Advanced Studies in Pure Mathematics 8, 1986
Complex Analytic Singularities
pp. 437–460

On the Resolution of the Three Dimensional Brieskorn Singularities

Mutsuo Oka

§ 1. Introduction

Let $f(z_0, \cdots, z_n)$ be a germ of an analytic function at the origin with an isolated critical point at $z=0$ and $f(0)=0$. We assume that the Newton boundary $\Gamma(f)$ is nondegenerate. Let $V=f^{-1}(0)$ and let Σ^* be a simplicial subdivision of the dual Newton diagram. Then there is a resolution $\pi: \tilde{V} \to V$ which is associated with Σ^*. For each strictly positive vertex P of Σ^* such that $\dim \Delta(P) \geqq 1$, there is a corresponding exceptional divisor $E(P)$. The purpose of this paper is to study the above resolution and to study the geometry of $E(P)$ in the case that $n=3$ and $f(z)=z_0^{a_0}+z_1^{a_1}+z_2^{a_2}+z_3^{a_3}$ with P being the weight vector of f. In Section 2, we will recall basic notations and the construction of the resolution of $V=f^{-1}(0)$. In Section 3, we will prove an isomorphism theorem about the exceptional surface $E(P)$ (Theorem (3.6)) which is one of the main results of this paper. In Section 4, we give a necessary and sufficient condition about $a=(a_0, a_1, a_2, a_3)$ for $E(P)$ to be a rational surface or a $K3$-surface. (Theorem (4.1) and Theorem (4.2)). There are 14 cases for $E(P)$ to be a rational surface and 22 cases for $E(P)$ to be a $K3$-surface up to Theorem (3.6). In Section 5, we will give the proof of Theorem (4.1) and Theorem (4.2).

§ 2. Preliminaries

Let $f(z)=\sum_\nu a_\nu z^\nu$ be the Taylor expansion of f. The Newton polygon $\Gamma_+(f)$ is the convex hull of $\cup\{\nu+(R^+)^{n+1}; a_\nu \neq 0\}$ and the union of its compact faces is denoted by $\Gamma(f)$ which is called the Newton boundary of f. Let N^+ be the set of the positive vectors of R^{n+1} which are considered to be in the dual space of R^{n+1} through the Euclidean inner product. For each $P \in N^+$, let $d(P)$ be the minimal value of $\{P(x); x \in \Gamma_+(f)\}$ and let $\Delta(P)=\{x \in \Gamma_+(f); P(x)=d(P)\}$. Two vectors P and Q in N^+ are said to be equivalent if and only if $\Delta(P)=\Delta(Q)$. The

Received December 26, 1984.
Revised May 29, 1985.

dual Newton diagram $\Gamma^*(f)$ is the polyhedral decomposition of N^+ by the above equivalence relation.

For given primitive integral vectors $P_1, \cdots, P_k$ of N^+, $\det(P_1, \cdots, P_k)$ is the greatest common divisor of all $k\times k$ minors of the $n\times k$ matrix $(P_1, \cdots, P_k)$. (Each vector of N^+ is considered to be a column vector.) The following lemma is a stronger version of Lemma (3.8) for $k=2$ of [11].

Lemma (2.1). *Let P, Q, R be the primitive integral vectors of N^+ such that* $\det(P, Q)=\det(P, R)=1$ *and let* $c=\det(P, Q, R)$. *Assume that* $c>1$.

(i) *There exist unique integers $k>0$, c_i, d_i $(i=1, \cdots, k)$ and unique primitive integral vectors $T_1, \cdots, T_k$ in the triangle $T(P, Q, R)$ such that $c=c_0>c_1>\cdots>c_k=1$, $0\leqq d_i<c_{i-1}$ and $T_i=(R+c_iT_{i-1}+d_iP)/c_{i-1}$ for $i=1, \cdots, k$. $(T_0=Q)$.*

(ii) *Let $T_1, \cdots, T_k$ be as above. Then there exist unique integers $\hat{c}_i$, $\hat{d}_i$ $(i=1, \cdots, k)$ such that $c=\hat{c}_0>\hat{c}_1>\cdots>\hat{c}_k=1$, $0\leqq\hat{d}_i<\hat{c}_{i-1}$ and*

$$T_{k-i+1}=(Q+\hat{c}_iT_{k-i+2}+\hat{d}_iP)/\hat{c}_{i-1}$$

for $i=1, \cdots, k$. $(T_{k+1}=R)$.

Proof. The assertion (i) is immediate from Lemma (3.8) of [11]. Let us call T_1 the first vertex of the triangle $T(P, Q, R)$ from Q to R. In the proof of Lemma (3.8) of [11], we have proved that T_1 is characterized by the integral vector which can be written as $T_1=xR+yQ+zP$ for non-negative rational numbers x, y, z such that $0=x, y, z<1$ and $\det(P, Q, R)=1$. Now we prove the assertion (ii) by the induction on k. If $k=1$, the assertion is obvious by (i). Assume that $k>1$. Note that $\det(P, R, T_1)=c_1$. Let $\hat{T}_1$ be the first vertex of $T(P, R, T_1)$ from R to T_1. Then

Assertion (2.2). *$\hat{T}_1$ is the first vertex of $T(P, R, Q)$ from R to Q.*

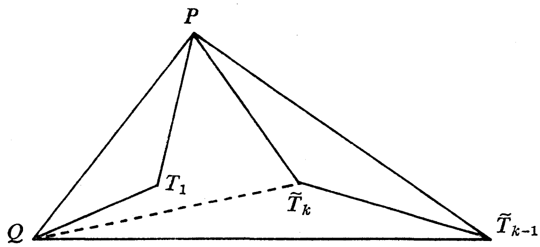
Proof of Assertion (2.2). We can write

$$\hat{T}_1=(T_1+\hat{c}R+\hat{d}P)/c_1 \quad \text{for} \quad 0\leqq\hat{c}, \hat{d}<c_1.$$

Expressing T_1 in P, Q and R, we get $\hat{T}_1=xQ+yR+zP$ where $x=1/c$, $y=(1+c\hat{c})/cc_1$ and $z=(d_1+c\hat{d})/cc_1$. As $c_1>1$ by the assumption, $\hat{c}$, $\hat{d}\leqq c_1-1$. Thus $y\leqq(cc_1-c+1)/cc_1<1$ and $z\leqq(cc_1-c+d_1)/cc_1<1$. By the above remark, this implies the assertion, because $\det(P, R, \hat{T}_1)=1$ by the definition.

Let $\hat{T}_1, \cdots, \hat{T}_t$ be the vertices of $T(P, R, T_1)$ from R to T_1 by (i). By the induction's hypothesis, we have $t=k-1$ and $T_k=\hat{T}_1, \cdots, T_2=\hat{T}_{k-1}$. Let $\tilde{T}_1, \cdots, \tilde{T}_m$ be the vertices of $T(P, R, Q)$ from R to Q by (i).

By the assertion we have $\hat{T}_1=\tilde{T}_1$ and T_1 is the first vertex of $T(P, Q, \hat{T}_1)$ from Q to $\hat{T}_1$. Assume that $\hat{T}_i=\tilde{T}_i$ and T_1 is the first vertex of $T(P, Q, \hat{T}_i)$. As $\hat{T}_{i+1}$ and $\tilde{T}_{i+1}$ are the first vertices of $T(P, \hat{T}_i, T_1)$ and $T(P, \hat{T}_i, Q)$ respectively, we have, by Assertion (2.2), that $\hat{T}_{i+1}=\tilde{T}_{i+1}$ and T_1 is the first vertex of $T(P, Q, \hat{T}_{i+1})$. Thus by the induction on i, $\hat{T}_i=\tilde{T}_i$ for $i=1, \cdots, k-1$ and T_1 is the first vertex of $T(P, Q, \hat{T}_{k-1})$. See the following figure.



As T_1 is the unique vertex of $T(P, Q, T_2)$ from Q to T_2, we have $T_1=\tilde{T}_k$ and $m=k$. Thus the assertion (ii) is proved.

Definition (2.3). We call $T_1, \cdots, T_k$ the *canonical vertices of* $T(P, Q, R)$ *around* P.

Let Σ^* be a simplicial subdivision of $\Gamma^*(f)$. We briefly recall the construction of the resolution $\pi : \tilde{V}\to V$ which is associated with Σ^*. For each n-simplex $\sigma=(P_0, \cdots, P_n)=(p_{ij})$, we associate an affine space C_σ^{n+1} with coordinate $y_\sigma=(y_{\sigma,0}, \cdots, y_{\sigma,n})$ and a birational morphism $\pi_\sigma : C_\sigma^{n+1}\to C^{n+1}$ which is defined by $\pi_\sigma(y_\sigma)=z$ where $z=(z_0, \cdots, z_n)$ and $z_i=\prod_{j=0}^n y_{y_\sigma,j}^{p_{ij}}$. X is the union of C_σ^{n+1} for σ which are glued along the images of π_σ. The projection $\hat{\pi} : X\to C^{n+1}$ is defined by $\hat{\pi}\,|\,C_\sigma^{n+1}=\pi_\sigma$. Let $\tilde{V}$ be the proper transform of V. Then the associated projection $\pi : \tilde{V}\to V$ is a resolution of V ([5], or § 4 of [11]). In C_σ^{n+1}, $\tilde{V}$ is defined by $f_\sigma(y_\sigma)=f(\pi_\sigma(y_\sigma))/\prod_{i=0}^n y_{y_\sigma i}^{d(P_i)}$. For a vertex P of Σ^* with $\dim \Delta(P)\geqq 1$, there is a corresponding divisor $E(P)$ which is defined as follows.

