\colon \Omega_f \times \Omega_f \to \boldsymbol{C}, \quad (\varphi(x)dx, \psi(x)dx) \longmapsto \operatorname{Res}\left[\begin{matrix} \varphi(\underline{x})\psi(\underline{x})d\underline{x} \\ \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z} \end{matrix}\right].$$

(3.3) In the following Table 4, we give a weighted homogeneous polynomial $f(x, y, z, \underline{\lambda})$ of degree h in three variable x, y and z of degrees a, b and c respectively with parameters $\underline{\lambda} = (\lambda_1, \cdots, \lambda_{m_0})$ satisfying the condition (3.2.1) of (3.2) Theorem 2, for each regular system of weights $(a, b, c; h)$ with $\varepsilon = 1, 0, -1$ (which are classified in Tables 1, 2 and 3 in § 2).

The meaning of the parameter $\underline{\lambda}$ will be explained in (3.6). The data $m_+ + m_0 + m_-$ and $\mu_+ + \mu_0 + \mu_-$ will be explained in (4.1.2) and (4.5.3).

Table 4.

1. *Case* $\varepsilon = 1$

Notation	Polynomial	$m_+ + m_0 + m_-$	$\mu_+ + \mu_0 + \mu_-$
A_l	$x^{l+1} + yz$	$l+0+0$	$0+0+l$
D_l	$x^{l-1} + xy^2 + z^2$	$l+0+0$	$0+0+l$
E_6	$x^4 + y^3 + z^2$	$6+0+0$	$0+0+6$
E_7	$x^3y + y^3 + z^2$	$7+0+0$	$0+0+7$
E_8	$x^5 + y^3 + z^2$	$8+0+0$	$0+0+8$

2. *Case* $\varepsilon=0$

Notation	Polynomial		$m_+ + m_0 + m_-$	$\mu_+ + \mu_0 + \mu_-$
$\tilde{E}_6$	$zy^2-x(x-z)(x-\lambda z)$	$\lambda\neq 0, 1$	$7+1+0$	$0+2+6$
$\tilde{E}_7$	$xy(x-y)(x-\lambda y)+z^2$	$\lambda\neq 0, 1$	$8+1+0$	$0+2+7$
$\tilde{E}_8$	$y(y-x^2)(y-\lambda x^2)+z^2$	$\lambda\neq 0, 1$	$9+1+0$	$0+2+8$

3. *Case* $\varepsilon=-1$

Notation	Polynomial		$m_+ + m_0 + m_-$	$\mu_+ + \mu_0 + \mu_-$
E_{12}	$x^7+y^3+z^2$		$11+0+1$	$2+0+10$
E_{13}	$yx^5+y^3+z^2$		$12+0+1$	$2+0+11$
E_{14}	$x^8+y^3+z^2$		$13+0+1$	$2+0+12$
E_{11}	$x^5+xy^3+z^2$		$10+0+1$	$2+0+9$
Z_{12}	$yx^4+xy^3+z^3$		$11+0+1$	$2+0+10$
Z_{13}	$x^6+xy^3+z^2$		$12+0+1$	$2+0+11$
W_{12}	$x^5+y^4+z^2$		$11+0+1$	$2+0+10$
W_{13}	$yx^4+y^4+z^2$		$12+0+1$	$2+0+11$
Q_{10}	$x^4+y^3+xz^2$		$9+0+1$	$2+0+8$
Q_{11}	$x^3y+y^3+xz^2$		$10+0+1$	$2+0+9$
Q_{12}	$x^5+y^3+xz^2$		$11+0+1$	$2+0+10$
S_{11}	$x^4+y^2z+z^2x$		$10+0+1$	$2+0+9$
S_{12}	$x^3y+y^2z+z^2x$		$11+0+1$	$2+0+10$
U_{12}	$x^4+y^3+z^3$		$11+0+1$	$2+0+10$
$R_{269}(J_{3,0})$	$y(y-x^3)(y-\lambda x^3)+z^2$	$\lambda\neq 0, 1$	$14+1+1$	$2+0+14$
$R_{297}(Z_{1,0})$	$xy(y-x^2)(y-\lambda x^2)+z^2$	$\lambda\neq 0, 1$	$13+1+1$	$2+0+13$
$R_{245}(Q_{2,0})$	$y(y-x^2)(y-\lambda x^2)+z^2$	$\lambda\neq 0, 1$	$12+1+1$	$2+0+12$
$R_{236}(W_{1,0})$	$(y^2-x^3)(y^2-\lambda x^3)+z^2$	$\lambda\neq 0, 1$	$13+1+1$	$2+0+13$
$R_{234}(S_{1,0})$	$x(y-x^2)(y-\lambda x^2)+yz^2$	$\lambda\neq 0, 1$	$12+1+1$	$2+0+12$
$R_{233}(U_{1,0})$	$x^3y+z(z-y)(z-\lambda y)$	$\lambda\neq 0, 1$	$12+1+1$	$2+0+12$
$R_{223}(V_{1,0})$	$y(y-x)(y-\lambda_1 x)(y-\lambda_2 x)+xz^2$	$\lambda_i\neq 0, 1,\ \lambda_1\neq\lambda_2$	$12+2+1$	$2+0+13$
$R_{225}(N_{16})$	$xy(x-y)(y-\lambda_1 x)(y-\lambda_2 x)+z^2$	$\lambda_2\neq 0, 1,\ \lambda_1\neq\lambda_2$	$13+2+1$	$2+0+14$
R_{146}	$y(y-x^4)(y-\lambda x^4)+z^2$	$\lambda\neq 0, 1$	$19+1+2$	$2+2+18$
R_{135}	$xy(y-x^3)(y-\lambda x^3)+z^2$	$\lambda\neq 0, 1$	$18+1+2$	$2+2+17$
R_{134}	$y(y-x^3)(y-\lambda x^3)+xz^2$	$\lambda\neq 0, 1$	$17+1+2$	$2+2+16$
R_{124}	$y(y-\lambda x^2)(y-\lambda_1 x^2)(y-\lambda_2 x^2)+z^2$	$\lambda_i\neq 0, 1,\ \lambda_1\neq\lambda_2$	$17+2+2$	$2+2+17$

R_{123}	$y^2z+xz^2+xy(x^2-\lambda_1 y)(x^2-\lambda_2 y)$	$16+2+2$	$2+2+16$
R_{122}	$yz(y-z)+x^4y$	$15+3+2$	$2+2+16$
R_{113}	$xy(x-y)(x-\lambda_1 y)(x-\lambda_2 y)(x-\lambda_3 y)+z^2$ $\lambda_i\neq 0, 1, \lambda_i\neq\lambda_j$	$19+3+3$	$2+4+19$
R_{112}	$z^2x+zy^3+y(x-\lambda_1 y)(x-\lambda_2 y)(x-\lambda_3 y)(x-\lambda_4 y)$ $\lambda_i\neq\lambda_j$	$17+4+3$	$2+4+18$
	$z^2x+y(x-\lambda_1 y)(x-\lambda_2 y)(x-\lambda_4 y)$ $\lambda_i\neq 0, \lambda_i\neq\lambda_j$	$17+4+3$	$2+4+18$
R_{111}	$f_4(x, y, z, \underset{\sim}{\lambda})$	$17+6+4$	$2+6+19$

where f_4 is a homogeneous polynomial of degree 4 in (x, y, z) with 6-dimensional parameter $(\underset{\sim}{\lambda})$, which defines a 6-dimensional family of smooth quartic curves.

(3.4) *Note.* It is remarkable that many of the polynomials in Table 4 above are already known as the relation polynomials among invariant functions for certain discrete groups as follows.

Case. $\varepsilon=1$

The polynomials for the types E_6, E_7 and E_8 first appeared in H.A. Schwarz [57] and then the cases A_l and D_l were completed by F. Klein [26] [19].

Namely the polynomial appeared as a relation among three homogeneous generators x, y and z of the ring of invariant polynomials in two variables (u, v) of $\boldsymbol{C}^2$, on which a finite subgroup Γ of $SL(2, \boldsymbol{C})$ is acting. Here Γ is the binary icosahedral group for E_8, the binary octahedral group for E_7, the binary tetrahedral group for E_6, a binary dihedral group for D_l and a cyclic group for A_l.

As is well known, these notations A_l, D_l, E_6, E_7, E_8 are exactly those for the Dynkin diagrams for finite root systems having no multiple bonds (cf. [5] [8]).

Case. $\varepsilon=0$

The polynomials appeared as equations for simple elliptic singularities in [47].

Namely they appeared as a relation among three homogeneous generators x, y and z of the ring of regular functions on a negative line bundle over an elliptic curve defined in the Legendre normal form. Here the Chern class of the line bundle is equal to -3, -2 and -1 for $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$, respectively. In other words, the polynomials may be regarded as relations among three homogeneous generators of the ring of invariant holomorphic function on $\boldsymbol{C}^2$ under the action of the Heisenberg groups corresponding to $-1, -2, -3 \in \mathrm{Ext}^2(\boldsymbol{Z}^2, \boldsymbol{Z}) \simeq \boldsymbol{Z}$. If the Chern class is less or equal than -4, then the invariant ring has more than 4 generators so that we disclude the case from our consideration.

These three cases correspond to the three Dynkin diagrams of types $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ in [53], which are exactly the diagrams for extended affine root systems without multiple bonds and of codimension one.

Case. $\varepsilon = -1$

The first 14 and the following 6 polynomials appeared as weighted homogeneous equations for 14 exceptional unimodal singularities and some of the bimodal singularities by V.I. Arnold [2] [3]. Then I.V. Dolgachev has shown that the singularities are canonical triangle or quadrangle singularties [13] [14].

