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Infinitely Very Near Singular Points

Tadao Oda*

To the memory of Professor Yasuo Akizuki

Introduction

The purpose of this paper is to explain the current state of Hironaka's program for the characteristic-free resolution of singularities of excellent schemes. More specifically, we collect together results on infinitely near points and infinitely very near points due to Bennett [B]. Giraud [G_1], [G_2]. Hironaka [H_1], [H_4]. Herrmann-Orbanz [HO], Oda [O_4] and Singh [S_1], [S_2], [S_3]. We remove unnecessary restrictions and give unified proofs for most of them, using the ideas found in the above papers. Hopefully, we thus get insight into possible further study of the program.

Hironaka's program consists in

(I) finding good numerical invariants for singularities which always improve or at least remain the same under any permissible blowing up, and

(II) finding a finite succession of permissible blowing ups which actually improves these numerical invariants at singular points of a given excellent scheme.

Hironaka successfully carried out this program for excellent surfaces (see [H_2] for a sketch). It might be possible to carry out (II) either

(II_1) in a mesh of inductions on the dimension as in Hironaka's proof [H_1] in characteristic zero, or

(II_2) in formulating a good game on Newton polyhedra and then finding a winning strategy for it.

Spivakovsky dealt with a prototype of this game formulated by Hironaka and in [S_5] found a winning strategy for a simpler version, while in [S_4] he showed that a winning strategy need not exist for a harder version. Nothing else seems to be known about (II) at the moment.

Our main concern in this paper is (I) for general excellent schemes. We have already sketched in [O_3], without proof, the current state of (I)

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in the technically much simpler case of hypersurface singularities.

Good numerical invariants for singularities found so far are essentially refined versions of those in [H_1] for characteristic zero. Complications arise in positive characteristics, however, due to the existence of *nonlinear additive forms*. They do not cause much trouble for surfaces as Hironaka showed in [H_2]. But already for threefolds, the simple result due to Zariski [Z, Lemma 7.1] for characteristic zero breaks down in positive characteristics. Hironaka [H_3] formulated this phenomenon in terms of group schemes, now called *Hironaka group schemes*. We have studied them as well as their higher order analogues in [O_1], [O_2], [O_4].

Here, we do not deal with Hironaka group schemes, but rather use the techniques developed so far for them to study the ridge of tangent cones and normal cones, which is more directly related to singular points. The *ridge* of a cone in a linear space is the largest subgroup scheme of the ambient linear space which leaves the cone stable under translation. Section 1 is devoted to the elementary but basic results on the ridge of a cone.

In Section 2. We collect together known results on the *Hilbert-Samuel functions* of a Noetherian local ring.

Then in Section 3, we introduce three numerical invariants known so far for a singularity and state basic stability theorems for them. We prove most of them in Section 4.

These numerical invariants are defined locally. Hence we restrict ourselves to the following situation: X is a scheme embedded in a regular ambient scheme Z. For a point x of X not necessarily closed, we introduce three numerical invariants as follows:

The *first* is the sequence $\{H_x^{(j)}(X)$; all nonnegative integers $j\}$ of higher order Hilbert-Samuel functions of the local ring $\mathcal{O}_{X,x}$ of X at x. They are integer-valued functions on the set of nonnegative integers and defined for $j=0$ by

$$H_x^{(0)}(X)(l) := \dim_{\kappa(x)}(\mathfrak{m}_{X,x})^l/(\mathfrak{m}_{X,x})^{l+1} \qquad \text{for } l\geqq 0,$$

where $\mathfrak{m}_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$ and $\kappa(x)$ is the residue field, and for $j\geq 1$ inductively by

$$H_x^{(j)}(X)(l) := \sum_{0\leq i\leq l} H_x^{(j-1)}(X)(i) \qquad \text{for } l\geq 0.$$

Let $\Pi\colon Z'\to Z$ be the blowing up along a center $Y\subset X$ *permissible* for X, i.e., Y is regular and X is normally flat along Y. The restriction of Π to the *strict transform* X' of X in Z' is the blowing up of X along Y. For a point x' of X' with $\Pi(x')=x$, we have the *first stability theorem* (Theorem 3.1), which asserts

$$H_x^{(j)}(X) \geq H_{x'}^{(j+d)}(X') \qquad \text{for all } j \geq 0,$$

that is,

$$H_x^{(j)}(X)(l) \geq H_{x'}^{(j+d)}(X')(l) \qquad \text{for all } l \geq 0 \text{ and } j \geq 0,$$

where $d := \text{tr. deg}_{\kappa(x)}\kappa(x')$ is the transcendence degree of the residue field extension.

Following Giraud [G_2], we call x' an *infinitely near point* of x if the equality holds for some (hence any) $j \geq 0$. The study of these higher order Hilbert-Samuel functions was started by Bennett [B] and improved by Hironaka [H_4], Singh [S_1], [S_2], [S_3], Giraud [G_1], [G_2] and Herrmann-Orbanz [HO] into its present form.

When X is a hypersurface in Z, the multiplicity of X at x gives the same information as the Hilbert-Samuel functions. In the general case, Hironaka [H_1], [H_4] introduced the *second* numerical invariant to be a countable sequence

$$\nu_x^*(X, Z) = (\nu(1), \nu(2), \cdots, \nu(r), \infty, \infty, \cdots)$$

of nonnegative integers $\nu(1) \leq (2) \leq \cdots \leq \nu(r)$ and ∞. Pick a *standard base* $\{g_1, \cdots, g_r\}$ for the $\mathcal{O}_{Z,x}$-ideal J with $\mathcal{O}_{X,x} = \mathcal{O}_{Z,x}/J$ and let $\nu(j)$ be the $\mathfrak{m}_{Z,x}$-adic order of g_j (cf. Sections 2 and 3).

As we recall in Theorem 3.2, [H_4] showed that

$$\nu_x^*(X, Z) \geq \nu_{x'}^*(X', Z')$$

in the lexicographic order (see also [S_3] and [HO]). Moreover, the equality holds if and only if x' is an infinitely near point of x.

The *third* numerical invariant is the nonnegative integer

$$\tau_x(X, Z) := \text{tr. deg}_{\kappa(x)}\mathfrak{A}_x(X, Z),$$

the transcendence degree over $\kappa(x)$ of the algebra $\mathfrak{A}_x(X, Z)$ defined as follows (cf. Sections 1 and 3): the associated graded ring

$$\text{gr}_x(Z) := \bigoplus_{i \geq 0} (\mathfrak{m}_{Z,x})^i / (\mathfrak{m}_{Z,x})^{i+1}$$

is a polynomial ring over $\kappa(x)$, say $\text{gr}_x(Z) = \kappa(x)[z_1, \cdots, z_n]$. An *additive form* h in $\text{gr}_x(Z)$ is a polynomial of the form

$$h = a_1 z_1^{p^e} + a_2 z_2^{p^e} + \cdots + a_n z_n^{p^e}$$

for elements $a_1, \cdots, a_n$ in $\kappa(x)$ and a nonnegative integer e, where p is the *characteristic exponent* of $\kappa(x)$. The *initial ideal* $\text{in}_x(X, Z)$ is defined to be the kernel of the canonical surjection $\mathrm{gr}_x(Z)\to\mathrm{gr}_x(X)$. Then $\mathfrak{A}_x(X, Z)$ is the *smallest* $\kappa(x)$-subalgebra of $\mathrm{gr}_x(Z)$ generated over $\kappa(x)$ by additive forms such that $\mathrm{in}_x(X, Z)$ is generated as an ideal by the intersection $\mathrm{in}_x(X, Z)\cap\mathfrak{A}_x(X, Z)$. In characteristic zero, we have $p=1$, hence additive forms are linear forms. Thus $\tau_x(X, Z)$ is the smallest number of variables in $\mathrm{gr}_x(Z)$ necessary to describe the ideal $\mathrm{in}_x(X, Z)$ as defined in [H_1].

Giraud [G_2] showed that if x' is an infinitely near point of x, then the *second stability theorem* (cf. Theorem 3.3) holds, i.e.,

$$\tau_x(X, Z)\leq\tau_{x'}(X', Z').$$

Since his concern was slightly different from ours here, he imposed the condition to the effect that $\mathcal{O}_{Z,x}$ is a *complete* local ring *containing a field.* As we see in Section 4, we can modify his proof and find these restrictions to be unnecessary in our situation.

Following Giraud [G_2], we call x' an *infinitely very near point* of x if x' is an infinitely near point of x and if, moreover, the equality $\tau_x(X, Z)=\tau_{x'}(X', Z')$ holds, i.e., *all three available numerical invariants stay the same* at x'.

As we see in Theorems 3.5 and 3.6 as well as Corollary 3.7, we can choose a standard base for the initial ideal $\mathrm{in}_x(X, Z)$ and one for $\mathrm{in}_{x'}(X', Z')$ related with each other in the following nice manner (6), if x' is an *infinitely very near point* of x. (Compare this result with [H_4, Proposition 21] which deals with the case of infinitely near points and which uses the Hironaka group scheme.)

(1) Let us denote

$$\nu_x^*(X, Z)=\nu_{x'}^*(X', Z')=:(\nu(1), \cdots, \nu(r), \infty, \infty, \cdots)$$
$$\tau_x(X, Z)=\tau_{x'}(X', Z')=:\tau.$$

Let us choose a regular system of parameters $(u_0, \cdots, u_n, v_1, \cdots, v_s)$ of $\mathfrak{m}_{Z,x}$ so that $\mathcal{O}_{Y,x}=\mathcal{O}_{Z,x}/(u_0, \cdots, u_n)\mathcal{O}_{Z,x}$ and that $(u_0, \cdots, u_n)\mathcal{O}_{Z',x'}=u_0\mathcal{O}_{Z',x'}$. Hence $\mathfrak{m}_{Z,x}\mathcal{O}_{Z',x'}=(u_0, v_1, \cdots, v_s)\mathcal{O}_{Z',x'}$.

(2) If we denote by $y_i:=\mathrm{in}_x(u_i, Z)$ the $\mathfrak{m}_{Z,x}$-adic initial form of u_i for $0\leq i\leq n$, then we have canonically

$$\kappa(x)\otimes_{\mathcal{O}_Y}\mathrm{gr}_Y(Z)=\kappa(x)[y_0, \cdots, y_n]\subset\mathrm{gr}_x(Z),$$

and the left hand side contains $\mathfrak{A}_x(X, Z)$. We can find algebraically independent additive forms $h_1, \cdots, h_\tau$ of degrees $q(1)=p^{e(1)}, \cdots, q(\tau)=p^{e(\tau)}$ with $e(1)\leq e(2)\leq\cdots\leq e(\tau)$ in the variables $y_0, \cdots, y_n$ such that

$$\mathfrak{A}_x(X, Z)=\kappa(x)[h_1, \cdots, h_\tau].$$

(3) $\mathcal{O}_{Z',x'}/\mathfrak{m}_{Z,x}\mathcal{O}_{Z',x'}$ is the local ring of $\Pi^{-1}(x)$ at its point x' and coincides with the subring consisting of the homogeneous elements of degree zero in the localization of $\kappa(x)[y_0, \cdots, y_n]$ with respect to a homogeneous prime ideal not containing y_0. The canonical surjective homomorphism $\mathrm{gr}_{x'}(Z') \to \mathrm{gr}_{x'}(\Pi^{-1}(x))$ has kernel generated by the $\mathfrak{m}_{Z',x'}$-adic initial forms $\mathrm{in}_{x'}(u_0, Z')$, $\mathrm{in}_{x'}(v_1, Z')$, $\cdots$, $\mathrm{in}_x(v_s, Z')$, which are linearly independent linear forms in $\mathrm{gr}_{x'}(Z')$. The above surjection turns out to induce an isomorphism of $\kappa(x')$-algebras

$$\mathfrak{A}_{x'}(X', Z') \xrightarrow{\sim} \mathfrak{A}_{x'}(X' \cap \Pi^{-1}(x), \Pi^{-1}(x)).$$

(4) For each $1 \leq i \leq \tau$, the initial form

$$h_i'' := \mathrm{in}_{x'}(h_i/y_0^{q(i)}, \Pi^{-1}(x))$$

turns out to be an additive form of degree $q(i)$ in $\mathrm{gr}_{x'}(\Pi^{-1}(x))$ with coefficients in the subfield $\kappa(x)F^{e(i)}(\kappa(x'))$ of $\kappa(x')$, where $F^{e(i)}$ is the $q(i)$-th power Frobenius map. Moreover, we have

$$\mathfrak{A}_{x'}(X' \cap \Pi^{-1}(x), \Pi^{-1}(x)) = \kappa(x')[h_1'', \cdots, h_\tau''].$$

(5) For each $1 \leq i \leq \tau$, there exists a unique additive form h_i' in $\mathrm{gr}_{x'}(Z')$ mapped to h_i'' under $\mathrm{gr}_{x'}(Z') \to \mathrm{gr}_{x'}(\Pi^{-1}(x))$ such that

$$\mathfrak{A}_{x'}(X', Z') = \kappa(x')[h_1', \cdots, h_\tau'].$$

(6) There exist $\kappa(x)$-coefficient polynomials $\psi_1, \cdots, \psi_r$ in τ variables of weights $q(1), \cdots, q(\tau)$ such that ψ_j is isobaric of weight $\nu(j)$ for $1 \leq j \leq r$ satisfying the following properties:

(i) The subset $\{\psi_j(h_1, \cdots, h_\tau);\ 1 \leq j \leq r\}$ of $\mathfrak{A}_x(X, Z)$ is a standard base for $\mathrm{in}_x(X, Z)$.

(ii) The subset $\{\psi_j(h_1'', \cdots, h_\tau'');\ 1 \leq j \leq r\}$ of $\mathfrak{A}_{x'}(X' \cap \Pi^{-1}(x))$ is a standard base for $\mathrm{in}_{x'}(X' \cap \Pi^{-1}(x), \Pi^{-1}(x))$.

(iii) The subset $\{\psi_j(h_1', \cdots, h_\tau');\ 1 \leq j \leq r\}$ of $\mathfrak{A}_{x'}(X', Z')$ is a standard base of $\mathrm{in}_{x'}(X', Z')$.

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§ 1. Homogeneous subgroup schemes and the ridge of a cone

In this section, we collect together basic facts on homogeneous subgroup schemes of a fixed vector group V. The ridge of a cone in V is an example of homogeneous subgroup schemes of V and will play an important role in the resolution of singularities, as we see in Sections 3 and 4. Other examples are Hironaka group schemes associated to homogeneous prime ideals of S introduced in [H_3], and their higher order versions studied in [O_2] and [O_3].

We fix a field k of *characteristic exponent* p, i.e., $p=1$ if k is of characteristic zero, while p is the characteristic of k if the latter is a nonzero prime. Let $S=k[z_1, \cdots, z_n]$ be a polynomial ring in n variables.

For each nonnegative integer ν, we denote by S_ν the k-subspace consisting of homogeneous polynomials of degree ν so that $S=\bigoplus_{\nu\geq 0} S_\nu$ is naturally a graded ring. An *additive form* in S is a homogeneous polynomial of the form

$$h=a_1z_1^{p^e}+a_2z_2^{p^e}+\cdots+a_nz_n^{p^e}$$

with elements $a_1, \cdots, a_n$ in k and a nonnegative integer e. For each nonnegative integer e, we denote by L_e the k-subspace in S_{p^e} of additive forms of degree p^e.