Let $\sigma=(P_0, \cdots, P_n)$ be an n-simplex of Σ^* such that $P_0=P$.

$$\begin{aligned} E(P)\cap C_\sigma^{n+1} &=\{y_\sigma ; y_{\sigma,0}=f_\sigma(y_\sigma)=0\} \\ &=\{y_\sigma \in C_\sigma^n ; g_{\Delta(P)}(y_{\sigma,1}, \cdots, y_{\sigma,n})=0\}, \end{aligned}$$

where $g_{\Delta(P)}(y_{\sigma,1}, \cdots, y_{\sigma,n})=f_{\Delta(P)}(\pi_\sigma(y_\sigma))/\prod y_{y_\sigma,i}^{d(P_i)}$ and $C_\sigma^n=C_\sigma^{n+1}\cap(y_{\sigma,0}=0)$. In [11], we have shown that $E(P)$ is a compact exceptional divisor if and only if P is a strictly positive vertex and its birational class depends only on the coefficients $\{a_\nu\}$ such that $\nu\in\Delta(P)$ and it does not depend on the choice of Σ^* (Corollary (5.4) of [11]).

§ 3. An isomorphism theorem for $E(P)$

Let $a_0, \cdots, a_3$ be positive integers and let $a=(a_0, \cdots, a_3)$. Let $f_a(z) = z_0^{a_0}+\cdots+z_3^{a_3}$. In this section and the following sections, we study the algebraic surfaces as the central exceptional divisors of the resolution of $V=f^{-1}(0)$ where $f=f_a$. First we define positive integers d, r_i, r_{ij} and $\hat{a}_i$ $(0\leqq i<j\leqq 3)$ as follows. Let $d=\text{g.c.d.}(a_0, \cdots, a_3)$ and let $r_i=\text{g.c.d.}\{a_j; j\neq i\}/d$. Then $r_0, \cdots, r_3$ are mutually coprime. Thus a_i is divisible by $d\prod_{j\neq i} r_j$ and we can write $a_i=\tilde{a}_i d\prod_{j\neq i} r_j$ for some positive integer $\tilde{a}_i$, for $i=0, \cdots, 3$. Let $r_{ij}=\text{g.c.d.}(\tilde{a}_k, \tilde{a}_l)$ where $\{i, j, k, l\}=\{0, \cdots, 3\}$. Note that $\text{g.c.d.}(r_{ij}, r_{ik})=\text{g.c.d.}(r_{ij}, r_{kl})=1$ for mutually distinct i, j, k, l because $\text{g.c.d.}\{\tilde{a}_j; j\neq i\}=1$ by the definition of $\tilde{a}_j$. Thus we can write

$$\tilde{a}_i=\hat{a}_i \prod_{\substack{j<k\\ j,k\neq i}} r_{jk} \quad \text{and} \tag{3.1}$$

$$a_i=d\hat{a}_i \prod_{j\neq i} r_j \prod_{\substack{j<k\\ j,k\neq i}} r_{jk}. \tag{3.2}$$

As $\text{g.c.d.}(a_0, \cdots, a_3)=d$ and $\text{g.c.d.}\{a_j; j\neq i\}=dr_i$, we get by (3.2) that $\text{g.c.d.}(\hat{a}_i, r_i)=\text{g.c.d.}(\hat{a}_i, r_{ij})=1$. We also have $\text{g.c.d.}(\hat{a}_i, \hat{a}_j)=\text{g.c.d.}(r_i, r_{jk}) =1$ for $i\neq j, k$ because $\text{g.c.d.}(\tilde{a}_i, \tilde{a}_j)=r_{kl}$ and $\text{g.c.d.}(a_i, a_j, a_k)=dr_l$. Thus we obtain

Proposition (3.3). *Let d, r_i, r_{ij}, $\hat{a}_i$ $(0\leqq i, j\leqq 3)$ be as above. Then*

$$a_i=d\hat{a}_i \prod_{j\neq i} r_j \prod_{\substack{j<k\\ j,k\neq i}} r_{jk} \tag{3.4}$$

and each of the following pairs are mutually coprime: (r_i, r_j), (r_i, r_{jk}), $(\hat{a}_i, \hat{a}_j)$, $(\hat{a}_i, r_i)$, $(\hat{a}_i, r_{ij})$, (r_{ij}, r_{ik}) and (r_{ij}, r_{kl}) where i, j, k and l are mutually distinct and $r_{ij}=r_{ji}$ for brevity's sake.

The above notations are used throughout this paper. Note that the least common multiple of $a_0, \cdots, a_3$ is $d\prod_{i=0}^{3}(r_i\hat{a}_i)\prod_{i<j} r_{ij}$. Let $P={}^t(P_0, \cdots, P_3)$ be the primitive weight vector of f_a. Then by the above remark,

$$p_i=r_i \prod_{j\neq i} (r_{ij}\hat{a}_j). \tag{3.5}$$

Here we use the notation $r_{ij}=r_{ji}$.

Let Σ^* be a simplicial subdivision and let $\pi: \tilde{V}\to V$ be the associated resolution. We call $E(P)$ the central exceptional divisor for f_a. Its birational class does not depend on Σ^*. We are going to prove the following theorem.

Theorem (3.6). *Let $a=(a_0, \cdots, a_3)$ and $b=(b_0, \cdots, b_3)$ be strictly positive integral vectors. Assume that they have the same d, r_i, r_{ij} which are defined in Proposition* (3.1). *Then the central exceptional divisors for f_a and f_b are birationally equivalent.*

Proof. We write $a \approx b$ if their central exceptional divisors are birationally isomorphic. We first show that the assertion can be reduced to the following case.

$$b_i=a_i \quad \text{for} \quad i=0, 1, 2 \quad \text{and} \quad b_3=sa_3 \tag{3.7}$$

where s is an integer which is coprime with each of r_3, $\hat{a}_i$ and r_{i3} for $i=0, 1, 2$. Assume that the assertion is true for a and b satisfying (3.7). Let $a'=(a'_0, \cdots, a'_3)$ where $a'_i=a_i/\hat{a}_i$. Then we have

$$a' \approx (a_0, a'_1, a'_2, a'_3) \approx (a_0, a_1, a'_2, a'_3) \approx (a_0, a_1, a_2, a'_3) \approx (a_0, a_1, a_2, a_3).$$

Thus $a \approx a'$. Similarly we have $b \approx b'$. By the assumption, $a' \approx b'$. Therefore $a \approx b$.

From now on, we assume (3.7). Let $P={}^t(p_0, \cdots, p_3)$ be the weight vector of f_a which is defined by (3.4). Then the weight vector $P(b)$ of f_b is ${}^t(sp_0, sp_1, sp_2, p_3)$. We will prove the theorem by showing that there exist canonical simplicial subdivisions Σ_a^* and Σ_b^* of $\Gamma^*(f_a)$ and $\Gamma^*(f_b)$ and 3-simplexes σ and σ' respectively so that the corresponding affine equations for $E(P)$ and $E(P(b))$ with respect to σ and σ' coincide. We first consider Σ_a^*. The dual Newton diagram $\Gamma^*(f_a)$ has four other vertices $P_0=(1, 0, 0, 0), \cdots, P_3=(0, 0, 0, 1)$ and P is situated at the "barycenter" of the 3-simplex with vertices $P_0, \cdots, P_3$. Note that

$$\det(P, P_i)=\hat{a}_i \quad (i=0, \cdots, 3). \tag{3.8}$$

We take the canonical primitive sequence on $\overline{PP_i}$. (See (3.5) of [11] for the definition.) Let P_i^1 be the first vertex on $\overline{PP_i}$ from P. By (3.8) and Lemma (3.3) of [11], we can write

$$P_i^1=(P_i+\alpha_i P)/\hat{a}_i \tag{3.9}$$

where α_i is the unique integer such that $0 \leqq \alpha_i < \hat{a}_i$ and

$$\alpha_i p_i + 1 \equiv 0 \text{ modulo } \hat{a}_i. \tag{3.10}$$

We also assume that Σ_a^* is canonical on $T(P, P_i^1, P_j^1)$ around P. Let T_{01}^1 be the first vertex of $T(P, P_0^1, P_1^1)$ from P_0^1. Note that

$$\det(P, P_0^1, P_1^1)=\det(P, P_0, P_1)/\hat{a}_0\hat{a}_1=r_{23}. \tag{3.11}$$

Thus we can write T^1_{01} by Lemma (2.1) as

$$(3.12)\qquad T^1_{01}=(P^1_1+\beta P^1_0+\gamma P)/r_{23},\quad 0\leqq\beta,\ \gamma<r_{23}.$$

Note that

$$(3.13)\qquad \begin{aligned}\det(P, P^1_0, T^1_{01}, P^1_2)&=\det(P, P^1_0, P^1_1, P^1_2)/r_{23}\\&=\det(P, P_0, P_1, P_2)/\hat{a}_0\hat{a}_1\hat{a}_2 r_{23}\\&=r_3 r_{03} r_{13}.\end{aligned}$$

