Advanced Studies in Pure Mathematics 8, 1986
Complex Analytic Singularities
pp. 405–436

On the Resolution of the Hypersurface Singularities

Mutsuo Oka

Dedicated to Professor M. Nakaoka on his 60th birthday

§ 1. Introduction

Let $f(z_0, \cdots, z_n)$ be a germ of an analytic function at the origin such that $f(0)=0$ and f has an isolated critical point at the origin. We assume that the Newton boundary of f is non-degenerate. Let V be the germ of the hypersurface $f^{-1}(0)$ at the origin. Let $\Gamma^*(f)$ be the dual Newton diagram and let Σ^* be a simplicial subdivision. It is well-known that there is a canonical resolution $\pi: \tilde{V} \to V$ which is associated with Σ^* ([8]). However the process to get Σ^* from $\Gamma^*(f)$ is not unique and a "bad" Σ^* produces unnecessary exceptional divisors. The purpose of this paper is to study this resolution through a canonical simplicial subdivision.

In Section 3, we will show that there is a canonical way to get a simplicial subdivision from $\Gamma^*(f)$. (Lemma (3.3) and Lemma (3.8))

In Section 4, we will recall the construction of the resolution $\pi: \tilde{V} \to V$ which is associated with a given simplicial subdivision Σ^*.

In Section 5, we will study the topology of the exceptional divisors using the canonical stratifications.

In Section 6, we will show the following: Assume that $n=2$. Then the resolution graph Γ of the resolution of V is obtained by a canonical surgery from $S_2\Gamma^*(f)$ (=the two-skeleton of $\Gamma^*(f)$ which is considered as a graph by a plane section). Let P be a vertex of Σ^* such that $\Delta(P)$ is a two-dimensional face of $\Gamma(f)$. Then the genus of the exceptional divisor $E(P)$ is equal to the number of the integral points in the interior of $\Delta(P)$. The other exceptional divisors are rational. (See Theorem (6.1) of § 6.)

In Section 7, we will study the fundamental group of the exceptional divisor $E(P)$. Assume that $n>2$ and $\Delta(P)$ is an n-simplex. Then we will show that $\pi_1(E(P))$ is a finite cyclic group and its order is determined by $\Gamma^*(f)$ (Theorem (7.3)).

In Section 8, we will study the divisors of the exceptional divisor $E(P)$ in the case of $n=3$.

Received December 26, 1984.

In Section 9, we will study the canonical divisors of the resolution space $\tilde{V}$ and of the exceptional divisors $E(P)$. (Theorem (9.1) and Theorem (9.2))

This paper consists of the following sections:

§2. Newton boundary and the dual Newton diagram

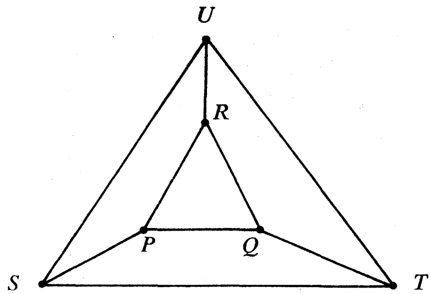
Let $f(z_0, \cdots, z_n)=\sum_\nu a_\nu z^\nu$ be the the Taylor expansion of f where $z^\nu= z_0^{\nu_0}\cdots z_n^{\nu_n}$ as usual. Recall that the Newton boundary $\Gamma(f)$ is the union of the compact faces of $\Gamma_+(f)$ where $\Gamma_+(f)$ is the convex hull of the union of the subsets $\nu+(R^+)^{n+1}$ of R^{n+1} for ν such that $a_\nu\neq 0$. For any (closed) face Δ of $\Gamma(f)$, we associate a polynomial $f_\Delta(z)=\sum_{\nu\in\Delta} a_\nu z^\nu$. We say that f is *non-degenerate* on Δ if

$$\frac{\partial f_\Delta}{\partial z_0}=\cdots=\frac{\partial f_\Delta}{\partial z_n}=0$$

has no solution in $(C^*)^{n+1}$. We say that f is *non-degenerate* if f is non-degenerate on any face Δ of $\Gamma(f)$ ([9], [16]).

Let N^+ be the space of positive vectors of the dual space $\hat{R}^{n+1}\cong R^{n+1}$. We denote the vectors in N^+ by column vectors. For any vector $A={}^t(a_0, \cdots, a_n)$ of N^+, we associate the linear function A on $\Gamma_+(f)$ which is defined by $A(x)=\sum_{i=0}^n a_i x_i$. Let $d(A)$ be the minimal value of A on $\Gamma_+(f)$ and let $\Delta(A)=\{x\in\Gamma_+(f); A(x)=d(A)\}$. We introduce an equivalence relation $\sim$ in N^+ by $A\sim B$ if and only if $\Delta(A)=\Delta(B)$. For any face Δ of dimension k of $\Gamma_+(f)$, there is an equivalence class Δ^* which is defined by $\Delta^*=\{A\in N^+; \Delta(A)=\Delta\}$. Note that $\dim \Delta^*=n-k$. (The cone of Δ^* has the dimension $n-k+1$). The collection of Δ^* gives a polyhedral decomposition $\Gamma^*(f)$ of N^+ which we call *the dual Newton diagram* of f. As each cell of $\Gamma^*(f)$ is a cone, we identify $\Gamma^*(f)$ with its projection on the hyperplane $L=\{x_0+\cdots+x_n=1\}$. We may assume that a vertex $P={}^t(p_0,\cdots,p_n)$ of $\Gamma^*(f)$ is a primitive integral vector. If P is strictly positive, i.e. $p_i>0$ for each i, $\Delta(P)$ is a compact face of $\Gamma(f)$.

Example (2.1). Let $f(x, y, z)=x^4+y^4+z^4+xyz$. Then $\Gamma(f)$ has three two-dimensional faces and $\Gamma^*(f)$ is the following.



$P={}^t(2, 1, 1)$
$Q={}^t(1, 2, 1)$
$R={}^t(1, 1, 2)$
$S={}^t(1, 0, 0)$
$T={}^t(0, 1, 0)$
$U={}^t(0, 0, 1)$

We say that a polyhedral decomposition Σ^* of $\Gamma^*(f)$ is a *simplicial subdivision* if the following conditions are satisfied ([8], [20]).
(i) Σ^* is a subdivision of $\Gamma^*(f)$ by the cones over the simplexes $\sigma=(P_0, \cdots, P_k)$ where $P_0, \cdots, P_k$ are primitive integral vectors which can be extended to a basis of Z^{n+1}. The intersection of two simplexes is a simplex. Each boundary of a simplex is a simplex.
(ii) Assume that $\Gamma(f)^I$ is non-empty where

$$\Gamma(f)^I=\{x \in \Gamma(f);\ x_i \neq 0 \ \text{only if}\ i \in I\}$$

and I is a subset of $\{0, \cdots, n\}$. Then $\sigma_I=\{P \in N^+;\ p_i=0 \text{ if } i \text{ is not in } I\}$ is a simplex.

Remark (2.2). We can assume that $\Gamma(f)^{\{i\}}$ is non-empty by adding monomials z_i^N of sufficiently high degree, if necessary. In this case, the vertices which are not strictly positive are $E_i={}^t(0, \cdots, \overset{i}{\breve{1}}, \cdots, 0)$ $(i=0, \cdots, n)$.

§ 3. Canonical simplicial subdivision

Let $P_i={}^t(p_{0i}, p_{1i}, \cdots, p_{ni})$ $(i=1, \cdots, k)$ be given integral vectors of N^+. We define a non-negative integer $\det(P_1, \cdots, P_k)$ by the greatest common divisor of all $k\times k$ minors of the matrix (p_{ji}) and we call $\det(P_1, \cdots, P_k)$ the *determinant* of $P_1, \cdots, P_k$.

Lemma (3.1). *Let $A=(a_{ij})$ be a unimodular matrix. Then* $\det(P_1, \cdots, P_k)=\det(AP_1, \cdots, AP_k)$.

The proof is an easy exercise of linear algebra.

Lemma (3.2). *Let $P_1, \cdots, P_k$ be given integral vectors such that* $\det(P_1, \cdots, P_k)=1$. *Then there exist integral vectors $P_{k+1}, \cdots, P_{n+1}$ such that* $\det(P_1, \cdots, P_{n+1})=1$.

Proof. Let M be the subgroup of Z^{n+1} generated by $P_1, \cdots, P_k$. Then by the structure theorem of a finitely generated abelian group, there is a subgroup M' of rank k such that $M \subset M'$ and M' is a direct summand of Z^{n+1}. Then the assumption $\det(P_1, \cdots, P_k)=1$ clearly implies that $M=M'$.

(I) Division of $S_2\Gamma^*(f)$.

Let $P={}^t(p_0, \cdots, p_n)$ and $Q={}^t(q_0, \cdots, q_n)$ be given integral vectors of N^+.

Lemma (3.3). *Let $c=\det(P, Q)$ and assume that $c>1$.*
(i) *Any integral vector P_1 on the line segment $\overline{PQ}$ such that* $\det(P, P_1)=1$ *can be written as $P_1=(Q+c_1P)/c$ for some integer $c_1>0$. c_1 is unique modulo c.*
(ii) *There exists a unique c_1 such that $0<c_1<c$.*

Proof. By Lemma (3.1) and Lemma (3.2), we may assume that $Q={}^t(1, 0, \cdots, 0)$. Then c is nothing but g.c.d. $(p_1, \cdots, p_n)$. Let $P_1=\lambda P+\mu Q$ for $\lambda \geqq 0$, $\mu \geqq 0$ and assume that P_1 is an integral vector satisfying $\det(P, P_1)=1$. As $\det(P, P_1)=\mu \det(P, Q)=\mu c=1$, we have $\mu=1/c$. As P_1 is an integral vector, $\lambda p_i \in Z$ for $i=1, \cdots, n$. This implies that λ can be written as $\lambda=c_1/c$ where c_1 is an integer such that $c_1p_0+1\equiv 0$ modulo c. The last equation has a unique solution in $0<c_1<c$ as g.c.d. $(c, p_0)=$ g.c.d. $(p_0, \cdots, p_n)=1$.

Remark (3.4). By the abuse of language, we say that P_1 is on the line segment $\overline{PQ}$ if $P_1=\lambda P+\mu Q$ for some non-negative numbers λ and μ.

Definition (3.5). Let $\overline{PQ}$ be a line segment of $S_2\Gamma^*(f)$ (=the two-skeleton of $\Gamma^*(f)$). We say that the sequence of primitive integral vectors $P_1, \cdots, P_k$ is the *canonical primitive sequence* of $\overline{PQ}$ if the following conditions are satisfied.
(i) If $c=\det(P, Q)>1$, there are non-negative integers c_i $(i=0, \cdots, k+1)$ such that

$$c=c_0>c_1>\cdots>c_k=1>c_{k+1}=0$$

and

$$P_{i+1}=(Q+c_{i+1}P_i)/c_i \quad (i=0, \cdots, k)\ (P_0=P,\ P_{k+1}=Q).$$

(ii) If $c=1$, $n=2$ and P and Q are strictly positive, $k=1$ and $P_1=P+Q$. (This condition is to have a good resolution.) Otherwise $k=0$.

The existence of the canonical primitive sequence is obvious by Lemma (3.3).

Lemma (3.6). *Assume that $c=\det(P, Q)>1$ and let $P_1, \cdots, P_k$ be the canonical primitive sequence of $\overline{PQ}$. Let c_i be as above and let $m_i=(c_{i-1}+c_{i+1})/c_i$ $(i=1, \cdots, k)$. Then each m_i is an integer such that $m_i \geqq 2$ and*

$$\frac{c}{c_1}=m_1-\cfrac{1}{m_2-\cfrac{\ddots}{-\cfrac{1}{m_k}}}.$$

Let $P_i={}^t(p_{0i}, \cdots, p_{ni})$. Then

$$m_i=(p_{ji-1}+p_{ji+1})/p_{ji} \text{ for any } j=0, \cdots, n.$$

Proof. We prove the assertion by the induction on k. Assume that $k=1$. Then $P_1=(P+Q)/c$. Thus $m_1=(c+0)/c_1=c$ and $c=(p_j+q_j)/p_{ji}$. Assume that $k>1$. As $P_1=(Q+c_1P)/c$ and $P_2=(Q+c_2P_1)/c_1$, we have that

$$\det(P, P_2)=\det(P, Q+c_2P_1)/c_1=\det(P, Q)(1+c_2/c)/c_1=m_1.$$

Thus m_1 is an integer and $m_1 \geqq 2$. As $P_2, \cdots, P_k$ is the canonical primitive sequence of $\overline{P_1Q}$, by the induction's hypothesis m_i $(i=2, \cdots, k)$ are integers greater than or equal to 2 and we have

$$m_1-\cfrac{1}{m_2-\cfrac{\ddots}{-\cfrac{1}{m_k}}}=\frac{c+c_2}{c_1}-\frac{1}{\cfrac{c_1}{c_2}}=\frac{c}{c_1}$$

completing the proof of the first assertion. The second assertion is immediate from the equality;

$$(c_{i-1}+c_{i+1})P_i=c_i(P_{i-1}+P_{i+1}).$$

Remark (3.7). By the same argument, the assertion of Lemma (3.6) is true for every primitive sequence $P_1, \cdots, P_k$ on $\overline{PQ}$ such that $\det(P_i, P_{i+1})=1$ except that we have $m_i \geqq 1$ instead of $m_i \geqq 2$. They are canonical if and only if $m_i \geqq 2$ for $i=1, \cdots, k$ by the first expression of m_i. In particular, $P_k, \cdots, P_1$ is the canonical primitive sequence of $\overline{QP}$ if and only if $P_1, \cdots, P_k$ is the canonical primitive sequence of $\overline{PQ}$.