We denote by F the p-th power *Frobenius map*. Then the direct sum

$$L:=\bigoplus_{e\geq 0} L_e$$

is naturally a (left) graded module over the *twisted polynomial ring* $k[F]$ over k in F with the relation $Fa=a^pF$ for each a in k. Here we regard $aF^e \in k[F]$ to be of degree e for each a in k and for each nonnegative integer e.

$V:=\operatorname{Spec}(S)$ is naturally an n-dimensional vector group scheme over k, i.e., a group scheme under the usual addition, together with the usual scalar multiplication action of the one-dimensional multiplicative group scheme G_m. $k[F]$ can be naturally identified with the ring of endomorphisms $\operatorname{End}(G_a)$ of the one-dimensional additive group scheme G_a, where F acts as the Frobenius endomorphism. Then L can be naturally identified with the $k[F]$-module $\operatorname{Hom}(V, G_a)$ consisting of the homomorphisms of group schemes from V to G_a. A *homogeneous subgroup scheme* G of V is a closed subgroup scheme of V stable under the scalar action of G_m. Thus for an ideal J of S, the subscheme $G=\operatorname{Spec}(S/J)$ is a homogeneous subgroup scheme if and only if J is a homogeneous ideal generated by a finite number of additive forms, or equivalently, J is generated by $J\cap L$.

The subring S^G of elements in S invariant under the translation action of G on S is a graded k-subalgebra of S and $\mathrm{Spec}(S^G)$ is the quotient group scheme V/G. Note that S^G is the graded k-subalgebra generated over k by

$$J \cap L = L^G = \mathrm{Hom}(V/G, G_a),$$

where $L^G = L \cap S^G$ and the extreme right hand side is the set of homomorphisms of group schemes from V/G to G_a naturally regarded as a graded $k[F]$-module.

By $G \mapsto L^G$ and $G \mapsto S^G$, we have one-to-one correspondences among homogeneous subgroup schemes in V, graded $k[F]$-submodules of L and *certain* graded k-subalgebras of S, which we now characterize and which will play important roles later.

For that purpose, denote by $\mathrm{Hyp}(V)$ the *hyperalgebra* of V, i.e., the set of invariant differential operators of finite order from S into itself over k. In down-to-earth terms, consider the *Taylor expansion* of $f \in S$

$$f(z+z^*) = \textstyle\sum_l (\partial_l f)(z)(z^*)^l$$

with $l = (l_1, \cdots, l_n)$ running through n-tuples of nonnegative integers, where for a new set of variables $z_1^*, \cdots, z_n^*$ we let $(z^*)^l = (z_1^*)^{l_1}(z_2^*)^{l_2} \cdots (z_n^*)^{l_n}$ in the usual multi-index convention. Then $f \mapsto \partial_l f$ is a differential operator from S into itself over k of order $|l| := l_1 + \cdots + l_n$ and is invariant with respect to the additive group law. We have the *composition law*

$$\partial_l \circ \partial_{l'} = (l, l')\partial_{l+l'}$$

with the multi-binomial coefficient

$$(l, l') := \textstyle\prod_{1 \le i \le n} (l_i + l'_i)!/(l_i!)(l'_i!),$$

and $l + l' := (l_1 + l'_1, \cdots, l_n + l'_n)$. For f and g in S we have *Leibnitz's rule*

$$\partial_l(fg) = \textstyle\sum_{l'+l''=l} (\partial_{l'} f)(\partial_{l''} g).$$

$\mathrm{Hyp}(V)$ is then the k-vector space consisting of finite k-linear combinations of $\{\partial_l;$ all n-tuples $l\}$ with the multiplication and comultiplication defined respectively by the composition and Leibnitz's rules.

The following characterization is of utmost importance later:

Lemma 1.1 (Hironaka [H_3, Corollary 2.3] and Giraud [G_1, Corollaire I.5.5]). *For a graded k-subalgebra U of S, the following are equivalent*:

(i) *There exists a (necessarily unique) homogeneous subgroup scheme G of V such that $U = S^G$.*

(ii) *U is generated as a k-subalgebra by additive forms.*
(iii) *U is stable under the action of* Hyp (V) *on S.*

Proof. (ii) implies (i), if we let $G=\mathrm{Spec}\,(S/J)$, where J is the ideal generated by $U\cap L$. (i) obviously implies (iii), since $\mathrm{Spec}\,(S^G)=V/G$.

As for the implication from (iii) to (ii), consider the k-subalgebra U' of U generated by $U\cap L$. As in [H_3, Corollary 2.3] and [G_1, Lemme I.5.4.3], we can easily show $U'=U$ by looking at the leading exponents of monomials in $z_1, \cdots, z_n$ with respect to the lexicographic order. q.e.d.

Remark. (cf. Hironaka [H_3, (1.2)] and Giraud [G_1, Proposition I.5.4]). For a homogeneous subgroup scheme G in V, we can linearly change the variables in S so that

$$S^G=k[h_1, \cdots, h_\tau]$$

with additive forms in "triangular shape"

$$h_i=z_i^{p^{e(i)}}+\sum_{i+1\leq j\leq n} a_{ij}z_j^{p^{e(i)}}, \quad i=1, \cdots, \tau,$$

where a_{ij} are elements of k and $e(1)\leq e(2)\leq \cdots \leq e(\tau)$ are nonnegative integers. In particular, S^G is purely transcendental over k and S is flat over S^G. Since S^G is generated by L^G as a k-algebra, we have only to note that L^G is $k[F]$-free and to choose a basis as $h_1, \cdots, h_\tau$. Note that $\tau=\dim V/G=\mathrm{tr.\,deg}_k S^G$.

An important example of homogeneous subgroup schemes in V arises as the ridge of a cone as follows:

A cone $C=\mathrm{Spec}\,(S/I)$ in V is a closed subscheme defined by a homogeneous ideal I of S. The *ridge* (faîte in French, due to P. Gabriel, cf. [G_1, I.3]) of C is the largest subgroup scheme $A=A(C)$ in V such that $A+C\subset C$. Namely, A is defined as a subgroup functor of V by

$$A(k'):=\{v'\in V(k');\ v'+(C\times_k k')\subset(C\times_k k')\}$$

for all k-algebras k'. It is easily shown to be representable by a homogeneous subgroup scheme of V, as we see in Proposition 1.2 below. The condition $A+C\subset C$ means that I is generated as an S-ideal by a finite number of elements of positive degree in S^A. On the other hand, S^A is generated as a k-algebra by a finite number of elements in L^A. Thus the consideration of the ridge of C amounts to that of the *smallest* k-subalgebra $k[h_1, \cdots, h_\tau]$ of S generated by additive forms $h_1, \cdots, h_\tau$ such that I is generated as an S-ideal by $I\cap k[h_1, \cdots, h_\tau]$. Note that $A(C)$ depends only on C and is independent of the particular embedding in a linear space V. However, $S^{A(C)}$ does depend on V, since $V/A(C)=\mathrm{Spec}\,(S^{A(C)})$.

Definition. Let I be a homogeneous ideal of S. A *standard base* of I is a minimal system of generators $\{f_1, \cdots, f_r\}$ such that each f_j is homogeneous and that

$$\deg f_1 \leq \deg f_2 \leq \cdots \leq \deg f_r.$$

A standard base $\{f_1, \cdots, f_r\}$ of I is said to be *normalized* with respect to the variables $z=(z_1, \cdots, z_n)$, if for each $1 \leq j \leq r$, the coefficient of f_j with respect to the monomial z^l vanishes when l is in the exponent $\exp((f_1, \cdots, f_{j-1}))$ of the ideal $(f_1, \cdots, f_{j-1})$. Here, the *exponent* $\exp(f) = \exp(f; z)$ of a nonzero homogeneous element f of S is the n-tuple l, if the monomial z^l appears in f with nonzero coefficient and if l is the greatest among such with respect to the lexicographic order of n-tuples. Then the *exponent* of a homogeneous S-ideal J is defined to be the set

$$\exp(J) = \exp(J; z) := \{\exp(f);\ \text{nonzero homogeneous } f \text{ in } J\}.$$

Remark. For a homogeneous ideal I of S, the sequence

$$\nu^*(I) := (\deg f_1, \deg f_2, \cdots, \deg f_r, \infty, \infty, \cdots)$$

is known to be independent of the choice of a standard base of I (cf. [H_1, Chapter III, § 1, Lemma 1, p. 205]). A standard base of I exists and can be easily modified to become normalized with respect to $(z_1, \cdots, z_n)$. For a homogeneous S-ideal J, its exponent $\exp(J)$ is a semigroup ideal in the semigroup of n-tuples of nonnegative integers, i.e., if l is in $\exp(J)$, then $l+m$ is in $\exp(J)$ for each n-tuple m of nonnegative integers.

Proposition 1.2 (Giraud [G_1, Propositions I.5.3 and III.2.10] and [G_2, Lemme 1.6]). *Let $C = \operatorname{Spec}(S/I)$ be a cone in V. Then the ridge $A(C)$ of C exists and is a homogeneous subgroup scheme of V satisfying the following properties*:

(i) *If a standard base $\{f_1, \cdots, f_r\}$ of I is normalized with respect to the variables $z=(z_1, \cdots, z_n)$ of S, then it is* quasi-normalized *with respect to z in the sense that*

$$\partial_l f_j = 0$$

for all $1 \leq j \leq r$ and all $l \in \exp(I; z)$ with $|l| < \deg f_j$, where ∂_l is the Taylor coefficient differential operator with respect to z for each n-tuple l of nonnegative integers.

(ii) *If a standard base $\{f_1, \cdots, f_r\}$ of I is quasi-normalized with respect to z, then $S^{A(C)}$ is the k-algebra of S generated by*

$$\{\partial_l f_j;\ 1 \leq j \leq r,\ l \text{ not in } \exp(I; z),\ |l| < \deg f_j\},$$

hence by $\mathrm{Hyp}(V)f_1+\cdots+\mathrm{Hyp}(V)f_r$, *where* $\mathrm{Hyp}(V)$ *is the hyperalgebra of invariant differential operators on* V *with the* k*-basis* $\{\partial_l$; all n-tuples l of nonnegative integers$\}$.

(iii) *In particular, any standard base of* I *quasi-normalized with respect to any set of variables of* S *is contained in* $S^{A(C)}$.

Proof. (i) Let $c_j(m)$ be the coefficient of f_j with respect to z^m. Then

$$(\partial_l f_j)(z)=\textstyle\sum_m (l, m)c_j(l+m)z^m,$$

where the summation is over all n-tuples m of nonnegative integers. If l is in $\exp(I; z)$ with $|l|<\deg f_j$, then l, hence $l+m$ for any m, are in $\exp((f_1, \cdots, f_{j-1}); z)$, since $\{f_1, \cdots, f_r\}$ is a standard base of I. Thus $c_j(l+m)=0$ for all m, since $\{f_1, \cdots, f_r\}$ is normalized with respect to z.

(ii) $A=A(C)$ is the subgroup functor of V defined by $A(k')=\{a\in V(k')\cong(k')^n;\ f_j(a+z)-f_j(z)\in k'\otimes_k I,\ 1\leq j\leq r\}$ for all k-algebras k'. Let $S^*=k[z_1^*, \cdots, z_n^*]$ and its homogeneous ideal I^* be copies of S and I, respectively. We then have

$$f_j(z+z^*)-f_j(z^*)=\textstyle\sum_{|l|<\deg f_j}(\partial_l f_j)(z)(z^*)^l$$

for $1\leq j\leq r$, where $z^*=(z_1^*, \cdots, z_n^*)$. Obviously, the classes of $\{(z^*)^l;\ l$ not in $\exp(I^*; z^*)\}$ modulo I^* form a k-basis of S^*/I^*. The image of $f_j(z+z^*)-f_j(z^*)$ in $S\otimes_k(S^*/I^*)$ coincides with that of

$$\textstyle\sum_l (\partial_l f_j)(z)(z^*)^l,$$

the summation being over all n-tuples l not in $\exp(I^*; z^*)$ with $|l|<\deg f_j$. Thus A is represented by the closed subscheme of V defined by the S-ideal generated by $P:=\{\partial_l f_j;\ 1\leq j\leq r,\ l$ not in $\exp(I; z)$ with $|l|<\deg f_j\}$, hence by $P':=\{\partial_l f_j;\ 1\leq j\leq r,\ |l|<\deg f_j\}$, since $\{f_1, \cdots, f_r\}$ is quasi-normalized with respect to z. However, P' is contained in S^A, since $(\partial_l f_j)(z+z^*)-(\partial_l f_j)(z^*)=\sum_{|m|<\deg f_j-|l|}(l, m)(\partial_{l+m}f_j)(z)(z^*)^m$, the right hand side of which vanishes on A. Since $\partial_l f_j$ is in k if $|l|\geq\deg f_j$, we conclude that S^A contains $P'':=\mathrm{Hyp}(V)f_1+\cdots+\mathrm{Hyp}(V)f_r$. The elements of positive degree in P'' generate the S-ideal defining A and P'' is stable under the action of $\mathrm{Hyp}(V)$. Hence by Lemma 1.1, S^A is the k-subalgebra generated by P''. q.e.d.

The following improvement of Proposition 1.2 due to Giraud will play an important role in Section 4:

Let $A(C)$ be the ridge of a cone $C=\mathrm{Spec}(S/I)$. As we saw in the remark after Lemma 1.1, we can choose variables in S so that

$$S=k[\xi_1, \cdots, \xi_\tau, \eta_1, \cdots, \eta_\sigma], \quad S^{A(C)}=k[h_1, \cdots, h_\tau]$$

with additive forms in triangular shape

$$\begin{aligned} h_i &= h_i(\xi, \eta) = a_i(\xi) + b_i(\eta), \\ a_i(\xi) &= \xi_i^{q(i)} + \textstyle\sum_{i<j\leq\tau} a_{ij}\xi_j^{q(i)}, \\ b_i(\eta) &= \textstyle\sum_{1\leq j\leq\sigma} b_{ij}\eta_j^{q(i)}, \end{aligned}$$

for $1\leq i\leq\tau$, where a_{ij} and b_{ij} are in k and $q(i)=p^{e(i)}$ with nonnegative integers $e(1)\leq e(2)\leq\cdots\leq e(\tau)$. For simplicity, let us denote $\xi:=(\xi_1, \cdots, \xi_\tau)$, $\eta:=(\eta_1, \cdots, \eta_\sigma)$, $h:=(h_1, \cdots, h_\tau)$, $a:=(a_1, \cdots, a_\tau)$ and $b:=(b_1, \cdots, b_\sigma)$. Consider the polynomial subring $S^\#:=k[\eta]$ of S and the polynomial residue ring $S':=S/(\eta_1, \cdots, \eta_\sigma)=k[\xi']$, where $\xi':=(\xi'_1, \cdots, \xi'_\tau)$. Here the projection $S\to S'$ sends $f(\xi, \eta)$ in S to $f(\xi', 0)$ in S'. We denote by I' the image of I in S'. For each τ-tuple μ (resp. σ-tuple ν) of nonnegative integers, let us denote by ξ^μ, ∂'_μ (resp. η^ν, ∂''_ν) the corresponding monomials in ξ (resp. in η) as well as the Taylor coefficient differential operator with respect to ξ^μ (resp. with respect to η^ν).