We subdivide the 3-dimensional simplex $(P, P^1_0, T^1_{01}, P^1_2)$ by Lemma (3.8) of [11] so that $\sigma=(P, P^1_0, T^1_{01}, R)$ is a 3-simplex of Σ^*_a where

$$(3.14)\qquad R=(P^1_2+\delta P^1_0+\varepsilon T^1_{01}+\mu P)/r_3 r_{03} r_{13}$$

where $0\leqq\delta$, ε, $\mu<r_3 r_{03} r_{13}$. We consider the defining equation g_σ of $E(P)$ in C^3_σ. As $g_\sigma(y_\sigma)\prod y^{m_i}_{\sigma,i}=f(\pi_\sigma(y_\sigma))$, where $m_0=d(P)$, $m_1=d(P^1_0)$, $m_2=d(T^1_{01})$ and $m_3=d(R)$, we have

$$(3.15)\qquad g_\sigma(y_\sigma)=y^{u_1}_{\sigma,1}y^{v_1}_{\sigma,2}y^{w_1}_{\sigma,3}+y^{u_2}_{\sigma,2}y^{v_2}_{\sigma,3}+y^{u_3}_{\sigma,3}+1$$

where

$$(3.16)\qquad u_1=P^1_0(A_0)-d(P^1_0)=a_0/\hat{a}_0=d\,r_1 r_2 r_3 r_{12} r_{13} r_{23},$$

$$(3.17)\qquad u_2=T^1_{01}(A_1)-d(T^1_{01})=a_1/\hat{a}_1 r_{23}=d\,r_0 r_2 r_3 r_{02} r_{03},$$

$$(3.18)\qquad u_3=R(A_2)-d(R)=a_2/\hat{a}_2 r_3 r_{03} r_{13}=d\,r_0 r_1 r_{01},$$

$$(3.19)\qquad v_1=T^1_{01}(A_0)-d(T^1_{01})=\beta u_1/r_{23},$$

$$(3.20)\qquad v_2=R(A_1)-d(R)=\varepsilon u_2/r_3 r_{03} r_{13}$$

and

$$(3.21)\qquad w_1=R(A_0)-d(R)=(\delta u_1+\varepsilon v_1)/r_3 r_{03} r_{13}.$$

Here $A_0=(a_0, 0, 0, 0)$, $\cdots$, $A_3=(0, 0, 0, a_3)$. Now we consider Σ^*_b. Recall that the weight vector $P(b)$ is ${}^t(sp_0, sp_1, sp_2, p_3)$. We assume that Σ^*_b is canonical on $\overline{P(b)P_i}$, $i=0, \cdots, 3$. The first vertex $P^1_i(b)$ of $\overline{P(b)P_i}$ can be written as

$$(3.22)\qquad P^1_i(b)=(P_i+\alpha_i(b)P(b))/\hat{a}_i$$

where $0\leqq\alpha_i(b)<\hat{a}_i$ and $i=0, 1, 2$ because $\det(P(b), P_i)=\hat{a}_i$ for $i=0, 1, 2$. As $p_j\equiv 0$ modulo $\hat{a}_i$ for $i\neq j$ and $P(b)=sP-(s-1)p_3P_3$, $\alpha_i(b)$ is the solution of

(3.23) $\alpha_i(b)s \equiv \alpha_i$ modulo $\hat{a}_i$ and $0 \leqq \alpha_i(b) < \hat{a}_i$.

(The existence of the solution of (3.23) is derived from the assumption that g.c.d.$(s, \hat{a}_i)=1$ for $i=0, 1, 2$.) Note that

(3.24) $P_i^1(b)-P_i^1 \in Z\langle P, P_3\rangle$ and

(3.25) $(P_i^1(b))_3 \equiv 0$ modulo $r_3r_{03}r_{13}r_{23}$

for $i=0$, 1 and 2 where $Z\langle P, P_3\rangle$ is the Z-module generated by P and P_3 and $(P_i^1(b))_3$ is the last coordinate of $P_i^1(b)$. We also assume that Σ_b^* is canonical on $T(P(b), P_i^1(b), P_j^1(b))$ around $P(b)$. Let $T_{01}^1(b)$ and $R(b)$ be defined in the same way as for T_{01}^1 and R and let $\sigma'=(P(b), P_0^1(b), T_{01}^1(b), R(b))$ be the desired 3-simplex. More precisely, we have

(3.26) $T_{01}^1(b)=(P_1^1(b)+\beta(b)P_0^1(b)+\gamma(b)P(b))/r_{23}$

where $0 \leqq \beta(b), \gamma(b) < r_{23}$ and

(3.27) $R(b)=(P_2^1(b)+\delta(b)P_0^1(b)+\varepsilon(b)T_{01}^1(b)+\mu(b)P(b))/r_3r_{03}r_{13}$

where $0 \leqq \delta(b), \varepsilon(b), \mu(b) < r_3r_{03}r_{13}$.

Assertion (3.28). *$\beta(b)=\beta$, $\delta(b)=\delta$ and $\varepsilon(b)=\varepsilon$.*

Proof of (3.28). Let γ' be an integer such that $\gamma's \equiv \gamma$ modulo r_{23}. Then we have by (3.23) and (3.24) that

$$(P_1^1(b)+\beta P_0^1(b)+\gamma'P(b))-r_{23}T_{01}^1 \in Z\langle P, P_3\rangle.$$

(Recall that $P(b)-sP=(1-s)p_3P_3$.) Then we can find an integer $\gamma(b)$ such that

$$P_1^1(b)+\beta P_0^1(b)+\gamma(b)P(b)-r_{23}T_{01}^1$$

is contained in $Z\langle P_3, r_{23}P\rangle$ and $0 \leqq \gamma(b) < r_{23}$. Let

(3.29) $T_{01}^1(b)'=(P_1^1(b)+\beta P_0^1(b)+\gamma(b)P(b))/r_{23}$.

Then it is obvious that $(T_{01}^1(b)')_i$ is an integer except $i=3$. However the last coordinate is also an integer by (3.25). Thus by the uniqueness of $T_{01}^1(b)$, we have that $T_{01}^1(b)'=T_{01}^1(b)$. Thus $\beta(b)=\beta$. As we can write $T_{01}^1(b)-T_{01}^1=xP_3+mP$ for some integer m and a rational number x, the integrality of $T_{01}^1(b)$ implies that x is an integer. Thus we get by (3.25),

(3.30) $T_{01}^1(b)-T_{01}^1 \in Z\langle P, P_3\rangle$ and

(3.31) $(T_{01}^1(b))_3 \equiv 0$ modulo $r_3r_{03}r_{13}$

by (3.29) where $(T^1_{01}(b))_3$ is the last coordinate of $T^1_{01}(b)$.

Now we consider (3.27). There is an integer μ' such that $\mu' s \equiv \mu$ modulo $r_3 r_{03} r_{13}$ by the assumption in (3.7). The difference of $P^1_2(b) + \delta P^1_0(b) + \varepsilon T^1_{01}(b) + \mu' P(b)$ and $r_3 r_{03} r_{13} R$ is contained in $Z\langle P, P_3 \rangle$ by (3.30) and (3.24). By the assumption on s, we can find an integer $\mu(b)$ such that $0 \leqq \mu(b) < r_3 r_{03} r_{13}$ and

$$(3.32) \qquad r_3 r_{03} r_{13}(R(b)' - R) \in Z\langle r_3 r_{03} r_{13} P, P_3 \rangle$$

where $R(b)'$ is defined by

$$(3.33) \qquad R(b)' = (P^1_2(b) + \delta P^1_0(b) + \varepsilon T^1_{01}(b) + \mu(b) P)/r_3 r_{03} r_{13}.$$

By (3.32), (3.25) and (3.31), $R(b)'$ is an integral vector. Thus we get $R(b)' = R(b)$ which implies that $\delta(b) = \delta$ and $\varepsilon(b) = \varepsilon$. This proves Assertion (3.28).

We have seen in (3.15)–(3.21) that the defining equation of $E(P(b))$ with respect to σ is determined by d, r_i, r_{ij}, β, δ, and ε. By the same argument and by Assertion (3.28), we conclude that the defining equation $g_{\sigma'}$ of $E(P(b))$ with respect to σ' is equal to g_σ in (3.15). This completes the proof.

Example (3.34). Let $a = (a_0, \cdots, a_3)$ and assume that g.c.d.$(a_i, a_j) = 1$ for $i \neq j$. Then $E(P)$ is birationally isomorphic to P^2.

Example (3.35). Let $a = (da_0, \cdots, da_3)$ where $\{a_i\}$ are mutually coprime. Then $E(P)$ is birationally isomorphic to the projective surface $\{X_0^d + \cdots + X_3^d = 0\}$ in P^3.

§ 4. The condition for $E(P)$ to be a rational or a $K3$-surface

In this section, we will study the algebraic surface as the central exceptional divisor of the resolution of the Brieskorn variety $V = f^{-1}(0)$ of dimension three. More precisely, we will study the necessary and sufficient condition for $E(P)$ to be either a rational or a $K3$-surface. Thus we may assume by Theorem (3.6) that $\hat{a}_i = 1$ for $i = 0, \cdots, 3$. The notations are the same as in Section 3. We assume that Σ^* is canonical on the line segment $\overline{PP_i}$. As $\hat{a}_i = 1$, we do not have any other vertex on $\overline{PP_i}$. We also assume that Σ^* is canonical around P. Let T^k_{ij} $(k = 1, \cdots, \nu_{ij})$ be the canonical vertices on $T(P, P_i, P_j)$ from P_i around P. Note that $\nu_{ij} = 0$ if and only if $r_{kl} = 1$ where $\{i, j, k, l\} = \{0, 1, 2, 3\}$. Let $C(P_i) = E(P_i) \cap E(P)$ and $C(T^k_{ij}) = E(T^k_{ij}) \cap E(P)$. Let H be the number of $\{(i, j);\ i < j$ and $r_{ij} > 1\}$. For brevity's sake, we denote $\chi(E(P))$ by χ, the affine equation $g_\sigma(y_\sigma)$ by $g(y)$ and the canonical divisor K_P by K. The following Theorem (4.1) and Theorem (4.2) are our main results of this paper.

Theorem (4.1). *$E(P)$ is a rational surface if and only if $a=(a_0, \cdots, a_3)$ is one of the following.*

(I) $H=0$.

(1) $a=(1, r, r, r)$ $(r_0=r)$. $\chi=3$ *and* $K=-3C(P_3)$ *and* $E(P)\cong P^2$. $g(y)=y_1+y_2^r+y_3^r+1$.

(2) $a=(2, 2, 2, 2)$ $(d=2)$. $\chi=4$, $K=-2C(P_3)$, $E(P)\cong P^1\times P^1$ *and* $g(y)=y_1^2+y_2^2+y_3^2+1$.

(3) $a=(3, 3, 3, 3)$ $(d=3)$. $\chi=9$, $K=-C(P_3)$ *and* $g(y)=y_1^3+y_2^3+y_3^3+1$. *$E(P)$ is isomorphic to the projective cubic surface* $\{X_0^3+\cdots+X_3^3=0\}$ *in* P^3.

(4) $a=(2, 4, 4, 4)$ $(d=r_0=2)$. $\chi=10$, $K=-C(P_3)$ *and* $g(y)=y_1^2+y_2^4+y_3^4+1$.

(5) $a=(2, 3, 6, 6)$ $(r_0=3, r_1=2)$. $\chi=11$, $K=-C(P_3)$ *and* $g(y)=y_1^2+y_2^3+y_3^6+1$.