(II) Division of $S_k\Gamma^*(f)$ ($k \geqq 3$).

Lemma (3.8). *Let Δ be a k-simplex with primitive integral vertices P_i ($i=0, \cdots, k$). Assume that $c=\det(P_0, \cdots, P_k)>1$ and* $\det(P_0, \cdots, P_{k-1})=1$.

(i) *Let R be an integral vector in the triangle Δ such that* $\det(P_0, \cdots, P_{k-1}, R)=1$. *Then we can write*

$$R=(c_0P_0+\cdots+c_{k-1}P_{k-1}+P_k)/c$$

for some non-negative integers $c_0, \cdots, c_{k-1}$. They are unique modulo c.

(ii) *There exists a unique R such that $0 \leqq c_i < c$ for each $i=1, \cdots, k-1$.*

Proof. We assume that $R=\sum_{i=0}^{k} d_iP_i$ for non-negative rational numbers $d_0, \cdots, d_k$. As $\det(P_0, \cdots, P_{k-1}, R)=1=cd_k$, we have that $d_k=1$. As $\det(P_0, \cdots, R, \cdots, P_k)$ is an integer and it is equal to d_ic, we can write $d_i=c_i/c$ for some non-negative integer and c_i is unique modulo c. To prove the existence, we may assume, by Lemma (3.1) and Lemma (3.2), that $P_0={}^t(1, 0, \cdots, 0), \cdots, P_{k-1}={}^t(0, \cdots, \overset{k}{\breve{1}}, \cdots, 0)$ and $P_k={}^t(p_0, \cdots, p_k, 0, \cdots, 0)$. Then c is nothing but p_k. The integrability of R implies

$$c_i+p_i\equiv 0 \qquad \text{modulo } c \text{ for} \quad i=0, \cdots, k-1.$$

Thus there exists a unique c_i such that $0 \leqq c_i < c$, completing the proof of Lemma (3.8).

Remark (3.9). (i) Note that R divides Δ into $k+1$ k-simplexes $(P_0 \cdots, R, \cdots, P_k)$ with the respective determinant $c_0, \cdots, c_{k-1}$ and 1.

(ii) If $\det(P_0, \cdots, P_{i-1}, P_{i+1}, \cdots, P_k)<c$, *then* $c_i>0$. *In particular, R is not on the* $(k-1)$*-simplex spanned by* $P_0, \cdots, P_{i-1}, P_{i+1}, \cdots, P_k$.

Proof. Assume that $c_0=0$ for brevity's sake. Then $\det(P_1, \cdots, P_{k-1}, R)=\det(P_1, \cdots, P_k)/c$ which implies that $\det(P_1, \cdots, P_k)$ is divisible by c. Thus $\det(P_1, \cdots, P_k)=c$. In this case, the subdivision of Δ is the cone of the subdivision of $(k-1)$-simplex $(P_1, \cdots, P_k)$.

(iii) Assume that $c_i>1$. As $\det(P_0, \cdots, P_{k-1}, R)=1$, $\det(P_0, \cdots, P_{i-1}, P_{i+1}, \cdots, P_{k-1}, R)=1$. Thus we can apply Lemma (3.8) to the simplex

$(P_0, \cdots, R, \cdots, P_k)$ to divide it into smaller simplexes. Therefore by the induction on c we can subdivide Δ into k-simplexes with determinant 1. We call such a subdivision a canonical subdivision of Δ. By (ii), a canonical subdivision is canonical on its faces.

Now we consider the simplicial subdivision of $\Gamma^*(f)$. We first subdivide $S_2\Gamma^*(f)$ by the canonical primitive sequences. Assume that $S_{k-1}\Gamma^*(f)$ is subdivided into simplexes with the respective determinant 1. Let ξ be a $(k-1)$-dimensional cell of $S_k\Gamma^*(f)$. We first subdivide ξ into $(k-1)$-simplexes $\xi_1, \cdots, \xi_s$ without adding any other vertices. We may assume that this subdivision is compatible with the subdivision of $S_{k-1}\Gamma^*(f)$. Assume that $\xi_1, \cdots, \xi_{m-1}$ are subdivided into simplexes with the determinant 1 so that they are compatible each other and compatible with the subdivision of $S_{k-1}\Gamma^*(f)$. Take ξ_m. If a $(k-2)$-dimensional face of Δ has determinant 1, we apply Lemma (3.8) to subdivide ξ_m into simplexes with determinant 1. In this process, no vertices are added on $\xi_m \cap S_{k-1}\Gamma^*(f)$ by Remark (3.9). We may also assume by Remark (3.9) that this subdivision is compatible with the subdivisions of $\xi_1, \cdots, \xi_{m-1}$. If the determinant of every face of ξ_m is greater than 1, we first take a canonical subdivision of a $(k-2)$-face and take the cone subdivision of ξ_m and apply Lemma (3.8) to subdivide each of the simplexes. By the induction on m, we can subdivide ξ into simplexes with the determinant 1. Thus applying this argument to every $(k-1)$-cell of $S_k\Gamma^*(f)$, we can subdivide $S_k\Gamma^*(f)$ into simplicial complexes which are compatible with the subdivision of $S_{k-1}\Gamma^*(f)$.

Remark (3.10). There does not exist a unique way to subdivide a k-cell into k-simplexes. See [8], [15] and [21] for further information.

§ 4. Resolution of V

Let f be an analytic function with an isolated critical point at the origin. We assume that f has a non-degenerate Newton boundary. Let Σ^* be a given simplicial subdivision of $\Gamma^*(f)$. For each n-simplex $\sigma = (P_0, \cdots, P_n)$ where $P_j = {}^t(p_{0j}, \cdots, p_{nj})$, we associate the $(n+1)$-dimensional Euclidean space C_σ^{n+1} with the coordinate $y_\sigma = (y_{\sigma,0}, \cdots, y_{\sigma,n})$ and the birational mapping $\hat{\pi}_\sigma\colon C_\sigma^{n+1} \to C^{n+1}$ which defined by $\hat{\pi}_\sigma(y_\sigma) = (z_0, \cdots, z_n)$ and $z_i = y_{\sigma,0}^{p_{i0}} \cdots y_{\sigma,n}^{p_{in}}$. By the abuse of the notation, we write $z = (y_\sigma)^\sigma$. Let X be the union of C_σ^{n+1} for σ which are glued along the images of π_σ. Let $\hat{\pi}\colon X \to C^{n+1}$ be the projection map and let $\tilde{V}$ be the proper transform of V. It is well known that $\pi\colon \tilde{V} \to V$ is a resolution of V where π is the restriction of $\hat{\pi}$ to $\tilde{V}$ ([8]).

Let $d_i = d(P_i)$ and $\Delta_i = \Delta(P_i)$. By the definition of the simplicial subdivision, we have

$$\bigcap_{i=0}^{n} \Delta_i = \{Q\} \tag{4.1}$$

for some vertex Q of $\Gamma(f)$. We define

$$g_{\Delta_i}(y_\sigma) = f_{\Delta_i}(\hat{\pi}_\sigma(y_\sigma)) / \prod_{j=0}^{n} y_{\sigma,j}^{d_j}.$$

By the definition, $g_{\Delta_i}(y_\sigma)$ is a function of n variables $y_{\sigma,j}$ $(j \neq i)$. If P_i is strictly positive, g_{Δ_i} is a polynomial with a non-zero constant. We can write

$$\hat{\pi}_\sigma^* f(y_\sigma) = \prod_{i=0}^{n} y_{\sigma,i}^{d_i} f_\sigma(y_\sigma).$$

By the definition of $\tilde{V}$, $\tilde{V} \cap C_\sigma^{n+1}$ is defined by $f_\sigma = 0$ and

$$\{y_\sigma \in C_\sigma^{n+1}; f_\sigma(y_{\sigma,0}, \cdots, y_{\sigma,n}) = y_{\sigma,i} = 0\}$$
$$= \{y_\sigma \in C_\sigma^{n+1};\ y_{\sigma,i} = g_{\Delta_i}(y_\sigma) = 0\}.$$

Thus if P_j is strictly positive, we have

$$\tilde{V} \cap \{y_{\sigma,j} = 0\} \neq \phi \tag{4.2}$$

if and only if $\dim \Delta_j > 0$.

Remark (4.3). Recall that $S_k\Gamma^*(f)$ is the union of the cells of $\Gamma^*(f)$ whose dimension is less than or equal to k. (The dimension of a cell decreases by 1 if we consider the projection into a hyperplane.) Note that P is in $S_n\Gamma^*(f)$ if and only if $\dim \Delta(P) \geqq 1$.

Corollary (4.4). *Assume that* $\sigma \cap S_n\Gamma^*(f) = \phi$. *Then* $\tilde{V} \cap C_\sigma^{n+1} \subset (C_\sigma^*)^{n+1}$.

Let P be a vertex of Σ^* such that $\dim \Delta(P) \geqq 1$ and let $\sigma = (P_0, \cdots, P_n)$ be an n-simplex such that $P_n = P$. We define

$$E(P; \sigma) = \{y_\sigma;\ y_{\sigma,n} = 0,\ g_{\Delta(P)}(y_{\sigma,0}, \cdots, y_{\sigma,n-1}) = 0\}.$$

$E(P; \sigma)$ is a smooth divisor of $\tilde{V} \cap C_\sigma^{n+1}$ in the neighbourhood of $\pi_\sigma^{-1}(0)$ by the non-degeneracy assumption of the Newton boundary $\Gamma(f)$. By the definition of π_σ, we have

(4.5) $\pi_\sigma(E(P; \sigma)) = \{0\}$ if and only if P is strictly positive.

Now we will study the gluing map between C_σ^{n+1} and C_τ^{n+1} where

$\tau=(Q_0, \cdots, Q_n)$. We can write

$$Q_i=\sum_{j=0}^{n}\lambda_{ji}P_j \qquad \text{for} \quad i=0, \cdots, n.$$

Then $y_\sigma=\hat{\pi}_\sigma^{-1}\cdot\hat{\pi}_\tau(y_\tau)=\hat{\pi}_{\sigma^{-1}\tau}(y)$ where $\sigma^{-1}\tau$ is the matrix $\Lambda=(\lambda_{ij})$. Namely we have

$$(4.6) \qquad y_{\sigma,i}=y_{\tau,0}^{\lambda_{i0}}\cdots y_{\tau,n}^{\lambda_{in}} \qquad (i=0, \cdots, n).$$

In particular, if $Q_n=P_n=P$, we have $\lambda_{in}=0$ except for $i=n$. Let $\Lambda'=(\lambda_{ij})_{0\leqq i,j<n}$ and let $y'_\tau=(y_{\tau,0}, \cdots, y_{\tau,n-1})$ and $y'_\sigma=(y_{\sigma,0}, \cdots, y_{\sigma,n-1})$. Then $y'_\sigma=(y'_\tau)^{\Lambda'}$ and $y_{\sigma,n}=y_{\tau,0}^{\lambda_{n0}}\cdots y_{\tau,n}^{\lambda_{nn}}$ and $\lambda_{nn}=1$. Thus $E(P;\tau)$ is birationally glued with $E(P;\sigma)$. Thus the union of $E(P;\sigma)$ for n-simplexes σ such that σ contains P as its vertices is a divisor of $\tilde{V}$ and we denote this by $E(P)$. If P is a strictly positive vertex, $E(P)$ is a compact divisor such that $\pi(E(P))=\{0\}$. The topology of $E(P)$ will be studied in the following sections.

We say that vertices $P_0, \cdots, P_{k-1}$ of Σ^* are *adjacent* if there is an n-simplex which contains $P_0, \cdots, P_{k-1}$ as its vertices.