Corollary 1.3 (Giraud [G_2, Lemme 1.7, 3.3.2 and 3.3.3]. *In the above notations, we have the following for the cone* $C=\mathrm{Spec}\,(S/I)$:

(i) *C is flat over* $\mathrm{Spec}\,(S^\#)$ *so that* $\{\eta_1, \cdots, \eta_\sigma\}$ *is a regular sequence for S/I. The exponent* $\exp(I; \xi, \eta)$ *of I with respect to* (ξ, η) *consists of all* (μ, ν), *where ν runs through all σ*-tuples *of nonnegative integers, while μ runs through the exponent* $\exp(I'; \xi')$ *of the S'-ideal I' with respect to ξ'. Moreover, if* $\{f_1, \cdots, f_r\}$ *is a standard base of I, then* $\{f_1(\xi', 0), \cdots, f_r(\xi', 0)\}$ *is a standard base of I'.*

(ii) *Suppose* $\{f_1, \cdots, f_r\}$ *is a standard base of I and is contained in* $S^{A(C)}$. *Then* $\{f_1, \cdots, f_r\}$ *is quasi-normalized with respect to* (ξ, η) *in the sense of Proposition* 1.2 *if and only if* $\partial'_m f_j=0$ *for all* $1\leq j\leq r$ *and all* $m\in\exp(I'; \xi')$ with $|m|<\deg f_j$.

(iii) *If* $\{f_1, \cdots, f_r\}$ *is a standard base of I quasi-normalized with respect to* (ξ, η), *then* $S^{A(C)}$ *is generated as a k-algebra by*

$$\{\partial'_m f_j;\ 1\leq j\leq r,\ \tau\text{-tuples } m \text{ not in } \exp(I'; \xi') \text{ with } |m|<\deg f_j\}.$$

Moreover, $\{f_1(\xi', 0), \cdots, f_r(\xi', 0)\}$ *is a standard base of I' quasi-normalized with respect to ξ', and the ridge $A(C')$ of the cone* $C':=\mathrm{Spec}\,(S'/I')$ *satisfies*

$$(S')^{A(C')}=k[h_1(\xi', 0), \cdots, h_\tau(\xi', 0)].$$

Proof. (i) Recall that $V:=\mathrm{Spec}\,(S)$ and $A:=A(C)$. Then the closed embedding $C/A=\mathrm{Spec}\,(S^A/(I\cap S^A))\to V/A=\mathrm{Spec}\,(S^A)$ gives rise to the equality $C=V\times_{(V/A)}(C/A)=\mathrm{Spec}\,(S/(I\cap S^A)S)$. The restriction $C\to V^{\#}:=\mathrm{Spec}\,(S^{\#})$ of the canonical morphism $V\to V^{\#}$ decomposes as

$$C=V\times_{(V/A)}(C/A)\to\{V^{\#}\times(V/A)\}\times_{(V/A)}(C/A)=V^{\#}\times(C/A)\to V^{\#}.$$

The first morphism is flat, since it is the base extension by $C/A\to V/A$ of the canonical morphism $V=\mathrm{Spec}\,(k[\xi,\eta])\to V^{\#}\times(V/A)=\mathrm{Spec}\,(k[\eta,h])$, which is obviously flat in view of the triangular shape of h. The second morphism is also flat, since it is the base extension by $V^{\#}\to\mathrm{Spec}\,(k)$ of $C/A\to\mathrm{Spec}\,(k)$. Consequently, $C\to V^{\#}$ is flat, i.e., S/I is $S^{\#}$-flat, and the regular sequence $\{\eta_1,\cdots,\eta_\sigma\}$ for $S^{\#}$ remains one for S/I. Hence we see that the kernel of $I\to I'$ coincides with $(\eta_1,\cdots,\eta_\sigma)I$. Then for $\exp(f')\in\exp(I';\xi')$ with a nonzero homogeneous $f'\in I'$, there exists a homogeneous f in I such that $f'=f(\xi',0)$ and that $\exp(f)=(\exp(f'),0)$. Therefore, $(\exp(f'),\nu)$ is in $\exp(I;\xi,\eta)$ for all ν. As we see in Corollary 2.4, $\{f_1(\xi',0),\cdots,f_r(\xi',0)\}$ is a standard base of I' if so is $\{f_1,\cdots,f_r\}$ for I, since $\{\eta_1,\cdots,\eta_\sigma\}$ is a regular sequence for S/I.

(ii) Suppose $f_1,\cdots,f_r$ are elements of $S^{A(C)}=k[h_1,\cdots,h_\tau]$. For $1\le j\le r$, τ-tuples μ and σ-tuples ν, we claim that $\partial'_\mu\partial''_\nu f_j$ is a k-linear combination of $\partial'_m f_j$, where m runs through τ-tuples of nonnegative integers satisfying $|m|=|\mu|+|\nu|$. We would then get (ii), since $(m,0)$ is in $\exp(I;\xi,\eta)$ if and only if m is in $\exp(I';\xi')$ by (i).

To prove the above claim, let us introduce a new set of variables $\xi^*=(\xi_1^*,\cdots,\xi_\tau^*)$ and $\eta^*=(\eta_1^*,\cdots,\eta_\sigma^*)$. We have $a(\xi+\xi^*)=a(\xi)+a(\xi^*)$ and $b(\eta+\eta^*)=b(\eta)+b(\eta^*)$, since a_i and b_i are additive forms. There exist polynomials $\psi_1,\cdots,\psi_r$ in τ variables $h=(h_1,\cdots,h_\tau)$ such that

$$f_j(\xi,\eta)=\psi_j(h(\xi,\eta))\qquad\text{for } 1\le j\le r.$$

Thus we have

$$\begin{aligned}f_j(\xi+\xi^*,\eta+\eta^*)&=\psi_j(h(\xi,\eta)+a(\xi^*)+b(\eta^*))\\&=\textstyle\sum_\lambda(D_\lambda\psi_j)(h(\xi,\eta))\{a(\xi^*)+b(\eta^*)\}^\lambda\end{aligned}$$

with $\lambda=(\lambda_1,\cdots,\lambda_\tau)$ running through τ-tuples of nonnegative integers, where $\{a(\xi^*)+b(\eta^*)\}^\lambda:=\prod_{1\le i\le\tau}(a_i(\xi^*)+b_i(\eta^*))^{\lambda_i}$ and where $\psi_j\mapsto D_\lambda\psi_j$ is the Taylor coefficient differential operator for τ variables h. Let us denote $\lambda q:=(\lambda_1 q(1),\cdots,\lambda_\tau q(\tau))$ for each τ-tuple λ. Then we have

$$\{a(\xi^*)+b(\eta^*)\}^\lambda=\textstyle\sum_{\mu,\nu}c_{\lambda,\mu,\nu}(\xi^*)^\mu(\eta^*)^\nu$$

with $c_{\lambda,\mu,\nu}\in k$, where μ and ν run through τ-tuples and σ-tuples of nonnegative integers such that $|\lambda q|=|\mu|+|\nu|$. Since $a(\xi^*)$ is in triangular shape, we see that $C_{\lambda,\lambda q,0}=1$ and that $c_{\lambda,\mu,\nu}=0$ if μ is lexicographically greater than λq. Comparing the coefficients with those of the Taylor expansion of $f_j(\xi, \eta)$, we have

$$(\partial'_\mu\partial''_\nu f_j)(\xi, \eta)=\sum_{|\lambda q|=|\mu|+|\nu|} c_{\lambda,\mu,\nu}(D_\lambda\psi_j)(h(\xi, \eta)),$$

for each τ-tuple μ and σ-tuple ν, where λ runs through τ-tuples satisfying $|\lambda q|=|\mu|+|\nu|$. If we let $\eta^*=0$, then

$$(\partial'_m f_j)(\xi, \eta)=\sum_{|\lambda q|=|m|} c_{\lambda,m,0}(D_\lambda\psi_j)(h(\xi, \eta))$$

for each τ-tuple m. Since $c_{\lambda,\lambda q,0}=1$ and $c_{\lambda,m,0}=0$ if m is lexicographically greater than λq, we can solve each $(D_\lambda\psi_j)(h(\xi, \eta))$ as a k-linear combination of $(\partial'_m f_j)(\xi, \eta)$ with m running through τ-tuples of nonnegative integers satisfying $|m|=|\lambda q|$. Hence the above claim is verified.

(iii) is obvious by (i) and (ii). q.e.d.

Remark. To a homogeneous prime ideal $\mathfrak{p}$ of S is associated the Hironaka subgroup scheme $B(\mathfrak{p})$ of V as follows (cf. Hironaka [H$_3$]): For each nonnegative integer ν, let $U_\nu(\mathfrak{p})$ be the subset of S_ν consisting of zero and all those nonzero $f\in S_\nu$ for which the hypersurface Proj (S/Sf) has multiplicity ν at $\mathfrak{p}$ regarded as a point of Proj (S). By the Jacobian criterion (cf. Oda [O$_1$, Proposition 2.2 (i)]), $f\in S_\nu$ is in $U_\nu(\mathfrak{p})$ if and only if

$$\mathrm{Diff}_{\nu-1}(S)f\subset\mathfrak{p},$$

where $\mathrm{Diff}_{\nu-1}(S)$ is the set of absolute (i.e., over the ring of integers) differential operators of S into itself of order at most $\nu-1$. Then $U(\mathfrak{p}):=\bigoplus_{\nu\geq 0} U_\nu(\mathfrak{p})$ is easily seen to be a k-subalgebra of S stable under Hyp (V). Hence by Lemma 1.1, there exists a unique homogeneous subgroup scheme $B(\mathfrak{p})$, the *Hironaka subgroup scheme associated to* $\mathfrak{p}$, in V such that $U(\mathfrak{p})=S^{B(\mathfrak{p})}$. [O$_3$, Proposition 4.1] associates to $\mathfrak{p}$ the sequence of *higher order Hironaka subgroup schemes.*

The following gives information on the ridge of the intersection of a cone with a linear subspace:

Lemma 1.4 ([Giraud [G$_1$, Corollaire I.6.9.2] and [G$_2$, Lemme 5.5.2]). *Let* $V'=\mathrm{Spec}\,(S')$ *be a codimension d linear subspace of* V, i.e., S' *is the residue ring of S modulo an ideal generated by d linearly independent linear forms. For a cone* $C=\mathrm{Spec}\,(S/I)$ *in* V, *let* $A=A(C)$ *be its ridge and let* $A'=A(C')$ *be the ridge of* $C':=V'\cap C=\mathrm{Spec}\,(S'/I')$. *Then* $\dim V/A\geq \dim V'/A'$ *holds and the transcendence degree over k satisfies*

$$\mathrm{tr.\,deg}_k\, S^A\geq \mathrm{tr.\,deg}_k (S')^{A'}.$$

Moreover, the following are equivalent:

(i) $\dim V/A = \dim V'/A'$.

(ii) *The equality* $\mathrm{tr.\,deg}_k S^A = \mathrm{tr.\,deg}_k (S')^{A'}$ *holds.*

(iii) *The closed embedding* $V' \to V$ *induces an isomorphism of quotient group schemes* $V'/A' \xrightarrow{\sim} V/A$.

(iv) *The canonical surjection* $S \to S'$ *induces an isomorphism* $S^A \xrightarrow{\sim} (S')^{A'}$, *which necessarily sends additive forms to additive forms and the ideal* $I \cap S^A$ *to* $I' \cap (S')^{A'}$.

Proof. The scheme-theoretic intersection $V' \cap A$ is contained in A' by their functorial definition. Thus we have a surjection α: $V'/(V' \cap A) \to V'/A'$, hence an injection α^*: $(S')^{A'} \to (S')^{V' \cap A}$. On the other hand, the canonical isomorphism $V'/(V' \cap A) = (V' + A)/A$ gives rise to a closed embedding β: $V'/(V' \cap A) \to V/A$, hence a surjection β^*: $S^A \to (S')^{V' \cap A}$. In particular, we have

$$\dim V/A \geq \dim V'/(V' \cap A) \geq \dim V'/A' \quad \text{and}$$

$$\mathrm{tr.\,deg}_k S^A \geq \mathrm{tr.\,deg}_k (S')^{V' \cap A} \geq \mathrm{tr.\,deg}_k (S')^{A'}.$$

(iii) holds if and only if $V = V' + A$ and $A' = V' \cap A$, while (iv) holds if and only if α^* and β^* are isomorphisms. Thus (iii) and (iv) are obviously equivalent and implies the obviously equivalent statements (i) and (ii). Suppose (ii) holds. Then S^A, $(S')^{V' \cap A}$ and $(S')^{A'}$ have the same transcendence degree over k. In particular, β^* is a surjection between finitely generated integral domains of the same transcendence degree over k, hence is an isomorphism and we get $V = V' + A$. Since $V' \cap A \subset A'$, we have the following inequalities for the Hilbert-Samuel functions of A, $V' \cap A$ and A' at the origin O (cf. Sections 2 and 3):

$$H_O^{(0)}(A) \leq H_O^{(d)}(V' \cap A) \leq H_O^{(d)}(A'),$$

where the first inequality is by the theorem of Bennett, Hironaka, Giraud and Singh we recall in the next section as Corollary 2.3, while the second inequality is obvious.

We now claim that $H_O^{(0)}(A) = H_O^{(d)}(A')$. Then it would follow that $H_O^{(d)}(V' \cap A) = H_O^{(d)}(A')$, which implies $V' \cap A = A'$. Together with the equality $V = V' + A$ above, we thus get (iii).

To show $H_O^{(0)}(A) = H_O^{(d)}(A')$, we may assume k to be *perfect*, since the formation of the ridge is compatible with base change. Then the "directrixes" $D := A_{\mathrm{red}} \subset V$ and $D' := A'_{\mathrm{red}} \subset V'$ are the largest linear subspace such that $D + C \subset C$ and $D' + C' \subset C'$, respectively. Obviously, we have $V' \cap D \subset D'$. Since $V = V' + A$ and $\dim (V' \cap A) = \dim A'$, we see that $V=V'+D$ and $V'\cap D=D'$. Hence we have an isomorphism $V'/D'\rightrightarrows V/D$, which induces a closed embedding $C'/D'\to C/D$. Thus for $r:=\dim D'$, we have

$$H_O^{(d+r)}(C'/D')\leq H_O^{(d+r)}(C/D).$$

The left hand side is equal to $H_O^{(d)}(C')$ again by Corollary 2.3, since C' is noncanonically isomorphic to the product $(C'/D')\times D'$. Similarly, the right hand side is equal to $H_O^{(0)}(C)$, since C is noncanonically isomorphic to $(C/D)\times D$ and $\dim D=d+r$. Since $C'=V'\cap C$ by definition, we have $H_O^{(0)}(C)\leq H_O^{(d)}(C')$ again by Corollary 2.3. Hence $H_O^{(d+r)}(C'/D')=H_O^{(d+r)}(C/D)$ and the closed embedding $C'/D'\to C/D$ is necessarily an isomorphism. Since $C'\cong(C'/D')\times D'$ and $C\cong(C/D)\times D\cong(C/D)\times D'\times(D/D')\cong C'\times(D/D')$, we see easily that A is noncanonically isomorphic to $A'\times(D/D')$. Hence $H_O^{(0)}(A)=H_O^{(d)}(A')$ once again by Corollary 2.3.

q.e.d.

§ 2. Hilbert-Samuel functions and series

In this section, we collect together results on the Hilbert-Samuel functions and series we use later. Of particular importance is the result due to Bennett, Hironaka, Giraud and Singh (Theorem 2.1 and Corollaries 2.3 and 2.4).

Throughout this section, we fix a Noetherian local ring $\mathfrak{O}$ with the maximal ideal $\mathfrak{M}$ and the residue field $\mathfrak{K}=\mathfrak{O}/\mathfrak{M}$.