Namely the polynomials are the relations among three homogeneous generators x, y and z of the invariant subring under the action of a finite group Γ/Γ' on the ring of regular function on the anticanonical bundle over a compact smooth curve $\boldsymbol{H}/\Gamma'$, where Γ is a Fuchsian group, having a triangle or a quadrangle as a half of the fundamental domain, acting on the upper half plane $\boldsymbol{H}$, and where Γ' is a finite index normal subgroup of Γ without fixed points on $\boldsymbol{H}$.

Particularly in the case E_{12} of $(a/h, b/h, c/h) = (1/2, 1/3, 1/7)$, $\boldsymbol{H}/\Gamma'$ is the curve of F. Klein of genus 3 s.t. $\Gamma/\Gamma' = \mathrm{Aut}(\boldsymbol{H}/\Gamma') \simeq \mathrm{PSL}(2, F_7)$ is the simple group of order 168: the maximal possible order for a genus 3 curve. This case was studied by F. Klein [26].

Including these 14+6 cases, the following theorem holds.

(3.5) **Theorem 4.** *Let $f(x, y, z)$ be a weighted homogeneous polynomial in R_h satisfying* (3.2.1), *for a regular system of weights with $\varepsilon = -1$. Then there exists a Fuchsian group $\Gamma \subset \mathrm{SL}(2, \boldsymbol{R})/\{\pm 1\}$ of the first kind, which is related to f in the following way.*

The binary extension $\tilde{\Gamma} \subset \mathrm{SL}(2, \boldsymbol{R})$ of Γ acts on the space $\tilde{\boldsymbol{H}} := \{(u, v) \in \boldsymbol{C}^2 : \mathrm{Im}(u/v) > 0\}$ properly discontinuously in the natural manner,

$$g(u, v) := (au + bv, cu + dv) \qquad \textit{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma},$$

s.t. there exist three homogeneous generators x, y and z for the ring of invariant holomorphic functions on $\tilde{\boldsymbol{H}}$ under the action of $\tilde{\Gamma}$, whose relations are generated by

$$f(x, y, z) = 0.$$

(Here a function $x(u, v)$ on $\tilde{\boldsymbol{H}}$ is homogeneous of degree a iff $(u\partial/\partial u + v\partial/\partial v)x = ax$.)

Theorem 5. *Conversely, if a polynomial $f(x, y, z)$ is obtained as in Theorem 4 for a suitable Fuchsian group of the first kind, then by putting $a:=\deg x$, $b:=\deg y$, $c:=\deg z$ and $h:=\deg f$, the $(a, b, c; h)$ is a regular system of weights, whose index ε is equal to -1.*

The proofs of these theorems will be given in (5.5).

P. Wagreich has classified the index of the Fuchsian group G of the first kind, whose ring A_G of automorphic forms is generated by 3 elements, and gave the table of the degrees of the generators and the relation polynomial [64, Theorem (3.1), (4.6), Table 1].

Therefore the above theorems assert that the table for regular systems of weights with $\varepsilon=-1$ coincides with the table given by Wagreich. Note that furthermore the above theorems reprove the theorem of Wagreich, since in our formulation any polynomials f of R_h satisfying (3.2.1) corresponds to a Fuchsian group of the first kind.

(3.6) Parameters $\lambda_1, \cdots, \lambda_{m_0}$.

(3.6.1) Let $$G:=\left\{(P, Q, R) \in R_a \times R_b \times R_c : \det \frac{\partial(P, Q, R)}{\partial(x, y, z)} \neq 0\right\}$$

be the group of automorphisms of the graded ring R (3.1.1), whose Lie algebra and the dimension is given by

(3.6.2) $$\mathfrak{G}:=\left\{P\frac{\partial}{\partial x}+Q\frac{\partial}{\partial y}+R\frac{\partial}{\partial z} : P \in R_a,\ Q \in R_b,\ R \in R_c\right\}.$$

(3.6.3) $$\dim_C G=\dim_C \mathfrak{G}=N(a)+N(b)+N(c).$$

Let $(a, b, c; h)$ be regular and let U be the domain in R_h of all polynomials $f \in R_h$ which define isolated singular points at 0, as in (3.2) Lemma 1. The group G acts on R_h and hence on U in an obvious manner.

Assertion 1. *Except in the case of type A_l,*

i) *the isotropy subgroup of G at any point of U is finite.*

ii) *U/G is an analytic variety of dimension equal to $m_0:=a_{-\varepsilon}$* (cf. (2.6)).

2. *The parameters $\lambda_1, \cdots, \lambda_{m_0}$ in the polynomials in Table 4 describe an m_0-dimensional section of $U \to U/G$.*

Proof. **1.** i) Since the group and its action are algebraic, we have only to show that any isotropy group has dimension 0.

If the isotropy subgroup at f had a positive dimension, then there exists $\theta=P(\partial/\partial x)+Q(\partial/\partial y)+R(\partial/\partial z) \in \mathfrak{G}$ s.t. $\theta f=0$, i.e.

$$P\frac{\partial f}{\partial x}+Q\frac{\partial f}{\partial y}+R\frac{\partial f}{\partial z}=0.$$

Since $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$ is a regular sequence, there eixst $A, B, C \in R$ (or in the localization of R by the maximal ideal), s.t.

$$*)\quad \begin{cases} P=C\dfrac{\partial f}{\partial y}-B\dfrac{\partial f}{\partial z}, \\ Q=A\dfrac{\partial f}{\partial z}-C\dfrac{\partial f}{\partial x}, \\ R=B\dfrac{\partial f}{\partial x}-A\dfrac{\partial f}{\partial y}. \end{cases}$$

By comparing homogeneous parts of the relation, one may assume that A, B and C are homogeneous with $\deg A=b+c-h=\varepsilon-a$, $\deg B=c+a-h=\varepsilon-b$, and $\deg C=a+b-h=\varepsilon-c$.

If one of these degrees, say $\varepsilon-c$, is non-negative, then $\varepsilon\geq c>0$. According to the classification in Table 4, this is possible only when $(a, b, c; h)$ is of type A_l.

ii) The same argument as in i) shows that the orbits of G in U are closed so that the action of G on U is proper. Due to a theorem of Holmann [20], U/G has the structure of an analytic space, whose dimension is

$$\begin{aligned} \dim U/G &= \dim U - \dim G \\ &= m(h)-m(a)-m(b)-m(c) \\ &= a_{-\varepsilon}=: m_0 \end{aligned}$$

(cf. (2.6) Assertion).

2. This is a consequence of rather cumbersome case-by-case calculations, which we omit in this paper. q.e.d.

§ 4. Universal unfoldings and Hamiltonian systems

In this paragraph, we recall and fix the notions for universal unfoldings and related subjects to explain our main motivation (4.3) of the present paper. Some hurrying readers may skip this paragraph and come back if it is necessary as a reference.

(4.1) **Universal unfoldings.** Let us introduce a polynomial $F(x, y, z, \underline{\mu}, \underline{\lambda}, \underline{\nu})$ for every regular system of weights (classified in Section 2), which is called the *universal unfolding of the singularity f* by R. Thom [61].

(4.1.1) $$F(\underline{x}, \underline{\mu}, \underline{\lambda}, \underline{\nu}):=f(\underline{x}, \underline{\lambda})+\sum_{j=1}^{m_+}\mu_j G_j(\underline{x}, \underline{\lambda})+\sum_{j=1}^{m_-}\nu_j H_j(\underline{x}, \underline{\lambda})$$

Here

i) $f(\underline{x}, \underline{\lambda})$ is a polynomial in Table 4.

ii) $G_1, \cdots, G_{m_+}$ (resp. $H_1, \cdots, H_{m_-}$) are homogeneous polynomials in $R \otimes_C C[\underline{\lambda}]$, representing certain homogeneous $C[\underline{\lambda}]_g$ free basis of

$$R \otimes C[\underline{\lambda}]_g \Big/ \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

of degrees $<h$ (resp. degrees $>h$). Here $g = g(\lambda)$ is a polynomial in λ, whose zero locus describes the exceptional values of λ given in the Table 4 and $C[\lambda]_g$ is the localization by g.

Recall (3.2) Lemma 2 and (1.4) Definition, so that

$$R \otimes C[\underline{\lambda}]_g \Big/ \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

is a graded $C[\underline{\lambda}]_g$-free module of rank μ and it splits into a direct sum of submodules $M_{<h}$, M_h and $M_{>h}$, where $M_{<h}$ (resp. $M_{>h}$, resp. $M_{>h}$) consists of elements of degree $<h$ (resp. $=h$, resp. $>h$) whose rank is equal to m_+ (resp. m_0, resp. m_-) given by,

$$\begin{aligned} m_+ &:= \sum_{j<h+\varepsilon} a_j = \sum_{j>-\varepsilon} a_j, \\ m_0 &:= a_{h+\varepsilon} = a_{-\varepsilon}, \\ m_- &:= \sum_{j>h+\varepsilon} a_j = \sum_{j<-\varepsilon} a_j, \end{aligned} \tag{4.1.2}$$

such that

$$\mu = m_+ + m_0 + m_-. \tag{4.1.3}$$

We shall assume

$$G_1 = 1. \tag{4.1.4}$$

By definition $F(\underline{x}, \underline{\mu}, \underline{\lambda}, \underline{\nu})$ is a weighted homogeneous polynomial of degree h with respect to the weights

$$\begin{aligned} &\deg \mu_j := h - \deg G_j > 0 && \text{for } j = 1, \cdots, m_+, \\ &\deg \lambda_j := 0 && \text{for } j = 1, \cdots, m_0, \\ &\deg \nu_j ;= h - \deg H_j < 0 && \text{for } j = 1, \cdots, m_-. \end{aligned} \tag{4.1.5}$$

Note **1.** Here we modified slightly the original definition of the universal unfolding by R. Thom [61].