Lemma (4.7). *Let P_i $(i=0, \cdots, k-1)$ be mutually distinct vertices of Σ^* with* $\dim \Delta(P_i)\geqq 1$ *for $i=0, \cdots, k-1$. We assume that P_0 is a strictly positive vertex. Then the intersection $E(P_0)\cap\cdots\cap E(P_{k-1})$ is non-empty if and only if $\{P_i\}$ $(i=0, \cdots, k-1)$ are adjacent and* $\dim \cap_i \Delta(P_i)\geqq 1$. *$\cap_i E(P_i)$ is a compact manifold of dimension $n-k$.*

Proof. Note that $E(P_i)\cap \boldsymbol{C}_\sigma^{n+1}$ is non-empty only if P_i is a vertex of σ. Thus if $\Delta=\cap_i \Delta(P_i)$ is non-empty, there exists an n-simplex $\sigma=(P_0, \cdots, P_n)$. We have

$$\bigcap_i E(P_i)\cap \boldsymbol{C}_\sigma^{n+1}$$
$$=\{y_\sigma \in \boldsymbol{C}_\sigma^{n+1};\ y_{\sigma,i}=0 \ \ (i=0, \cdots, k-1) \ \ g_\Delta(y_{\sigma,k}\cdots y_{\sigma,n})=0\}.$$

Thus this is non-empty if and only if $\dim \Delta\geqq 1$. $\cap_i E(P_i)$ is compact as it is a closed subspace of the compact divisor $E(P_0)$. The smoothness is immediate from the non-degeneracy assumption of $\Gamma(f)$.

It is easy to see that the divisor $E(P)$ is connected if $\dim \Delta(P)>1$. However

Lemma (4.8). *Assume that P is a strictly positive and* $\dim \Delta(P)=1$. *Then $E(P)$ has $(r(\Delta(P))+1)$ connected components where $r(\Delta(P))$ is the number of the integral points of the relatively interior of $\Delta(P)$. Each component is rational.*

Proof. We can find a simplex $\sigma=(P_0, \cdots, P_n)$ such that $P=P_n$ and $\Delta(P_i)\supset\Delta(P)$ for $i=0, \cdots, n-2$ and $\Delta(P_{n-1})$ is one of the boundary of $\Delta(P)$. Let $f_{\Delta(P)}(z)=\sum_{i=0}^{r+1}a_i z^{\nu_i}$ where ν_i $(i=0, \cdots, r+1)$ are the integral points on $\Delta(P)$ in this order and a_0 and a_{r+1} are non-zero. Then $E(P;\sigma)$ is defined by $y_{\sigma,n}=0$ and $g_{\Delta(P)}(y_{\sigma,n-1})=0$. As the number of the integral points on $\Delta(P)$ and on the support of $g_{\Delta(P)}(y_{\sigma,n-1})$ is equal, we may assume that

$$g_{\Delta(P)}(y_{\sigma,n-1})=\sum_{i=0}^{r+1}a_i\, y_{\sigma,n-1}^i.$$

Thus the non-degeneracy assumption on $\Delta(P)$ implies that $E(P;\sigma)$ is the disjoint union of $r+1$ $(n-1)$-dimensional planes

$$L(\sigma)_i=\{y_{\sigma,n-1}=\xi_i \text{ and } y_{\sigma,n}=0\}$$

where ξ_i $(i=0, \cdots, r+1)$ are non-zero and mutually distinct. As $E(P)$ is a non-singular algebraic variety, this implies the assertion. We can directly see this as follows. Let $\tau=(Q_0, \cdots, Q_n)$ be an n-simplex such that $\Delta(Q_j)\supset\Delta(P)$ for $j<s$ and $Q_s=P$ and $\Delta(Q_k)$ is a single point for $k>s$ for some s. We can find a simplex $\theta=(R_0, \cdots, R_n)$ such that $R_j=Q_j$ for $j<s$, $R_n=P$ and $\Delta(R_k)\supset\Delta(P)$ for $k<n-1$. Watching the gluing map carefully, we can see that $E(P;\tau)\subset E(P;\theta)$. Thus $E(P)$ is covered by $E(P;\sigma)$ where σ is of the above type. Assume that σ and θ are as above. Then the gluing matrix $\Lambda=(\lambda_{ij})$ of C_σ^{n+1} and C_θ^{n+1} satisfies $\lambda_{in}=0$ for $i<n$ and $\lambda_{nn}=1$. As $\{P_0, \cdots, P_{n-2}, P_n\}$ and $\{R_0, \cdots, R_{n-2}, R_n\}$ generate the same Z module, we have that $\lambda_{(n-1)i}=0$ for $i<n-1$ and $\lambda_{(n-1)(n-1)}=\varepsilon$ where ε is 1 or -1 according to whether R_{n-1} is on the same side of P_{n-1} or not with respect to $\Delta(P)^*$. Thus the component $y_{\sigma,n-1}=\xi_i$ corresponds to the component $y_{\theta,n-1}^{\varepsilon}=\xi_i$. Thus the union of $E(P;\sigma)$ for σ is a disjoint union of $r+1$ rational varieties as desired.

§ 5. Topology of the exceptional divisors

Let $g(u_1, \cdots, u_n)$ be a polynomial with support $S(g)$. We say that g is *globally non-degenerate* ($=$0- *non-degenerate* in [20]) if the equation

$$g_\Delta(u)=\frac{\partial g_\Delta(u)}{\partial u_1}=\cdots=\frac{\partial g_\Delta(u)}{\partial u_n}=0$$

has no solution in $(C^*)^n$ for any face Δ of $S(g)$. In [17], we have proved

Theorem (5.1). *Let g be a globally non-degenerate polynomial. Then*
(i) $\chi((C^*)^n-g^{-1}(0))=(-1)^n n!$ n-dim. volume $S(g)$.

(ii) *If the dimension of* $S(g)$ *is greater than or equal to* 3, $\pi_1((C^*)^n - g^{-1}(0))$ *is a free abelian group of rank* $n+1$.

By the additivity of the Euler characteristics and (i) of Theorem (5.1), we have

Corollary (5.2) ([20]). *Let* g *be as above and let* $V^* = g^{-1}(0) \cap (C^*)^n$. *Then* $\chi(V^*) = (-1)^{n+1} n!$ n-dim. volume $S(g)$. (*Here* n-dim. volume *implies the n-dimensional volume.*)

In this section, we study the topology of exceptional divisors of the resolution $\pi: \tilde{V} \to V$ constructed in Section 4. Let $\sigma = (P_0, \cdots, P_{k-1})$ be a $(k-1)$-simplex of Σ^*. We define $E(\sigma) = E(P_0, \cdots, P_{k-1})$ by $\bigcap_{i=0}^{k-1} E(P_i)$ and $E(\sigma)^* = E(P_0, \cdots, P_{k-1})^*$ by $E(\sigma) - \bigcap_{Q \neq P_i} E(Q)$. We define $\Delta(\sigma) = \Delta(P_0, \cdots, P_{k-1}) = \bigcap_{i=0}^{k-1} \Delta(P_i)$. We fix a strictly positive vertex P such that $\dim \Delta(P) \geqq 1$. The collection of $E(\sigma)^*$ for σ which contains P as a vertex gives a canonical stratification of $E(P)$.

Theorem (5.3). (i) *Assume that* $\tau = (P_0, P_1, \cdots, P_{k-1})$ *be a* $(k-1)$-*simplex of* Σ^*. *Let* $\sigma = (P, P_1, \cdots, P_n)$ *be an n-simplex such that* $\tau \subset \sigma$. *Then*

$$\chi(E(\tau)^*) = (-1)^{n-k+1}(n-k)! \ (n-k)\text{- dim. volume } S(g_{\Delta(\tau)}(y_\sigma)).$$

In particular, the Euler characteristic $\chi(E(\tau)^*)$ *is non-zero if and only if* $\dim \Delta(\tau) = n-k$.

(ii) *The birational class of* $E(\tau)$ *depends only on the coefficients of* f *on* $\Delta(\tau)$. *It does not depend on the particular choice of* Σ^* *either.*

(iii) $\chi(E(P)) = \sum \chi(E(\tau)^*)$ *where the sum is taken for simplexes* τ *which contain P.*

Corollary (5.4). (i) $\chi(E(P)^*) = (-1)^{n+1}(n+1)!$ $(n+1)$-dim. volume $C(0, \Delta(P))/d(P)$ *where* $C(0, \Delta(P))$ *is the cone of* $\Delta(P)$ *with the origin.*

(ii) *The birational class of* $E(P)$ *depends only on the coefficients of* f *on* $\Delta(P)$. *If* $\dim \Delta(P) = r$ *is smaller than n, there exists a compact algebraic manifold* $M(P)$ *of dimension* $r-1$ *such that* $E(P)$ *is birationally equivalent to* $P^{n-r} \times M(P)$.

The proof of Theorem (5.3) and Corollary (5.4) occupies the rest of this section. Let $\sigma = (P_0, \cdots, P_n)$ be a simplex of Σ^* and let $\tau = (P_0, \cdots, P_{k-1})$. By the definition, $E(\tau)^* \subset C_\sigma^{n+1}$ and $E(\tau)^*$ is equal to

$$\{(y_{\sigma,k}, \cdots, y_{\sigma,n});\ g_{\Delta(\tau)}(y_\sigma) = 0 \text{ and } y_{\sigma,j} \neq 0 \text{ for } j \geqq k\}.$$

The polynomial $g_{\Delta(\tau)}(y_\sigma)$ is defined by the equation

$$f_{\Delta(\tau)}(\pi_\sigma(y_\sigma))=\prod_{i=0}^{n} y_{\sigma,i}^{d(P_i)} g_{\Delta(\tau)}(y_\sigma).$$

Thus it is easy to see that $g_{\Delta(\tau)}$ is globally non-degenerate as f is non-degenerate on $\Delta(\tau)$. (Compare with Lemma (5.2) of [17]). Thus the assertion of (i) of Theorem (5.3) is immediate from Corollary (5.2). The assertion (iii) of Theorem (5.3) is also obvious by the additivity of the Euler characteristics.

Assume that $P=P_0$ and $\dim \Delta(P)=n$. Then $E(P)^*$ is defined by

$$y_{\sigma,0}=g_{\Delta(P)}(y_{\sigma,1},\cdots,y_{\sigma,n})=0 \quad\text{and}\quad y_{\sigma,i}\neq 0 \quad\text{for}\quad i=1,\cdots,n$$

where

$$f_{\Delta(P)}(\pi_\sigma(y_\sigma))=\prod_{i=0}^{n} y_{\sigma,i}^{d(P_i)} g_{\Delta(P)}(y_\sigma).$$

Thus we have the equality

$$\begin{aligned}(n+1)!\ \text{volume}\ C(0,\Delta(P))&=(n+1)!\ \text{volume}\ C(0,S(\pi_\sigma^* f_{\Delta(P)}))\\&=(n+1)!\ \text{volume}\ S(g_{\Delta(P)})d(P)/(n+1)\\&=n!\ \text{volume}\ S(g_{\Delta(P)})d(P).\end{aligned}$$

This proves the assertion (i) of Corollary (5.4).