Definition. The sequence $\{H_{\mathfrak{O}}^{(j)};\ j$ nonnegative integers$\}$ of the *Hilbert-Samuel functions* of the local ring $\mathfrak{O}$ is defined inductively as follows: the 0-th Hilbert-Samuel function is the integer-valued function on the set of nonnegative integers given by

$$H_{\mathfrak{O}}^{(0)}(l):=\dim_{\mathfrak{K}}(\mathfrak{M}^l/\mathfrak{M}^{l+1}),$$

for nonnegative integers l. For a positive integer j, the j-th Hilbert-Samuel function is an integer-valued function on the set of nonnegative integers determined from the $(j-1)$-st one by

$$H_{\mathfrak{O}}^{(j)}(l):=\textstyle\sum_{0\leq i\leq l} H_{\mathfrak{O}}^{(j-1)}(i) \qquad \text{for } l\geq 0.$$

The corresponding *Hilbert-Samuel series* are the formal power series in a variable t with integer coefficients defined by

$$h_{\mathfrak{O}}^{(j)}(t):=\textstyle\sum_{l\geq 0} H_{\mathfrak{O}}^{(j)}(l)t^l.$$

The Hilbert-Samuel functions and series give the same information, but one is sometimes more convenient than the other. Note that they depend only on the *associated graded ring* $\mathrm{gr}_{\mathfrak{M}}(\mathfrak{O}):=\bigoplus_{l\geq 0}\mathrm{gr}^l_{\mathfrak{M}}(\mathfrak{O})$, where $\mathrm{gr}^l_{\mathfrak{M}}(\mathfrak{O}):=\mathfrak{M}^l/\mathfrak{M}^{l+1}$. The first Hilbert-Samuel function

$$H^{(1)}_{\mathfrak{O}}(l)=\sum_{0\leq i\leq l}\dim_{\mathfrak{K}}(\mathfrak{M}^i/\mathfrak{M}^{i+1})=\mathrm{length}_{\mathfrak{O}}(\mathfrak{O}/\mathfrak{M}^{l+1})$$

is usually called just the Hilbert-Samuel function of $\mathfrak{O}$. We have

$$H^{(j)}_{\mathfrak{O}}(l)=H^{(j+1)}_{\mathfrak{O}}(l)-H^{(j+1)}_{\mathfrak{O}}(l-1)$$

for all $j\geq 0$ and $l\geq 1$. Thus $H^{(j)}_{\mathfrak{O}}$ is the difference of $H^{(j+1)}_{\mathfrak{O}}$, and $H^{(j)}_{\mathfrak{O}}$ is the "j-th order sum function" of $H^{(0)}_{\mathfrak{O}}$. As for the Hilbert-Samuel series, we have

$$h^{(j)}_{\mathfrak{O}}(t)=h^{(j-1)}_{\mathfrak{O}}(t)/(1-t)$$

for all $j\geq 1$. Hence

$$h^{(j)}_{\mathfrak{O}}(t)=h^{(0)}_{\mathfrak{O}}(t)/(1-t)^j \qquad \text{for } j\geq 0,$$

and we can use this to define $h^{(j)}_{\mathfrak{O}}$ also for negative j.

We need to compare below the Hilbert-Samuel functions and series of two different local rings. For that purpose, we introduce the following order as in [B], [H_4], [G_1], [G_2], [S_1], etc.

Definition. Let H and H' be integer-valued functions on the set of nonnegative integers. We define $H\geq H'$ to mean

$$H(l)\geq H'(l) \qquad \text{for all nonnegative integers } l.$$

For the corresponding formal power series $h(t):=\sum_{l\geq 0}H(l)t^l$ and $h'(t):=\sum_{l\geq 0}H'(l)t^l$, we also denote this fact by $h\geq h'$.

Remark. The above order $H\geq H'$ is only a partial order, but is stronger than the lexicographic order indexed by nonnegative integers l. $H\geq H'$ implies $H^{(j)}\geq H'^{(j)}$ for nonnegative integers j, where $H^{(j)}$ and $H'^{(j)}$ are the "j-th order sum functions" determined from H and H', respectively, exactly as in the case of Hilbert-Samuel functions. It also implies $h^{(j)}\geq h'^{(j)}$ for the corresponding "j-th order series".

For a local homomorphism $\mathfrak{O}\to\mathfrak{O}'$ of local rings with the maximal ideals $\mathfrak{M}$ and $\mathfrak{M}'$, we have the induced homomorphism of graded rings $\mathrm{gr}_{\mathfrak{M}}(\mathfrak{O})\to\mathrm{gr}_{\mathfrak{M}'}(\mathfrak{O}')$. When $\mathfrak{O}'=\mathfrak{O}/\mathfrak{J}$ for an $\mathfrak{O}$-ideal $\mathfrak{J}$, we denote the kernel of the surjective homomorphism by

$$\mathrm{in}_{\mathfrak{M}}(\mathfrak{J}):=\ker[\mathrm{gr}_{\mathfrak{M}}(\mathfrak{O})\to\mathrm{gr}_{\mathfrak{M}/\mathfrak{J}}(\mathfrak{O}/\mathfrak{J})]$$

and call it the $\mathfrak{M}$-*adic initial ideal* of $\mathfrak{J}$. If φ is an element of $\mathfrak{M}^l$ not in $\mathfrak{M}^{l+1}$, then we denote by $\mathrm{in}_{\mathfrak{M}}(\varphi)$ the class of φ in $\mathrm{gr}^l_{\mathfrak{M}}(\mathfrak{O})$ and call it the $\mathfrak{M}$-*adic initial form* of φ and $l:=\mathrm{ord}_{\mathfrak{M}}(\varphi)$ the $\mathfrak{M}$-*adic order* of φ.

The following theorem plays an important role in the resolution of singularities. It was originally given by Bennett [B, (4.1), Theorem (1)] for $j\geq 1$ and $\nu(1)=\cdots=\nu(d)=1$, and by Hironaka [H_4, Propositions 5 and 6] for $j\geq 1$, $d=1$, $\nu(1)=1$. Singh [S_1, Theorem 1, Corollaries 1, 2 and 3] later improved it to include the case $j=0$ and $\nu(1),\cdots,\nu(d)$ arbitrary, from which all the other cases follow. Giraud [G_1, Lemme I.3.9 and Proposition I.6.6] dealt with the question in a more general setup and stated the result in its present form in [G_2, Lemme de transversalité, 1.2].

Theorem 2.1 (Bennett, Hironaka, Giraud and Singh). *Let $\mathfrak{O}$ be a Noetherian local ring with the maximal ideal $\mathfrak{M}$. Let $\varphi_1,\cdots,\varphi_d$ be its elements satisfying $\varphi_i\in\mathfrak{M}^{\nu(i)}$ for $1\leq i\leq d$ and positive integers $\nu(i)$. Let $\mathfrak{O}':=\mathfrak{O}/(\varphi_1,\cdots,\varphi_d)$ be the residue local ring of $\mathfrak{O}$ modulo its ideal $(\varphi_1,\cdots,\varphi_d)$ with the maximal ideal $\mathfrak{M}'$. Then the following holds for the Hilbert-Samuel series*:

$$h^{(j+d)}_{\mathfrak{O}'}(t)\geq h^{(j)}_{\mathfrak{O}}(t)\cdot\prod_{1\leq i\leq d}(1-t^{\nu(i)})/(1-t)$$

for all nonnegative integers j. Let Φ_i be the class of φ_i in $\mathrm{gr}^{\nu(i)}_{\mathfrak{M}}(\mathfrak{O})$. Then the equality holds above for some (*hence any*) *nonnegative integer j if and only if $\Phi_1,\cdots,\Phi_d$ form a regular sequence in $\mathrm{gr}_{\mathfrak{M}}(\mathfrak{O})$. In this case, the canonical homomorphism*

$$\mathrm{gr}_{\mathfrak{M}}(\mathfrak{O})/(\Phi_1,\cdots,\Phi_d)\to\mathrm{gr}_{\mathfrak{M}'}(\mathfrak{O}')$$

is an isomorphism.

By induction on d and j, we can reduce the proof of Theorem 2.1 to the case $d=1$ and $j=0$. We follow the formulation of Hironaka [H_4, Proposition 6] and Singh [S_1, Theorem 1, Corollaries 2 and 3].

Proposition 2.2. *Let φ be an element of $\mathfrak{M}^\nu$ for a positive integer ν. For each nonnegative integer l, let*

$$\mathfrak{a}_l:=\{x\in\mathfrak{O};\ x\varphi\in\mathfrak{M}^{l+1}\}.$$

Then $\mathfrak{a}_l\supset\mathfrak{M}^{l-\nu+1}$ and we have

$$H^{(1)}_{\mathfrak{O}/\mathfrak{O}\varphi}(l)-\sum_{0\leq i\leq\nu-1}H^{(0)}_{\mathfrak{O}}(l-i)=\mathrm{length}_{\mathfrak{O}}(\mathfrak{a}_l/\mathfrak{M}^{l-\nu+1})$$

for all $l\geq 0$, hence

$$h^{(1)}_{\mathfrak{O}/\mathfrak{O}\varphi}(t)-h^{(0)}_{\mathfrak{O}}(t)(1-t^{\nu})/(1-t)=\sum_{l\geq 0}\operatorname{length}_{\mathfrak{O}}(\mathfrak{a}_l/\mathfrak{M}^{l-\nu+1})t^l.$$

Moreover, the following are equivalent:

(i) $H^{(1)}_{\mathfrak{O}/\mathfrak{O}\varphi}(l)=\sum_{0\leq i\leq \nu-1}H^{(0)}_{\mathfrak{O}}(l-i)$ for all $l\geq 0$.

(ii) $h^{(1)}_{\mathfrak{O}/\mathfrak{O}\varphi}(t)=h^{(0)}_{\mathfrak{O}}(t)(1-t^{\nu})/(1-t)$.

(iii) $\mathfrak{a}_l=\mathfrak{M}^{l-\nu+1}$ for all $l\geq 0$.

(iv) $\mathfrak{O}\varphi\cap\mathfrak{M}^{l+1}=\varphi\mathfrak{M}^{l-\nu+1}$ for all $l\geq 0$.

(v) *The class Φ of φ in* $\operatorname{gr}^{\nu}_{\mathfrak{M}}(\mathfrak{O})$ *is not a zero divisor in* $\operatorname{gr}_{\mathfrak{M}}(\mathfrak{O})$.

(vi) *The class Φ of φ in* $\operatorname{gr}^{\nu}_{\mathfrak{M}}(\mathfrak{O})$ *is not a zero divisor in* $\operatorname{gr}_{\mathfrak{M}}(\mathfrak{O})$ *and the canonical homomorphism* $\operatorname{gr}_{\mathfrak{M}}(\mathfrak{O})/(\Phi)\to\operatorname{gr}_{\mathfrak{M}/\mathfrak{O}\varphi}(\mathfrak{O}/\mathfrak{O}\varphi)$ *is an isomorphism.*

Proof. Let us denote $\mathfrak{O}':=\mathfrak{O}/\mathfrak{O}\varphi$ with the maximal ideal $\mathfrak{M}':=\mathfrak{M}/\mathfrak{O}\varphi$. Since $\mathfrak{O}'/(\mathfrak{M}')^{l+1}=\mathfrak{O}/(\mathfrak{O}\varphi+\mathfrak{M}^{l+1})$, we have an exact sequence

$$0\to(\mathfrak{O}\varphi+\mathfrak{M}^{l+1})/\mathfrak{M}^{l+1}\to\mathfrak{O}/\mathfrak{M}^{l+1}\to\mathfrak{O}'/(\mathfrak{M}')^{l+1}\to 0.$$

The multiplication by φ induces an isomorphism

$$\mathfrak{O}/\mathfrak{a}_l\overset{\sim}{\to}\mathfrak{O}\varphi/(\mathfrak{O}\varphi\cap\mathfrak{M}^{l+1})=(\mathfrak{O}\varphi+\mathfrak{M}^{l+1})/\mathfrak{M}^{l+1}.$$

Since φ is in $\mathfrak{M}^{\nu}$, we see that $\mathfrak{a}_l\supset\mathfrak{M}^{l-\nu+1}$, and we have an exact sequence

$$0\to\mathfrak{a}_l/\mathfrak{M}^{l+1}\to\mathfrak{O}/\mathfrak{M}^{l+1}\to\mathfrak{O}/\mathfrak{a}_l\to 0.$$

All these are modules of finite length over $\mathfrak{O}$. Thus looking at the lengths over $\mathfrak{O}$, we have $H^{(1)}_{\mathfrak{O}'}(l)=\operatorname{length}_{\mathfrak{O}}(\mathfrak{a}_l/\mathfrak{M}^{l+1})=\operatorname{length}_{\mathfrak{O}}(\mathfrak{a}_l/\mathfrak{M}^{l-\nu+1})+\sum_{0\leq i\leq\nu-1}H^{(0)}_{\mathfrak{O}}(l-i)$. Hence (i), (ii), (iii), (iv) and (v) are obviously equivalent. (vi) obviously implies (v), while (iv) implies (vi), since

$$\begin{aligned}\operatorname{gr}^l_{\mathfrak{M}'}(\mathfrak{O}')&=(\mathfrak{O}\varphi+\mathfrak{M}^l)/(\mathfrak{O}\varphi+\mathfrak{M}^{l+1})=\mathfrak{M}^l/\mathfrak{M}^l\cap(\mathfrak{O}\varphi+\mathfrak{M}^{l+1})\\&=\mathfrak{M}^l/\{\mathfrak{M}^{l+1}+(\mathfrak{O}\varphi\cap\mathfrak{M}^l)\}=\mathfrak{M}^l/(\mathfrak{M}^{l+1}+\varphi\mathfrak{M}^{l-\nu})\\&=\operatorname{gr}^l_{\mathfrak{M}}(\mathfrak{O})/\Phi\cdot\operatorname{gr}^{l-\nu}_{\mathfrak{M}}(\mathfrak{O}).\end{aligned}$$

q.e.d.

When $\nu(1)=\nu(2)=\cdots=\nu(d)=1$, Theorem 2.1 has the following form, which can be found in [B], [H$_4$], [G$_1$], [G$_2$] and [S$_1$] quoted above and only in which form we use this result later:

Corollary 2.3. *Let $\varphi_1,\cdots,\varphi_d$ be elements of the maximal ideal $\mathfrak{M}$ of a Noetherian local ring $\mathfrak{O}$. For the residue local ring $\mathfrak{O}':=\mathfrak{O}/(\varphi_1,\cdots,\varphi_d)$ of $\mathfrak{O}$ modulo $(\varphi_1,\cdots,\varphi_d)$ with the maximal ideal $\mathfrak{M}'$, we have the following inequality for the Hilbert-Samuel functions* (*resp. series*): *For all $j\geq 0$,*

$$H^{(j+d)}_{\mathfrak{O}'}\geq H^{(j)}_{\mathfrak{O}}\quad(\text{resp. } h^{(j+d)}_{\mathfrak{O}'}\geq h^{(j)}_{\mathfrak{O}}).$$

Let Φ_i be the class of φ_i in $\mathrm{gr}^1_{\mathfrak{M}}(\mathfrak{O})$. Then the equality holds above for some (hence any) nonnegative integer j if and only if $\Phi_1, \cdots, \Phi_d$ form a regular sequence in $\mathrm{gr}_{\mathfrak{M}}(\mathfrak{O})$. In this case, the canonical homomorphism

$$\mathrm{gr}_{\mathfrak{M}}(\mathfrak{O})/(\Phi_1, \cdots, \Phi_d) \rightarrow \mathrm{gr}_{\mathfrak{M}'}(\mathfrak{O}')$$

is an isomorphism.