(II) $H=1$.

(6) $a=(1, r, rs, rs)$ *where* $s>1$. $(r_0=r, r_{01}=s)$. $\chi=4$, $K=-(s+2)C(P_3)-2C(T_{23}^1)$ *and* $g(y)=y_1+y_2^r+y_3^{rs}+1$, $E(P)\cong S_s$.

(7) $a=(2, 2, 2r, 2r)$ *where* $r>1$. $(d=2, r_{01}=r)$. $\chi=4+2r$, $K=-2C(P_3)-C(T_{23}^1)$ *and* $g(y)=y_1^2+y_2^2+y_3^{2r}+1$.

(8) $a=(2, 2, r, 2r)$ *where* $r>1$. $(r_2=2, r_{01}=r)$. $\chi=6+r$, $K=-3C(P_3)-2C(T_{23}^2)-C(T_{23}^1)$ *and* $g(y)=y_1^2+y_2^2+y_3^r+1$.

(9) $a=(3, 3, 2, 6)$ $(r_2=3, r_{01}=2)$. $\chi=10$, $K=-2C(P_3)-C(T_{23}^1)$ *and* $g(y)=y_1^3+y_2^3+y_3^2+1$.

(III) $H=2$.

(10) $a=(1, rt, rs, rst)$ *where* g.c.d.$(s, t)=1$ *and* $s, t>1$. $(r_0=r, r_{01}=s, r_{02}=t)$. $\chi=3+\nu_{23}+\nu_{13}$, $K<0$ *and* $g(y)=y_1+y_2^{rt}+y_3^{rs}+1$.

(11) $a=(s, s, r, r)$ *where* $s, t>1$ *and* g.c.d.$(s, r)=1$. $(r_{01}=r, r_{23}=s)$. $\chi=4+rs$, $K=-2C(P_3)-C(T_{23}^1)+(s-2)C(P_2)$ *and* $g(y)=y_1^s y_2+y_2+y_3^r+1$.

(12) $a=(2, 3, 4, 12)$ $(r_1=2, r_{01}=2, r_{02}=3)$. $\chi=12$, $K=-2C(P_3)-C(T_{23}^1)-C(T_{13}^1)$ *and* $g(y)=y_1^2+y_2^3+y_3^4+1$.

(IV) $H\geqq 3$.

(13) $a=(3, 2, 5, 30)$ $(r_{01}=5, r_{12}=3, r_{02}=2)$. $\chi=14$, $K=-2C(P_3)-C(T_{13}^1)-C(T_{03}^1)-C(T_{23}^1)$ *and* $g(y)=y_1^3+y_2^2+y_3^5+1$.

(14) $a=(1, rst, rqt, rqs)$ *where* s, t, q *are mutually coprime and* $s, t, q>1$. $(r_0=r, r_{01}=q, r_{02}=s, r_{03}=t)$.
$\chi=3+\nu_{12}+\nu_{13}+\nu_{23}$ *and* $g(y)=y_1+y_2^{rst}y_3^{\varepsilon rs}+y_3^{rq}+1$.

Theorem (4.2). *$E(P)$ is a $K3$-surface if and only if $a=(a_0, \cdots, a_3)$ is one of the following.*

(I) $H=0$.

(1) $a=(4, 4, 4, 4)$ $(d=4)$. $\chi=24$, $K=0$ *and* $g(y)=y_1^4+y_2^4+y_3^4+1$. *$E(P)$ is isomorphic to the projective surface of degree 4* $\{X_0^4+\cdots+X_3^4=0\}$ *in* P^3.

(2) $a=(2, 6, 6, 6)$ $(d=2, r_0=3)$.
$\chi=24$, $K=0$ *and* $g(y)=y_1^2+y_2^6+y_3^6+1$.

(II) $H=1$.

(3) $a=(3, 3, 6, 6)$ $(d=3, r_{01}=2)$.
$\chi=24$, $K=0$ *and* $g(y)=y_1^3+y_2^3+y_3^6+1$.

(4) $a=(2, 3, 12, 12)$ $(r_0=3, r_1=2, r_{01}=2)$.
$\chi=24$, $K=0$ *and* $g(y)=y_1^2+y_2^3+y_3^{12}+1$.

(5) $a=(2, 4, 8, 8)$ $(d=r_0=r_{01}=2)$.
$\chi=24$, $K=0$ *and* $g(y)=y_1^2+y_2^4+y_3^8+1$.

(6) $a=(5, 5, 2, 10)$ $(r_2=5, r_{01}=2)$.
$\chi=24$, $K=0$ *and* $g(y)=y_1^5+y_2^5+y_3^2+1$.

(7) $a=(3, 3, 4, 12)$ $(r_2=3, r_{01}=4)$.
$\chi=24$, $K=0$ *and* $g(y)=y_1^3+y_2^3+y_3^4+1$.

(8) $a=(3, 3, 5, 15)$ $(r_2=3, r_{01}=5)$.
$\chi=25$, $K=C(P_3)$ *and* $g(y)=y_1^3+y_2^3+y_3^5+1$.

(9) $a=(4, 4, 3, 12)$ $(r_2=4, r_{01}=3)$.
$\chi=25$, $K=C(P_3)$ *and* $g(y)=y_1^4+y_2^4+y_3^3+1$.

(III) $H=2$.

(10) $a=(2, 4, 6, 12)$ $(d=r_{02}=2, r_{01}=3)$.
$\chi=24$, $K=0$ *and* $g(y)=y_1^2+y_2^4+y_3^6+1$.

(11) $a=(2, 3, 8, 24)$ $(r_1=2, r_{01}=4, r_{02}=3)$.
$\chi=24$, $K=0$ *and* $g(y)=y_1^2+y_2^3+y_3^8+1$.

(12) $a=(2, 5, 4, 20)$ $(r_1=2, r_{01}=2, r_{02}=5)$.
$\chi=24$, $K=0$ *and* $g(y)=y_1^2+y_2^5+y_3^4+1$.

(13) $a=(2, 7, 4, 28)$ $(r_1=r_{01}=2, r_{02}=7)$.
$\chi=26$, $K=2C(P_3)+C(T_{23}^1)$ *and* $g(y)=y_1^2+y_2^7+y_3^4+1$.

(14) $a=(2, 5, 6, 30)$ $(r_1=2, r_{01}=3, r_{02}=5)$. $\chi=27$,
$K=3C(P_3)+2C(T_{23}^2)+C(T_{23}^1)$ *and* $g(y)=y_1^2+y_2^5+y_3^6+1$.

(15) $a=(2, 3, 10, 30)$ $(r_1=2, r_{01}=5, r_{02}=3)$.
$\chi=25$, $K=C(P_3)$ *and* $g(y)=y_1^2+y_2^3+y_3^{10}+1$.

(16) $a=(3, 2, 9, 18)$ $(r_1=r_{01}=3,\ r_{02}=2)$.
$\chi=24$, $K=0$ *and* $g(y)=y_1^3+y_2^2+y_3^9+1$.

(17) $a=(3, 6, 4, 4)$ $(r_0=r_{01}=2,\ r_{23}=3)$.
$\chi=24$, $K=0$ *and* $g(y)=y_1^3y_2^2+y_2^2+y_3^4+1$.

(IV) $H\geqq 3$.

(18) $a=(3, 2, 7, 42)$ $(r_{01}=7,\ r_{12}=3,\ r_{02}=2)$.
$\chi=24$, $K=0$ *and* $g(y)=y_1^3+y_2^2+y_3^7+1$.

(19) $a=(3, 2, 11, 66)$ $(r_{01}=11,\ r_{12}=3,\ r_{02}=2)$. $\chi=27$,
$K=4C(P_3)+2C(T_{13}^1)+C(T_{03}^1)$ *and* $g(y)=y_1^3+y_2^2+y_3^{11}+1$.

(20) $a=(2, 3, 10, 15)$ $(r_{01}=5,\ r_{02}=3,\ r_{13}=2)$.
$\chi=24$, $K=0$ *and* $g(y)=y_1^2y_3+y_2^3+y_3^5+1$.

(21) $a=(2, 3, 14, 21)$ $(r_{01}=7,\ r_{02}=3,\ r_{13}=2)$.
$\chi=25$, $K=C(P_3)$ *and* $g(y)=y_1^2y_3+y_2^3+y_3^7+1$.

(22) $a=(2, 5, 6, 15)$ $(r_{01}=3,\ r_{02}=5,\ r_{13}=2)$.
$\chi=25$, $K=C(P_3)$ *and* $g(y)=y_1^2y_3+y_2^5+y_3^3+1$.

We will prove Theorem (4.1) and Theorem (4.2) simultaneously. Their proofs occupy the rest of this section as well as the following section. We first prepare some basic lemmas.

Recall that the first vertex T_{ij}^1 from P_i of $T(P, P_i, P_j)$ can be written by Lemma (2.1) as

$$T_{ij}^1=(P_j+\beta_{ij}P_i+\gamma_{ij}P)/r_{ij}^* \tag{4.3}$$

where $r_{ij}^*=r_{kl}$ and $\{i, j, k, l\}=\{0, 1, 2, 3\}$ and $0\leqq\beta_{ij}, \gamma_{ij}<r_{ij}^*$. Fix (i, j) such that $r_{ij}^*>1$ and let

$$\frac{r_{ij}^*}{\beta_{ij}}=m_1-\cfrac{1}{m_2-\cfrac{}{\ddots-\cfrac{1}{m_{b_{ij}}}}}$$

be the continuous fraction representation where $m_i>1$. Then by Theorem (8.5) of [11], we have that

Lemma (4.4). (i) *$b_{ij}=\nu_{ij}$ and $C(T_{ij}^k)$ is a disjoint union of $dr_ir_jr_{ij}$ copies of rational curves and $C(T_{ij}^k)^2=-m_kdr_ir_jr_{ij}$.*

(ii) $C(P_i)^2=dr_i^2\prod_{t\neq i}(r_{it}/r_{it}^*)-\sum_{t\neq i}\beta_{it}dr_ir_t(r_{it}/r_{it}^*)$.