Now we prove (ii) of Theorem (5.3). Let Σ^*, be another simplicial subdivision of $\Gamma^*(f)$ and let $\pi'\colon \tilde{V}'\to V$ be the associated resolution. We denote the exceptional divisors in this resolution by $E'(P)$, $E'(\tau)$ etc. Let $\sigma=(P_0,\cdots,P_n)$ be a simple x of Σ^* and let $\sigma'=(Q_0,\cdots,Q_n)$ be a simplex of $\Sigma^{*\prime}$. We assume that there is an integer k, $0<k<n$, such that $\Delta(\tau)=\Delta(\tau')$ and $\dim \Delta(\tau)=n+1-k$ where $\tau=(P_0,\cdots,P_{k-1})$ and $\tau'=(Q_0,\cdots,Q_{k-1})$. $E(\tau)$ is defined in C_σ^{n+1} by

$$y_{\sigma,0}=\cdots=y_{\sigma,k-1}=0 \text{ and } g_{\Delta(\tau)}(y_{\sigma,k},\cdots,y_{\sigma,n})=0.$$

$E'(\tau')$ is defined in $C_{\sigma'}^{n+1}$ by

$$y_{\sigma',0}=\cdots=y_{\sigma',k-1}=0 \text{ and } \hat{g}_{\Delta(\tau')}(y_{\sigma',k},\cdots,y_{\sigma',n})=0$$

where

$$\hat{g}_{\Delta(\tau')}(y_{\sigma'})=f_{\Delta(\tau')}(\pi'_{\sigma'}(y_{\sigma'}))/\prod_{i=0}^{n} y_{\sigma',i}^{d(Q_i)}.$$

By the assumption, the Z-modules generated by $\{P_0,\cdots,P_{k-1}\}$ and $\{Q_0,\cdots,Q_{k-1}\}$ respectively are equal and they are equal to the submodule of Z^n which is generated by the integral points of Closure $(\Delta(\tau)^*)$. Therefore the matrix $\Lambda=\sigma^{-1}\sigma'$ satisfies that $\lambda_{ji}=0$ for $j\geqq k$ and $i<k$. Let Λ_2 be the unimodular matrix defined by $\Lambda_2=(\lambda_{ij})_{i,j\geqq k}$. Write C_σ^{n+1} as $C_\sigma^k\times C_\sigma^{n+1-k}$ and $y_\sigma=(y_1, y_2)$ where $y_1=(y_{\sigma,0}, \cdots, y_{\sigma,k-1})$ and $y_2=(y_{\sigma,k}, \cdots, y_{\sigma,n})$. Similarly we write $C_{\sigma'}^{n+1}$ as $C_{\sigma'}^k\times C_{\sigma'}^{n+1-k}$ and $y_{\sigma'}=(y_1', y_2')$. By the definition, we have

$$f_{\Delta(\tau)}(\pi_\sigma(y_{\sigma'}^\Lambda))=f_{\Delta(\tau')}(\pi'_{\sigma'}(y_{\sigma'})).$$

As $y_{\sigma'}^\Lambda=(y_1, y_2)$ and $y_2=(y_2')^{\Lambda_2}$, we have

$$g_{\Delta(\tau)}((y_2')^{\Lambda_2})=\hat{g}_{\Delta(\tau')}(y_2')\prod_{i=k}^{n}(y_{\sigma',i})^{\alpha_i}$$

for some integers $\alpha_k, \cdots, \alpha_n$. The last equality implies that the birational mapping φ: $C_{\sigma'}^{n+1-k}\to C_\sigma^{n+1-k}$ which is defined by $y_2=(y_{2'})^{\Lambda_2}$ induces the birational mapping of $E'(\sigma')$ and $E(\sigma)$. This completes the proof of the assertion (ii) of Theorem (5.3).

Now we will prove (ii) of Corollary (5.4). Let P be a strictly positive vertex such that $\dim\Delta(P)=r$ and $0<r<n$. Let $\sigma=(P_0, \cdots, P_n)$ be a simplex such that $P=P_{n-r}$ and $\Delta(P_i)\supset\Delta(P)$ for $i=0, \cdots, n-r-1$. Then $E(P)^*$ is defined by

$$y_{\sigma,n-r}=0,\ y_{\sigma,i}\neq 0\ (i\neq n-r) \text{ and } g_{\Delta(P)}(y_{\sigma,n-r+1}, \cdots, y_{\sigma,n})=0$$

which is isomorphic to $(C^*)^{n-r}\times E(P_0, \cdots, P_{n-r})^*$. Thus we can take $E(P_0. \cdots, P_{n-r})$ as $M(P)$. This completes the proof of Theorem (5.3) and Corollary (5.4).

§ 6. Surface singularities

In this section, we study the case $n=2$ in detail. Let π: $\tilde{V}\to V$ be the resolution of V constructed in Section 4. Let E_i $(i=1, \cdots, k)$ be the irreducible components of the exceptional divisor $\pi^{-1}(0)$. The resolution graph Γ is defined in the following way. For each E_i, we associate a vertex v_i with weight m_i which is the self-intersection number of E_i in $\tilde{V}$. When E_i is not a rational curve, we also put the genus $g(E_i)$ to v_i. If $E_i\cap E_j$ is non-empty, we join v_i and v_j by a line segment.

Recall that we identify $S_2\Gamma^*(f)$ with a graph which is the hyperplane section of $S_2\Gamma^*(f)$. Let Δ be a two dimensional face of $\Gamma(f)$. We define an integer $g(\Delta)$ as the number of the integral points on the interior of Δ. Let Ξ be a one dimensional face of $\Gamma(f)$. Recall that $r(\Xi)$ is defined as the number of the integral points on the interior of Ξ. Our main result of this section is

Theorem (6.1). *Let $\pi: \tilde{V} \to V$ be the resolution of V which is associated with Σ^*. Then we have*

(i) *If P is a strictly positive vertex of Σ^* such that* $\dim \Delta(P)=2$, *the genus of $E(P)$ is $g(\Delta(P))$.*

(ii) *If P is a strictly positive vertex with* $\dim \Delta(P)=1$, *$E(P)$ is a disjoint union of $(r(\Delta(P))+1)$ copies of rational curves.*

(iii) *Assume that Σ^* is canonical in the sense of* (3.5). *Then the resolution graph is obtained by the following surgery of $S_2\Gamma^*(f)$: Let $\overline{PQ}$ be a line segment of $S_2\Gamma^*(f)$ and assume that P is strictly positive. Let $c=$* $\det(P, Q)$. *If $c>1$, let c_1 be the unique integer such that $P_1=(Q+c_1P)/c$ is an integral vector and $0<c_1<c$. (Lemma* (3.3)). *Let*

$$\frac{c}{c_1}=m_1-\cfrac{1}{m_2-\cfrac{}{\ddots-\cfrac{1}{m_k}}}$$

where each $m_i \geqq 2$. We insert $r(\Delta(P)\cap\Delta(Q))+1$ copies of the following chain of rational curves

$$\overset{-m_1}{\text{——}\cdot\text{——}}\overset{-m_2}{\cdot\text{——}}\cdots\text{——}\overset{-m_k}{\cdot}\text{——}$$

between P and Q. If $c=1$ and Q is also strictly positive, the above chain is replaced by $\overset{-1}{\text{——}\cdot\text{——}}$. *If $c=1$ and Q is not strictly positive, we do nothing. Those vertices which are not strictly positive are omitted after the surgery.*

(iv) *Assume that* $\dim \Delta(P)=2$. *Let $Q_1, \cdots, Q_s$ be the vertices of $\Sigma^* \cap S_2\Gamma^*(f)$ which are adjacent to P. Let $P={}^t(p_0, p_1, p_2)$ and $Q_i={}^t(q_{0i}, q_{1i}, q_{2i})$. Then the self-intersection number of $E(P)$ is*

$$\frac{-\sum_{i=1}^{s}(r(\Delta(P)\cap\Delta(Q_i))+1)q_{ji}}{p_j}$$

for any $j=0, 1, 2$.

Proof. To prove (i) of Theorem (6.1), we need the following Lemma.

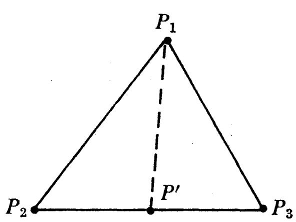
Lemma (6.2). *Let Δ be a compact convex polyhedron in R^2 with integral vertices $P_1, \cdots, P_k$. Then*

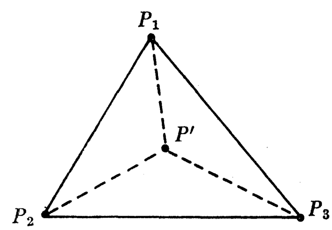
$$2 \ volume \ \Delta = 2g(\Delta) + \sum_{i=1}^{k} (r(\Delta_i)+1) - 2$$

where $\partial\Delta = \cup_{i=1}^{k} \Delta_i$.

Proof. Step 1. Assume that $k=3$ and the integral points of Δ are P_1, P_2 are P_3. By a parallel translation if necessary, we may assume that $P_1 = {}^t(0, 0)$. Then P_2 and P_3 are primitive integral vectors as $\overline{P_iP_j}$ $(i \neq j)$ contains no other integral points than P_i and P_j by the assumption. Assume that $c = \det(P_2, P_3) > 1$. Then by Lemma (3.3), there is a positive integer c_1 such that $Q = (P_2 + c_1P_3)/c$ is an integral point and $0 < c_1 < c$. Thus Q is an integral point of Δ and $Q \neq P_i$ for $i = 1, 2, 3$. This is a contradiction to the assumption. Thus $\det(P_2, P_3) = c = 1$. This implies 2 volume $\Delta = 1$. Thus the assertion is true for this case.

Step 2. Assume that $k=3$ and that either $r(\Delta_1)$ or $g(\Delta)$ is greater than 1. Then we can find an integral point P' on Δ so that Δ is divided into two or three triangles as in the following figures.





It is easy to see that the right side of the assertion in Lemma (6.2) is additive under the above division. Thus the assertion is reduced to Step 1 by a finite subdivision.

Step 3. Assume that $k > 3$. We prove the assertion by the induction on k. We assume that the assertion is true for polyhedra with $k-1$ vertices. We divide Δ into two polyhedra by adding the line segment $\overline{P_1P_{k-1}}$ to Δ. As the right side of the equality in Lemma (6.2) is also additive under this subdivision, the assertion is reduced to the induction's hypothesis. This completes the proof of Lemma (6.2).

Let P be a strictly positive vertex of $S_2\Gamma^*(f)$ such that $\dim \Delta(P) = 2$. Let Δ_i $(i = 1, \cdots, s)$ be the boundaries of $\Delta(P)$ and let Q_i $(i = 1, \cdots, s)$ be the vertices of $\Sigma^* \cap S_2\Gamma^*(f)$ which are adjacent to P and $\Delta(P) \cap \Delta(Q_i) = \Delta_i$. Let $\sigma = (P, P_2, P_3)$ be any 2-simplex of Σ^*. Then by Theorem (5.3), we have

$$\begin{aligned}\chi(E(P)^*) &= -2 \text{ volume } S(g_{\Delta(P)}) \\ &= -2g(S(g_{\Delta(P)}) - \sum_{i=1}^{k} (r(\Delta_i)+1) + 2.\end{aligned}$$

Here we used the fact that $g(\Delta)=g(S(g_{\Delta(P)}))$ and $r(\Delta_i)=r(S(g_{\Delta_i}))$ etc. By Theorem (5.3), we have

$$\chi(E(P))=\chi(E(P)^*)+\sum_{i=1}^{k}\chi(E(P, Q_i))$$

which is equal to $-2g(\Delta(P))+2$, completing the proof of (i) of Theorem (6.1). The assertion (ii) of Theorem (6.1) is immediate from Lemma (4.8).

We assume now that Σ^* is canonical. The assertion about the graph is obvious by Section 4 except the assertion about the self-intersection numbers. Let $\overline{PQ}$ be a line segment of $S_2\Gamma^*(f)$ such that P is strictly positive. (Then $\dim \Delta(P)=2$.) Let

$$c=c_0>c_1\cdots>c_k=1>c_{k+1}=0$$

be as in Definition (3.5). Then $\overline{PQ}$ has k vertices P_i $(i=1, \cdots, k)$ which are inductively defined by

$$P_{i+1}=(Q+c_{i+1}P_i)/c_i$$

where $P_0=P$ and $P_{k+1}=Q$. Let $\sigma_i=(P_i, P_{i+1}, R_i)$ be a fixed two simplex of Σ^* for each $i=0, \cdots, k$. We know that $E(P_i)$ is the union of $r(\Delta(P)\cap \Delta(Q))+1$ disjoint rational curves. We consider the holomorphic function $\varphi_j=\pi^*z_j$ on $\tilde{V}$ for fixed j. Let $P_i={}^t(p_{0i}, p_{1i}, p_{2i})$ and $R_i={}^t(r_{0i}, r_{1i}, r_{2i})$. Then in the chart $\boldsymbol{C}^3_{\sigma i}$,

$$\varphi_j(y_{\sigma_i})=y_{\sigma i,0}^{p_{ji}}\, y_{\sigma i,1}^{p_{ji+1}}\, y_{\sigma i,2}^{r_{ji}}.$$

Thus we get

$$(\varphi_j)=\sum_{i=0}^{k+1}p_{ji}E(P_i)+D$$

where D is a divisor which does not intersect with $E(P_i)$. By Theorem (2.6) or [10], we have

$$(6.3) \qquad (\varphi_j)\cdot E(P_i)=0$$

which implies

$$p_{ji-1}E(P_{i-1})\cdot E(P_i)+p_{ji}E(P_i)^2+p_{ji+1}E(P_i)\cdot E(P_{i+1})=0$$

for $i=1, \cdots, k$. We can write $E(P_i)=\bigcup_{s=1}^{r} E_{is}$ $(r=r(\Delta(P, Q))+1)$ so that

$$E_{i-1s}\cdot E_{it}=\begin{cases}1 & \text{if } s=t\\ 0 & \text{if } s\neq t.\end{cases}$$

As $E(p_{i-1})\cdot E(P_i)=r(\Delta(P, Q))+1$, we obtain from (6.3) that

$$-E_{is}^2=(p_{j\,i-1}+p_{j\,i+1})/p_{ji}$$

which is equal to m_i where m_i is as in Lemma (3.6). The case where $c=1$ and P and Q are strictly positive can be treated in the same way. This proves (iii) of Theorem (6.1). The assertion (iv) of Theorem (6.1) can also be proved by the same argument using the equality $(\varphi_j)\cdot E(P)=0$.