Namely we added the base $G_1 = 1$ in (4.1.1), so that the number of parameters in F is equal to μ instead of $\mu-1$, which is called the codimension by R. Thom.

This modification is necessary to introduce a *Hamiltonian system*, which leads us to *the flat structure* on the space of the parameters $(\underline{\mu}, \underline{\lambda}, \underline{\nu})$ (cf. (4.3)).

Note **2.** The equation $F(x, y, z, 0, \underline{\lambda}, \underline{\nu})=0$ defines a family of hypersurfaces with an isolated singular point at the origin, whose Milnor number (cf. [32]) is constantly μ. The dimension m_0+m_- of the parameters is called the inner modality by V.I. Arnold [2].

(4.2) **The flat family** φ: $X\to S$.

Using the universal unfolding (4.1.1), let us define a flat family of affine algebraic surfaces,

$$\varphi: X^{\mu+2} \longrightarrow S^{\mu} \tag{4.2.1}$$

(the superscript denoting the dimension of the variety),

where

i) *S^{μ} is a smooth affine variety, given by*

$$S^{\mu}:=\boldsymbol{C}^{m_+}\times S_0^{m_0}\times \boldsymbol{C}^{m_-} \tag{4.2.2}$$

where $\boldsymbol{C}^{m_+}$ and $\boldsymbol{C}^{m_-}$ are the spaces of the coordinates $(\mu_1, \cdots, \mu_{m_+})$ and $(\nu_1, \cdots, \nu_{m_-})$ respectively, and

$$\{S_0:=\{(\underline{\lambda}) \in \boldsymbol{C}^{m_0}: g(\underline{\lambda})\neq 0\}. \tag{4.2.3}$$

ii) *$X^{\mu+2}$ is a smooth affine hypersurface in $Z:=\boldsymbol{C}^3\times S$, defined by*

$$X:=\{(\underline{x}, \underline{\mu}, \underline{\lambda}, \underline{\nu}) \in \boldsymbol{C}^3\times S: F(\underline{x}, \underline{\mu}, \underline{\lambda}, \underline{\nu})=0\}. \tag{4.2.4}$$

iii) *The map φ: $X\to S$ is induced from the projection,*

$$p: Z^{\mu+3}:=\boldsymbol{C}^3\times S^{\mu}\to S^{\mu}. \tag{4.2.5}$$

By definition,

i) *φ is flat.*

ii) *The fiber $X_t:=\varphi^{-1}(t)$ over a point $t=(\underline{\mu}, \underline{\lambda}, \underline{\nu}) \in S$ is an affine hypersurface in $\boldsymbol{C}^3$ defined by the equation $F(x, y, z, \underline{\mu}, \underline{\lambda}, \underline{\nu})=0$.*

iii) *The spaces S, Z and X admit the action of $c \in \boldsymbol{C}^*$ defined as the multiplication of the power of c for each coordinate $\underline{x}$, $\underline{\mu}$, $\underline{\lambda}$ and $\underline{\nu}$ according as the degree, so that the map φ is equivariant with respect to the $\boldsymbol{C}^*$-action.* (Compare [45, 2.3].)

(4.3) The above defined family $\varphi: X \to S$ has a structure called a Hamiltonian system ([50], [51]) or a gauge structure ([41]), which leads us to a study of a period mapping by a use of a primitive form ([51], § 3]).

As explained in the introduction the present paper is motivated for a study of the inverse map of the period mapping. Let us briefly formulate the problem and explain the current stage of the answer to the problem.

The primitive form induces the following two mappings.

i) **A flat embedding** ([51, (3.3), 1)]),

$$S^{\mu} \longrightarrow \Omega_f^{\mu}$$

which is a C^*-equivariant locally biregular embedding of S into a domain in a weighted vector space Ω_f carrying a non-degenerate bilinear form J (the residue pairing) (cf. (3.2) *Note* **2**). (The embedding is defined by an integrable torsion free connection ∇ with $\nabla J=0$, which is the leading term of the Gauß-Manin connection for the family $X \to S$.)

ii) **A period mapping** ([51, (5.7)]),

$$\tilde{S}^{\mu} \longrightarrow E^{\mu}$$

which is a locally biregular embedding of the "*monodromy covering space*" $\tilde{S}$ of S into "*the exponent shifted*" (*co-*)*homology group* E with the intersection from I. (The period mapping is defined by the solutions of "the exponent shifted" Gauß-Manin connection for the family $X \to S$.)

Problem. *Describe the inverse map from the period domain in* E *to* Ω_f, *which makes the diagram* $\begin{array}{ccc} \tilde{S} & \to & E \\ \downarrow & & \downarrow \\ S & \to & \Omega_f \end{array}$ *commutative.*

Precisely, the problem asks to construct a homogeneous generater system of degrees $m_1+\varepsilon, \cdots, m_{\mu}+\varepsilon$ of the ring of monodromy invariant automorphic functions on the domain of the period in E, which are identified with a linear coordinate system of Ω_f.

Case $\varepsilon=1$. In this case the monodromy groups are the finite Weyl groups. Including these cases, for all finite Coxeter groups, the map $E \to \Omega_f$ is constructed explicitly in [49] case by case and in [48] without using the classification of the groups (cf. [21] [22] [25] [36] [37] [67]).

Recently P. Orlik found some structure similar to the flat structure for certain unitary reflexion groups. ([40])

Case $\varepsilon=0$. In this case the map $E \to \Omega_f$ is completely determined and described in terms of *extended affine root system* [53], whose flat invariants will be studied in a forthcoming paper. It has close relationship with the representation theory of affine Kac-Moody Lie algebras [23] [24] [29]. Another explicit description of flat coordinates is given in [37].

There is not yet a systematic study of the flat structure for the case $\varepsilon=-1$ and the present note is a first attempt in this direction.

A partial result on the root lattice is given in [9] [10] [30] [55].

(4.4) **Discriminant.** As a first step toward the analysis of the Hamiltonian system, let us recall the notion of the discriminant. Notations are as in (4.3).

Assertion 1. *Consider two maps,*

$$\hat{\Phi}:=\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, p\right):\ Z^{\mu+3}\to \boldsymbol{C}^3\times S^{\mu},$$
(4.4.1)
$$\Phi:=\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, \pi\circ\varphi\right):\ X^{\mu+2}\to \boldsymbol{C}^3\times T^{\mu-1}$$

where $T:=S_0\times \boldsymbol{C}^{m_+-1}$ and $\pi: S\to T$ is the natural projection forgetting μ_1.

They are flat finite maps of degree μ, so that $\hat{\Phi}_*\mathcal{O}_Z$ (resp. $\hat{\Phi}_*\mathcal{O}_X$) are $\mathcal{O}_{C^3\times S}$- (resp. $\mathcal{O}_{C^3\times S}$-) free modules of rank μ.

Proof. Obvious, since the maps are defined as the perturbations of the regular sequence $\partial F/\partial x(\underline{x}, 0, \underline{\lambda}, \underline{\nu})$, $\partial F/\partial y(\underline{x}, 0, \underline{\lambda}, \underline{\nu})$, $\partial F/\partial z(\underline{x}, 0, \underline{\lambda}, \underline{\nu})$ and $\lambda_1, \cdots, \lambda_{m_0}$ and $\nu_1, \cdots, \nu_{m_-}$ in the lower degree terms parametrized by $\mu_1, \cdots, \mu_{m_+}$. q.e.d.

Define two varieties by

(4.4.2)
$$\hat{C}_F^{\mu}:=\hat{\Phi}^{-1}(O\times S)\subset Z$$
$$C_{\varphi}^{\mu-1}:=\Phi^{-1}(O\times T)\subset X.$$

Note that $\hat{C}_F$ and C_φ are exactly the critical loci of the maps (F, p): $Z^{\mu+3}\to \boldsymbol{C}\times S^{\mu}$ and φ: $X^{\mu+2}\to S^{\mu}$ respectively.

As subvarieties in $Z^{\mu+3}$, we have the relation,

(4.4.3)
$$C_\varphi=\hat{C}_F\cap X$$

and therefore a short exact sequence,

(4.4.4)
$$0\longrightarrow\mathcal{O}_{\hat{C}_F}\overset{F}{\longrightarrow}\mathcal{O}_{\hat{C}_F}\longrightarrow\mathcal{O}_{C_\varphi}\longrightarrow 0.$$

Here F means the multiplication by F.

Note that $p\,|\,\hat{C}_F=\hat{\Phi}\,|\,\hat{C}_F=\hat{\Phi}\,|\,\hat{\Phi}^{-1}(0\times S)$ is flat finite of degree μ and $\Phi p\,|\,C_\varphi=\varphi\,|\,C_\varphi$. By taking the direct image $R^{\cdot}p_*$ of (4.4.4), one obtains an exact sequence,

$$(4.4.5) \qquad 0 \longrightarrow p_*\mathcal{O}_{\hat{C}_F} \xrightarrow{p_*F} p_*\mathcal{O}_{\hat{C}_F} \longrightarrow \varphi_*\mathcal{O}_{C_\varphi} \longrightarrow 0.$$

Here $p_*\mathcal{O}_{\hat{C}_F}$ *is a* $\mathcal{O}_S$*-free module of rank* μ, *so that* (4.4.5) *gives a* $\mathcal{O}_S$*-free resolution of* $\varphi_*\mathcal{O}_{C_\varphi}$. *Then the discriminant* $\varDelta$ *for the Hamiltonian system is,*

$$(4.4.6) \qquad \varDelta(\underline{\mu}, \underline{\lambda}, \underline{\nu}) := \det(p_*F)$$

where det *means the determinant of a* $\mathcal{O}_S$*-matrix presentation of the* $\mathcal{O}_S$*-homomorphism with respect to some* $\mathcal{O}_S$*-free basis of* $p_*\mathcal{O}_{C_F}$. ([50, (2.6.4)]).