Let $\mathcal{O}$ be a regular local ring with the maximal ideal $\mathfrak{m}$ and the residue field k. Thus $\mathrm{gr}_{\mathfrak{m}}(\mathcal{O})$ is a polynomial ring over k. Elements $g_1, \cdots, g_r$ in $\mathcal{O}$ are said to form a *standard base* of an $\mathcal{O}$-ideal J if their $\mathfrak{m}$-adic initial forms $f_1 := \mathrm{in}_{\mathfrak{m}}(g_1), \cdots, f_r := \mathrm{in}_{\mathfrak{m}}(g_r)$ form a standard base of the $\mathfrak{m}$-adic initial ideal $\mathrm{in}_{\mathfrak{m}}(J) := \ker[\mathrm{gr}_{\mathfrak{m}}(\mathcal{O}) \rightarrow \mathrm{gr}_{\mathfrak{m}/J}(\mathcal{O}/J)]$ in the sense of Section 1, i.e., $\{f_1, \cdots, f_r\}$ is a minimal system of generators of the homogeneous ideal $\mathrm{in}_{\mathfrak{M}}(J)$ such that $\deg f_1 \leq \deg f_2 \leq \cdots \leq \deg f_r$. Following [$H_1$, Chapter III], we define the numerical invariant $\nu^*(J, \mathcal{O})$ to be the sequence

$$\nu^*(J, \mathcal{O}) := (\deg f_1, \deg f_2, \cdots, \deg f_r, \infty, \infty, \cdots).$$

We known that $\{g_1, \cdots, g_r\}$ then actually generates the $\mathcal{O}$-ideal J (cf. [H_1, Lemma 6, p. 190 and Corollary, p. 208]) and that $\nu^*(J, \mathcal{O})$ depends only on $\mathrm{in}_{\mathfrak{m}}(J)$ and is independent of the particular standard base of J used to define it. This numerical invariant $\nu^*(J, \mathcal{O})$ also plays an important role in the resolution of singularities, as we see in the following sections.

We use the result in Corollary 2.3 sometimes in the following form which can be found in Hironaka [H_4, Lemma 7], Giraud [G_1, Scholie I.6.7 and Proposition 6.8] and [G_2, Corollaire 1.4 and Lemme 5.4.3].

Corollary 2.4. *Let $\mathcal{O}$ be a regular local ring with the maximal ideal $\mathfrak{m}$. Suppose $\{v_1, \cdots, v_d\}$ is a part of a regular system of parameters of $\mathfrak{m}$ so that $\mathcal{O}' := \mathcal{O}/(v_1, \cdots, v_d)$ is a regular local ring with the maximal ideal $\mathfrak{m}' := \mathfrak{m}/(v_1, \cdots, v_d)$. Denote by J' the image in $\mathcal{O}'$ of an $\mathcal{O}$-ideal J contained in $\mathfrak{m}$.*

(i) *We have $H^{(j)}_{\mathcal{O}/J} \leq H^{(j+d)}_{\mathcal{O}'/J'}$ for any $j \geq 0$. The equality holds for some (hence any) $j \geq 0$ if and only if $\mathrm{in}_{\mathfrak{m}}(v_1), \cdots, \mathrm{in}_{\mathfrak{m}}(v_d)$ form a regular sequence for the $\mathrm{gr}_{\mathfrak{m}}(\mathcal{O})$-module $\mathrm{gr}_{\mathfrak{m}/J}(\mathcal{O}/J)$. In this case, we have a canonical isomorphism*

$$\mathrm{gr}_{\mathfrak{m}/J}(\mathcal{O}/J)/(\mathrm{in}_{\mathfrak{m}}(v_1), \cdots, \mathrm{in}_{\mathfrak{m}}(v_d))\,\mathrm{gr}_{\mathfrak{m}/J}(\mathcal{O}/J) \xrightarrow{\sim} \mathrm{gr}_{\mathfrak{m}'/J'}(\mathcal{O}'/J').$$

(ii) *Suppose $H^{(0)}_{\mathcal{O}/J} = H^{(d)}_{\mathcal{O}'/J'}$. If $g_1, \cdots, g_r$ form a standard base of J, then their images $g'_1, \cdots, g'_r$ in $\mathcal{O}'$ form a standard base of J'. In particular, $\nu^*(J, \mathcal{O}) = \nu^*(J', \mathcal{O}')$ holds.*

(iii) *Suppose* $v_1, \cdots, v_d$ *form a regular sequence for the* $\mathcal{O}$-*module* $\mathcal{O}/J$. *If* $\nu^*(J, \mathcal{O})=\nu^*(J', \mathcal{O}')$ *holds, then we have* $H^{(0)}_{\mathcal{O}/J}=H^{(d)}_{\mathcal{O}'/J'}$.

Proof. (i) follows from Corollary 2.3 applied to $\mathfrak{O}:=\mathcal{O}/J$ and the images $\varphi_1, \cdots, \varphi_d$ in $\mathfrak{O}$ of $v_1, \cdots, v_d$.

By induction on d, we can reduce the proof of (ii) and (iii) to the case $d=1$. Then we have the following commutative diagram with exact rows and surjective columns:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{in}_{\mathfrak{m}}(J) & \longrightarrow & \mathrm{gr}_{\mathfrak{m}}(\mathcal{O}) & \longrightarrow & \mathrm{gr}_{\mathfrak{m}/J}(\mathcal{O}/J) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{in}_{\mathfrak{m}'}(J') & \longrightarrow & \mathrm{gr}_{\mathfrak{m}'}(\mathcal{O}') & \longrightarrow & \mathrm{gr}_{\mathfrak{m}'/J'}(\mathcal{O}'/J') & \longrightarrow & 0 \end{array}$$

The middle column is surjective with kernel $\mathrm{in}_{\mathfrak{m}}(v_1)\cdot\mathrm{gr}_{\mathfrak{m}}(\mathcal{O})$. By (i), we have $H^{(0)}_{\mathcal{O}/J}=H^{(1)}_{\mathcal{O}'/J'}$ if and only if $\mathrm{in}_{\mathfrak{m}}(v_1)$ is not a zero divisor for $\mathrm{gr}_{\mathfrak{m}/J}(\mathcal{O}/J)$ on the right hand side, or equivalently, the left hand column has kernel $\mathrm{in}_{\mathfrak{m}}(v_1)\cdot\mathrm{in}_{\mathfrak{m}}(J)$.

For the proof of (ii), let $\{g_1, \cdots, g_r\}$ be a standard base of J and let $f_j:=\mathrm{in}_{\mathfrak{m}}(g_j)$ for $1\leq j\leq r$. Thus their images $f'_j:=\mathrm{in}_{\mathfrak{m}'}(g'_j)$ for $1\leq j\leq r$ generate $\mathrm{in}_{\mathfrak{m}'}(J')$. Since $\deg f_j=\deg f'_j$ if $f'_j\neq 0$, it suffices to show that $\{f'_1, \cdots, f'_r\}$ is a minimal system of generators of $\mathrm{in}_{\mathfrak{m}'}(J')$. If not, there would exist j and homogeneous $\beta_1, \cdots, \beta_{j-1}$ in $\mathrm{gr}_{\mathfrak{m}}(\mathcal{O})$ such that $\deg\beta_i=\deg f_j-\deg f_i$ for $1\leq i\leq j-1$ and that $f_j-\sum_{1\leq i\leq j-1}\beta_i f_i$ is mapped to zero in $\mathrm{in}_{\mathfrak{m}'}(J')$. Hence it would be in $\mathrm{in}_{\mathfrak{m}}(v_1)\cdot\mathrm{in}_{\mathfrak{m}}(J)$ and be of the form $\mathrm{in}_{\mathfrak{m}}(v_1)\cdot\sum_{1\leq i\leq j-1}\alpha_i f_i$ for homogeneous α_i in $\mathrm{gr}_{\mathfrak{m}}(\mathcal{O})$ with $\deg\alpha_i=\deg f_j-\deg f_i-1$, since $\{f_1, \cdots, f_r\}$ is a standard base of $\mathrm{in}_{\mathfrak{m}}(J)$. Thus

$$f_i=\sum_{1\leq i\leq j-1}(\beta_i+\alpha_i\cdot\mathrm{in}_{\mathfrak{m}}(v_1))f_i,$$

a contradiction to the minimality of $\{f_1, \cdots, f_r\}$.

For the proof of (iii), let $\{g_1, \cdots, g_r\}$ be a standard base of J. Thus $\{f_1, \cdots, f_r\}$ with $f_j:=\mathrm{in}_{\mathfrak{m}}(g_j)$ is a standard base of $\mathrm{in}_{\mathfrak{m}}(J)$. Let g'_j be the image of g_j in J' and let f'_j be the image of f_j in $\mathrm{in}_{\mathfrak{m}'}(J')$. Since $\nu^*(J, \mathcal{O})=\nu^*(J', \mathcal{O}')$ by assumption, $\{g'_1, \cdots, g'_r\}$ and $\{f'_1, \cdots, f'_r\}$ are standard bases of J' and $\mathrm{in}_{\mathfrak{m}'}(J')$, respectively, with $\nu(j):=\mathrm{ord}_{\mathfrak{m}}(g_j)=\mathrm{ord}_{\mathfrak{m}'}(g'_j)=\deg f_j=\deg f'_j$ for $1\leq j\leq r$. By what we saw above, it suffices to show that any homogeneous f of a given degree ν in $\mathrm{in}_{\mathfrak{m}}(J)\cap\mathrm{in}_{\mathfrak{m}}(v_1)\cdot\mathrm{gr}_{\mathfrak{m}}(\mathcal{O})$ is necessarily contained in $\mathrm{in}_{\mathfrak{m}}(v_1)\cdot\mathrm{in}_{\mathfrak{m}}(J)$. There exists $g\in J$ such that $\mathrm{in}_{\mathfrak{m}}(g)=f$. Since the image of f in $\mathrm{in}_{\mathfrak{m}'}(J')$ vanishes by assumption, the image g' of g in J' is contained in $(\mathfrak{m}')^{\nu+1}$. Thus there exist a_j in $\mathfrak{m}^{\nu+1-\nu(j)}$ for $1\leq j\leq r$ such that their images a'_j in $(\mathfrak{m}')^{\nu+1-\nu(j)}$ satisfy $g'=\sum_{1\leq j\leq r}a'_j g'_j$. Hence $\tilde{g}:=g-\sum_{1\leq j\leq r}a_j g_j$ is in $J\cap(v_1\mathcal{O})$. Since v_1 is not a zero divisor for the $\mathcal{O}$-module $\mathcal{O}/J$ by assumption, $\tilde{g}$ is in v_1J. Thus $f=\mathrm{in}_{\mathfrak{m}}(g)=\mathrm{in}_{\mathfrak{m}}(\tilde{g})$ is in $\mathrm{in}_{\mathfrak{m}}(v_1)\cdot\mathrm{in}_{\mathfrak{m}}(J)$. q.e.d.

Remark. Geometrically, Corollary 2.4, (i) means the following: Inside the regular ambient scheme $Z:=\mathrm{Spec}(\mathcal{O})$, the intersection of the closed subscheme $X:=\mathrm{Spec}(\mathcal{O}/J)$ with the regular closed subscheme $Z':=\mathrm{Spec}(\mathcal{O}')$ is $X\cap Z'=\mathrm{Spec}(\mathcal{O}'/J')$. In the tangent space $T_{\mathfrak{m}}(Z):=\mathrm{Spec}(\mathrm{gr}_{\mathfrak{m}}(\mathcal{O}))$ of Z at $\mathfrak{m}$, the tangent space $T_{\mathfrak{m}}(Z'):=\mathrm{Spec}(\mathrm{gr}_{\mathfrak{m}'}(\mathcal{O}'))$ is a linear subspace which transversally intersects the tangent cone $C_{\mathfrak{m}}(X):=\mathrm{Spec}(\mathrm{gr}_{\mathfrak{m}/J}(\mathcal{O}/J))$ of X at $\mathfrak{m}$ so that the tangent cone $C_{\mathfrak{m}}(X\cap Z'):=\mathrm{Spec}(\mathrm{gr}_{\mathfrak{m}'/J'}(\mathcal{O}'/J'))$ of $X\cap Z'$ at $\mathfrak{m}$ satisfies

$$C_{\mathfrak{m}}(X\cap Z')=C_{\mathfrak{m}}(X)\cap T_{\mathfrak{m}}(Z')$$

scheme-theoretically.

The Hilbert-Samuel functions appear also in numerical criteria for other properties. Here is another example asserting the *upper-semicontinuity* of the Hilbert-Samuel functions and characterizing the *permissibility*, which collect together those results found in Hironaka [H_1, Theorem 2, p. 195], Bennett [B, Theorems (2) and (3)] and Singh [S_2, Theorem 1 and papers quoted on p. 20]. For the proof, we refer the reader to these papers.

Theorem 2.5. *Let $\mathfrak{O}$ be a Noetherian local ring. If $\mathfrak{Q}$ is a prime ideal of $\mathfrak{O}$, then for $c:=\dim\mathfrak{O}/\mathfrak{Q}$ we have the upper-semicontinuity*

$$H_{\mathfrak{O}}^{(j)}\geq H_{\mathfrak{O}_{\mathfrak{Q}}}^{(j+c)} \qquad \text{for all } j\geq 0.$$

Suppose $\mathfrak{O}/\mathfrak{Q}$ is regular and let $\mathfrak{M}$ be the maximal ideal of $\mathfrak{O}$ with the residue field $\mathfrak{K}:=\mathfrak{O}/\mathfrak{M}$. Then the following are equivalent:

(i) $\mathrm{gr}_{\mathfrak{Q}}(\mathfrak{O})$ *is flat over* $\mathfrak{O}/\mathfrak{Q}$, i.e., $\mathfrak{Q}$ *is permissible.*

(ii) *The canonical surjective homomorphism* $\mathrm{gr}_{\mathfrak{M}}(\mathfrak{O})\to\mathrm{gr}_{\mathfrak{M}/\mathfrak{Q}}(\mathfrak{O}/\mathfrak{Q})$ *splits. A splitting and the canonical homomorphism* $\mathrm{gr}_{\mathfrak{Q}}(\mathfrak{O})\otimes_{\mathfrak{O}/\mathfrak{Q}}\mathfrak{K}\to\mathrm{gr}_{\mathfrak{M}}(\mathfrak{O})$ *then give rise to an isomorphism*

$$(\mathrm{gr}_{\mathfrak{Q}}(\mathfrak{O})\otimes_{\mathfrak{O}/\mathfrak{Q}}\mathfrak{K})\otimes_{\mathfrak{K}}\mathrm{gr}_{\mathfrak{M}/\mathfrak{Q}}(\mathfrak{O}/\mathfrak{Q})\xrightarrow{\sim}\mathrm{gr}_{\mathfrak{M}}(\mathfrak{O}).$$

(iii) *The equality $H_{\mathfrak{O}}^{(j)}=H_{\mathfrak{O}_{\mathfrak{Q}}}^{(j+c)}$ holds for some* (*hence any*) $j\geq 0$.