In practice, the following is more convenient to compute $g(E(P))$.

Corollary (6.4). *Let P be a strictly positive vertex of Σ^* with* $\dim \Delta(P)=2$. *Then*

$$2-2g(E(P))=\frac{-6}{d(P)}\text{ volume } C(0, \Delta(P))+\sum_{i=1}^{k}(r(\Delta_i)+1)$$

where $\partial\Delta(P)=\Delta_1\cup\cdots\cup\Delta_k$.

Now we give several examples of the resolution.

(I) Pham-Brieskorn variety

Let $f(x, y, z)=x^{a_0}+y^{a_1}+z^{a_2}$ where $a_i\geqq 2$. Let $d=\text{g.c.d.}\,(a_0, a_1, a_2)$ and let $r_i=\text{g.c.d.}\,(a_{i-1}, a_{i+1})/d$ where $a_{i+3}=a_i$. Then $r_i(i=0, 1, 2)$ are mutually coprime and we can write

(6.5) $$a_i=dr_{i-1}r_{i+1}\hat{a}_i \qquad (i=0, 1, 2)$$

for some integers $\hat{a}_i$ $(i=0, 1, 2)$. $S_2\Gamma^*(f)$ is as in Figure (6.6).

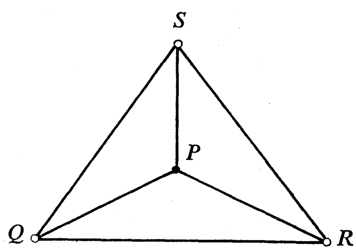


Figure (6.6)

Here $P={}^t(r_0\hat{a}_1\hat{a}_2, r_1\hat{a}_0\hat{a}_2, r_2\hat{a}_0\hat{a}_1)$, $Q={}^t(1,0,0)$, $R={}^t(0, 1, 0)$ and $S={}^t(0, 0, 1)$. Thus the resolution graph is star-shaped and all the vertices are rational except possibly $E(P)$. This is well known by [18]. By Theorem (6.1) and Corollary (6.4), we have

Lemma (6.7). *The genus of $E(P)$ is*

$$d\{dr_0r_1r_2-(r_0+r_1+r_2)\}/2+1.$$

In particular, $E(P)$ *is rational (assuming* $r_0 \leqq r_1 \leqq r_2$*) if and only if*

(i) $d=r_0=r_1=1$ *or*

(ii) $d=2$, $r_0=r_1=r_2=1$. *Note that* (i) *and* (ii) *are equivalent to*

(i)′ a_2 *is coprime with* a_0 *and* a_1 *or*

(ii)′ g.c.d. $(a_i, a_j)=2$ *for* $i \neq j$, *(Compare with* [3].)

Example (6.8). Let $(a_0, a_1, a_2)=(2, 3, 5)$. Then $P={}^t(15, 10, 6)$. The following are necessary data for the surgery.

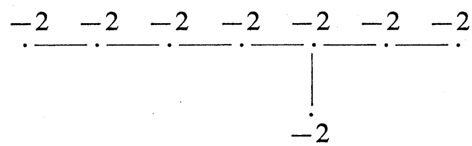
(1) $\overline{PQ}$: $\det(P, Q)=2$ and $(P+Q)/2={}^t(8, 5, 3)$.

(2) $\overline{PR}$: $\det(P, R)=3$ and $(R+2P)/3={}^t(10, 7, 4)$ and $3/2=2-1/2$.

(3) $\overline{PS}$: $\det(P, S)=5$ and $(S+4P)/5={}^t(12, 8, 5)$ and

$$\frac{5}{4}=2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2}}}.$$

$E(P)$ is rational by Lemma (6.7) and $-E(P)^2$ is $(8+10+12)/15=2$. Thus the resolution graph is:



Example (6.9). Let $(a_0, a_1, a_2)=(2s, 3s, 5s)$. Then we have the same dual Newton diagram. $E(P)$ has genus $(s-1)(s-2)/2$ and $-E(P)^2=2s$. Each branch of the resolution graph is replaced by s copies.

(II) $T_{p,q,r}$ **singularities** ([1])

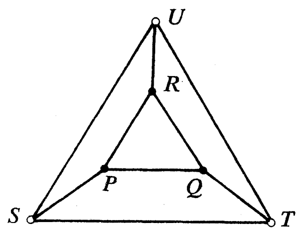
Let $f(x, y, z)=x^p+y^q+z^r+xyz$ where $1/p+1/q+1/r<1$. $\Gamma^*(f)$ has three strictly positive vertices $P={}^t(p_0, p_1, p_2)$, $Q={}^t(q_0, q_1, q_2)$ and $R={}^t(r_0, r_1, r_2)$ which correspond to y^q+z^r+xyz, z^r+x^p+xyz and x^p+y^q+xyz respectively. They satisfy

(6.10) $$p_1q=p_2r=p_0+p_1+p_2$$

(6.11) $$q_0p=q_2r=q_0+q_1+q_2$$

(6.12) $$r_0p=r_1q=r_0+r_1+r_2.$$

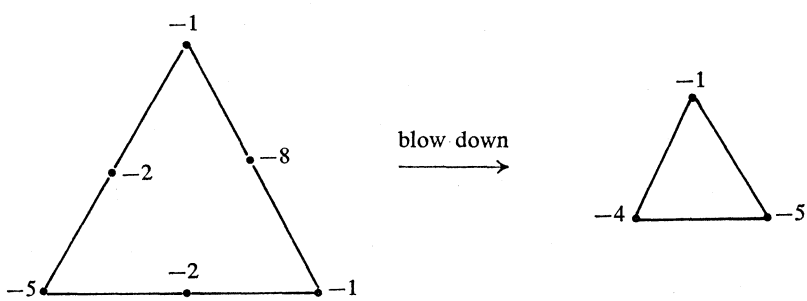
The dual Newton diagram is as follows.



where $S={}^t(1, 0, 0)$, $T={}^t(0, 1, 0)$ and $U={}^t(0, 0, 1)$. It is easy to see that the genera of $\Delta(P)$, $\Delta(Q)$ and $\Delta(R)$ are zero. As $\det(P, S)=\text{g.c.d.}(p_1, p_2)=1$ by (6.10). Similarly we have that $\det(Q, T)=1$ and $\det(R, U)=1$. Note also that $r(\Delta(P, Q))=0$, $r(\Delta(Q, R))=0$ and $r(\Delta(R, P))=0$. Thus by Theorem (6.1), we have

Proposition (6.13). *The resolution graph of $T_{p,q,r}$ is a cyclic chain of rational curves.*

Example (6.14). *Let* $(p, q, r)=(3, 4, 4)$. Then the resolution graph is



§ 7. Fundamental group of $E(P)$

Let P be a strictly positive vertex of a fixed simplicial subdivision Σ^* and we assume that $n>2$ and $\Delta(P)$ is an n-simplex of $\Gamma(f)$, i.e. $\Delta(P)$ is spanned by $(n+1)$-vertices. In this section, we will show that the fundamental group of $E(P)$ is a finite cyclic group whose order is independent of the choice of Σ^*. First we show

Lemma (7.1). *Assume $n>2$. Let $\sigma=(P, Q_1, \cdots, Q_n)$ be an n-simplex of Σ^*. Then the inclusion map j: $E(P; \sigma)^* \to (\boldsymbol{C}_\sigma^*)^n$ induces an isomorphism of the fundamental groups where $(\boldsymbol{C}_\sigma^*)^n=\{y_\sigma \in \boldsymbol{C}_\sigma^{n+1};\ y_{\sigma,0}=0$ and $y_{\sigma,i}\neq 0$ for $i\neq 0\}$.*

Proof. When we move the coefficients of f on $\Delta(P)$ keeping the non-degeneracy condition, then the corresponding exceptional divisor $E(P)$ moves diffeomorphically. Thus $E(P)^*$ also moves diffeomorphically. Thus we may assume that

$$g_{\Delta(P)}(y_{\sigma,1}, \cdots, y_{\sigma,n}) = c + \sum_{i=1}^{n} a_i y_\sigma^{\mu_i}$$

where c and a_i $(i=1, \cdots, n)$ are non-zero. Here 0 and μ_i $(i=1, \cdots, n)$ are the vertices which span $S(g_{\Delta(P)})$. We consider the weighted homogeneous polynomial $h(y_\sigma) = \sum_{i=1}^{n} a_i y_\sigma^{\mu_i}$. Then we have a canonical fibration

$$h\colon (\boldsymbol{C}^*)^n - h^{-1}(0) \to \boldsymbol{C}^*$$

and $E(P)^* = h^{-1}(-c)$. In Theorem (5.3) of [17], we have proved the map $b\colon (\boldsymbol{C}^*)^n - h^{-1}(0) \to (\boldsymbol{C}^*)^n \times \boldsymbol{C}^*$, which is defined by $b(y_\sigma) = (y_\sigma, h(y_\sigma))$, induces an isomorphism of the fundamental groups. Compared with the exact sequence of the homotopy groups of the above fibration, the assertion is now immediate from Theorem (5.3) of [17].

Let $\tau = (P, Q_1, \cdots, Q_n)$ be an n-simplex of Σ^*. We say that τ is *good* if $\dim \Delta(Q_i) > 0$ for $i = 1, \cdots, n-1$. Let $\xi = (P, R_1, \cdots, R_n)$ be any n-simplex and assume that $\dim \Delta(R_i) > 0$ if and only if $i \leqq k$. It is easy to see that there is a good p-simplex $\hat{\xi} = (P, \hat{R}_1, \cdots, \hat{R}_n)$ such that $\hat{R}_i = R_i$ for $i = 1, \cdots, k$, By the definition of $E(P; \xi)$, we have the inclusion $E(P; \xi) \subset E(P; \hat{\xi})$. Thus we need only good simplexes to calculate $\pi_1(E(P))$ through the Van Kampen theorem.

Let $\tau = (P, Q_1, \cdots, Q_n)$ be a good simplex of Σ^* and let $e_{\tau,i}$ $(i=1, \cdots, n)$ be the canonical generators of $\pi_1(E(P; \tau)^*) \cong \pi_1((C^*)^n) \cong Z^n$. Note that $e_{\tau,i}$ $(i=1, \cdots, n-1)$ are trivial in $\pi_1(E(P; \tau))$ because $E(P; \tau) \cap \{y_{\tau,i} = 0\}$ is non-empty. Thus we get

$$\pi_1(E(P; \tau)) \cong Z \tag{7.2}$$

where Z is generated by $e_{\tau,n}$.

We fix a good simplex $\tau = (P, Q_1, \cdots, Q_n)$ from now on. For a vertex Q of Σ^*, we define $A_\tau(Q)$ by the determinant of the matrix $(P, Q_1, \cdots, Q_{n-1}, Q)$. The main theorem of this section is

Theorem (7.3). *$\pi_1(E(P))$ is a finite cyclic group of order d where d is the greatest common divisior of $\{A_\tau(Q)\}$ where Q is adjacent to P in Σ^* and* $\dim \Delta(Q) > 0$. *d is independent of the choice of Σ^*.*

Proof. Let $\xi = (P, R_1, \cdots, R_n)$ be a good simplex of Σ^* and let $\Lambda = (\lambda_{ij})$ be the gluing matrix. Namely $R_i = \sum_{j=0}^{n} \lambda_{ji} Q_j$ for $i = 1, \cdots, n$. Note that $\lambda_{n,i} = A_\tau(R_i)$. Let $e_{\xi,i}$ $(i=1, \cdots, n)$ be the canonical generators of

$\pi_1(E(P;\xi)^*)$. Through the gluing map, $e_{\xi,i}$ corresponds to $\sum_{j=1}^{n}\lambda_{ji}e_{\tau,j}$. As $e_{\xi,i}$ is trivial in $\pi_1(E(P;\xi))$ for $i=1,\cdots,n-1$, we have

$$(7.4) \qquad \pi_1(E(P;\xi)\cup E(P;\tau))\cong Z/d_\xi Z$$

where d_ξ is the greatest common divisor of $A_\tau(R_i)$ for $i=1,\cdots,n-1$. For any vertex of Σ^* which is adjacent to P and $\dim \Delta(Q)>0$, there is a good simplex σ such that Q is a vertex of σ. Thus the first assertion of the theorem is immediate from the above argument.