Assertion 2. *$\varDelta$ is a monic polynomial of degree μ in μ_1, i.e.*

$$(4.4.7) \qquad \varDelta = \mu_1^\mu + A_1\mu_1^{\mu-1} + \cdots + A_{\mu-1}\mu_1 + A_\mu$$
$$\text{for } A_i \in \mathcal{O}_T \text{ and degree } (A_i) = \mu - i.$$

3. *The zero locus D of $\varDelta$, which is equal to the critical set $\varphi(C)$ in S, has constant multiplicity μ along the subspace $O \times S_0 \times \boldsymbol{C}^{m-}$.*

(4.5) **The Milnor fibration.** Outside the discriminant locus $D \subset S$, the restriction of the map φ in (4.2.1),

$$(4.5.1) \qquad \varphi \,|\, X - \varphi^{-1}(D) \colon X - \varphi^{-1}(D) \rightarrow S - D$$

is a locally trivial smooth fiber bundle, whose general fiber is homotopic to a joint of μ copies of 2-spheres, which we shall call the Milnor fibration [32].

Let X_t be a general fiber of φ. Then the ball $B_r := \{(x, y, z) \in \boldsymbol{C}^3 \colon |x|^2 + |y|^2 + |z|^2 \le r^2\}$ intersects X_t transversally at its boundary for $r \gg |t|$, so that the fiber X_t retracts to $Y := X_t \cap B_r$, where Y is a compact real 4-manifold with boundary. The Poincaré duality $P \colon H_2(Y, \partial Y) \simeq H^2(Y)$ combined with the excision morphism $E \colon H_2(Y) \rightarrow H_2(Y, \partial Y)$ defines a symmetric bilinear form,

$$(4.5.2) \qquad I \colon H_2(Y) \times H_2(Y) \rightarrow \boldsymbol{Z}, \qquad I(u, v) := \langle PEu, v \rangle$$

which is the so called *intersection form* on $H_2(Y) \simeq H_2(X)$ [loc. cit.], whose radical$:= \{u \in H_2(Y) \colon I(u, v) = 0 \text{ for } \forall v \in H_2(Y)\}$ is the image of $H_2(\partial Y)$ in $H_2(Y)$.

($\because$ Recall the exact sequence,

$$0 \rightarrow H_2(\partial Y) \rightarrow H_2(Y) \xrightarrow{E} H_2(Y, \partial Y) \rightarrow H_1(\partial Y) \rightarrow 0.)$$

Let us denote by (μ_+, μ_0, μ_-) the singnature of I. (i.e. $\mu_\pm$ is the maximal rank of subspaces of $H_2(Y)\otimes \boldsymbol{R}$ on which I is positive or negative definite and $\mu_0:=\operatorname{rank}(\operatorname{rad} I)$.)

Formula

$$\begin{aligned}
&\mu_+:=2\sum_{j<0} a_j=2\sum_{j>h} a_j=\sum_{j<0} a_j+\sum_{j>h} a_j,\\
(4.5.3)\qquad &\mu_0:=2a_0=2a_h=a_0+a_h,\\
&\mu_-:=\sum_{0<j<h} a_j.
\end{aligned}$$

Proof. Let $p\colon \tilde{X}\to X_t$ be the minimal good resolution of the singular point of $X_t:=\varphi^{-1}(t)$ for $t\in O\times S_0\times \boldsymbol{C}^{m-}$ with the exceptional set $E:=p^{-1}(0)$ (cf. for instance [39]). Since $\mu_0=\operatorname{rank} H_2(\partial Y)=\operatorname{rank} H_1(E)$ and E consists of rational curves and the central curve E_0 forming a star graph (cf. [loc. cit.]. or § 5 (5.3)), we have $\mu_0=2$ (genus of E_0). It is not hard to see that the genus of E_0 is equal to a_0 (cf [loc. cit.] or § 5 (5.3)).

Using the limit mixed Hodge structure on $H^2(Y)$, Steenbrink has shown ([60]), $\mu_0+\mu_+=\#\{$exponents which do not lie in the interval $(0, h)\}$ $=\sum_{j\le 0} a_j+\sum_{j\ge h} a_j$ (cf. (3.2) Note 1). q.e.d.

Note. The geometric genus p_g of the singular point X_t is defined by $p_g:=\operatorname{rank} R^1\pi_*(\mathcal{O}_{\tilde{X}})_t$. Then we have the formula,

$$2p_g=\mu_0+\mu_+,$$

due to M. Saito [56, § 1. Theorem 1] and A. Durfee [16, Proposition (3.1)].

§ 5. Resolution of singularities

Resolution of the singularity with a good $\boldsymbol{C}^*$ action has been studied by several authors [39] [13] [44] [38].

In this paragraph, we describe the weighted graphs for the minimal good resolution (as defined in [39] (2.4)) of the singular point of the surface $X_\lambda:=\{f(\underline{x}, \underline{\lambda})=0\}$ defined by the polynomial f given in Table 4.

(5.1) The singularity defined by $f(x)=0$ for a polynomial with $\varepsilon=1$, is known to be a rational double point, whose exceptional set of the minimal good resolution consists of smooth rational curves of self intersection number -2, intersecting among themselves transversally in the form of the Dynkin diagram. ([4] [6] [17] [35])

(5.2) The singularity defined by $f(\underline{x}, \lambda)=0$ for a polynomial with $\varepsilon=0$, is known to be a simple elliptic singularity, whose exceptional set of the minimal good resolution consists of a smooth elliptic curve of self intersection number -3, -2 or -1 according as the notation for the singularity is $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$ (cf. [46] [63]).

(5.3) *Case* $\varepsilon=-1$. The singularities defined by the first 14 polynomials in Table 4 for $\varepsilon=-1$ are known to be exceptional unimodular singularities, whose resolution graphs are stars of types $(1,0;p,1;q,1;r,1)$ for some $p, q, r \in \boldsymbol{N}$ with $1/p+1/q+1/r<1$ (cf. I.V. Dolgachev [13]). Similar description of the resolution graphs for the next 6 singularties is given in [14]. (Cf. [63] [42])

Using the general method of resolution (for instance [39] [44] [38]) it is not hard to determine all the resolution graphs for the singularities with $\varepsilon=-1$ as follows.

Theorem 6. *Let* $p\colon \tilde{X}\to X_{\underline{\lambda}}:=\{(x, y, z)\in \boldsymbol{C}^3 : f(x, y, z, \underline{\lambda})=0\}$ *be the minimal good resolution of the singular point of* $X_{\underline{\lambda}}$ *which is defined by a polynomial in Table* 4 *for* $\varepsilon=-1$, $\underline{\lambda}\in S_0$.

Then the exceptional set $E:=p^{-1}(0)$ *is a union of a smooth curve* E_0 *of genus* $g:=a_0$, *called the central curve, and smooth rational curves* $E_1, \cdots, E_r$, *intersecting among themselves as follows*:

$$
(5.3.1)\qquad \begin{aligned}
&E_0^2=-(a_1-a_0+1), &&\\
&E_0\cdot E_j=1 &&\text{for } 1\le j\le r,\\
&E_i\cdot E_j=0 &&\text{for } 1\le i<j\le r,\\
&E_i^2=-p_i &&\text{for } 1\le i\le r,
\end{aligned}
$$

where the set $A:=\{p_1, \cdots, p_r\}$ *is given by*

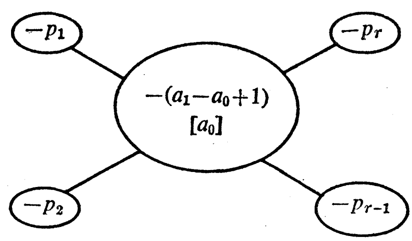
$$
(5.3.2)\qquad A:=\bigcup_{\substack{k=1\\ c_k\nmid h}}^{3}\{c_k\}\cup \bigcup_{\substack{k,l=1\\ \gcd(c_k,c_l)>1}}^{3}\{\underbrace{\gcd(c_k, c_l), \cdots, \gcd(c_k, c_l)}_{m(c_k,c_l)-1 \text{ copies}}\}.
$$

Here we denote by c_1, c_2, c_3 *the weights* a, b, c *and* $m(c_k, c_l)=m(h\colon c_k, c_l)$ (*cf.* (1.5.1)).

There is a relation,

$$
(5.3.3)\qquad 2g-2+E_0^2+r=0 \quad \textit{and} \quad r=3(1-a_0)+a_1.
$$

The resolution graph



The following Table 5 gives the explicit list of the values g, E_0^2, p_1, $\cdots$, p_r and the value $2g-2+\sum_{i=1}^{r}(1-1/p_i)$.

Table 5.