Furthermore, if $\mathfrak{O}=\mathcal{O}/J$, $\mathfrak{M}=\mathfrak{m}/J$, $\mathfrak{Q}=\mathfrak{q}/J$ for a regular local ring $\mathcal{O}$ with the maximal ideal $\mathfrak{m}$ and a prime ideal $\mathfrak{q}$, then (i), (ii), (iii) *are also equivalent to the following*:

(iv) *There exists a standard base $\{g_1,\cdots,g_r\}$ of J such that the $\mathfrak{m}$-adic and $\mathfrak{q}$-adic orders of g_j coincide, i.e., $\mathrm{ord}_{\mathfrak{m}}(g_j)=\mathrm{ord}_{\mathfrak{q}}(g_j)$, for all $1\leq j\leq r$.*

Remark. Geometrically, (ii) of Theorem 2.5 means the following: Let $X := \mathrm{Spec}(\mathfrak{O})$ with its regular closed subscheme $Y := \mathrm{Spec}(\mathfrak{O}/\mathfrak{Q})$. Thus the tangent cone $C_{\mathfrak{M}}(X) := \mathrm{Spec}(\mathrm{gr}_{\mathfrak{M}}(\mathfrak{O}))$ of X at $\mathfrak{M}$ contains the tangent space $T_{\mathfrak{M}}(Y) := \mathrm{Spec}(\mathrm{gr}_{\mathfrak{M}/\mathfrak{Q}}(\mathfrak{O}/\mathfrak{Q}))$ of Y at $\mathfrak{M}$. Then the canonical surjection $C_{\mathfrak{M}}(X) \to C_{\mathfrak{M}}(X, Y)$ splits and we have a noncanonical isomorphism

$$C_{\mathfrak{M}}(X, Y) \times T_{\mathfrak{M}}(Y) \cong C_{\mathfrak{M}}(X),$$

where $C_{\mathfrak{M}}(X, Y) := \mathrm{Spec}(\mathrm{gr}_{\mathfrak{Q}}(\mathfrak{O}) \otimes_{\mathfrak{O}/\mathfrak{Q}} \mathfrak{K})$ is the normal cone of X along Y at $\mathfrak{M}$.

In the special case where $\mathfrak{O}$ in Theorem 2.5 is the local ring $\mathcal{O}_{W,w}$ at a point w of a scheme W of finite type over a field K, we can prove the upper-semicontinuity of the Hilbert-Samuel functions by relating them to coherent $\mathcal{O}_W$-modules:

When K is *perfect*, Bennett [B, Chapter III, § 2, especially Proposition (2.2)] showed that

$$\mathcal{O}_{W,w}/(\mathfrak{m}_{W,w})^{l+1} = \kappa(w) \otimes_{\mathcal{O}_W} \mathcal{P}^l_{W/K} \qquad \text{for } l \geq 0,$$

for a *closed* point w of W, hence

$$H^{(1)}_{\mathcal{O}_{W,w}}(l) = \dim_{\kappa(w)}\{\kappa(w) \otimes_{\mathcal{O}_W} \mathcal{P}^l_{W/K}\} \qquad \text{for } l \geq 0,$$

where $\mathcal{P}^l_{W/K}$ is the coherent $\mathcal{O}_W$-module of principal parts on W over K of orders up to l and $\kappa(w) := \mathcal{O}_{W,w}/\mathfrak{m}_{W,w}$ is the residue field at w regarded as an $\mathcal{O}_W$-module. The right hand side of this latter equality is well-known to be an upper-semicontinuous function in w.

Giraud [G_2, Théorème 2.12] extended this result to one valid for general K by introducing a suitable subfield $k \subset K$, as we now recall.

For that purpose, we need to introduce some notations. A field extension L/K is said to be *differentiably finite* if the kernel $\Upsilon_{L/K}$ and the cokernel $\Omega^1_{L/K}$ of the canonical homomorphism (cf. [EGA, Chapter $0_{\mathrm{IV},1}$, § 21])

$$L \otimes_K \Omega^1_K \longrightarrow \Omega^1_L$$

are finite dimensional over L, where, for instance, Ω^1_L denotes the module of Kähler differentials of L over the prime field. In this case, we define the *index* by

$$t(L/K) := \dim_L \Omega^1_{L/K} - \dim_L \Upsilon_{L/K}.$$

If L/K is of finite type, then *Cartier's equality* holds:

$$t(L/K) = \text{tr. deg}_K L.$$

On the other hand, if L/K is separable with $\Omega^1_{L/K}$ finite dimensional over L, we have $\Upsilon_{L/K} = 0$ and

$$t(L/K) = \dim_L \Omega^1_{L/K}.$$

In general, if L/K and K/k are differentiably finite, then so is L/k and the additivity of the index holds:

$$t(L/k) = t(L/K) + t(K/k).$$

In particular, if L/K is of finite type and if K/k is separable with $\Omega^1_{K/k}$ finite dimensional over K as in the following theorem, then we have

$$t(L/k) = \text{tr. deg}_K L + \dim_K \Omega^1_{K/k}.$$

Theorem 2.6. (Bennett and Giraud). *Let W be a scheme of finite type over a field K. Then there exists a subfield $k \subset K$ with K/k separable and with $\dim_K \Omega^1_{K/k}$ finite such that the following holds: For each nonnegative integer l, the $\mathcal{O}_W$-module $\mathcal{P}^l_{W/k}$ of principal parts on W over k of orders up to l is $\mathcal{O}_W$-coherent and for each $w \in W$, we have*

$$H^{(1+t(\kappa(w)/k))}_{\mathcal{O}_{W,w}}(l) = \dim_{\kappa(w)} \{\kappa(w) \otimes_{\mathcal{O}_W} \mathcal{P}^l_{W/k}\},$$

where $t(\kappa(w)/k)$ is the index over k of the residue field $\kappa(w)$ of W at w.

For the proof of this theorem, we refer the reader to [G_2].

Remark. Given w in W, we can actually choose k as above so that, furthermore, $\kappa(w)/k$ is separable (cf. [G_2, Lemme 2.10]).

§ 3. Stability theorems under a permissible blowing up

In this section, we introduce three numerical invariants to measure a singularity, and collect together known results on their behavior under a permissible blowing up. We prove most of them in the next section.

Let $\mathcal{O}$ be a regular local ring with the maximal ideal $\mathfrak{m}$ and the residue field $k := \mathcal{O}/\mathfrak{m}$. For an ideal $J \subset \mathfrak{m}$, we consider the closed subscheme $X := \text{Spec}(\mathcal{O}/J)$ embedded in the regular scheme $Z := \text{Spec}(\mathcal{O})$. X contains the closed point x of Z corresponding to the maximal ideal $\mathfrak{m}$. We deal with the following three numerical invariants for the singularity of X at x:

(1) The sequence $\{H^{(j)}_x(X)$; all nonnegative integers $j\}$ of the Hilbert-Samuel functions of the local ring $\mathcal{O}_{X,x} = \mathcal{O}/J$ of X at x each of which is an integer-valued function on the set of nonnegative integers defined by

$$H_x^{(j)}(X) := H_{\mathcal{O}/J}^{(j)}, \qquad j \geq 0,$$

the right hand side being the one defined in Section 2. They are independent of the embedding of X in Z and in fact depend only on the tangent cone

$$C_x(X) := \operatorname{Spec}(\operatorname{gr}_x(X))$$

of X at x, where $\operatorname{gr}_x(X) := \operatorname{gr}_{\mathfrak{m}/J}(\mathcal{O}/J)$.

(2) The countably infinite sequence consisting of a finite number of nonnegative integers and ∞ defined by

$$\nu_x^*(X, Z) := \nu^*(J, \mathcal{O}),$$

the right hand side being the one defined immediately before Corollary 2.4.

(3) The tangent cone $C_x(X)$ is canonically embedded in the tangent space $T_x(Z) := \operatorname{Spec}(\operatorname{gr}_x(Z))$ of Z at x, where $\operatorname{gr}_x(Z) := \operatorname{gr}_{\mathfrak{m}}(\mathcal{O})$. Then we let $A_x(X)$ to be the *ridge* of $C_x(X) \subset T_x(Z)$ in the sense of Section 1. Denote

$$\mathfrak{A}_x(X, Z) := \operatorname{gr}_x(Z)^{A_x(X)},$$

the ring of invariants in the polynomial ring $\operatorname{gr}_x(Z)$ under the translation action of $A_x(X)$. As we remarked in Section 1, $\mathfrak{A}_x(X, Z)$ is the *smallest* k-subalgebra of $\operatorname{gr}_x(Z)$ generated by additive forms such that the initial ideal $\operatorname{in}_x(X, Z) := \operatorname{in}_{\mathfrak{m}}(J)$ is generated as a $\operatorname{gr}_x(Z)$-ideal by $\operatorname{in}_x(X, Z) \cap \mathfrak{A}_x(X, Z)$. We then define the third numerical invariant to be the nonnegative integer

$$\tau_x(X, Z) := \operatorname{tr.deg}_k \mathfrak{A}_x(X, Z) = \dim \{T_x(Z)/A_x(X)\}.$$

Our concern is the behavior of these three numerical invariants under a permissible blowing up: For a prime ideal $\mathfrak{q}$ of $\mathcal{O}$ with $J \subset \mathfrak{q} \subset \mathfrak{m}$ and with $\mathcal{O}/\mathfrak{q}$ *regular*, consider the regular closed subscheme $Y := \operatorname{Spec}(\mathcal{O}/\mathfrak{q})$ of X. Y is said to be *permissible* for X at x, if X is *normally flat along* Y at x, that is, $\operatorname{gr}_Y(X) := \operatorname{gr}_{\mathfrak{q}/J}(\mathcal{O}/J)$ is a flat module over $\mathcal{O}_{Y,x} = \mathcal{O}/\mathfrak{q}$. Denote $\operatorname{gr}_Y(Z) := \operatorname{gr}_{\mathfrak{q}}(\mathcal{O})$. Then we may and *do* regard

$$S := k \otimes_{\mathcal{O}/\mathfrak{q}} \operatorname{gr}_Y(Z)$$

canonically as a polynomial k-subalgebra of $\operatorname{gr}_x(Z) = \operatorname{gr}_{\mathfrak{m}}(\mathcal{O})$ generated by linear forms. In fact, S coincides with the ring of invariants in $\operatorname{gr}_x(Z)$ under the translation action of the tangent space $T_x(Y) := \operatorname{Spec}(\operatorname{gr}_x(Y))$ of Y at x regarded as a linear subspace of $T_x(Z)$. As we recalled in Theorem 2.5 and the remark immediately after that, the permissibility guarantees that $\mathrm{in}_x(X, Z)$ is generated as a $\mathrm{gr}_x(Z)$-ideal by $\mathrm{in}_x(X, Z)\cap S$, that is, $C_x(X)$ is stable under the translation by $T_x(Y)$. Hence we have $T_x(Y)\subset A_x(X)$ and

$$\mathfrak{A}_x(X, Z)\subset S.$$

The normal space of Z along Y at x is

$$N_x(Z, Y):=T_x(Z)/T_x(Y)=\mathrm{Spec}\,(S)$$

and canonically contains the normal cone

$$C_x(X, Y):=C_x(X)/T_x(Y)=\mathrm{Spec}\,(S/\{S\cap \mathrm{in}_x(X, Z)\})$$

of X along Y at x. The permissibility of Y means that we have a non-canonical isomorphism

$$C_x(X)\cong C_x(X, Y)\times T_x(Y).$$

Thus $A_x(X)/T_x(Y)$ is the ridge of $C_x(X, Y)$.

Let $\Pi: Z'\to Z$ be the blowing up of Z along the permissible center Y, i.e.,

$$Z':=\mathrm{Proj}\,(\mathrm{Rees}\,(\mathfrak{q}))$$

with the Rees algebra $\mathrm{Rees}\,(\mathfrak{q}):=\bigoplus_{i\geq 0}\mathfrak{q}^i$. Such a blowing up is called a *permissible blowing up*. Denote by X' the *strict* (also called proper) transform of X in Z'. Thus the restriction $X'\to X$ of Π to X' is the blowing up of X along Y. Consider a (not necessarily closed) point x' in the fiber $\Pi^{-1}(x)$ and denote by $\mathcal{O}':=\mathcal{O}_{Z',x'}$ the local ring of Z' at x' with the maximal ideal $\mathfrak{m}':=\mathfrak{m}_{Z',x'}$ and the residue field $k':=\mathcal{O}'/\mathfrak{m}'$. We denote by J' the strict transform of J, i.e., the ideal of $\mathcal{O}'$ such that $\mathcal{O}_{X',x'}=\mathcal{O}'/J'$. We consider the three numerical invariants for $X'\subset Z'$ at x':

$$\{H_{x'}^{(j)}(X'): j\geq 0\},\quad \nu_{x'}^*(X', Z')\quad \text{and}\quad \tau_{x'}(X', Z').$$

Theorem 3.1 (The first stability theorem) (Bennett [B, Theorem (0E)], Hironaka [H_4, Theorem I] and Singh [S_1, Main Theorem]). *If we denote $d:=\mathrm{tr.\,deg}_k k'$, then the Hilbert-Samuel functions satisfy*

$$H_x^{(j)}(X)\geq H_{x'}^{(j+d)}(X')\qquad \text{for all } j\geq 0$$

in the sense of the order in Section 2.

Definition. x' is said to be an *infinitely near point* of X, or to be inifinitely near to x, if the equality

$$H_x^{(j)}(X) = H_{x'}^{(j+d)}(X'), \qquad d = \text{tr.}\deg_k k'$$

holds for some (hence any) nonnegative integer j.

Theorem 3.2 (Hironaka [H_4, Theorems II and III], Singh [S_3, Corollary 3.4] and Herrmann-Orbanz [HO, Theorem]). *We have*

$$\nu_x^*(X, Z) \geq \nu_{x'}^*(X', Z')$$

with respect to the lexicographic order. The equality holds if and only if x' is an infinitely near point of x.

Remark. As Singh [S_3, Examples 1 and 2] noted, neither of $H_x^{(j)}(X)$ and $\nu_x^*(X, Z)$ determines the other. In spite of this, the equality in Theorem 3.1 is equivalent to that in Theorem 3.2. [S_3] and [HO] introduced a new numerical invariant $\nu_x^{**}(X, Z)$ which determines $H_x^{(j)}(X)$ and which is an infinite matrix with the first row equal to $\nu_x^*(X, Z)$.

Theorem 3.3 (The second stability theorem) (Giraud [G_2, Théorème 5.5.3]). *If x' is an infinitely near point of x, then*

$$\tau_x(X, Z) \leq \tau_{x'}(X', Z').$$

Definition. x' is said to be an *infinitely very near point* of x, or to be infinitely very near to x if x' is infinitely near to x and the equality $\tau_x(X, Z) = \tau_{x'}(X', Z')$ holds.

Thus by Theorem 3.2, the following are equivalent:

(i) x' is infinitely very near to x.

(ii) $H_x^{(j)}(X) = H_{x'}^{(j+d)}(X')$ for some $j \geq 0$ and $\tau_x(X, Z) = \tau_{x'}(X', Z')$.

(iii) $\nu_x^*(X, Z) = \nu_{x'}^*(X', Z')$ and $\tau_x(X, Z) = \tau_{x'}(X', Z')$.

(iv) $H_x^{(j)}(X) = H_{x'}^{(j+d)}(X')$ for any $j \geq 0$, $\nu_x^*(X, Z) = \nu_{x'}^*(X', Z')$ and $\tau_x(X, Z) = \tau_{x'}(X', Z')$.