Now we prove that d is independent of the choice of Σ^*. Let $P_1,\cdots,P_{n+1}$ be the vertices of $\Gamma^*(f)$ which correspond to n-dimensional faces of $\Gamma(f)$ which are adjacent to $\Delta(P)$ i.e., $\partial\Delta(P)=\bigcup_{i=1}^{n+1}(\Delta(P_i)\cap\Delta(P))$. Let Ξ_{ij} be the n-dimensional cell of $\Gamma^*(f)$ which contains P and P_k for k such that $k\neq i,j$. Note that $\Delta(R)=\Delta(P)\cap\{\bigcap_{k\neq i,j}\Delta(P_k)\}$ for any vertex R of Interior (Ξ_{ij}) and $\dim\Delta(R)=1$. We can take a good simplex $\tau=(P,Q_1,\cdots,Q_n)$ such that $Q_1,\cdots,Q_{n-1}\in$ Closure $(\Xi_{n-1,1})$ and

$$(7.5) \qquad Q_1=\sum_{j=1}^{i}a_{ij}P_j+b_jP, \qquad i=1,\cdots,n-1$$

where a_{ij} and b_i ($i=1,\cdots,n$ and $j\leqq i$) are non-negative rational numbers. As $\det(P,Q_1,\cdots,Q_i)=1$ for $1\leqq i\leqq n$, we can easily see by the induction on i that

$$(7.6) \qquad a_{ii}=\det(P,P_1,\cdots,P_{i-1})/\det(P,P_1,\cdots,P_i).$$

By (7.6), a_{ii} ($i=1,\cdots,n$) are independent of the subdivision Σ^*. Let Q be a primitive integral vector of Σ^* with $\dim\Delta(Q)>1$.

By (7.5) and (7.6), we have

$$(7.7) \qquad \begin{aligned} A_\tau(Q)&=\det(P,Q_1,\cdots,Q_{n-1},Q)\\ &=\det(P,P_1,\cdots,P_{n-1},Q)/\det(P,P_1,\cdots,P_{n-1}).\end{aligned}$$

The last equality says that $A_\tau(Q)$ depends only on Q. Let $\xi_{ij}=(P,R_1,\cdots,R_n)$ be a good n-simplex such that $R_k\in$ Closure (Ξ_{ij}) for $k=1,\cdots,n-1$. Then any integral vector Q on Ξ_{ij}, which is not necessarily a vertex of Σ^*, is contained in a Z-submodule generated by $P,R_1,\cdots,R_{n-1}$. Thus the ideals in Z generated by $\{A_\tau(R_1),\cdots,A_\tau(R_{n-1})\}$ and by $\{A_\tau(Q)$ for all integral vectors $Q\in\Xi_{ij}\}$ respectively are equal. Thus the second assertion of the theorem is immediate from (7.7). This completes the proof of Theorem (7.3).

Corollary (7.8). *Assume that $\Delta(P)$ is an n-simplex. Then the first Betti number of $E(P)$ is zero . In particular, the irregularity of $E(P)$ is also zero.*

Example (7.9). Let $f(z)=z_0^{a_0}+\cdots+z_3^{a_3}$. Let $P={}^t(p_0,\cdots,p_3)$ be the weight vector of f. $\Gamma^*(f)$ has four other vertices $P_0={}^t(1,0,0,0),\cdots,P_3={}^t(0,0,0,1)$. Let Σ^* be a simplicial subdivision of $\Gamma^*(f)$ and let τ be as in the proof of Theorem (7.3). Let P_i^1 be the vertex of Σ^* which is on the line segment $\overline{PP_i}$ and P_i^1 is adjacent to P. Then P_i^1 can be written as

$$\begin{aligned}P_i^1&=(P_i+c_iP)/\det(P,P_i)\\&=(P_i+c_iP)/\text{g.c.d.}\{p_j;j\neq i\}\end{aligned}$$

where c_i is a non-negative integer (Lemma (3.3)). By (7.7), $A_\tau(P_i^1)=\det(P,P_0,P_1,P_i^1)/\det(P,P_0,P_1)$. Thus we have

$$\begin{aligned}A_\tau(P_2^1)&=p_3/\text{g.c.d.}(p_2,p_3)\ \text{g.c.d.}(p_0,p_1,p_3)\\A_\tau(P_3^1)&=p_2/\text{g.c.d.}(p_2,p_3)\ \text{g.c.d.}(p_0,p_1,p_2).\end{aligned}$$

As $p_3/\text{g.c.d.}(p_2.p_3)$ and $p_2/\text{g.c.d.}(p_2,p_3)$ are coprime, we have that $d=1$. Namely

Proposition (7.10). *The central divisor $E(P)$ of the Brieskorn variety is simply connected.*

The following example shows that $\pi_1(E(P))$ is not trivial in general.

Example (7.11). Let $f(z)=\sum_{i=0}^3(z_i^2z_{i+1}z_{i+2}^4+z_i^{11})$ where $z_{i+4}=z_i$ and $n=3$. $\Gamma(f)$ has five compact 3-dimensional faces which are the support of $\sum_{i=0}^3 z_i^2z_{i+1}z_{i+2}^4$ and $\sum_{i=1}^3 z_{j+i}^2z_{j+i+1}z_{j+i+2}^4+z_j^{11}+z_{j+3}^{11}$ $(j=0,\cdots,3)$. The corresponding vertices in $\Gamma^*(f)$ are $P,P_0,\cdots,P_3$ where $P={}^t(1,1,1,1)$, $P_0={}^t(1,2,3,1)$, $P_1={}^t(1,1,2,3)$, $P_2={}^t(3,1,1,2)$, $P_3={}^t(2,3,1,1)$. For example,

$$z_1^2z_2z_3^4+z_2^2z_3z_0^4+z_3^2z_0z_1^4+z_0^{11}+z_3^{11}$$

is a weighted homogeneous polynomial of degree 11 by the weight P_0. Geometrically, P is at the barycenter of $P_0,\cdots,P_3$. As $\det(P,P_i)=\det(P,P_i,P_j)=1$ for any $i\neq j$, we do not need any other vertices on the triangles $T(P,P_i,P_j)$ to get a simplicial subdivision Σ^*. We take $\tau=(P,P_0,P_1,R)$ where $R=(P_2+2P_0+3P_1+2P)/5={}^t(2,2,3,3)$. As $A_\tau(P_2)=A_\tau(P_3)=5$ and $A_\tau(P_i)=0$ $(i=0,1)$, we have that $d=5$. Therefore $\pi_1(E(P))\cong Z/5Z$.

§ 8. Exceptional divisors of the three dimensional singularities

In this section, we will study the topology of exceptional divisors $E(P)$ of the three dimensional singularities. Thus we assume that $n=3$.

Let P be a strictly positive vertex of $\Sigma^*(f)$ such that $\dim \varDelta(P)=3$. Let $\varDelta_1, \cdots, \varDelta_s$ be the two-dimensional faces of $\varDelta(P)$ and let $\Xi_1, \cdots, \Xi_q$ be one-dimensional faces of $\varDelta(P)$. Each Ξ_k is an intersection of two of $\varDelta_j$. Assume that $\Xi_k=\varDelta_i \cap \varDelta_j$. Then this implies that $\varDelta_i^*, \varDelta_j^* \subset \bar{\Xi}_k^*$ where $\bar{\Xi}_k^*$ is the closure of $\Xi_k^*=\{Q; \varDelta(Q)=\Xi_k\}$. Let P_i be the unique vertex of $\bar{\varDelta}_i^*$ which is adjacent to P. Let $T_k^1, \cdots, T_k^{\nu_k}$ be the vertices on $\bar{\Xi}_k^*$ which are adjacent to P and not on $\bar{\varDelta}_i^*$ and $\bar{\varDelta}_j^*$. See Figure (8.1).

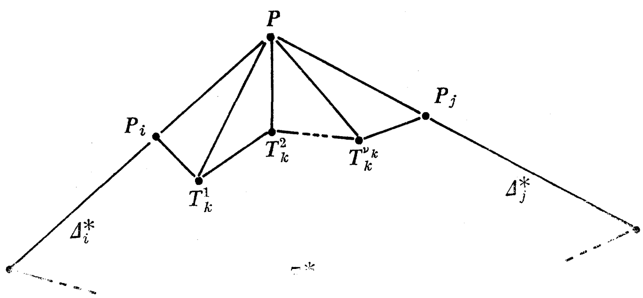


Figure (8.1)

Definition (8.2). Let $c_k=\det(P, P_i, P_j)$. We say that $T_k^1, \cdots, T_k^{\nu_k}$ are *canonical* at P if T_k^l is inductively defined by

$$(8.3) \qquad T_k^l=(P_j+c_{k,l}T_k^{l-1}+d_{k,l}P)/c_{k,l-1}$$

where

$$(8.4) \qquad 0\leqq c_{k,l},\ d_{k,l}<c_{k,l-1} \qquad (l=1, \cdots, \nu_k)$$

and $c_{k,0}=c_k$, $T_k^0=P_i$, $c_{k,\nu_k}=1$. See Lemma (3.8). For a vertex Q of Σ^* with $\dim \varDelta(Q)\geqq 1$, we define a divisor $C(Q)$ of $E(P)$ by $C(Q)=E(P)\cap E(Q)$. This is non-empty if and only if Q is adjacent to P. Let $\{a, b\}$ be a pair of integers such that $0\leqq a<b\leqq 3$. Let $P={}^t(p_0, \cdots, p_3)$ and $Q={}^t(q_0, \cdots, q_3)$. We define $|P, Q|_{a,b}$ by (a, b)-minor $(p_aq_b-p_bq_a)$ of 4×2 matrix (P, Q).

Theorem (8.5). (i) *$C(T_k^l)$ is a union of $r((\Xi_k)+1)$ copies of rational curves and the genus of $C(P_i)$ is $g(\varDelta_i)$.*

(ii) *The Euler characteristics $\chi(E(P))$ is equal to*

$$24 \text{ volume } C(0, \varDelta(P))/d(P)-2\sum_{i=1}^{s} g(\varDelta_i)+2s+\sum_{k=1}^{q}\nu_k(r(\Xi_k)+1).$$

(iii) *Let T_k^t $(t=1, \cdots, \nu_k)$ be the vertices on $\bar{\Xi}_k^*$ as above Let $-n_l$ be the self-intersection number of a component of $C(T_k^l)$ and let*

$$T_k^1=(P_j+\bar{c}_{k,1}P_i+\bar{d}_{k,1}P)/c_k$$

where $c_k=\det(P, P_i, P_j)$. *Then we have that* $n_m\geqq 1$ $(m=1, \cdots, \nu_k)$ *and*

$$\frac{c_k}{\bar{c}_{k,1}}=n_1-\cfrac{1}{n_2-\cfrac{\ddots}{\quad -\cfrac{1}{n_{\nu_k}}}}.$$

(iv) *Assume that* $\{T_k^t\}$ $(t=1, \cdots, \nu_k)$ *are canonical sequence in the sense of Definition* (8.2). *Thus* $\bar{c}_{k,1}=c_{k,1}$. *Then* $n_l\geqq 2$. *In particular,* ν_k *and* $\{n_l\}$ $(l=1, \cdots, \nu_k)$ *are determined by* c_k *and* $c_{k,1}$ *through the continuous fraction representation of* $c_k/c_{k,1}$.

(v) *The self-intersection number* $C(P_i)^2$ *is equal to*

$$-\sum_Q (r(P, P_i, Q)+1)|P, Q|_{a,b}/|P, P_i|_{a,b}$$

where the sum is taken for Q *such that* (P, P_i, Q) *is a simplex of* Σ^* *and* $\dim\Delta(P, P_i, Q)\geqq 1$ *and* $r(P, P_i, Q)=r(\Delta(P, P_i, Q))$. (*We assume that* a, b *are so chosen that* $|P, P_i|_{a,b}\neq 0$).

Proof. Let $\sigma=(P, P_i, T_k^1, R)$ be a 3-simplex of Σ^*. (If $\nu_k=0$, T_k^1 should be replaced by P_j.) Then we have seen in Section 4 that $C(P_i)$ is defined by $g_{\Delta_i}(y_{\sigma,2}, y_{\sigma,3})=0$. Note that $C(P_i)\cdot C(T_k^1)$ consists of $r(\Xi_k)+1$ points which are solutions of $g_{\Delta_i}(0, y_{\sigma,3})=g_{\Xi_k}(y_{\sigma,3})=0$. Thus we have

$$\chi(C(P_i))=\chi(C(P_i)^*)+\sum_{\Delta_i\supset\Xi_k}(r(\Xi_k)+1).$$

The first term is equal to

$$-2g(\Delta_i)-\sum_{\Delta_i\supset\Xi_k}(r(\Xi_k)+1)+2$$

by Lemma (6.2) and Theorem (5.3) (i) and the invariance of the number of the integral points on a polyhedron by a unimodular matrix. Thus we have that $\chi=2-2g(\Delta_i)$ which says that the genus of $C(P_i)$ is $g(\Delta_i)$. The rationality of $C(T_k^l)$ is derived by a similar argument or Lemma (4.8). Now the assertion (ii) is immediate from the additivity of the Euler characteristic and Corollary (5.4).