Notation	g	E_0^2	r	$p_1, \cdots, p_r$	$2g-2+\sum(1-1/p_i)$
E_{12}	0	-1	3	2, 3, 7	1/42
E_{13}	0	-1	3	2, 4, 5	1/20
E_{14}	0	-1	3	3, 3, 4	1/12
Z_{11}	0	-1	3	2, 3, 8	1/24
Z_{12}	0	-1	3	2, 4, 6	1/12
Z_{13}	0	-1	3	3, 3, 5	2/15
W_{12}	0	-1	3	2, 5, 5	1/10
W_{13}	0	-1	3	3, 4, 4	1/6
Q_{10}	0	-1	3	2, 3, 9	1/18
Q_{11}	0	-1	3	2, 4, 7	3/28
Q_{12}	0	-1	3	3, 3, 6	1/6
S_{11}	0	-1	3	2, 5, 6	2/15
S_{12}	0	-1	3	3, 4, 5	13/60
U_{12}	0	-1	3	4, 4, 4	1/4
$R_{269}(J_{3,0})$	0	-2	4	2, 2, 2, 3	1/6
$R_{247}(Z_{1,0})$	0	-2	4	2, 2, 2, 4	1/4
$R_{245}(Q_{2,0})$	0	-2	4	2, 2, 2, 5	3/10
$R_{236}(W_{1,0})$	0	-2	4	2, 2, 3, 3	1/3
$R_{234}(S_{1,0})$	0	-2	4	2, 2, 3, 4	5/12
$R_{233}(U_{1,0})$	0	-2	4	2, 3, 3, 3	1/2
$R_{225}(N_{16})$	0	-3	5	2, 2, 2, 2, 2	1/2
$R_{223}(V_{1,0})$	0	-3	5	2, 2, 2, 2, 3	2/3
R_{146}	1	-1	1	2	1/2
R_{135}	1	-1	1	3	2/3
R_{134}	1	-1	1	4	3/4
R_{124}	1	-2	2	2, 2	1
R_{123}	1	-2	2	2, 3	7/6
R_{122}	1	-3	3	2, 2, 2	3/2
R_{113}	2	-2	0		2
R_{112}	2	-3	1	2	5/2
R_{111}	3	-4	0		4

Proof. For the sake of completeness, we describe the construction of resolution explicitly as follows. Let $\pi: \hat{C}^3(a, b, c) \to C^3$ be the weighted blowing up of C^3 defined as follows. Let $P(a, b, c) := (C^3 - \{0\})/\sim$ be the weighted projective space, where $\sim$ is defined by $(x, y, z) \sim (t^a x, t^b y, t^c z)$ for $t \in C^*$. Then $\hat{C}^3(a, b, c)$ is the closure of the graph $G \subset C^3 \times P(a, b, c)$ of the natural map $C^3 - \{0\} \to P(a, b, c)$ and π is induced from the projection to the first factor. We have an isomorphism $\pi^{-1}(0) \simeq P(a, b, c)$.

Let us denote by $\hat{X}_\lambda \subset \hat{C}^3$ the strict transform of $X_\lambda \subset C^3$. Then the exceptional set $\hat{E}_0 := \hat{X}_\lambda \cap \pi^{-1}(0)$ is isomorphic to the smooth curve in $P(a, b, c)$ defined by the weighted homogeneous equation $f(x, y, z, \underline{\lambda}) = 0$, whose genus g is equal to a_0.

Assertion. *$\hat{X}_\lambda$ has at most cyclic quotient singular points at $\hat{E}_0 \cap (l_x \cup l_y \cup l_z)$, where l_x, l_y and l_z are coordinate axises of $P(a, b, c)$ defined by $x=0$, $y=0$ and $z=0$ respectively.*

The set $\hat{E}_0 \cap (l_x \backslash l_y \backslash l_z)$ consists of $N(b, c: h) - 1$ points, which are cyclic quotient singular points $\hat{X}_\lambda$ of type $(p, -\varepsilon)$ for $p = \gcd(b, c)$. The point $l_y \cap l_z$ belongs to $\hat{E}_0$ iff $m(a: h) = 0$. Then the point is a cyclic quotient singular point of $\hat{X}_\lambda$ of type $(a, -\varepsilon)$.

For a definition of a cyclic quotient singular point of type (p, q), see [44, § 2].

Proof. Let us define a chart $\mathscr{U}$ of $\hat{C}^3$ as the image of

$$\begin{array}{ccc} 1 \times C^3 & \longrightarrow & \hat{C}^3(a, b, c) \\ \cup\!\!\!\!\!\!\: & & \cup \\ (1, y, z, w) & \longmapsto & (w^a, w^b y, w^c z) \times (1: y: z). \end{array}$$

The image $\mathscr{U}$ is isomorphic to the quotient variety $1 \times C^3 / Z_a$, where $Z_a \simeq$ the set of a-th roots of unity acts on $1 \times C^3$ by $(1, y, z, w) \mapsto (\zeta^{-a}, \zeta^{-b} y, \zeta^{-c} z, \zeta w)$ for $\zeta \in Z_a$. Therefore $\hat{X}_\lambda \cap \mathscr{U}$ is isomorphic to the quotient variety Y/Z_a, where Y is the smooth hypersurface in $1 \times C^3$,

$$Y := \{(1, y, z, w) \in C^3 : f(1, y, z) = 0\}$$

and Z_a action on Y is induced from that on $1 \times C^3$.

If a point $\underline{x} = (1, y, z, w) \in Y$ is fixed by $\zeta \in Z_a$ for a $\zeta \neq 1$, then $w = 0$. Furthermore if $yz \neq 0$, then $\zeta^b = 1$ and $\zeta^c = 1$. This is a contradiction to $\zeta \neq 1$, since $\gcd(a, b, c) = 1$. Thus fixed points on Y appear along the coordinate line $y = 0$ or $z = 0$.

The type of the action at a fixed point $\underline{x} \in Y$ is determined as follows. First let us show that two functions $f(1, y, z)$ and w from a part of a local coordinate for $1 \times C^3$ at $\underline{x}$, since the rank of

$$\frac{\partial(f(1,y,z),w)}{\partial(y,z,w)}=\begin{bmatrix}\frac{\partial f}{\partial y}, & \frac{\partial f}{\partial z}, & 0\\ 0, & 0, & 1\end{bmatrix}$$

is equal to two. ($\because$ Otherwise $\partial f/\partial y=\partial f/\partial z=f=0$ at $\underline{x}$ and hence $\partial f/\partial x(1,y,z)=hf(1,y,z)-by(\partial f/\partial y)(1,y,z)-cz(\partial f/\partial z)(1,y,z)=0$. This contradicts the isolatedness of the singular point of X_λ.)

The Jacobian determinant of the action of $\zeta\in$ *Isotropy at* $\underline{x}\subset Z_a$ on Y at $\underline{x}$ is calculated by,

$$\det\left(\frac{\partial(\zeta^{-b}y,\zeta^{-c}z,\zeta w)}{\partial(y,z,w)}\right)\Big/\frac{\partial f(\zeta^{-a},\zeta^{-b}y,\zeta^{-c}z)}{\partial f(1,y,z)}=\zeta^{-b-c+1+h}=\zeta^{a+1-\varepsilon}=\zeta^{1-\varepsilon}.$$

Since ζ acts on one coordinate w of Y by multiplication, ζ acts on v (=the local coordinate of Y at $\underline{x}$, s.t. ζ acts on (w,v) linear diagonally) by the multiplication of $\zeta^{-\varepsilon}$, (i.e. $\zeta\times(w,v)\to(\zeta w,\zeta^{-\varepsilon}v)$ at $\underline{x}\in Y$). q.e.d.

It is well known by E. Brieskorn [6] that the graph of the minimal good resolution of a cyclic quotient singular point of type $(p,1)$ is $(-p)$ and that of type $(p,-1)$ is $\underbrace{(-2)—(-2)—\cdots—(-2)}_{p-1}$. By resolving the singular points of $\hat{X}_\lambda$ minimally, we obtain the description in the theorem, where E_0 is the strict transform of $\hat{E}_0$.

The formula (5.3.3) is a consequence of the adjunction relation on $\tilde{X}_\lambda$ as follows.

Let ω_0 be a meromorphic 2-form on $\tilde{X}_\lambda$, which is a lifting of the two form $\mathrm{Res}_{X_\lambda}\left[\frac{dxdydz}{f(x,y,z,\lambda)}\right]$ on X_λ. Then ω_0 has simple poles along E_i for $i=1,\cdots,r$ and a double pole along E_0, i.e.

$$(5.3.4)\qquad [\omega_0]=-2E_0-\sum_{i=1}^{r}E_i.$$

($\because$ As above, let (w,v) be local coordinates of Y at a fixed point $\underline{x}$ of Z_p. The lifting of

$$\mathrm{Res}_{X_\lambda}\left[\frac{d(w^a)d(w^by)d(w^cz)}{f(w^a,w^by,w^cz)}\right]=a\frac{dw}{w^{1-\varepsilon}}\mathrm{Res}\left[\frac{dydz}{f(1,y,z)}\right]$$

on Y is locally expressed as $u(w,v)dwdv/w^{1-\varepsilon}$ for some unit u. Let $E_i\subset\tilde{X}_\lambda$ be the exceptional set of the resolution of $\underline{x}$ in $\hat{X}_\lambda$. Then at the point $E_0\cap E_i$, one may choose $W:=w^p$ and $U:=w/v$ as local coordinates of $\tilde{X}_\lambda$ s.t. E_i: $W=0$ and E_0: $U=0$. Then

$$\omega = \text{pull back of } u\frac{dwdv}{w^2}$$
$$= -\frac{u}{p}\frac{dWdU}{WU^2}.$$

This implies (5.3.4)).

The adjunction formula on $\tilde{X}$, applied to the curve E_0 implies

$$2g-2=[\omega_0]E_0+E_0^2.$$

By substituting $[\omega_0]$ by (5.3.4), we obtain the formula (5.3.3).

This completes the proof of Theorem 6. q.e.d.

Note. Artin's fundamental cycle (cf. [4]) Z_0 and Z_0^2, $p_a(Z_0):=1-\chi(\mathcal{O}_{Z_0})$ are easily calculated as follows (cf. [42] [63] [27] [61] [68]).