(iii) *$C(P_i)$ is a rational curve if and only if*

(1) *there exists an integer k such that $k\neq i$ and* g.c.d.(a_k, a_j) $=1$ *for any j such that $j\neq i, k$ or*

(2) g.c.d.$(a_j, a_l)=2$ *for any j, l such that $j\neq l$ and $j, l\neq i$.*

Note that

$$\frac{r^*_{ij}}{\beta_{ji}}=m_{b_{ij}}-\cfrac{1}{m_{b_{ij}-1}-\cfrac{\ddots}{\ \ -\cfrac{1}{m_1}}}.$$

The following is also due to Theorem (8.5) of [11].

Lemma (4.5). *Let $\chi(E(P))$ be the Euler characteristic of $E(P)$. Then*

$$\chi(E(P))=d^3\prod_{i=0}^{3} r_i^2\prod_{i<j} r_{ij}-d^2\prod_{i=0}^{3} r_i\sum_{i=0}^{3}\Big(r_i\prod_{\substack{k\\ k\neq i}} r_{ik}\Big)+d\sum_{i<j}(\nu_{ij}+1)r_ir_jr_{ij}. \tag{4.6}$$

We take $\sigma=(P, P_0, T^1_{01}, R)$ as a fixed 3-simplex where

$$T^1_{01}=(P_1+\beta P_0+\gamma P)/r_{23},\quad 0\leqq\beta,\ \gamma<r_{23} \tag{4.7}$$

and

$$R=(P_2+\delta P_0+\varepsilon T^1_{01}+\mu P)/r_3r_{03}r_{13} \tag{4.8}$$

where $0\leqq\delta$, ε, $\mu<r_3r_{03}r_{13}$ and T^1_{01} is assumed to be P_1 if $r_{23}=1$. ($\beta=\beta_{01}$, $\gamma=\gamma_{01}$ in (4.3)).

Let ω be the meromorphic 2-form on $E(P)$ such that $\omega|E(P)\cap C^3_\sigma$ is $dy_{\sigma,1}\wedge\cdots\wedge dy_{\sigma,3}/dg_\sigma$ and let

$$(\omega)=K=\sum_{i=1}^{3} n_iC(P_i)+\sum_{i,j,k} n(T^k_{ij})C(T^k_{ij}). \tag{4.9}$$

Then by Theorem (9.9) of [11] and (4.7) and (4.8), we have

$$n_1=r_{23}-\beta-1\geqq 0,\quad 0\leqq n_1<r_{23}, \tag{4.10}$$

$$n_2=r_3r_{03}r_{13}-\delta-\varepsilon-1,\quad |n_2|<r_3r_{03}r_{13}, \tag{4.11}$$

$$n_3=d\,r_0r_1r_2r_{01}r_{02}r_{12}-\{p_0+p_1(n_1+1)+p_2(n_2+1)\}/p_3-1. \tag{4.12}$$

We introduce meromorphic function φ_{ij} by

$$\varphi_{ij}=\pi^*(z_i^{p'_j}/z_j^{p'_i}) \tag{4.13}$$

where $p'_j=p_j/r_{ij}$ and $p'_i=p_i/r_{ij}$. For a vertex $Q={}^t(q_0,\cdots,q_3)$ of Σ^*, we define $|Q, P|_{ij}=q_ip_j-q_jp_i$. As we have seen in Section 8 of [11], we have, (for fixed i and j)

(4.14) $$(\varphi_{ij})=n'_i C(P_i)+n'_j C(P_j)+\sum_{k,l,s} n'(T^s_{kl})C(T^s_{kl})$$

where

(4.15) $$n'(T^s_{kl})=|T^s_{kl}, P|_{ij}/r_{ij}$$

and

(4.16) $$n'_i=p_j/r_{ij} \quad \text{and} \quad n'_j=-p_i/r_{ij}.$$

Note that $n'_i=|P_i, P|_{ij}/r_{ij}$ and $n'_j=|P_j, P|_{ij}/r_{ij}$.

Recall that the geometric genus p_g of $E(P)$ is the dimension of the vector space of the holomorphic 2-forms on $E(P)$, The following lemma plays an important role for the proof of Theorem (4.1) and Theorem (4.2).

Lemma (4.17). (i) *Assume that* $(\varphi^s_{ij}\omega)3$ *has non-negative coefficients on* $C(P_k)$ *for* $k=0, \cdots, 3$. *Then* $\varphi^s_{ij}\omega$ *is a holomorphic 2-form.*

(ii) $(\omega)=0$ *if the coefficients of* $C(P_k)$ *in* (ω) $(k=0, \cdots, 3)$ *are zero.*

Proof. Take any vertex T^m_{kl} and express it as $T^m_{kl}=xP_k+yP_l+zP$ where x, y, z are non-negative rational numbers. Note that T^m_{kl} takes its maximum $d(T^m_{kl})$ on $\Delta(P_k)\cap\Delta(P_l)\cap\Delta(P)$. Thus by Theorem (9.9) of [11], we have

(4.18) $$n(T^m_{kl})=xn_k+yn_l+x+y-1.$$

On the other hand, we have by (4.15),

(4.19) $$n'(T^m_{kl})=xn'_k+yn'_l.$$

Let h^m_{kl} be the coefficient of $C(T^m_{kl})$ in $(\varphi^s_{ij}\omega)$. Then we have

$$h^m_{kl}=n(T^m_{kl})+sn'(T^m_{kl})=x(n_k+sn'_k)+y(n_l+sn'_l)+x+y-1.$$

As $x, y\geqq 0$ and $(x, y)\neq(0, 0)$, it is easy to see that $h^m_{kl}\geqq 0$. Assume that $n_k+sn'_k=0$ for $k=0, \cdots, 3$. Then applying the above equality for T^1_{kl} which is defined by (4.3), we obtain $-1<n(T^1_{kl})<1$ which implies that $n(T^1_{kl})=0$ and $x+y=1$. Recall that

$$T^m_{kl}=x_m T^{m-1}_{kl}+y_m P_l+z_m P$$

for $0\leqq x_m, y_m, z_m<1$ by Lemma (2.1). Thus by a similar argument as above using an induction on m, we get $n(T^m_{kl})=0$ for any T^m_{kl}, completing the proof of the assertion (ii).

Let t be defined by

$$t=\begin{cases}1 & \text{if } n_2\geqq 0\\ 2 & \text{if } n_2<0.\end{cases} \tag{4.20}$$

This notation is also used in Section 5. We apply Lemma (4.17) for $\varphi_{ij}=\varphi_{23}$, to obtain

Lemma (4.21). (i) *Assume that* $n_3-(t-1)p_2/r_{23}\geqq 0$. *Then* $\varphi_{23}^{t-1}\omega$ *is holomorphic and* $p_g\geqq 1$.

(ii) *Assume that* $n_3-tp_2/r_{23}\geqq 0$. *Then* $\varphi_{23}^{t-1}\omega$ *and* $\varphi_{23}^{t}\omega$ *are holomorphic. In particular,* $p_g\geqq 2$.

Proof. The assertion is immediate from Lemma (4.17), (4.9)–(4.11) and from the following inequality: $n_2>-r_3r_{03}r_{13}=-p_3/r_{23}$.

Recall that a minimal rational surface is either P^2 or S_s ($s=0, 2, 3, \cdots$). S_s is characterized among $\{S_s\}$ ($s=0, 2, 3, \cdots$) by the following property: there is an irreducible curve E_s in S_s such that $E_s^2=-s$ (p. 519, [4]).

Let M be a compact algebraic surface. The Castelnuovo-Enriques criterion for the rationality (p. 536, [4]) implies

Lemma (4.22). *Assume that* $q(M)=0$ *and that* M *has a meromorphic 2-form* ω *such that* $-(\omega)>0$. *Then* M *is a rational surface.*

Proof. The assertion $-(\omega)>0$ implies that the plurigenera $P_s(M)=0$ for $s>0$. Thus M is a rational surface.

We say that M is a minimal $K3$-surface if $\pi_1(M)=\{0\}$ and the canonical line bundle K is trivial. We say that M is a $K3$-surface if M has a birational morphism to a minimal $K3$-surface. In any case, $p_g=1$ and $\chi(M)\geqq 24$.

Lemma (4.23). *Let* M *be an algebraic surface with a holomorphic 2-form* ω. *Let* $(\omega)=\Sigma_{i=0}^{\nu} m_iC_i$ *with* C_i *being irreducible.*

(i) *Assume that* $C\subset M$ *is an exceptional curve of the first kind. Then* $C=C_k$ *for some* k. *Let* $\overline{M}$ *be the surface when* C_k *is blown down. Then* ω *induces a holomorphic 2-form* $\overline{\omega}$ *so that* $(\overline{\omega})=\Sigma_{i\neq k} m_i\overline{C}_i$ *where* $\overline{C}_i$ *is the image of* C_i *in* $\overline{M}$.

(ii) *Assume that the virtual genus* $\pi(C_\nu)$ *is positive. Then* M *is not a* $K3$*-surface.* ($\pi(C)$ *is defined by* $1+(KC+C^2)/2$. *See* p. 471, [4]).

Proof. By a stronger version of the Castelnuovo-Enriques criterion (p. 505, [4]), C is exceptional if and only if $C^2<0$ and $C\cdot K<0$. If $C\neq C_i$ for any i, $C\cdot K\geqq 0$ which proves (i). Let $C=C_k$ be an exceptional divisor of the first kind and let $\overline{M}=M/C_k$. Then the image curve $\overline{C}_\nu$ satisfies the same condition as in (ii). Thus after possible blowing downs, C_ν can not be eliminated in (ω) by (i). This proves (ii).