Now we study the self-intersection numbers of $C(T_k^l)$. Let T_k^l $(l=1, \cdots, \nu_k)$ be as in Figure (8.1). By Lemma (3.8), we can write

(8.6) $$T_k^l=(P_j+\bar{c}_{k,l}T_k^{l-1}+\bar{d}_{k,l}P)/\bar{c}_{k,l-1}$$

for $l=1, \cdots, \nu_k$ where $\bar{c}_{k,0}=c_k$ and $\bar{c}_{k,\nu_k}=1$. Here $\bar{c}_{k,l}>0$ but $\bar{d}_{k,l}$ might be a negative integer in general. We consider the meromorphic function $\varphi=\pi^*(z_b^{p_a}/z_a^{p_b})$ on $E(P)$. Let $\sigma_l=(P, T_k^{l-1}, T_k^l, R_l)$ be a 3-simplex of Σ^*. Then it is easy to see that

$$\varphi(y_{\sigma_l})=\prod_{i=1}^{3} y_{\sigma_l,i}^{d_i}$$

where

$$d_1=|P, T_k^{l-1}|_{a,b}, \quad d_2=|P, T_k^l|_{a,b} \quad \text{and} \quad d_3=|P, R_l|_{a,b}$$

which implies that

$$(\varphi)=\sum_{l=0}^{\nu_k+1}|P, T_k^l|_{a,b}C(T_k^l)+D$$

where D is a linear sum of $C(Q)$ for which $C(Q)\cap C(T_k^m)$ is empty for $m=1, \cdots, \nu_k$. ($T_k^0=P_i$, $T_k^{\nu_k+1}=P_j$.) As $(\varphi)\cdot C(T_k^m)=0$ (Theorem (2.6) of [10]), we have

$$\begin{aligned}(8.7)\qquad & |P, T_k^{m-1}|_{a,b}C(T_k^{m-1})\cdot C(T_k^m)+|P, T_k^m|_{a,b}C(T_k^m)^2 \\ & +|P, T_k^{m+1}|_{a,b}C(T_k^{m+1})\cdot C(T_k^m)=0\end{aligned}$$

for $m=1, \cdots, \nu_k$. As $C(T_k^m)$ has $(r(\Xi_k)+1)$ components, (8.7) implies

$$(8.8)\qquad n_m=(|P, T_k^{m-1}|_{a,b}+|P, T_k^{m+1}|_{a,b})/|P, T_k^m|_{a,b}.$$

On the other hand, (8.6) implies that

$$(8.9)\qquad \bar{c}_{k,m-1}|P, T_k^m|_{a,b}=\bar{c}_{k,m}|P, T_k^{m-1}|_{a,b}+|P, P_j|_{a,b}.$$

We prove the assertion (iii) by the induction on ν_k.

(a) Assume that $\nu_k=1$. Then the assertion is immediate from (8.8) and (8.9).

(b) Assume that $\nu_k>1$ and

$$\frac{\bar{c}_{k,1}}{\bar{c}_{k,2}}=n_2-\cfrac{1}{n_3-\cfrac{\ddots}{-\cfrac{1}{n_{\nu_k}}}}.$$

Then we have

$$n_1-\cfrac{1}{n_2-\cfrac{}{\ddots-\cfrac{1}{n_{\nu_k}}}}=\frac{n_1\bar{c}_{k,1}-\bar{c}_{k,2}}{\bar{c}_{k,1}}.$$

From (8.9), we can obtain the equality

$$(8.10)\qquad (\bar{c}_{k,m-1}+\bar{c}_{k,m+1})|P,\ T_k^m|_{a,b}=\bar{c}_{k,m}(P,\ T_k^{m-1}|_{a,b}+|P,\ T_k^{m+1}|_{a,b})$$

which implies that

$$n_m=(\bar{c}_{k,m-1}+\bar{c}_{k,m+1})/\bar{c}_{k,m}.$$

Thus $n_m\geqq 1$ and

$$(n_1\bar{c}_{k,1}-\bar{c}_{k,2})/\bar{c}_{k,1}=c_k\bar{c}_{k,1}$$

which proves the assertion.

Now we prove the assertion (iv). Assume that $\bar{c}_{k,m}=c_{k,m}$ and

$$c_k=c_{k,0}>c_{k,1}>\cdots>c_{k,\nu_k}=1.$$

Then by (8.8) and (8.10), we have

$$n_m=(c_{k,m-1}+c_{k,m+1})/c_{k,m}>1$$

which implies $n_m\geqq 2$, proving the assertion. The assertion (v) is also easily obtained by the equality $(\varphi)\cdot C(P_i)=0$.

§ 9. Canonical divisors

Let π: $\tilde{V}\to V$ be the resolution of V associated with Σ^*. In this section, we study the canonical divisors $\tilde{K}$ of $\tilde{V}$ and K_p of $E(P)$ respectively.

(I) The canonical divisor $\tilde{K}$ of $\tilde{V}$.

Let $\hat{\pi}: X\to \boldsymbol{C}^{n+1}$ be the projection map constructed in Section 4. Recall that $\tilde{V}$ is a complex submanifold of codimension one of X. Let ω' be a meromorphic n-form on a neighbourhood of the origin of $\boldsymbol{C}^{n+1}$ such that

$$\omega'\wedge df=dz_0\wedge dz_1\wedge\cdots\wedge dz_n.$$

It is easy to see that the restriction ω of ω' to V is a meromorphic n-form which does not depend on the choice of ω'. We denote ω by $dz_0\wedge\cdots\wedge dz_n/df$. We want to know the local expression of the meromorphic n-form

$\pi^*(\omega)$ on $\tilde{V}$. Let $\sigma=(P_0, \cdots, P_n)$ be an n-simplex of Σ^* and let $P_i={}^t(p_{0i}, \cdots, p_{ni})$. Then we have

$$\hat{\pi}^*(dz_0\wedge\cdots\wedge dz_n)=\det(p_{ij})\prod_{i=1}^{n} y_{\sigma,i}^{\beta_i} dy_{\sigma,0}\wedge\cdots\wedge dy_{\sigma,n}$$

where $\beta_i=|P_i|-1$ and $|P_i|=\sum_{j=0}^{n} p_{ji}$. Similarly we have

$$\begin{aligned}\hat{\pi}^*(df)&=d(\hat{\pi}^*f)\\&=d\left[\prod_{i=0}^{n} y_{\sigma,i}^{d(P_i)} f_\sigma(y_\sigma)\right]\\&=d[\prod y_{\sigma,i}^{d(P_i)}]f_\sigma(y_\sigma)+\prod y_{\sigma,i}^{d(P_i)}df_\sigma.\end{aligned}$$

Here $f_\sigma=0$ is the defining equation of $\tilde{V}$ in C_σ^{n+1}. We get a meromorphic n-form $\tilde{\omega}_\sigma$ on C_σ^{n+1} by taking the "residue":

$$\begin{aligned}\tilde{\omega}_\sigma&=\hat{\pi}_\sigma^*(dz_0\wedge\cdots\wedge dz_n)/\hat{\pi}_\sigma^*df\\&=\prod y_{\sigma,i}^{\alpha(P_i)}(dy_{\sigma,0}\wedge\cdots\wedge dy_{\sigma,n}/df_\sigma)\end{aligned}$$

where $\alpha(P_i)=|P_i|-d(P_i)-1$. As we have the equality:

$$\tilde{\omega}_\sigma\wedge\hat{\pi}_\sigma^*df=\hat{\pi}_\sigma^*(dz_0\wedge\cdots\wedge dz_n),$$

we can easily see that the restriction of $\tilde{\omega}_\sigma$ to $\tilde{V}$ is equal to $\pi_\sigma^*(\omega)$ by the above property. Note that $dy_{\sigma,0}\wedge\cdots\wedge dy_{\sigma,n}/df_\sigma$ is a nowhere vanishing n-form on $\tilde{V}\cap C_\sigma^{n+1}$. Thus we obtain

Theorem (9.1). *$\tilde{K}=(\tilde{\omega})=\sum_P \alpha(P)E(P)$ where $\alpha(P)=|P|-d(P)-1$ and the sum is taken for the vertex $P\in\Sigma^*$ such that* $\dim \Delta(P)>0$.

Corollary (9.2). *The coefficient $\alpha(P)$ of $\tilde{K}$ does not depend on the choice of Σ^* which contains P as a vertex.*

By applying Theorem (9.1) to Theorem (1.5) of [5], we can calculate the signature of the Milnor fibre F of f in the case of $n=2$ from the Newton boundary $\Gamma(f)$.

It is well-known that the canonical divisor $\tilde{K}$ of the minimal resolution π: $\tilde{V}\to V$ of the isolated surface singularity satisfies that $-\tilde{K}\geqq 0$ where the equality holds only for rational double points. For a hypersurface singularity of dimension 2 with a non-degenerate Newton boundary, this can be proved by the following corollary.

Assume that $n=2$ and let $p={}^t(p_1, p_2, p_3)$ be a strictly positive vertex of Σ^* such that $\dim \Delta(P)=2$. Then

Corollary (9.3). *Let P be as above. Then $\alpha(P)\geqq 0$ if and only if $\Delta(P)\supset S(h)$ where h is one of the following weighted homogeneous polynomials and $S(h)$ is the support of h, up to a permutation of the coordinates.*

(A_k)

(i) $x^2+y^2+z^{k+1}$
(ii) $x^2+y^2+2yz^a \quad (k=2a-1)$
(iii) x^a+xy+z^{k+1}
(iv) $x^a+xy+yz^b \quad (k=ab-1)$
(v) $xy+x^{a+1}z^b+z^c \quad (b<c,\ k=c-1)$
(vi) $xy+x^az^c+yz^d \quad (a>0,\ k=ad+c-1)$
(vii) $xy+x^az^b+x^cz^d \quad (0<c<a,\ 0\leqq b<d)$

(D_k)

(i) $x^2+yz^2+y^{k-1} \quad (k>3)$
(ii) $x^2+y^3+z^3 \quad (D_4)$
(iii) $x^2+yz(z+2y^d) \quad (k=2d+1)$
(iv) $x^2+2xy^a+yz^2 \quad (k=2a+1)$

(E_6)

(i) $x^2+y^3+z^4$
(ii) $x^2+y^3+2xz^2$

(E_7) $x^2+y^3+yz^3$

(E_8) $x^2+y^3+z^5$

(M)

(i) $x^2+y^b(y^d+z),\quad b<d+2$
(ii) $x^2+3xy^a+y^bz,\quad b<a+1$

(N) $xy^a+y^bz^c(y^d+z^e),\quad d>(a-1)e.$

Proof. Assume that $\alpha(P)\geqq 0$. By Theorem (9.1), $\alpha(P)\geqq 0$ if and only if $|P|>d(P)$. Note that $|P|$ is the degree of the monomial xyz by the weight P. Thus $\Delta(P)$ contains no vertices (i, j, k) such that $i, j, k>0$. We assume that $p_1\geqq p_2, p_3$.

(I) Assume that $(1, 1, 0)$ (or $(1, 0, 1)$) is on $\Delta(P)$. It is easy to see that any 2-simplex, which contains $(1, 1, 0)$ and has a strictly positive weight, is one of (i)$\sim$(vii) of (A_k). Note that (ii)$\sim$(vi) reduces to (i) by suitable changes of coordinates. For example,

(iv): $x^a+xy+yz^b=x^a+y(x+z^b)=(X-Z^b)^a+XY=X(Y+\cdots)+(-1)^aZ^{ab}$.

(vi): $xy+x^az^c+yz^d=y(x+z^d)+x^az^c=XY+(X-Z^d)^aZ^c=X(Y+\cdots)+(-1)^aZ^{ab+c}$.

(vii) is reduced to A_1 of two variables by

$$xy+x^a z^b+x^c z^d=x(y+x^{a-1}z^b+x^{c-1}z^d).$$

Thus we assume that neither (1, 1, 0) nor (1, 0, 1) are on $\Delta(P)$ from now on.

(II) Assume that $(m, 0, 0) \in \Delta(P)$. Then $d(P)=mp_1<|P|$ implies that $m=2$ or 1. $m=1$ is omitted as $V=f^{-1}(0)$ is non-singular in this case. Thus $m=2$.

(1) Assume that $\Delta(P)$ contains two integral points $(0, b+d, c)$ and $(0, b, c+e)$ for $b, c\geqq 0$, $d, e>0$. This corresponds to $y^b z^c(y^d+z^e)$. Note that (2, 0, 0), (1, 1, 1) and (0, 2, 2) are colinear. We may assume that $b\geqq c$. By the assumption that $\alpha(P)\geqq 0$, we have

$$c<2 \quad \text{and} \quad d(2-c)>e(b+d-2).$$

The following cases are possible.