Resolution graph	Z_0	Z_0^2	$p_a(Z_0)$
$(1, 0; 2, 1; 3, 1; r, 1)$ $r \geq 6$	$6E_0+3E_1+2E_2+E_3$	$6-r$	1
$(1, 0; 2, 1; q, 1; r, 1)$ $q, r \geq 4$	$4E_0+2E_1+E_2+E_3$	$8-q-r$	1
$(1, 0; p, 1; q, 1; r, 1)$ $p, q, r \geq 3$	$3E_0+E_1+E_2+E_3$	$9-p-q-r$	1
$(2, 0; 2, 1; q, 1; r, 1; s, 1)$ $q, r, s \geq 2$	$2E_0+\sum_{i=1}^{4} E_i$	$6-q-r-s$	1
$(3, 0; 2, 1; 2, 1; 2, 1; 2, 1; r, 1)$ $t \geq 0$	$2E_0+\sum_{i=1}^{5} E_i$	$-t$	1
$(p_0, g; p_1, 1; \cdots; p_r, 1)$ $g \geq 1$	$\sum_{i=0}^{r} E_i$	$2r-\sum_{i=0}^{r} p_i$	$1+(p_0-r)/2$

(5.4) The following theorem due to H. Pinkham gives an insight into the proof below of Theorems 3 and 4 in (3.5).

Theorem. ([44, Theorem 1.1]) *Let X_0 be any normal two dimensional variety $/\boldsymbol{C}$ with a good $\boldsymbol{C}^*$-action fixing an isolated point $0 \in X_0$. Then there exists a smooth proper curve C, a finite group G of automorphisms of C and a G-invariant, ample invertible sheaf $\mathscr{L}$ of* rank 1 *on C such that*

a) *G acts on $X(C, \mathscr{L})$, freely except at the vertex, where $X(C, \mathscr{L})$ denotes the space obtained by contraction of the zero section of the line bundle $F(C, \mathscr{L})$ over C associated to the invertible sheaf $\mathscr{L}^{-1}$ over C.*

b) *X_0 is analytically isomorphic to the quotient of $X(C, \mathscr{L})$ by G.*

(The proof of the theorem essentially uses the solution of *Fenchel's conjecture* due to Bundgard-Nielsen [11] and Fox [18].)

Let us return to our case: the hypersurface X_0 defined by a polynomial for a regular system of weights $(a, b, c; h)$ with $\varepsilon:=a+b+c-h$, which admits a good C^*-action $t\times(x, y, z)\to(t^a x, t^b y, t^c z)$ for $t\in C^*$. Let us keep the notations of the proof of (5.3) Theorem 5.

Due to the universality of the weighted blowing up $\hat{X}_\lambda\to X_\lambda$, there exists a holomorphic map $F(C, \mathscr{L})\to\hat{X}_\lambda$, which makes the following diagram commutative.

$$\begin{array}{ccc} F(C, \mathscr{L}) & \xrightarrow{G} & \hat{X}_\lambda \\ \downarrow C & & \downarrow \hat{E}_0 \\ X(C, \mathscr{L}) & \xrightarrow{G} & X_\lambda \end{array}$$

Therefore $F(C, \mathscr{L})/G\simeq\hat{X}_\lambda$ and $C/G\simeq\hat{E}_0$. Then the Hurwitz formula for the covering $C\to\hat{E}_0$ implies,

$$(5.4.1) \qquad 2g(C)-2=(\#G)(2g(E_0)+2+\sum_{i=1}^{r}(1-1/p_i)).$$

Let ω be the meromorphic 2-form on $F(C, \mathscr{L})$ which is the pull-back of $\mathrm{Res}_{X_\lambda}\left[\frac{dxdydz}{f(x, y, z, \lambda)}\right]$ on X_λ. Then the natural action of $t\in C^*$ on the line bundle $F(C, \mathscr{L})$ induces,

$$(5.4.2) \qquad t^*\omega=t^\varepsilon\omega.$$

(5.5) *Proof of* (3.5) *Theorem* 4.
First let us show:

Lemma. *Let $F\to C$ be a line bundle over a smooth curve C. Suppose there exists a nowhere vanishing holomorphic 2-form ω on $F\backslash C$ (=the total space of F minus the zero section) and an integer $\varepsilon\in Z$, s.t.*

$$t^*\omega=t^\varepsilon\omega \qquad \textit{for } t\in C^*$$

where t^* means the action of t on the space of two forms on $F\backslash C$ induced from the geometric action of t on $F\backslash C$ by multiplication on fibers.

Then $F^{\otimes\varepsilon}$ is isomorphic to the canonical bundle of C.

Proof. Let x_i $(i\in I)$ be local coordinates for a collection of charts U_i $(i\in I)$ of C and let y_i $(i\in I)$ be the fiber coordinate for a trivialization of $F|_{U_i}$.

Let $a_{ji}(x)$ be the holomorphic function on $U_i\cap U_j$ which defines the transition rule $y_j=a_{ij}(x)y_i$ of the fiber coordinates for $i, j\in I$.

Let $\varphi_i(x_i, y_i)dx_idy_i$ be the local expression for ω on $F|_{U_i}$ in terms of the coordinates (x_i, y_i). Then by the assumption on ω, $t^*(\varphi_i(x_i, y_i)dx_idy_i)$

$=\varphi_i(x_i, ty_i)dx_i dty_i$ is equal to $t^{\varepsilon}\varphi_i(x_i, y_i)dx_i dy_i$. Hence $\varphi_i(x_i, ty_i)=t^{\varepsilon-1}\varphi_i(x_i, y_i)$. By substituting $y_i=1$ and $t=y_i$, we obtain,

$$\varphi_i(x_i, y_i)=y_i^{\varepsilon-1}\varphi_i(x, 1) \qquad \text{for } i \in I.$$

On the other hand, on the intersection $F|_{U_i}\cap F|_{U_j}$, two differential forms

$$\begin{aligned}\omega|_{U_j}&=\varphi_j(x_j, y_j)dx_j dy_j\\ &=\varphi_j(x_j, a_{ji}(x)y_i)\frac{dx_j}{dx_i}dx_i d(a_{ji}(x)y_i)\\ &=(a_{ji}(x))^{\varepsilon}y_i^{\varepsilon-1}\frac{dx_j}{dx_i}\varphi_j(x_j, 1)dx_i dy_i\end{aligned}$$

and

$$\omega|_{U_i}=\varphi_i(x_i, y_i)dx_i dy_i=y_i^{\varepsilon-1}\varphi_i(x_i, 1)dx_i dy_i$$

should coincide with each other.

Therefore one obtains the relation

$$a_{ji}(x)^{\varepsilon}\frac{dx_j}{dx_i}\varphi_j(x_j, 1)=\varphi_i(x_i, 1).$$

This means the collection $\{\varphi_i(x_i, 1)\}_{i\in I}$ defines a section of the line bundle $F^{\otimes(-\varepsilon)}K_C$ over C, where K_C is the canonical bundle of C. Again by the assumption on ω, $\varphi_i(x_i, 1)$ are nowhere vanishing. Thus the collection $\{\varphi_i(x_i, 1)\}_{i\in I}$ defines a trivialization of the line bundle $F^{\otimes(-\varepsilon)}K_C$. q.e.d.

Applying the lemma to (5.4.2) in our case $\varepsilon=-1$, we see that the line bundle $F(C, \mathscr{L})$ is isomorphic to the anticanonical bundle K_C^{-1} over C.

Since $\mathscr{L}$ is ample, the Hurwitz formula (5.4.1) implies,

$$2g(C)-2=(\#G)\left(2g-2+\sum_{i=1}^{r}(1-1/p_i)\right)>0,$$

so that the universal covering of C is the upper half plane $\boldsymbol{H}:=\{z\in \boldsymbol{C}: \mathrm{Im}(z)>0\}$. Let $\pi: \boldsymbol{H}\to C$ be the universal covering map and let $\boldsymbol{H}\times_{\pi}K_C^{-1}$ be the lifting of the anticanonical bundle from C to $\boldsymbol{H}$. By a suitable trivialization of the bundle, the group $\pi_1(C)\hookrightarrow \mathrm{SL}(2, \boldsymbol{R})/\pm 1$ acts on $\boldsymbol{H}\times_{\pi}K_C^{-1}$ as follows.

$$\boldsymbol{H}\times_{\pi}K_C^{-1}\longrightarrow \boldsymbol{H}\times_{\pi}K_C^{-1}$$

$$(z, y) \longmapsto \left(\frac{az+b}{cz+b}, (cz+d)^{-2}y\right).$$

Recall $\tilde{\boldsymbol{H}} := \{(u, v) \in \boldsymbol{C}^2 : \operatorname{Im}(u/v) > 0\}$ and define the following double covering

$$\tilde{\boldsymbol{H}} \longrightarrow \boldsymbol{H} \times_\pi K_C^{-1}$$

$$(u, v) \longmapsto (z, y), \quad z := u/v, \quad y := v^{-2}$$

so that the action of $\pi_1(C) \subset \mathrm{SL}(2, \boldsymbol{R})/\pm 1$ on $\boldsymbol{H} \times_\pi K_C^{-1}$ can be lifted to the linear action of the binary extension of $\pi_1(C)$ in $\mathrm{SL}(2, \boldsymbol{R})$ on $\tilde{\boldsymbol{H}}$.

This completes the proof of the Theorem 3.