Actually, we have further detailed information on infinitely very near points. To describe it, we need the following:

The fiber of $\Pi: Z' \to Z$ over x coincides with

$$\Pi^{-1}(x) = \text{Proj}\,(S),$$

the projective space associated to the normal space $N_x(Z, Y) = \text{Spec}\,(S)$ of Z along Y at x. Thus the point $x' \in \Pi^{-1}(x)$ corresponds to a homogeneous prime ideal $\mathfrak{p}$ of the polynomial ring S over k different from the ideal S_+ of polynomials without constant terms. We can choose the variables in S so that

$$S = k[y_0, \cdots, y_n], \qquad y_0 \text{ not in } \mathfrak{p}.$$

Let $R:=S_{\mathfrak{p}}$ be the localization with the maximal ideal M and the residue field $K:=R/M$. Hence R is the local ring of $N_x(Z, Y)$ at its point $\mathfrak{p}$. The local ring $\mathcal{O}'/\mathfrak{m}\mathcal{O}'$ of $\Pi^{-1}(x)$ at x' coincides with the subring of R consisting of the homogeneous elements of degree zero. Moreover, y_0 is a unit in R, and K is generated over k' by the image $\bar{y}_0$ of y_0 in K, that is, $K=k'(\bar{y}_0)$. In these notations, we first of all have the following:

Proposition 3.4 (Hironaka [H_4, Theorem IV]). *If x' is an infinitely near point of x, then any homogeneous element f in $\mathfrak{A}_x(X, Z)\subset S$ of degree ν necessarily belongs to M^{ν}. Hence f/y_0^{ν} is in $(\mathfrak{m}'/\mathfrak{m}\mathcal{O}')^{\nu}$.*

By sending such f to its M-adic initial form $\mathrm{in}_M(f)$ we have a (necessarily injective, see [O_1, Proposition 2.2, (i)]) homomorphism of graded rings

$$\mathrm{in}_M\colon \mathfrak{A}_x(X, Z)\longrightarrow \mathrm{gr}_{\mathfrak{p}}(N_x(Z, Y))=\mathrm{gr}_M(R).$$

On the other hand, we have the graded k'-subalgebras

$$\mathfrak{A}_{x'}(X', Z')\subset \mathrm{gr}_{x'}(Z')=\mathrm{gr}_{\mathfrak{m}'}(\mathcal{O}')$$

$$\mathfrak{A}_{x'}(X'\cap \Pi^{-1}(x), \Pi^{-1}(x))\subset \mathrm{gr}_{x'}(\Pi^{-1}(x))$$

$$\mathfrak{A}_{\mathfrak{p}}(C_x(X, Y), N_x(Z, Y))\subset \mathrm{gr}_{\mathfrak{p}}(N_x(Z, Y)).$$

Theorem 3.5 (Giraud [G_2, Théorème 5.5.3]). *x' is an infinitely very near point of x if and only if the following conditions are satisfied:*

(i) *x' is an infinitely near point of x.*

(ii) *The canonical injection $\mathrm{in}_M\colon \mathfrak{A}_x(X, Z)\to \mathrm{gr}_{\mathfrak{p}}(N_x(Z, Y))=\mathrm{gr}_M(R)$ induces an isomorphism of graded K-algebras*

$$\mathrm{in}_M\colon K\otimes_k \mathfrak{A}_x(X, Z)\xrightarrow{\sim}\mathfrak{A}_{\mathfrak{p}}(C_x(X, Y), N_x(Z, Y))$$

which sends additive forms to additive forms and which sends the ideal $K\otimes_k\{\mathrm{in}_x(X, Z)\cap \mathfrak{A}_x(X, Z)\}$ to the ideal $\mathrm{in}_{\mathfrak{p}}(C_x(X, Y), N_x(Z, Y))\cap \mathfrak{A}_{\mathfrak{p}}(C_x(X, Y), N_x(Z, Y))$.

(iii) *The canonical surjection $\mathrm{gr}_{x'}(Z')\to \mathrm{gr}_{x'}(\Pi^{-1}(x))$ induces an isomorphism of graded k'-algebras*

$$\mathfrak{A}_{x'}(X', Z')\xrightarrow{\sim}\mathfrak{A}_{x'}(X'\cap \Pi^{-1}(x), \Pi^{-1}(x))$$

which sends additive forms to additive forms and which sends the ideal $\mathrm{in}_{x'}(X', Z')\cap \mathfrak{A}_{x'}(X', Z')$ to the ideal $\mathrm{in}_{x'}(X'\cap \Pi^{-1}(x), \Pi^{-1}(x))\cap \mathfrak{A}_{x'}(X'\cap \Pi^{-1}(x), \Pi^{-1}(x))$.

As for the isomorphism in Theorem 3.5, (ii), we have still further information as follows:

Theorem 3.6 (Oda [O_4, Main Theorem]). *Suppose x' is an infinitely very near point of x and let $h \in \mathfrak{A}_x(X, Z)$ be an additive form of degree p^e. Then $\mathrm{in}_M(h)$ is an additive form belonging to $\mathfrak{A}_{\mathfrak{p}}(C_x(X, Y), N_x(Z, Y))$ and has coefficients in the subfield $kF^e(K)$ of K. Furthermore, $\mathrm{in}_{\mathfrak{m}'/\mathfrak{m}\mathcal{O}'}(h/y_0^{p^e})$ is an additive form belonging to $\mathfrak{A}_{x'}(X' \cap \Pi^{-1}(x), \Pi^{-1}(x))$ and has coeffcients in the subfield $kF^e(k')$ of k'. Here F^e denotes the p^e-th power Frobenius map on the fields K and k'.*

We do not reproduce the lengthy proof of this theorem here.

As a consequence of Theorem 3.6 and the proof of Theorem 3.5 in the next section, we get the following information as to the choice of standard bases for $\mathrm{in}_x(X, Z)$ and $\mathrm{in}_{x'}(X', Z')$: Suppose x' is an infinitely very near point of x and denote

$$\nu_x^*(X, Z) = \nu_{x'}^*(X', Z') =: (\nu(1), \cdots, \nu(r), \infty, \infty, \cdots)$$
$$\tau_x(X, Z) = \tau_{x'}(X', Z') =: \tau.$$

We can choose algebraically independent additive forms $h_1, \cdots, h_\tau$ in S of degrees $q(1) = p^{e(1)}, \cdots, q(\tau) = p^{e(\tau)}$ with $e(1) \leq \cdots \leq e(\tau)$ such that

$$\mathfrak{A}_x(X, Z) = k[h_1, \cdots, h_\tau].$$

Let us choose the variables $y = (y_0, \cdots, y_n)$ for S so that $h_1, \cdots h_\tau$ are in triangular shape as in the remark after Lemma 1.1, and let $\{f_1, \cdots, f_r\}$ be a standard base for $\mathrm{in}_x(X, Z)$ *quasi-normalized* with respect to these y in the sense of Proposition 1.2, for instance, normalized with respect to y. Then $f_1, \cdots, f_r$ are in $\mathfrak{A}_x(X, Z)$ so that there exist k-coefficient polynomials $\psi_1, \cdots, \psi_r$ in τ variables $h_1, \cdots, h_\tau$ of weights $q(1), \cdots, q(\tau)$ such that ψ_j is isobaric of weight $\nu(j)$ for $1 \leq j \leq r$ satisfying

$$f_j = \psi_j(h_1, \cdots, h_\tau) \qquad \text{for } 1 \leq j \leq r.$$

Corollary 3.7. *In the above notations, we have the following:*

(i) *The initial forms $\mathrm{in}_M(h_1), \cdots, \mathrm{in}_M(h_\tau)$ are algebraically independent additive forms in $\mathrm{gr}_{\mathfrak{p}}(N_x(Z, Y))$ such that*

$$\mathfrak{A}_{\mathfrak{p}}(C_x(X, Y), N_x(Z, Y)) = K[\mathrm{in}_M(h_1), \cdots, \mathrm{in}_M(h_\tau)]$$

and that $\mathrm{in}_M(h_i)$ has coefficients in $kF^{e(i)}(K)$ for each $1 \leq i \leq \tau$. Moreover, the subset $\{\psi_j(\mathrm{in}_M(h_1), \cdots, \mathrm{in}_M(h_\tau)); 1 \leq j \leq r\}$ of $\mathfrak{A}_{\mathfrak{p}}(C_x(X, Y), N_x(Z, Y))$ is a standard base of $\mathrm{in}_{\mathfrak{p}}(C_x(X, Y), N_x(Z, Y))$ quasi-normalized with respect to a suitable set of variables.

(ii) *Let* $h_i'' := \mathrm{in}_M(h_i/y_0^{q(i)})$ *for each* $1 \leq i \leq \tau$. *Then* $h_1'', \cdots, h_\tau''$ *are algebraically independent additive forms in* $\mathrm{gr}_{x'}(\Pi^{-1}(x))$ *such that*

$$\mathfrak{A}_{x'}(X' \cap \Pi^{-1}(x), \Pi^{-1}(x)) = k'[h_1'', \cdots, h_\tau'']$$

and that h_i'' *has coefficients in* $kF^{e(i)}(k')$ *for each* $1 \leq i \leq \tau$. *Moreover, the subset* $\{\psi_j(h_1'', \cdots, h_\tau''); 1 \leq j \leq r\}$ *of* $\mathfrak{A}_{x'}(X' \cap \Pi^{-1}(x), \Pi^{-1}(x))$ *is a standard base of* $\mathrm{in}_{x'}(X' \cap \Pi^{-1}(x), \Pi^{-1}(x))$ *quasi-normalized with respect to a suitable set of variables.*

(iii) *For each* $1 \leq i \leq \tau$, *there exists a unique additive form* h_i' *in* $\mathrm{gr}_{x'}(Z')$ *mapped to* h_i'' *under the canonical surjection* $\mathrm{gr}_{x'}(Z') \to \mathrm{gr}_{x'}(\Pi^{-1}(x))$ *such that*

$$\mathfrak{A}_{x'}(X', Z') = k'[h_1', \cdots, h_\tau'].$$

Moreover, the subset $\{\psi_j(h_i', \cdots, h_\tau'); 1 \leq j \leq r\}$ *of* $\mathfrak{A}_{x'}(X', Z')$ *is a standard base of* $\mathrm{in}_{x'}(X', Z')$ *quasi-normalized with respect to a suitable set of variables.*

§ 4. Proof of the stability theorems

In this section, we prove Theorems 3.1, 3.3 and 3.5, Corollary 3.7, Proposition 3.4 as well as a part of Theorem 3.2 simultaneously by first breaking them up into five stages and then recombining them at the end of this section. Our proof is essentially the same as that in [H_4] and [G_2].

We retain the notations in Section 3.

The following three stages are rather easy:

Proposition 4.1. *Let* O *be the origin of the tangent cone* $C_x(X)$. *Then we have the following equalities*:

(0_H) $H_x^{(l)}(X) = H_O^{(l)}(C_x(X))$ *for any* $l \geq 0$.

(0_ν) $\nu_x^*(X, Z) = \nu_O^*(C_x(X), T_x(Z))$.

(0_τ) $\mathfrak{A}_x(X, Z) = \mathfrak{A}_O(C_x(X), T_x(Z))$ *and* $\tau_x(X, Z) = \tau_O(C_x(X), T_x(Z))$.

Proof. By definition, these invariants on the left hand side depend only on the associated graded rings $\mathrm{gr}_x(X)$ and $\mathrm{gr}_x(Z)$, i.e., on $C_x(X)$ and $T_x(Z)$. Hence the equalities are obvious.

Proposition 4.2. *Let* O *denote the origin of the tangent cone* $C_x(X)$ *as well as that of the normal cone* $C_x(X, Y)$. *Then we have the following equalities*:

(1_H) $H_O^{(l)}(C_x(X)) = H_O^{(l+s)}(C_x(X, Y))$ *for any* $l \geq 0$,

where $s = \dim Y$, *hence* $s+1$ *is the codimension of* $\Pi^{-1}(x)$ *in* Z'.

(1_ν) $\nu_O^*(C_x(X), T_x(Z)) = \nu_O^*(C_x(X, Y), N_x(Z, Y))$.

(1_τ) $\mathfrak{A}_O(C_x(X), T_x(Z)) = \mathfrak{A}_O(C_x(X, Y), N_x(Z, Y))$ *and*

$\tau_O(C_x(X), T_x(Z)) = \tau_O(C_x(X, Y), N_x(Z, Y))$.

Proof. Since Y is permissible, we have a noncanonical isomorphism $C_x(X) \cong C_x(X, Y) \times T_x(Y)$ as we recalled in Theorem 2.5 and the remark immediately after that. Thus $A_x(X)/T_x(Y)$ is the ridge of the normal cone $C_x(X, Y)$ and we clearly have (1_τ). Since $T_x(Y)$ is an s-dimensional linear space, we have (1_H) and (1_ν) by Corollary 2.4.

Proposition 4.3. *Let* $\mathfrak{p}$ *be the homogeneous prime ideal of* $S := k \otimes_{\mathcal{O}_Y} \mathrm{gr}_Y(Z)$ *corresponding to the point* x' *of* $\Pi^{-1}(x) = \mathrm{Proj}\,(S)$. *Regard* $\mathfrak{p}$ *also as a point of* $N_x(X, Y) = \mathrm{Spec}\,(S)$. *Then the following equalities hold*:

(3_H) $H_{\mathfrak{p}}^{(l)}(C_x(X, Y)) = H_{x'}^{(l)}(X' \cap \Pi^{-1}(x))$ *for any* $l \geq 0$.

(3_ν) $\nu_{\mathfrak{p}}^*(C_x(X, Y), N_x(Z, Y)) = \nu_{x'}^*(X' \cap \Pi^{-1}(x), \Pi^{-1}(x))$.

(3_τ) $\tau_{\mathfrak{p}}(C_x(X, Y), N_x(Z, Y)) = \tau_{x'}(X' \cap \Pi^{-1}(x), \Pi^{-1}(x))$.

Proof. Again obvious by definition, since for the homogeneous ideal $I := S \cap \mathrm{in}_x(X, Z)$, we have $C_x(X, Y) = \mathrm{Spec}\,(S/I)$, while $X' \cap \Pi^{-1}(x) = \mathrm{Proj}\,(S/I)$.

We have two more stages which are more involved.

Proposition 4.4. (i) *We have the inequality*

(4_H) $H_{x'}^{(l+s+1)}(X' \cap \Pi^{-1}(x)) \geq H_{x'}^{(l)}(X')$ *for any* $l \geq 0$,

where $s+1$ *is the codimension of* $\Pi^{-1}(x)$ *in* Z', *hence* $s = \dim Y$.

(ii) *The equality in* (4_H) *holds for some (hence any)* $l \geq 0$ *if and only if the following equality holds*:

(4_ν) $\nu_{x'}^*(X' \cap \Pi^{-1}(x), \Pi^{-1}(x)) = \nu_{x'}^*(X', Z')$.

(iii) *If the equality in* (4_H) *holds for some (hence any)* $l \geq 0$, *then the following inequality holds*:

(4_τ) $\tau_{x'}(X' \cap \Pi^{-1}(x), \Pi^{-1}(x)) \leq \tau_{x'}(X', Z')$.

(iv) *Suppose the equality in* (4_H) *holds for some (hence any)* $l \geq 0$.

Then the equality in (4_τ) *holds if and only if the canonical surjection* $\mathrm{gr}_{x'}(Z') \to \mathrm{gr}_{x'}(\Pi^{-1}(x))$ *induces an isomorphism of graded* k'*-algebras*

$$\mathfrak{A}_{x'}(X', Z') \xrightarrow{\sim} \mathfrak{A}_{x'}(X' \cap \Pi^{-1}(x), \Pi^{-1}(x))$$

which sends additive forms to additive forms and which sends the ideal $\mathrm{in}_{x'}(X', Z') \cap \mathfrak{A}_{x'}(X', Z')$ *to*

$$\mathrm{in}_{x'}(X' \cap \Pi^{-1}(x), \Pi^{-1}(x)) \cap \mathfrak{A}_{x'}(X' \cap \Pi^{-1}(x), \Pi^{-1}(x)).$$

Remark. As Hironaka [H_4, Example (4.2)] noted, no general inequality holds between the left and right hand sides of (4_ν). This is one of the reasons why the proof of Theorem 3.2 does not parallel that of Theorem 3.1.