(i) $c=1, b=1, e=1$

(ii) $c=1,\ b=1,\ d=1,$

(iii) $c=0,\ e=1,\ b<d+2,$

(iv) $c=0, e=2,\ b=1$

(v) $c=0,\ b=0,\ d=3,\ e=3, 4, 5$ (or $e=3$, $d=3, 4, 5$).

(i) corresponds to (iii) of (D_k): $x^2+yz(2y^d+z)=x^2+y(z+y^d)^2-y^{2d+1}$. (ii) is reduced to (i) by changing y and z. (iii) corresponds to (i) of (M). (iv) corresponds to (i) of (D_k). (v) corresponds to $x^2+y^3+z^e$ $(e=3, 4, 5)$ which are (ii) of (D_k), (i) of (E_6) and (E_8).

(2) Assume that $\Delta(P)$ contains only one point on the (y, z)-plane. As $\dim \Delta(P)=2$, we may assume that $(1, a, 0)$ and $(0, b, e)$ are on $\Delta(P)$ and $e>0$. As $(0, 2a, 0)$ is on the plane which is spanned by $\Delta(P)$, we can use the discussion of (1) $(c=0,\ d=2a-b)$ to see that $\alpha(P)\geqq 0$ if and only if (iii)′ $e=1$, $b<a+1$, or (iv)′ $e=2$, $b=0$, or (v)′ $e=2$, $b=1$, or (vi)′ $c=0$, $b=0$, $a=2$, $e=3$. (iii)′ corresponds to (ii) of (M) and note that $x^2+2xy^a+y^b z=(x+y^a)^2-y^{2a}+y^b z$. (vi)′ corresponds to (ii) of (A_k). (v)′ corresponds to (v) of (D_k), (vi)′ corresponds to (ii) of (E_6).

(3) Assume that $\Delta(P)$ contains no point on the (y, z)-plane. Then $\Delta(P)$ contains (1, a, 0) and (1, 0, m) for some a, $m>1$. Thus the plane generated by $\Delta(P)$ contains $(0, 2a, 0)$ and $(0, 0, 2m)$. Thus we can use (1) to conclude that there is no such $\Delta(P)$.

(III) Assume that $\Delta(P)$ does not intersect with the x-axis. As $p_1\geqq p_2, p_3$ and $d(P)<\alpha(P)$, we may assume that $(1, a, 0)$ is on $\Delta(P)$ with $a>1$. Then it is easy to see that there is no point $(1, 0, m)$ on $\Delta(P)$. Thus there are two integral points $(0, b+d, c)$ and $(0, b, c+e)$ on $\Delta(P)$ such that

b, $c \geqq 0$ and d, $e > 0$. We need the condition:

$$p_1 + ap_2 = (b+d)p_2 + cp_3, \qquad dp_2 = ep_3$$

and $p_1 + ap_2 < p_1 + p_2 + p_3$. This is equivalent to $d > (a-1)e$ which corresponds to (N), completing the proof of Corollary (9.3).

(II) The canonical divisor K_p of $E(P)$

Now we consider the canonical divisor K_p of the exceptional divisor $E(P)$ for a fixed P. Let $\sigma = (P_0, \cdots, P_n)$ be a fixed n-simplex of Σ^* where $P_0 = P$. Let $\boldsymbol{C}_\sigma^n = \{y_\sigma \in \boldsymbol{C}_\sigma^{n+1};\ y_{\sigma,0} = 0\}$. $E(P)$ is defined by $g_\sigma(y_{\sigma,1}, \cdots, y_{\sigma,n}) = 0$ where

$$(9.4) \qquad g_\sigma(y_{\sigma,1} \cdots, y_{\sigma,n}) \prod_{i=0}^{n} y_{\sigma,i}^{d(P_i)} = f_{\Delta(P)}(\pi_\sigma(y_\sigma)).$$

We consider a holomorphic n-form ω_σ on $E(P) \cap \boldsymbol{C}_\sigma^n$ which is the restriction of an n-form $\hat{\omega}_\sigma$ on $\boldsymbol{C}_\sigma^n$ which satisfies

$$(9.5) \qquad \hat{\omega}_\sigma \wedge dg_\sigma = dy_{\sigma,1} \wedge \cdots \wedge dy_{\sigma,n}.$$

It is easy to see that ω_σ is nowhere vanishing and ω_σ does not depend on the choice of $\hat{\omega}_\sigma$. For brevity's sake, we write

$$\omega_\sigma = dy_{\sigma,1} \wedge \cdots \wedge dy_{\sigma,n} / dg_\sigma.$$

Let $\tau = (Q_0, \cdots, Q_n)$ be another n-simplex such that $Q_0 = P$. Let $Q_i = \sum_{j=0}^{n} \lambda_{ji} P_j$ for $i = 1, \cdots, n$ and let $\Lambda = (\lambda_{ji})$ $(1 \leqq i, j \leqq n)$. Then we have

$$(9.6) \qquad y_{\sigma,i} = \prod_{j=0}^{n} y_{\tau,j}^{\lambda_{ij}} \qquad (i = 0, \cdots, n)$$

$\lambda_{00} = 1$ and $\lambda_{j0} = 0$ for $j > 0$. By a similar calculation as in (I), we have

$$(9.7) \qquad dy_{\sigma,1} \wedge \cdots \wedge dy_{\sigma,n} = \det(\Lambda) \prod_{i=1}^{n} y_{\tau,i}^{\beta_i}\, dy_{\tau,1} \wedge \cdots \wedge dy_{\tau,n}$$

$$(9.8) \qquad g_\sigma(y_\sigma) = \prod_{i=1}^{n} y_{\tau,i}^{\gamma_i} g_\tau(y_\tau)$$

where $\beta_i = \sum_{j=1}^{n} \lambda_{ji} - 1$ and $\gamma_i = d(Q_i) - \sum_{j=0}^{n} d(P_j)\lambda_{ji}$. Let $A_\sigma = \bigcap_{j=0}^{n} \Delta(P_j)$. Then A_σ is a vertex of $\Gamma(f)$ and we have

$$\sum_{j=0}^{n} d(P_j)\lambda_{ji} = \sum_{j=0}^{n} \lambda_{ji} P_j(A_\sigma) = Q_i(A_\sigma).$$

Thus we get $\gamma_i = d(Q_i) - Q_i(A_\sigma)$. Let $\alpha_i = Q_i(A_\sigma) - d(Q_i) + \sum_{j=1}^{n} \lambda_{ji} - 1$

and ω_τ be the restriction of $\hat{\omega}_\tau$ to $E(P)\cap C_\tau^n$ where

$$\hat{\omega}_\tau=\sum_{i=1}^{n} y_{\tau,i}^{\alpha i}(dy_{\tau,1}\wedge\cdots\wedge dy_{\tau,n}/dg_\tau).$$

By (9.6), (9.7) and (9.8),

$$\hat{\omega}_\tau\wedge dg_\sigma=dy_{\sigma,1}\wedge\cdots\wedge dy_{\sigma,n}$$

on $E(P)\cap C_\tau^n\cap C_\tau^n$. Thus we get $\omega_\tau=\omega_\sigma$ on $E(P)\cap C_\tau^n\cap C_\sigma^n$. Therefore the collection of $\{\omega_\tau\}$ defines meromorphic n-form ω. Note that λ_{ji} depends only on σ and Q_i. Thus we obtain

Theorem (9.9). *$K_P=(\omega)=\sum\alpha(Q)C(Q)$ where the sum is taken for every vertex Q of Σ^* which is adjacent to P and* $\dim\Delta(Q)>1$. *$C(Q)$ is defined by $E(P)\cap E(Q)$ and*

$$\alpha(Q)=Q(A_\sigma)-d(Q)+\sum_{j=1}^{n}\lambda_j(Q)-1$$

where $Q=\sum_{j=0}^{n}\lambda_j(Q)P_j$.

Remark (9.10). Assume that $n=3$. $C(Q)$ is a smooth curve of genus $g(\Delta(P)\cap\Delta(Q))$ if $\dim(\Delta(P)\cap(Q))=2$. If $\dim(\Delta(P)\cap(Q))=1$, $C(Q)$ has $r(\Delta(P)\cap\Delta(Q))+1$ connected components. Each component is a rational curve (Theorem (8.5)).

Example (9.11). Let $n=3$ and $f(z)$ be $\sum_{i=0}^{3}(z_i^2z_{i+1}z_{i+2}^4+z_i^{11})$ as in Example (7.11). Let $P={}^t(1,1,1,1)$. P corresponds to the homogeneous part of degree 7. There are 4 branches $\overline{PP_i}$ in $\Gamma^*(f)$ at P where $P_0={}^t(1,2,3,1)$, $P_1={}^t(1,1,2,3)$, $P_2={}^t(3\ 1\ 1,2)$ and $P_3={}^t(2,3,1,1)$. As $\det(P,P_i,P_j)=1$ for $i\neq j$, we need no vertices on $T(P,P_i,P_j)$. Let $\sigma=(P,P_0,P_1,R)$ where $R=(P_2+2P_0+3P_1+2P)/5={}^t(2,2,3,3)$. Thus the affine equation of $E(P)$ in C_σ^3 is

$$y_1^5y_3^2+y_2^5y_3^3+y_3+1=0.$$

By Theorem (9.9), we have $K_P=-C(P_2)+2C(P_3)$. By Theorem (8.5), we have that $C(P_i)^2=1$ for $i=0,\cdots,3$ and $C(P_2)\cdot C(P_3)=1$. Thus $K_P^2=1$. On the other hand, $C(P_i)$ is a curve of genus 2 by Remark (9.10). Therefore the Euler characteristic $\chi(E(P))$ is

$$\chi(E(P))=\chi(E(P)^*)+\sum_{i=0}^{3}\chi(C(P_i))-\sum_{i\neq j}\chi(C(P_i)\cap C(P_j))$$
$$=25-8-6=11.$$

By Noether's formula, we get $p_g=0$. Thus $E(P)$ is an algebraic surface with $q=p_g=0$ and $\pi_1(E(P))\cong Z/5Z$. $E(P)$ is called a Godeaux surface ([19], [13]).

References

[1] V. I. Arnold, Critical points of smooth functions and their normal forms, Uspekhi. Mat. Nauk, **30**:5 MR no. 8701 (1975), 3–65.

[2] M. Artin, On isolated rational singularities of surfaces, Amer. J. Math., **88** (1966), 129–136.

[3] E. Brieskorn, Beispiele zur Differentialtopologie von Singularitaten, Invent. Math., **2** (1966), 1–14.

[4] A. H. Durfee, Fourteen characterization of double points, L'Enseignement Math.

[5] ——, The signature of smoothings of complex surface singularities, Math. Ann., **232** (1966), 85–98.

[6] P. Griffiths and J. Harris, Principles of Algebraic Geometry, A Wiley-Interscience Publication, 1978.

[7] F. Hirzebruch, The topology of normal singularities of an algebraic surface, Seminaire Bourbaki, **240** (1962–1963).

[8] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, Toroidal Embeddings, Lecture Note in Math., **339**, Springer (1973).

[9] A. G. Kouchnirenko, Polyedres de Newton et nombres de Milnor, Invent. Math., **32** (1976), 1–32.

[10] H. B. Laufer, Normal two-dimensional singularities, Ann. of Math. Studies, **71** (1971).

[11] ——, On μ for surface singularities, Proceeding of Symposia in Pure Math., **30** (1977), 45–49.

[12] J. Milnor, Singular points of complex hypersurface, Ann. Math. Studies, **61**, Princeton Univ. Press (1968).

[13] Y. Miyaoka, Tricanonical maps of numerical Godeaux surfaces, Invent. Math., **34** (1976), 99–111.

[14] D. Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, Publ. Math. l'Inst. des Hautes Etudes Sci. Paris, **9** (1961).

[15] M. Oka, On the resolution of three dimensional Brieskorn singularities, in this Proceeding.

[16] ——, On the bifurcation of the multiplicity and topology of the Newton boundary, J. Math. Soc. Japan, **31** (1979), 435–450.

[17] ——, On the topology of the Newton boundary II, J. Math. Soc. Japan, **32** (1980), 65–92.

[18] P. Orlik and P. Wagreich, Isolated singularities of algebraic surfaces with C^*-action, Ann. of Math., **93** (1971), 205–228.

[19] M. Reid, Surface with $p_g=0$, $K^2=1$, J. Fac. Sci. Univ. of Tokyo, **25** no. 1 (1978), 75–92.

[20] A. N. Varchenko, Zeta-Function of Monodromy and Newton's Diagram, Invent. Math., **37** (1976), 253–262.

[21] T. Oda, Lectures on torus embeddings and its applications, Tata Institute of Fund. Res. Bombay, 1978.

Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro-ku, Tokyo 152
Japan