Proof of (3.5) *Theorem* 5. Let Γ be any Fuchsian group of the first kind. Then due to Fox [18] and Bungaard, Nielsen [11], there exists a finite index normal subgroup Γ' of Γ which does not have a torsion element. Let us denote by $\tilde{\Gamma}'$ and $\tilde{\Gamma}$ the binary extensions of Γ' and Γ in $\mathrm{SL}(2, \boldsymbol{R})$, respectively. Obviously $\tilde{\Gamma}'$ acts freely on $\tilde{\boldsymbol{H}}$ so that $\tilde{\boldsymbol{H}}/\Gamma'$ is the anti-canonical bundle over a curve $\boldsymbol{H}/\Gamma'$ on which the finite group Γ/Γ' is still acting. Consider the 2-form ω on the anti-canonical bundle $\tilde{\boldsymbol{H}}/\tilde{\Gamma}'$ defined by $dudv = -\frac{1}{2}y^{-2}dzdy$ which is invariant under Γ/Γ' and satisfies

$$*) \qquad t^*\omega = t^{-1}\omega$$

for the $\boldsymbol{C}^*$-action on the fibers on $\tilde{\boldsymbol{H}}/\tilde{\Gamma}'$. Therefore ω descends to the space $\tilde{\boldsymbol{H}}/\tilde{\Gamma}$ satisfying the same rule as *). (Note that any of Γ/Γ' does not act trivially on $\boldsymbol{H}/\Gamma'$.)

Suppose that $\tilde{\boldsymbol{H}}/\tilde{\Gamma}$ is embedded in $\boldsymbol{C}^3$ by three automorphic forms x, y and z of degrees a, b and c, respectively so that the image is a hypersurface defined by a weighted homogeneous polynomial f of degree h. Let us denote by ω' the differential form on $\tilde{\boldsymbol{H}}/\tilde{\Gamma}'$ which is the lifting of $\operatorname{Res}\left[\frac{dxdydz}{f(x, y, z)}\right]$. Then ω' is nowhere vanishing and satisfies

$$**) \qquad t^*\omega' = t^\varepsilon\omega' \qquad \text{for } t \in \boldsymbol{C}^*$$

where $\varepsilon := a+b+c-h$.

Then $\varphi := \omega'/\omega$ defines a nowhere vanishing holomorphic function on $\tilde{\boldsymbol{H}}/\tilde{\Gamma}'$ ($\simeq$ the anti-canonical bundle over $\boldsymbol{H}/\Gamma'$) satisfying $t^*\varphi = t^{\varepsilon+1}\varphi$. This implies that $(\varepsilon+1)$-th power of the canonical bundle of $\boldsymbol{H}/\Gamma'$ is the trivial bundle, which is possible only when $\varepsilon+1=0$. Thus $(a, b, c; h)$ is a regular system of weights with $\varepsilon = -1$.

This completes the proof of Theorems 4 and 5.

(5.6) Let us recall that a Fuchsian group of the first kind Γ is determined (as an abstract group) by a data $(p_1, \cdots, p_r; g)$ called the signature of Γ, where $p_1, \cdots, p_r$ (called the exponents, which are integers ≥ 2 or ∞) are the set of ramification indexes of $\boldsymbol{H} \rightarrow \overline{\boldsymbol{H}/\Gamma}$ and g (≥ 0) is the genus of $\overline{\boldsymbol{H}/\Gamma}$. (Here $\overline{\boldsymbol{H}/\Gamma}$ is the complete curve by adding cusps if necessary,) ([19, pp. 182–190].) The group is described as follows.

$$\Gamma \simeq \Big\langle c_1, \cdots, c_r, a_1, \cdots, a_g, b_1, \cdots, b_g : c_\nu^{p_\nu} = 1, \quad \nu = 1, \cdots r, \ c_1 \cdots c_r \prod_{\lambda=1}^{g} (a_\lambda b_\lambda a_\lambda^{-1} b_\lambda^{-1}) = 1 \Big\rangle \tag{5.6.1}$$

The volume of the fundamental domain of the group is given by $\pi\mu(\Gamma)$, where

$$\mu(\Gamma) := 2g - 2 + \sum_{\nu=1}^{r} (1 - p_\nu^{-1}). \tag{5.6.2}$$

The following theorem is a supplement to the previous Theorems 4, 5 in (3.5) (cf. [39]).

Theorem 7. i) *The signature of the Fuchsian group for a regular system of weights with* $\varepsilon = -1$, *is exactly* $(p_1, \cdots, p_r; g)$ *given in* (5.3) *Theorem* 5, *which are exhibited in* (5.3) *Table* 5.

ii) *The volume* $\mu(\Gamma)$ *of the Fuchsian group for a regular system of weights* $(a, b, c; h)$ *is given by*

$$\mu(\Gamma) = h/abc. \tag{5.6.3}$$

Proof. i) This is an obvious consequence of the construction of the resolution.

ii) This is a direct consequence of a comparison of Tables 3 and 5. An intrinsic proof without the use of the tables is given in (6.2.4). q.e.d.

(5.7) *Note.* Let us put

$$d := \det (E_i \cdot E_j)_{i,j=0,\cdots,r}. \tag{5.7.1}$$

Using the data (5.3.1), one calculates easily,

$$d = (-1)^r \prod_{i=1}^{r} p_i \Big(E_0^2 + \sum_{j=1}^{r} 1/p_j \Big). \tag{5.7.2}$$

Therefore applying (5.3.3) and (5.6.2), one obtains a formula for the volume.

(5.7.3) $$\mu(\Gamma)=(-1)^{r+1}d\Big/\prod_{i=1}^{r} p_i$$

only in terms of the intersection matrix $(E_i\cdot E_j)_{i,j=0,\cdots,r}$.

§ 6. Compactification of the Milnor fiber

For a proof of Theorem 7 ii) (5.6.3), we introduce for $\varepsilon=1, 0, -1$ the compactification of Milnor fibers by adding a divisor at infinity, which has been studied by several authors. Here in this paper, we follow the idea of H. Pinkham [42–46], who has explained the strange duality due to V.I. Arnold [2].

(6.1) Let us recall the compactification of Pinkham below. We disclude the case of type A_l to avoid some complications.

Let $F(x, y, z, \underline{\mu}, \underline{\lambda}, \underline{\nu})$ be the universal unfolding as defined in (4.1.1) for a regular system of weights $(a, b, c; h)$.

Define a new polynomial

$$G(x, y, z, w, \underline{\mu}, \underline{\lambda}):=F(x, y, z, (w^{\deg\mu_1})\mu_1, \cdots, (w^{\deg\mu_{m+}})\mu_{m_+}, \underline{\lambda}, 0)$$

which is a weighted homogeneous polynomial of degree h in (x, y, z, w) with $\deg x=a$, $\deg y=b$, $\deg z=c$ and $\deg w=1$, with parameters $(\underline{\mu}, \underline{\lambda}) \in \boldsymbol{C}^{m_+}\times S_0$. For instance,

$$G(x, y, z, w, 1, 0, \cdots, 0, \underline{\lambda})=f(x, y, z, \underline{\lambda})+w^h.$$

For each value of the parameter $(\underline{\mu}, \underline{\lambda}) \in \boldsymbol{C}^{m_+}\times S_0$ we define a complete hypersurface

$$\bar{X}_{\underline{\mu}\underline{\lambda}}:=\{(x:y:z:w) \in \boldsymbol{P}(a, b, c, 1): G(x, y, z, w, \underline{\mu}, \underline{\lambda})=0\}$$

in the weighted projective space $\boldsymbol{P}(a, b, c, 1):=(\boldsymbol{C}^4-\{0\})/\sim$, where $\sim$ is defined by $(x, y, z, w)\sim(t^a x, t^b y, t^c z, tw)$ for $\forall t \in \boldsymbol{C}^*$.

The map $(x, y, z) \in \boldsymbol{C}^3\to(x:y:z:1) \in \boldsymbol{P}(a, b, c, 1)$ is injective so that we shall identify the image domain with $\boldsymbol{C}^3$. Then the complement $\boldsymbol{P}(a, b, c, 1)\backslash\boldsymbol{C}^3$ is naturally isomorphic to $\boldsymbol{P}(a, b, c)$. Thus when there is no confusion, we identify $\boldsymbol{P}(a, b, c)$ with the divisor of $\boldsymbol{P}(a, b, c, 1)$ at "infinity".

The "open part" $\bar{X}_{\underline{\mu}\underline{\nu}}\cap\boldsymbol{C}^3$ of $\bar{X}_{\underline{\mu}\underline{\lambda}}$ is naturally identified with the Milnor fiber $X_{\underline{\mu}\underline{\nu}\underline{0}}$ of (4.2.1). The "divisor at infinity" $\bar{X}_{\underline{\mu}\underline{\lambda}}\backslash X_{\underline{\mu}\underline{\lambda}\underline{0}}=\bar{X}_{\underline{\mu}\underline{\lambda}}\cap\boldsymbol{P}(a, b, c)$, denoted by $\bar{E}_\infty$, is isomorphic to the curve in $\boldsymbol{P}(a, b, c)$ defined by,

$$f(x, y, z, \underline{\lambda})=0$$

in an obvious way, which is isomorphic to the central curve E_0 in the resolution of $X_{\underline{\lambda}}$ (cf. Theorem 6).

Assertion ([44, 4.1]). *The singular points of* $\bar{X}_{\underline{\mu}\underline{\nu}}$ *along* $\bar{E}_\infty$ *appears at most at* $\bar{E}_\infty \cap (l_x \cup l_y \cup l_z)$, *which are cyclic quotient singularities of types* (p_i, ε) *for* $i=1, \cdots, r$.

Here l_x (*resp.* l_y, *resp.* l_z) *is the coordinate axis of* $\boldsymbol{P}(a, b, c)$ *defined by* $x=0$ (*resp.* $y=0$, *resp.* $z=0$), *and* p_i*'s are the value describe in Theorem* 4, *at the corresponding points on* E_0.

Proof. To illustrate the duality with the proof of (5.3) Theorem 6, we give an elementary proof of this assertion.