For the calculation of p_g, we have the Noether's formula:

$$12(1-q+p_g)=K^2+\chi. \tag{4.24}$$

§ 5. Proof of Theorem (4.1) and Theorem (4.2)

In this section, we assume that $E(P)$ is either a rational or a $K3$-surface. Thus

$$p_g \leqq 1. \tag{5.1}$$

We use the same notations as in Section 4. By Lemma (4.21), we must have

$$n_3-tp_2/r_{23}=n_3-t\,r_2r_{02}r_{12}<0. \tag{5.2}$$

On the other hand, we have by (4.10)–(4.12),

$$p_3(n_3-tp_2/r_{23}) \geqq d\prod_{i=0}^{3} r_i \prod_{i<j} r_{ij}-p_0-p_1r_{23}-2p_2p_3/r_{23}-p_3. \tag{5.3}$$

Recall that H is the number of $\{(i,j);\, r_{ij}>1,\, i<j\}$. We also define J by the number of $\{i;\, r_i>1\}$.

The proof is divided into several cases by H. As the proof is so long and boring, we first explain the outline of the proof and we will only give the proof for $H=0$ and 1 in detail and leaves the other cases to the reader. We first use (5.2) and (5.3) to pick up all the possible cases which satisfies (5.2). They are finite and we call them the exceptional cases. Then we use the results in Section 4 to study further details about these exceptional cases.

I. $\boldsymbol{H=0}$. ($r_{ij}=1$ for any i, j.)

Then $a_i=d\prod_{j\neq i} r_j$ and $P={}^t(r_0, \cdots, r_3)$. We may assume that

$$r_0 \geqq r_1 \geqq r_2 \geqq r_3. \tag{5.4}$$

By (5.3), we have

$$r_3(n_3-tr_2) \geqq d\,r_0r_1r_2r_3-r_0-r_1-2r_2r_3-r_3. \tag{5.5}$$

Assume that $r_3>1$. As the right side of (5.5) is a monotone increasing function of $r_0, \cdots, r_3$ and d under (5.4), we get by a rough estimation

$$r_3(n_3-tr_2) \geqq 2 \quad (d=1,\ r_i=2 \text{ for } i=0, \cdots, 3)$$

which contradicts (5.2). Thus $r_3=1$ and by (4.11) and (4.12), we get $n_2=0$, $t=1$ and

$$\begin{aligned} n_3-r_2 &= dr_0r_1r_2-r_0-r_1-2r_2-1 \\ &= \{(r_0-1)(r_1-1)-2\}+\{(dr_2-1)r_0r_1-2r_2\}. \end{aligned}$$

By the above expression and (5.2), we must have $r_2=1$. Thus we have

$$n_3-1=(r_0-1)(r_1-1)-4+(d-1)r_0r_1<0.$$

It is easy to see that this is the case if and only if
(I–i) $d=1$, $r_1=1$, i.e. $a=(1, r_0, r_0, r_0)$ or
(I–ii) $d=1$, $r_1=2$, $r_0=3$, i.e. $a=(2, 3, 6, 6)$ or
(I–iii) $d=2$, $r_0=r_1=1$, i.e. $a=(2, 2, 2, 2)$ of
(I–iv) $d=2$, $r_0=2$, $r_1=1$, i.e. $a=(2, 4, 4, 4)$ or
(I–v) $d=2$, $r_0=3$, $r_1=1$, i.e. $a=(2, 6, 6, 6)$ or
(I–vi) $d=3$, $r_0=1$, i.e. $a=(3, 3, 3, 3)$ or
(I–vii) $d=4$, $r_0=1$, i.e. $a=(4, 4, 4, 4)$.

We study the above 7 cases in more detail.

Case (I-i). Assume that $a=(1, r, r, r)$.

Then $P={}^t(r, 1, 1, 1)$ and $\sigma=(P, P_0, P_1, P_2)$ and $K=-3C(P_3)<0$. This implies that $E(P)$ is a rational surface by Lemma (4.22). $\chi(E(P))=3$ by Lemma (4.5). Therefore $E(P)$ is isomorphic to P^2. $g(y)=y_1+y_2^r+y_3^r+1$ by (3.15). This corresponds to (1) of Theorem (4.1).

Case (I-ii). Assume that $a=(2, 3, 6, 6)$, $(r_0=3, r_1=2)$.

Then $P={}^t(3, 2, 1, 1)$, $\sigma=(P, P_0, P_1, P_2)$, $K=-C(P_3)$ and $\chi=11$. $E(P)$ is rational by Lemma (4.22) and $g(y)=y_1^2+y_2^3+y_3^6+1$. This corresponds to (5) of Theorem (4.1).

Case (I-iii). Assume that $a=(2, 2, 2, 2)$, $(d=2)$.

Then $P={}^t(1, 1, 1, 1)$, $\sigma=(P, P_0, P_1, P_2)$, $K=-2C(P_3)$, $\chi=4$ and $g(y)=y_1^2+y_2^2+y_3^2+1$. As $\pi: \tilde{V}\to V$ is the blowing up of V at the origin, $E(P)$ is isomorphic to the projective surface $\{X_0^2+\cdots+X_3^2=0\}$ in P^3 which is isomorphic to $P^1\times P^1\cong S_0$. This corresponds to (2) of Theorem (4.1).

Case (I-iv). $a=(2, 4, 4, 4)$, $(d=2, r_0=2)$.

Then $P={}^t(2, 1, 1, 1)$, $\sigma=(P, P_0, P_1, P_2)$, $K=-C(P_3)$, $\chi=10$ and $g(y)$

$=y_1^2+y_2^4+y_3^4+1$. This corresponds to (4) of Theorem (4.1).

Case (I-v). $a=(2, 6, 6, 6)$, $(d=2, r_0=3)$.

Then $P={}^t(3, 1, 1, 1)$, $\sigma=(P, P_0, P_1, P_2)$, $K=0$ and $\chi=24$. Thus $E(P)$ has a trivial canonical bundle and is a minimal $K3$-surface. The affine equation $g(y)=y_1^2+y_2^6+y_3^6+1$. This corresponds to (2) of Theorem (4.2).

Case (I-vi). $a=(3, 3, 3, 3)$, $(d=3)$.

Then $P={}^t(1, 1, 1, 1)$, $\sigma=(P, P_0, P_1, P_2)$, $K=-C(P_3)$, $\chi=9$ and $g(y)=y_1^3+y_2^3+y_3^3+1$. $E(P)$ is a rational surface which is isomorphic to the projective cubic surface $\{X_0^3+\cdots+X_3^3=0\}$ in P^3. This corresponds to (3) of Theorem (4.1).

Case (I-vii). $a=(4, 4, 4, 4)$, $(d=4)$.

Then $P={}^t(1, 1, 1, 1)$, $\sigma=(P, P_0, P_1, P_2)$, $K=0$, $\chi=24$ and $g(y)=y_1^4+y_2^4+y_3^4+1$. This is a well known $K3$-surface. This corresponds to (1) of Theorem (4.2).

II. $\boldsymbol{H=1}$.

We assume that $r_{01}\neq 1$ and $r_{ij}=1$ for other i and j. $P={}^t(r_0r_{01}, r_1r_{01}, r_2, r_3)$ in general. Thus $\sigma=(P, P_0, P_1, R)$ and $R=P_2$ if $r_3=1$.

(II-1) Assume that $\boldsymbol{J=0}$, i.e. $r_i=1$ for any i.

Then $P={}^t(r_{01}, r_{01}, 1, 1)$, $\sigma=(P, P_0, P_1, R_2)$, and $n_2=0, n_3=(d-2)r_{01}-2$. At $t=1$, we have by (5.2)

$$(d-2)r_{01}-3<0.$$

This is the case if and only if

(II-i) $d=1$, i.e. $a=(1, 1, r_{01}, r_{01})$ or
(II-ii) $d=2$, i.e. $a=(2, 2, 2r_{01}, 2r_{01})$ or
(II-iii) $d=3$, $r_{01}=2$, i.e. $a=(3, 3, 6, 6)$.

We will study (II-i) later.

Case (II-ii). Let $a=(2, 2, 2r, 2r)$ with $r>1$.

Then $r_{01}=r$, $P={}^t(r, r, 1, 1)$ and $\sigma=(P, P_0, P_1, P_2)$ and $\nu_{23}=1$. Namely $T_{23}^1=(P_2+P_3+(r-1)P)/r$. $K=-2C(P_3)-C(T_{23}^1)$. Thus $E(P)$ is rational by Lemma (4.22) and $\chi=4+2r$ by Lemma (4.5). $g(y)=y_1^2+y_2^2+y_3^{2r}+1$ by (3.15). The correspondence is (7) of Theorem (4.1).

Case (II-iii). Let $a=(3, 3, 6, 6)$.

Then $r_{01}=2$, $P={}^t(2, 2, 1, 1)$ and $\sigma=(P, P_0, P_1, P_2)$ and $\nu_{23}=1$, i.e. $T_{23}^1=(P_2+P_3+P)/2$. $K=0$ and $\chi=24$. Thus $E(P)$ is a minimal $K3$-surface. $g(y)=y_1^3+y_2^3+y_3^6+1$. This corresponds to (3) of Theorem (4.2).

(II-2) Assume that $H=1$ and $J=1$.

By a change of coordinates if necessary, we have two possibilities.

$$(\alpha)\ r_0>1 \quad \text{or} \quad (\beta)\ r_2>1.$$

In any case, $P={}^t(r_0r_{01}, r_{01}, r_2, 1)$ and $\sigma=(P, P_0, P_1, P_2)$. Thus $n_2=0$ and $n_3=dr_0r_2r_{01}-(r_0+1)r_{01}-r_2-1$. We have from (5.2) that

$$n_3-r_2=dr_0r_2r_{01}-(r_0+1)r_{01}-2r_2-1<0. \tag{5.6}$$

Case (II-2-α). Assume that $r_0>1$ and $r_i=1$ $(i\neq 0)$.

Then

$$n_3-r_2=dr_0r_{01}-(r_0+1)r_{01}-3=r_{01}\{(d-1)r_0-1\}-3<0$$

if and only if
(II-iv) $d=1$, i.e. $a=(1, r_0, r_0r_{01}, r_0r_{01})$ or
(II-v) $d=2$ and $r_0=r_{01}=2$, i.e. $a=(2, 4, 8, 8)$.