Let us define a chart U of $\boldsymbol{P}(a, b, c, 1)$ as the image of

$$\begin{array}{ccc} 1\times \boldsymbol{C}^3 & \longrightarrow & \boldsymbol{P}(a, b, c, 1) \\ \cup & & \cup \\ (1, y, z, w) & \longmapsto & (1:y:z:w). \end{array}$$

The image U is isomorphic to the quotient variety $1\times \boldsymbol{C}^3/\boldsymbol{Z}_a$, where $\boldsymbol{Z}_a \simeq$ the set of a-th root of unity acts on $1\times \boldsymbol{C}^3$ by $(1, y, z, w)\mapsto(\zeta^a, \zeta^b y, \zeta^c z, \zeta w)$ for $\zeta \in \boldsymbol{Z}_a$. Therefore $\bar{X}_{\underline{\mu},\underline{\lambda}} \cap U$ is isomorphic to the quotient variety $Y/\boldsymbol{Z}_a$, where Y is the smooth hypersurface in $1\times \boldsymbol{C}^3$,

$$Y:=\{(1, y, z) \in 1\times \boldsymbol{C}^3 \colon G(1, y, z, w, \underline{\mu}, \underline{\lambda})=0\}$$

and the $\boldsymbol{Z}_a$-action on Y is induced from that on $1\times \boldsymbol{C}^3$.

If a point $\underline{x}=(1, y, z, w) \in Y$ is fixed by $\zeta \in \boldsymbol{Z}_a$ for a $\zeta\neq 1$, then $w=0$ and $yz=0$. This implies that the singular points of $\bar{X}_{\underline{\mu},\underline{\lambda}}$ appear along $\bar{E}_\infty \cap (l_x \cup l_y \cup l_z)$ as cyclic quotient singularities. Their types are determined as follows.

First not that the rank of $\partial(G(1, y, z, w), w)/\partial(y, z, w)$ is equal to two at a fixed point $\underline{x}$. Therefore we shall regard G and w as local coordinates of $1\times \boldsymbol{C}^3$ at $\underline{x}$.

The Jacobian determinant of the action of $\zeta \in \mathrm{Isotropy}_{\underline{x}} \subset \boldsymbol{Z}_a$ on Y at $\underline{x}$ is calculated by

$$\det \frac{\partial(\zeta^b y, \zeta^c z, \zeta w)}{\partial(y, z, w)} \Big/ \frac{\partial G(\zeta^a, \zeta^b y, \zeta^c z, \zeta w)}{\partial G(1, y, z, w)} = \zeta^{b+c+1-h} = \zeta^{a+1+\varepsilon} = \zeta^{1+\varepsilon}.$$

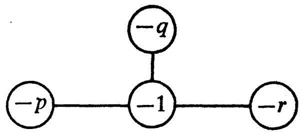
Since ζ acts by multiplication of ζ on one coordinate w of Y, it acts on the remaining coordinates by the multiplication of ζ^ε. This implies that the singularity is of type (p, ε), where p is the order of the isotropy subgroup.

The order p of the isotropy is calculated in the same way in the proof of Theorem 6. q.e.d.

(6.2) Let us denote by $\tilde{X}:=\tilde{X}_{\underline{\mu}\underline{\nu}}$ the minimal good resolution of the complete surface $\bar{X}_{\underline{\mu}\underline{\lambda}}$ at ∞ and denote by E_∞ the strict transform of $\bar{E}_\infty$.

1. *The case* $\varepsilon=1$. (cf. [54])

Due to the above description, the divisor $\tilde{X}-X_{\underline{\mu}\underline{\lambda}}$ at infinity has rational curves intersecting among themselves in the form



for some integers p, q, r (:= the lengths of the branches for the Dynkin diagram of the corresponding type.)

The canonical divisor $K_{\tilde{X}}$ of $\tilde{X}$ is given by

(6.2.1) $$K_{\tilde{X}}=-(2E_\infty+E_1+E_2+E_3)$$

and $\tilde{X}$ is a rational surface, whose second Betti number is equal to $\mu+4$.

One obtains the formula

(6.2.2) $$1/p+1/q+1/r-1=h/abc$$

where the left hand side is volume of the fundamental domain on the sphere S^2 under the action of the corresponding Klein group.

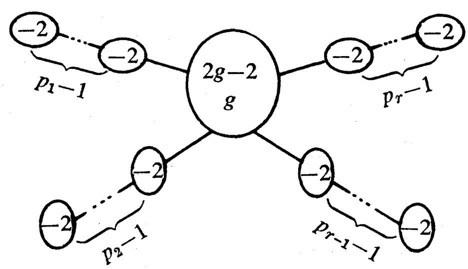
2. *The case* $\varepsilon=0$. (cf. [12] [28] [46])

In this case $\bar{E}_\infty$ is a smooth elliptic curve and $\bar{X}_{\underline{\mu}\underline{\lambda}}$ is smooth along $\bar{E}_\infty$. $\tilde{X}$ is known as a del Pezzo surface whose canonical divisor is given by

(6.2.3) $$K_{\tilde{X}}=-E_\infty.$$

3. *The case* $\varepsilon=-1$.

The divisor $\tilde{X}-X_{\underline{\mu}\underline{\lambda}}$ at infinity has the form



Assertion (compare [42], [15, 3.5.3]). *$\tilde{X}_{\underline{\mu}\underline{\lambda}}$ is a K3 surface for $(\underline{\mu}, \underline{\lambda}) \in C^{m+}\times S_0\times O\backslash D$.*

Proof. Consider the 3-form Ω on $\boldsymbol{C}^4$, given by

$$\Omega := \frac{(axdydz + bydzdx + czdxdy)dw + wdxdydz}{w^{1+\varepsilon}G(x, y, z, w, \mu, \lambda)}$$

which induces a 3-form on $\boldsymbol{P}(a, b, c, 1)$ denoted by the same letter Ω. For instance the restriction of Ω to the affine chart $\boldsymbol{C}^3 \times 1$ is equal to $dxdydz/G(x, y, z, 1, \underline{\mu}, \underline{\lambda})$. Thus the residue $\omega := \mathrm{Res}_{\bar{X}}(\Omega)$ defines a 2-form on $\bar{X}_{\underline{\mu\lambda}}$, which is holomorphic and nonvanishing except at infinity.

If $\varepsilon = -1$, from the expression for Ω, one sees easily that at infinity along $\bar{E}_\infty \subset \bar{X}_{\underline{\mu\lambda}}$, ω is regular and non-vanishing except at the singular points of $\bar{X}_{\underline{\mu\lambda}}$. Since the singular points of $\bar{X}_{\underline{\mu\lambda}}$ are rational double points (of type A_{p_i}), the lifting $\tilde{\omega}$ of ω to $\tilde{X}$ is holomorphic and nowhere vanishing. $\tilde{X}$ is simply connected since $\pi_1(X_{\underline{\mu\lambda}}) \to \pi_1(\tilde{X})$ is surjective and the Milnor fiber $X_{\underline{\mu\lambda}}$ is simply connected. q.e.d.

Formula.

(6.2.4) $$2g - 2 + \sum_{i=1}^{r} (1 - 1/p_i) = h/abc.$$

Proof. Consider $\bar{E}_\infty$ as a divisor in the singular surface $\bar{X}_{\underline{\mu\lambda}}$ defined as the compactification of the Milnor fiber $X_{\underline{\mu\lambda}}$.

The total transform E'_∞ on $\tilde{X}$ for $\bar{E}_\infty$ in the sense of D. Mumford [32, II] is easily calculated as,

$$E'_\infty := E_\infty + \sum_{i=1}^{r} \sum_{j=1}^{p_i - 1} \frac{p_i - j}{p_i} E_{ij}$$

where E_{ij} is the exceptional curve on the i-th branch at the j-th position from the E_∞.

Therefore by definition,

*) $$\bar{E}^2_\infty := E'^2_\infty = E^2_\infty + \sum_{i=1}^{r} (1 - 1/p_i) = 2g - 2 + \sum_{i=1}^{r} (1 - 1/p_i).$$

On the other hand as rational Cartier divisors on $\bar{X}_{\underline{\mu\lambda}}$, we have numerical equivalences $a\bar{E}_\infty \sim E_x$, $b\bar{E}_\infty \sim E_y$ and $c\bar{E}_\infty \sim E_z$, where E_x, E_y and E_z are divisors on $\bar{X}_{\underline{\mu\lambda}}$ defined by $x=0$, $y=0$ and $z=0$ respectively. If $c \mid h$, then $E_x \cdot E_y = h/c$ so that

**) $$\bar{E}^2_\infty = \left(\frac{1}{a} E_x\right) \cdot \left(\frac{1}{b} E_y\right) = \frac{1}{ab} E_x \cdot E_y = \frac{1}{ab} \frac{h}{c} = \frac{h}{abc}.$$

Other cases can be calculated similarly.

Comparing *) and **), we complete the proof. q.e.d.

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Supplement

After submitting this paper, the auther was informed from I. V. Dolgachev the following references.

[D] Dolgachev, I. V., On the link space of a Gorenstein Quasihomogeneous Surface singularities, Math. Ann., **265** (1983), 529–540.

[S] Sherbak, I. G., Algebras of automorphic forms with three generaters, Functional Anal. Appl., **12**, No. 2 (1977), 156–158.

One key step (1. Proposition 1.) for the proof of the main theorem in [D] is parallel to a key step ((5.5) Lemma) for the proof of (3.5) Theorems 3, 4 in the present paper, which treats a special case. Therefore assuming (3.2) Theorem and [D, 1. Remark 1.], the calculations of the table 3. of weights and table 5. of signatures are reduced to calculations of weights for certain quasihomogeneous polynomials defining isolated singular points, which is actually done in the paper [S] by a help of [14] [39].

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