Case (II-2-β). Assume that $r_2>1$ and r_i $(i\neq 2)$.

Then we have by (5.6) that

$$n_3-r_2=(r_2-2)(r_{01}-2)+(d-1)r_2r_{01}-5$$

and $n_3-r_2<0$ if and only if
(II-vi) $d=1$ and $r_2=2$, i.e. $a=(2, 2, r_{01}, 2r_{01})$ where r_{01} is odd and $r_{01}>1$ or
(II-vii) $d=1$ and $r_{01}=2$, i.e. $a=(r_2, r_2, 2, 2r_2)$ where r_2 is odd and $r_2>1$ or
(II-viii) $d=1$, $r_2=3$, $r_{01}=4$, i.e. $a=(3, 3, 4, 12)$ or
(II-ix) $d=1$, $r_2=3$, $r_{01}=5$, i.e. $a=(3, 3, 5, 15)$ or
(II-x) $d=1$, $r_2=4$, $r_{01}=3$, i.e. $a=(4, 4, 3, 12)$ or
(II-xi) $d=1$, $r_2=5$, $r_{01}=3$, i.e. $a=(5, 5, 3, 15)$.
(Note that g.c.d.$(r_2, r_{01})=1$ by Proposition (3.3).)

Now we study further details about the above exceptional cases.

Case (II-i, iv). Assume that $a=(1, r, rs, rs)$ where $s>1$.

Then $r_{01}=s$, $P={}^t(rs, s, 1, 1)$, $n_2=0$ and $n_3=-(s+2)$. $\nu_{23}=1$. In fact, $T^1_{23}=(P_2+P_3+(s-1)P)/s$ and $K=-(s+2)C(P_3)-2C(T^1_{23})$. Therefore $E(P)$ is rational by Lemma (4.22) and $\chi=4$ by Lemma (4.5). $C(P_3)^2=0$ and $C(T^1_{23})^2=-s$ by Lemma (4.4). We assert that $E(P)$ is minimal. In fact, for any irreducible curve C such that $C\neq C(P_3)$, $C(T^1_{23})$, we have $C\cdot K=0$ or $\leqq -2$ by the above expression of K. Thus by the genus formula (p. 505, [6]), C cannot be an exceptional curve of the first kind. By the classification of the minimal rational surface, $E(P)$ is isomorphic to S_s. $g(y)=y_1+y_2^r+y_3^{rs}+1$.

This corresponds to (6) of Theorem (4.1).

Case (II-v). Assume that $a=(2, 4, 8, 8)$, $(d=r_0=r_{01}=2)$.

Then $P={}^t(4, 2, 1, 1)$ and $\sigma=(P, P_0, P_1, P_2)$. $\nu_{23}=1$, i.e. $T^1_{23}=(P_2+P_3+P)/2$. $K=0$ and $\chi=24$. Thus $E(P)$ is a minimal $K3$-surface. $g(y)=y_1^2+y_2^4+y_3^8+1$. $E(P)$ corresponds to (5) Theorem (4.2).

Case (II-vi). Assume that $a=(2, 2, r, 2r)$ where r is an odd integer such that $r>1$.

Then $r_{01}=r$, $P={}^t(r, r, 2, 1)$, $\sigma=(P, P_0, P_1, P_2)$, $n_2=0$ and $n_3=-3$. Thus $K<0$ and $E(P)$ is rational by Lemma (4.22). $\nu_{23}=2$. Namely T^1_{23} $(P_3+2P_2+(r-1)P)/r$ and $T^2_{23}=(T^1_{23}+P_3)/2$. $K=-3C(P_3)-2C(T^2_{23})-C(T^1_{23})$ and $\chi=6+r$. $g(y)=y_1^2+y_2^2+y_3^r+1$.

This corresponds to (8) of Theorem (4.1).

Case (II-vii). Assume that $a=(r, r, 2, 2r)$ and $r>1$, odd.

Then $r_{01}=2$, $P={}^t(2, 2, r, 1)$, $\sigma=(P, P_0, P_1, P_2)$ and $n_2=0$ and $n_3=r-5$. $\nu_{23}=1$. Namely $T^1_{23}=(P_3+P_2+P)/2$ and we have

$$K=(r-5)\,C(P_3)+\frac{(r-5)}{2}\,C(T^1_{23}).$$

Note that $K^2=(r-5)^2/2$ by Lemma (4.4) and $\chi=r^2-r+4$, If $r=3$, $a=(3, 3, 2, 6)$ and $K=-2C(P_3)-C(T^1_{23})$. Thus $E(P)$ is a rational surface which corresponds to (9) of Theorem (4.1). If $r=5$, $a=(5, 5, 2, 10)$ and $K=0$. This case corresponds to the $K3$-surface corresponding to (6) of Theorem (4.2). Suppose that $r>5$. Then $E(P)$ is minimal by Lemma (4.23) and $p_g=(r^2-4r+11)/8-1\geqq 3$. In any case, $g(y)=y_1^r+y_2^r+y_3^2+1$.

Case (II-viii). Let $a=(3, 3, 4, 12)$, $(r_{01}=4, r_2=3)$.

Then $P={}^t(4, 4, 3, 1)$, $\sigma=(P, P_0, P_1, P_2)$ and $K=0$. $\nu_{23}=3$ as $T^1_{23}=$

$(P_3+3P_2+3P)/4$ and $4/3=2-\dfrac{1}{2-1/2}$ and $\chi=24$. $g(y)=y_1^3+y_2^3+y_3^4+1$. $E(P)$ is a K3-surface corresponding to (7) of Theorem (4.2).

Case (II-ix). Let $a=(3, 3, 5, 15)$, $(r_2=3, r_{01}=5)$.

Then $P={}^t(5, 5, 3, 1)$, $\sigma=(P, P_0, P_1, P_2)$ and $n_2=0, n_3=1$. $\nu_{23}=2$, i.e. $T_{23}^1=(P_3+3P_2+4P)/5$ and $5/3=2-1/3$. $\chi=25$ and $K=C(P_3)$. As $C(P_3)^2=-1$ and $C(P_3)$ is a rational curve by Lemma (4.4), we can blow down $C(P_3)$ and let $\overline{M}=E(P)/C(P_3)$. Our two-form ω induces a nowhere vanishing two form on $\overline{M}$. Thus $\overline{M}$ and $E(P)$ are K3-surfaces. $g(y)=y_1^3+y_2^3+y_3^5+1$.

The correspondence is (8) of Theorem (4.2).

Case (II-x). Let $a=(4, 4, 3, 12)$, $(r_3=4, r_{01}=3)$.

Then $P={}^t(3, 3, 4, 1)$, $\sigma=(P, P_0, P_1, P_2)$, $n_2=0$, $n_3=1$ and $\nu_{23}=1$. ($T_{23}^1=(P_3+P_2+2P)/3$.) $K=C(P_3)$ and $\chi=25$. As $C(P_3)$ is an exceptional curve of the first kind by Lemma (4.4), we get a minimal $K3$-surface by blowing down $C(P_3)$. $g(y)=y_1^4+y_2^4+y_3^3+1$. This corresponds to (9) of Theorem (4.2).

Case (II-xi). Let $a=(5, 5, 3, 15)$, $(r_2=5, r_{01}=3)$.

$P={}^t(3, 3, 5, 1)$, $\sigma=(P, P_0, P_1, P_2)$ and $n_3=3$ and $\nu_{23}=2$. Namely $T_{23}^1=(P_3+2P_2+2P)/3$ and $3/2=2-1/2$. $K=3C(P_3)+2C(T_{23}^2)+C(T_{23}^1)$. $\chi=45$, $C(P_3)^2=-3$ and $E(P)$ is minimal by Lemma (4.23) ($p_g=3$). This completes the case of $H=1$ and $J=1$.

(II-3) $H=1$ and $J=2$, $(r_{01}>1)$.

We may assume that

$$r_0\geqq r_1 \quad\text{and}\quad r_2\geqq r_3. \tag{5.7}$$

There are three possibilties up to the ordering of $\{a_i\}$: $(\alpha) r_0>r_1>1$ or $(\beta) r_0, r_2>1$ or $(\gamma) r_2>r_3>1$.

Case (II-3-α). $r_0>r_1>1$.

Then $P={}^t(r_0r_{01}, r_1r_{01}, 1, 1)$, $\sigma=(P, P_0, P_1, P_2)$, $n_2=0$, $n_3=dr_0r_1r_{01}-(r_0+r_1)r_{01}-2$. By (5.2), $n_3-1=r_{01}(dr_0r_1-r_0-r_1)-3<0$ if and only if (II-xii) $d=1$, $r_0=3$, $r_1=r_{01}=2$, i.e. $a=(2, 3, 12, 12)$.

Then $P={}^t(6, 4, 1, 1)$, $\sigma=(P, P_0, P_1, P_2)$, $K=0$, $\nu_{23}=1$ and $\chi=24$. Thus $E(P)$ is a minimal $K3$-surface corresponding to (4) of Theorem (4.2). $g(y)=y_1^2+y_2^3+y_3^{12}+1$.

Case (II-3-β) $r_0, r_2>1$.

Then $P={}^t(r_0r_{01}, r_{01}, r_2, 1)$, $n_2=0$,

$$n_3-r_2=dr_0r_2r_{01}-(r_0+1)r_{01}-2r_2-1.$$

Thus n_3-r_2 is a monotone increasing function for each $d\geqq 1$, $r_0, r_2, r_{01}>1$. If $d>1$, we have

$$n_3-r_2\geqq 2^4-3\cdot 2-2^2-1>0$$

which is a contradiction to (5.2). Thus $d=1$. If $r_{01}\geqq 3$,

$$n_3-r_2\geqq(3r_0-2)(r_2-1)-6>0$$

as g.c.d. $(r_0, r_2)=1$. Thus $d=1$ and $r_{01}=2$ and $n_3-r_2=2(r_0-1)(r_2-1)-5$. This is negative if and only if (II-xiii) $d=1$, $r_0=r_{01}=2$ and $r_2=3$, i.e. $a=(3.\ 6, 4, 12)$.

Then $P={}^t(4, 2, 3, 1)$, $\sigma=(P, P_0, P_1, P_2)$, $\nu_{23}=1$ and $\chi=34$ and $K=2C(P_3)+C(T^1_{23})$. $(T^1_{23}=(P_2+P_3+P)/2.)$ As $C(P_3)$ has a positive genus by Lemma (4.4), $E(P)$ is minimal and not a $K3$-surface.

Case (II-3-γ), $r_2>r_3>1$.

Then $a=(dr_2r_3, dr_2r_3, dr_3r_{01}, dr_2r_{01})$ and $P={}^t(r_{01}, r_{01}, r_2, r_3)$. By Lemma (4.4), $C(P_l)$ $(l=2, 3)$ is not rational. By (5.3), we can estimate

$$r_3(n_3-(t-1)r_2)\geqq dr_2r_3r_{01}-2r_{01}-r_2r_3-r_3\geqq 6,$$
$$(d=1, r_{01}=2, r_2=3, r_3=5).$$

Therefore by Lemma (4.21) and Lemma (4.23), $p_g\geqq 1$ and $E(P)$ is not a $K3$-surface.

(II-4) $H=1$ and $J\geqq 3$.

We may assume that $r_0\geqq r_1$ and $r_2\geqq r_3$. By (5.3), we have

$$(5.8)\qquad r_3(n_3-tr_2)\geqq dr_{01}\prod_{i=0}^{3} r_i-(r_0+r_1)r_{01}-2r_2r_3-r_3.$$

Assume that $r_3=1$. Then $r_i>1$ for $i=0, 1, 2$. As the right side of (5.8) is a monotone increasing function of each variable, by a rough estimation we get that

$$n_3-tr_2>16-8-4-1=3$$

which is a contradiction to (5.2). Assume that $r_3>1$. Then only r_1 can be 1 as $J\geqq 3$. By substituting $r_1=1$ in (5.8), we get that $r_3(n_3-r_2)\geqq 16-6-8-2=0$. This is a contradiction to (5.2). This completes the proof of Theorem (4.1) and Theorem (4.2) in the case that $H=1$.

III. $H=2$**.**

There are two possibilities.

(III-A) $r_{01}, r_{02}>1$ and $r_{ij}=1$ otherwise.
(III-B) $r_{01}, r_{23}>1$ and $r_{ij}=1$ otherwise.

(III-A) $H=2$ and $r_{01}>1$ and $r_{02}>1$. The following are exceptional cases.

(III-i) $a=(1, r_0r_{02}, r_0r_{01}, r_0r_{01}r_{02})$.
(III-ii) $a=(2, 4, 6, 12)$ $(d=2, r_{01}=3, r_{02}=2)$.
(III-iii) $r_{01}=2$, i.e. $a=(2, r_{02}, 4, 4r_{02})$ or
(III-iv) $r_{01}=3$, $r_{02}=5$, i.e. $a=(2, 5, 6, 30)$ or
(III-v) $r_{01}=4$, $r_{02}=3$, i.e. $a=(2, 3, 8, 24)$ or
(III-vi) $r_{01}=5$, $r_{02}=3$, i.e. $a=(2, 3, 10, 30)$.
(III-vii) $r_1=3$, $r_{01}=3$ and $r_{02}=2$, i.e. $a=(3, 2, 9, 18)$.

(III-i) corresponds to (10) of Theorem (4.1). (III-ii) corresponds to (10) of Theorem (4.2).

We study (III-iii). Assume that $a=(2, r, 4, 4r)$ with r odd>1. The case of $r=3$ corresponds to (12) of Theorem (4.1). The case of $r=5$ corresponds to (12) of Theorem (4.2). The case of $r=7$ corresponds to (13) of Theorem (4.2). Assume that $r\geqq 9$. Then $n_3-4\geqq 0$. As ω and $\varphi_{13}\omega$ are holomorphic two-forms by Lemma (4.17), $p_g\geqq 2$.

(III-iv) corresponds to (14) of Theorem (4.2). (III-v) corresponds to (11) of Theorem (4.2). (III-vi) corresponds to (15) of Theorem (4.2). (III-vii) corresponds to (16) of Theorem (4.2).

Case (III-B). Assume that $H=2$ and $r_{01}, r_{23}>1$.

The possible exceptional cases are

(III-viii) $a=(r_{23}, r_{23}, r_{01}, r_{01})$ $(d=1, r_i=1)$ or
(III-ix) $d=1$, $r_0=r_{01}=2$, i.e. $a=(r_{23}, 2r_{23}, 4, 4)$ or
(III-x) $d=1$, $r_0=2$, $r_{01}=3$, $r_{23}=5$, i.e. $a=(5, 10, 6, 6)$ or
(III-xi) $d=1$, $r_0=2$, $r_{01}=4$, $r_{23}=3$, i.e. $a=(3, 6, 8, 8)$ or
(III-xii) $d=1$, $r_0=r_{01}=3$, $r_{23}=2$, i.e. $a=(2, 6, 9, 9)$.
(III-xiii) $d=1$, $r_0=2$, $r_{01}=5$ $r_{23}=3$, i.e. $a=(3. 6, 10, 10)$.

(III-viii) corresponds to (11) of Theorem (4.1). (The rationality is immediate from the affine equation: $g(y)=y_1^{r_{23}}y_2+y_2+y_3^{r_{01}}+1=0$.) (III-x), (III-xi), (III-xii) and (III-Xiii) are cancelled as $p_g\geqq 2$ by calculation. For (III-ix), we have that $p_g=(r_{23}-1)/2$. Thus the case of $r_{23}=3$ corresponds to (17) of Theorem (4.2).

(IV) $H=3$.

There are three possibilities.
(A) $r_{01}, r_{02}, r_{03}>1$.
(B) $r_{01}, r_{12}, r_{02}>1$.
(C) $r_{01}, r_{02}, r_{13}>1$.

Case (A) contains no exceptional case.

Case (IV-B). Assume that $H=3$ and $r_{01}, r_{12}, r_{02}>1$.

The exceptional cases are:
(IV-B-i) $d=1$, $r_{02}=2$, $r_{12}=3$, $r_{01}=5$, i.e. $a=(3, 2, 5, 30)$ or
(IV-B-ii) $d=1$, $r_{02}=2$, $r_{12}=3$, $r_{01}=7$, i.e. $a=(3, 2, 7, 42)$ or
(IV-B-iii) $d=1$, $r_{02}=2$, $r_{12}=3$, $r_{01}=11$, i.e. $a=(3, 2, 11, 66)$.

The case (IV-B-i) corresponds to (13) of Theorem (4.1). The Case (IV-B-ii) corresponds to (18) of Theorem (4.2). The case (IV-B-iii) corresponds to (19) of Theorem (4.2).

Case (IV-C). Assume that $r_{01}, r_{02}, r_{13}>1$. This case contains the following exceptional cases.
(IV-C-i) $r_{01}=3$ and $r_{13}=2$, i.e. $a=(2, r_{02}, 6, 3r_{02})$ or
(IV-C-ii) $r_{01}=5$, $r_{13}=2$ and $r_{02}=3$, i.e. $a=(2, 3, 10, 15)$ or
(IV-C-iii) $r_{01}=7$, $r_{02}=3$ and $r_{13}=2$, i.e. $a=(2, 3, 14, 21)$.
(IV-C-iv) $r_{01}=4$, $r_{13}=3$ and $r_{02}=5$, i.e. $a=(3, 5, 12, 20)$ or
(IV-C-v) $r_{01}=4$, $r_{13}=3$ and $r_{02}=7$, i.e. $a=(3, 5, 12, 28)$.
(IV-C-vi) $r_{01}=5$, $r_{13}=3$ and $r_{02}=4$, i.e. $a=(3, 4, 15, 20)$.

The Case (IV-C-i) with $r_{02}=5$ corresponds to (22) of Theorem (4.2). If $r_{02}\neq 5$, $p_g>1$. The Case (IV-C-ii) corresponds to (20) of Theorem (4.2). The Case (IV-C-iii) corresponds to (21) of Theorem (4.2). The Cases (IV-C-iv, v, vi) are eliminated as $p_g>1$.

If $H\geqq 4$, there is no exceptional cases. This completes the proof of Theorem (4.1) and Theorem (4.2).

References

[1] M. Artin, On isolated rational singularities of surfaces, Amer. J. Math., **88** (1966), 129–136.
[2] E. Brieskorn, Beispiele zur Differentialtopologie von Singularitaten, Invent. Math., **2** (1966), 1–14.
[3] A. H. Durfee, The Signature of Smoothings of Complex Surface Singularities, Math. Ann., **232** (1966), 85–98.
[4] P. Griffiths and J. Harris, Principles of Algebraic Geometry, A Wiley-Interscience Publication (1978).
[5] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, Toroidal Embeddings, Lecture Note in Math., **339**, Springer (1973).
[6] A. G. Kouchnirenko, Polyedres de Newton et Nombres de Milnor, Invent. Math., **32** (1976), 1–32.
[7] H. B. Laufer, Normal Two-Dimensional Singularities, Ann. of Math. Studies, **71** (1971).
[8] J. Milnor, Singular Points of Complex Hypersurface, Ann. Math. Studies, **61**, Princeton Univ. Press (1968).
[9] Y. Miyaoka, Tricanonical Maps of Numerical Godeaux Surfaces, Invent. Math., **34** (1976), 99–111.
[10] D. Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, Publ. Math. I.H.E.S., Paris, **9** (1961).
[11] M. Oka, On the Resolution of Hypersurface Singularities, in this Proceeding.
[12] ——, On the topology of the Newton boundary II, J. Math. Soc. Japan, **32** (1980), 65–92.
[13] M. Reid, Surface with $p_g=0$, $K^2=1$, J. Fac. Sci. Univ. Tokyo, **25** no. 1 (1978), 75–92.
[14] A. N. Varchenko, Zeta-Function of Monodromy and Newton's Diagram, Invent. Math., **37** (1976), 253–262.

Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro-ku, Tokyo 152
Japan