Proof. We can find a regular system of parameters $\{u_0, \cdots, u_n, v_1, \cdots, v_s\}$ for $\mathfrak{m}$ such that $\mathfrak{q} = u_0\mathcal{O} + \cdots + u_n\mathcal{O}$ and that $\mathfrak{q}\mathcal{O}' = u_0\mathcal{O}'$. Thus $\mathfrak{m}\mathcal{O}' = u_0\mathcal{O}' + v_1\mathcal{O}' + \cdots + v_s\mathcal{O}'$. Note that $\mathfrak{q}\mathcal{O}'$ is the ideal in $\mathcal{O}'$ defining $\Pi^{-1}(Y)$ at x', while $\mathfrak{m}\mathcal{O}'$ is the one defining $\Pi^{-1}(x)$ at x'. The restriction $X' \to X$ of Π is the blowing up of X along Y. Moreover, by the permissibility of Y, we have $X' \cap \Pi^{-1}(Y) = \mathrm{Proj}\,(\mathrm{gr}_Y(X))$ with $\mathrm{gr}_Y(X)$ flat on Y. Hence $\{u_0, v_1, \cdots, v_s\}$, which is a part of a regular system of parameters for $\mathfrak{m}'$, is also a regular sequence for the $\mathcal{O}'$-module $\mathcal{O}'/J' = \mathcal{O}_{X',x'}$.

(i) follows from Corollary 2.4, (i), while (ii) follows from Corollary 2.4, (ii) and (iii), in view of what we saw above.

Furthermore, in this case we have

$$C_{x'}(X' \cap \Pi^{-1}(x)) = C_{x'}(X') \cap T_{x'}(\Pi^{-1}(x))$$

by the remark immediately after Corollary 2.4. Hence by Lemma 1.4, we have $\tau_{x'}(X' \cap \Pi^{-1}(x), \Pi^{-1}(x)) := \dim \{T_{x'}(\Pi^{-1}(x))/A_{x'}(X' \cap \Pi^{-1}(x))\} \leq \dim \{T_{x'}(Z')/A_{x'}(X')\} =: \tau_{x'}(X', Z')$, which is (iii). Again by Lemma 1.4, the equality holds here if and only if $\mathrm{gr}_{x'}(Z') \to \mathrm{gr}_{x'}(\Pi^{-1}(x))$ induces an isomorphism of graded k'-algebras

$$\mathfrak{A}_{x'}(X', Z') \xrightarrow{\sim} \mathfrak{A}_{x'}(X' \cap \Pi^{-1}(x), \Pi^{-1}(x))$$

between the rings of invariants, which has the required properties. q.e.d.

The final stage is the most involved. We prove them using several lemmas.

Proposition 4.5. *Let* $N_x(Z, Y) = \mathrm{Spec}\,(S)$ *be the normal space of* Z *along* Y *at* x *and let* $C_x(X, Y) = \mathrm{Spec}\,(S/I)$ *be the normal cone of* X *along* Y *at* x. *For the homogeneous prime ideal* $\mathfrak{p}$ *of* S *corresponding to the point*

$x' \in \Pi^{-1}(x) = \mathrm{Proj}(S)$, *let* $R := S_{\mathfrak{p}}$ *be the localization with the maximal ideal* M *and the residue field* $K := R/M$. *Then we have the following*:

(i) *The inequality*

$$(2_H) \quad H_O^{(l)}(C_x(X, Y)) \geq H_{\mathfrak{p}}^{(l+c)}(C_x(X, Y)) \qquad \textit{for all } l \geq 0$$

holds, where O *is the origin of* $C_x(X, Y) \subset N_x(Z, Y)$ *and* $c := \dim \mathrm{Spec}^{\cdot}(S/\mathfrak{p}) = \mathrm{tr.\,deg}_k K$.

(ii) *If the equality in* (2_H) *holds for some (hence any)* l, *then the equality*

$$(2_\nu) \quad \nu_O^*(C_x(X, Y), N_x(Z, Y)) = \nu_{\mathfrak{p}}^*(C_x(X, Y), N_x(Z, Y))$$

and the inequality

$$(2_\tau) \quad \tau_O(C_x(X, Y), N_x(Z, Y)) \leq \tau_{\mathfrak{p}}(C_x(X, Y), N_x(Z, Y))$$

hold. The equality in (2_τ) *holds in this case if and only if the map sending each homogeneous* f *in* S *to its* M*-adic initial form* $\mathrm{in}_M(f)$ *in* $\mathrm{gr}_M(R)$ *induces an isomorphism of graded* K*-algebras*

$$K \otimes_k \mathfrak{A}_O(C_x(X, Y), N_x(Z, Y)) \xrightarrow{\sim} \mathfrak{A}_{\mathfrak{p}}(C_x(X, Y), N_x(Z, Y))$$

which sends additive forms to additive forms and which sends the ideal $K \otimes_k \{\mathrm{in}_O(C_x(X, Y), N_x(Z, Y)) \cap \mathfrak{A}_O(C_x(X, Y), N_x(Z, Y))\}$ *to* $\mathrm{in}_{\mathfrak{p}}(C_x(X, Y), N_x(Z, Y)) \cap \mathfrak{A}_{\mathfrak{p}}(C_x(X, Y), N_x(Z, Y))$.

From now on, let us simply denote

$$N := N_x(Z, Y) = \mathrm{Spec}(S), \qquad C := C_x(X, Y) = \mathrm{Spec}(S/I)$$

and let $A := A_O(C)$ be the ridge of C at the origin O. By definition, R is the local ring of N at its point $\mathfrak{p}$. By Corollary 1.3, we can choose the variables in S so that

$$S = k[\xi_1, \cdots, \xi_\tau, \eta_1, \cdots, \eta_\sigma]$$
$$\mathfrak{A}_O(C, N) := S^A = k[h_1, \cdots, h_\tau]$$

with additive forms in triangular shape

$$h_i = h_i(\xi, \eta) = \xi_i^{q(i)} + \textstyle\sum_{i<j\leq\tau} a_{ij}\xi_j^{q(i)} + \sum_{1\leq j\leq\sigma} b_{ij}\eta_j^{q(i)}$$

for $1 \leq i \leq \tau$, where $q(i) = p^{e(i)}$ for nonnegative integers $e(1) \leq e(2) \leq \cdots \leq e(\tau)$ and a_{ij}, b_{ij} are elements of k. Note that $\tau = \tau_O(C, N) = \mathrm{tr.\,deg}_k S^A$ is the number of ξ_i's as well as that of h_i's. Let us choose, once for all, a standard base $\{f_1, \cdots, f_r\}$ of I normalized with respect to $\{\xi_1, \cdots, \xi_\tau, \eta_1, \cdots, \eta_\sigma\}$. Hence

$$\nu_x^*(X, Z) = \nu_O^*(C, N) = (\nu(1), \cdots, \nu(r), \infty, \infty, \cdots)$$

with $\nu(j) = \deg f_j$ for $1 \leq j \leq r$. Let $S \to S' := S/(\eta_1, \cdots, \eta_\sigma) = k[\xi'_1, \cdots, \xi'_\tau]$ be the projection to the residue polynomial ring sending $f(\xi, \eta)$ in S to $f(\xi', 0)$ where $\xi := (\xi_1, \cdots, \xi_\tau)$, $\eta := (\eta_1, \cdots, \eta_\sigma)$ and $\xi' := (\xi'_1, \cdots, \xi'_\tau)$. Denote by I' the image of I in S'. Then again by Corollary 1.3, S^A is generated as a k-algebra by $\{\partial'_m f_j$; all $1 \leq j \leq r$ and all τ-tuples m of nonnegative integers not in $\exp(I'; \xi')$ with $|m| < \nu(j)\}$, while $\partial'_m f_j = 0$ if m is in $\exp(I'; \xi')$ with $|m| < \nu(j)$. Here $\exp(I'; \xi')$ is the exponent of the ideal I' with respect to the variables ξ'. Hence there exist isobaric polynomials $\psi_1, \cdots, \psi_r$ of weights $\nu(1), \cdots, \nu(r)$, respectively, in τ variables $h_1, \cdots, h_\tau$ of respective weights $q(1), \cdots, q(\tau)$ such that

$$f_j = \psi_j(h_1, \cdots, h_\tau) \qquad \text{for } 1 \leq j \leq r,$$

while there exist polynomials $\varphi_1, \cdots, \varphi_\tau$ in variables $\zeta_{m,j}$ of weight $\nu(j) - |m|$ with $1 \leq j \leq r$ and with m running through τ-tuples of nonnegative integers not in $\exp(I'; \xi')$ satisfying $|m| < \nu(j)$ such that each φ_i is isobaric of weight $q(i)$ and that

$$h_i = \varphi_i(\cdots, \partial'_m f_j, \cdots) \qquad \text{for } 1 \leq i \leq \tau,$$

if we put $\partial'_m f_j$ in place of the variable $\zeta_{m,j}$.

We fix these notations throughout the rest of this section. As in Corollary 1.3, let us denote

$$S^\# := k[\eta_1, \cdots, \eta_\sigma] \quad \text{and} \quad N^\# := \mathrm{Spec}(S^\#).$$

The injection $S^\# \subset S$ induces a canonical morphism $N \to N^\#$ which sends O and $\mathfrak{p}$ to $O^\#$ and $\mathfrak{p}^\#$, respectively, where the prime ideal $\mathfrak{p}^\# := \mathfrak{p} \cap S^\#$ is regarded also as a point of $N^\#$, while $O^\#$ is the origin of the linear space $N^\#$ corresponding to the maximal ideal $S^\#_+$. We denote by $R^\#$ the local ring of $N^\#$ at $\mathfrak{p}^\#$ with the maximal ideal $M^\#$ and the residue field $K^\# := R^\#/M^\#$. Thus $R^\#$ is the localization of $S^\#$ with respect to $\mathfrak{p}^\#$ and $K^\#$ is the field of fractions of $S^\#/\mathfrak{p}^\#$.

We denote by $N' := \mathrm{Spec}(S')$, $C' := \mathrm{Spec}(S'/I')$ and A' the fibers over $O^\# \in N^\#$ of N and its closed subschemes C and A, respectively, i.e., the base extensions with respect to $\mathrm{Spec}(k) \to N^\#$ at the closed point $O^\#$. N' is a linear subspace of N passing through O and C' is a cone in it.

We have an occasion later to consider also the fibers N'', C'' and A'' over $\mathfrak{p}^\# \in N^\#$ of N, C and A, that is, the base extensions with respect to $\mathrm{Spec}(K^\#) \to N^\#$ at the point $\mathfrak{p}^\#$. Thus $N'' = \mathrm{Spec}(S'')$ with $S'' := S \otimes_{S^\#} K^\#$.

Lemma 4.6 (cf. [G_2, 3.3.2 and 3.3.3]). *In the above notations, we have the equalities*

$$H_O^{(0)}(C)=H_O^{(\sigma)}(C'),\qquad \nu_O^*(C,N)=\nu_O^*(C',N').$$

Furthermore, A' is the ridge of C' and $S\to S'$ induces an isomorphism of graded k-algebras $\mathfrak{A}_O(C,N)\xrightarrow{\sim}\mathfrak{A}_O(C',N')$ which sends additive forms to additive forms and $I\cap\mathfrak{A}_O(C,N)$ to $I'\cap\mathfrak{A}_O(C',N')$. In particular, we have the equality $\tau_O(C,N)=\tau_O(C',N')$.

The proof is obvious by Corollaries 1.3 and 2.4.

Lemma 4.7 (cf. [G_2, Lemme 5.2.2]). *In the above notations, suppose the equality $H_O^{(0)}(C)=H_{\mathfrak{p}}^{(c)}(C)$ in (2_H) holds. Then we have the following equalities for the M-adic orders*:

$$\begin{aligned}\mathrm{ord}_M(f_j)&=\nu(j):=\deg f_j, &&1\le j\le r\\ \mathrm{ord}_M(h_i)&=q(i):=\deg h_i, &&1\le i\le\tau.\end{aligned}$$

Proof. The equalities $\mathrm{ord}_M(f_j)=\nu(j)$ for $1\le j\le r$ imply $\mathrm{ord}_M(h_i)=q(i)$ for $1\le i\le\tau$. Indeed, there exist isobaric polynomials $\varphi_1,\cdots,\varphi_\tau$ of weights $q(1),\cdots,q(\tau)$ such that

$$h_i=\varphi_i(\cdots,\partial'_m f_j,\cdots)$$

and that $\partial'_m f_j$ is given the weight $\nu(j)-|m|$, which is exactly the M-adic order of $\partial'_m f_j$. Hence $\mathrm{ord}_M(h_i)=q(i)$.

Obviously, $\mathrm{ord}_M(f_j)\le\deg f_j$ by the Jacobian criterion, for instance. Hence it remains to show $\mathrm{ord}_M(f_j)\ge\deg f_j$.

For that purpose, let us apply Theorem 2.6 to the scheme C of finite type over k. We can find a subfield k' of k with k separable over k' and $\dim_k\Omega^1_{k/k'}$ finite. Then for any point $w\in C$, we have

$$H_w^{(1+t(\kappa(w)/k'))}(C)(l)=\dim_{\kappa(w)}\{\kappa(w)\otimes_{\mathcal{O}_C}\mathcal{P}^l_{C/k'}\}$$

for all $l\ge0$, where $t(\kappa(w)/k')=\mathrm{tr.deg}_k\,\kappa(w)+\dim_k\Omega^1_{k/k'}$. For $w=O$, we have $\kappa(O)=k$ and $t(\kappa(O)/k')=\dim_k\Omega^1_{k/k'}$, while for $w=\mathfrak{p}$, we have $\kappa(\mathfrak{p})=K$ and $t(\kappa(\mathfrak{p})/k')=c+t(\kappa(O)/k')$. Thus the equality $H_O^{(0)}(C)=H_{\mathfrak{p}}^{(c)}(C)$ in (2_H) implies

$$\dim_{\kappa(O)}\{\kappa(O)\otimes_{\mathcal{O}_C}\mathcal{P}^l_{C/k'}\}=\dim_{\kappa(\mathfrak{p})}\{\kappa(\mathfrak{p})\otimes_{\mathcal{O}_C}\mathcal{P}^l_{C/k'}\}$$

for all $l\ge0$. The equality in (2_H) also guarantees that x' is in $X'\cap\Pi^{-1}(x)=\mathrm{Proj}\,(S/I)$, hence $\mathfrak{p}$ is in C. Since $\mathfrak{p}$ is a homogeneous ideal as well, $D:=\mathrm{Spec}\,(S/\mathfrak{p})$ contains O and is contained in C. The above equality thus means that $\mathcal{O}_D \otimes_{\mathcal{O}_C} \mathscr{P}^l_{C/k'}$ is $\mathcal{O}_D$-flat for all $l \geq 0$.

We have canonical homomorphisms for the modules of principal parts

$$\mathcal{O}_N \xrightarrow{j^l_{N/k'}} \mathscr{P}^l_{N/k'} \longrightarrow \mathscr{P}^l_{C/k'},$$

for all $l \geq 0$, where $j^l_{N/k'}$ is the jet map or the Taylor expansion map which can be described as follows: Let $c_1, \cdots, c_t$ be elements of k such that their differentials $dc_1, \cdots, dc_t$ in $\Omega^1_{k/k'}$ form a k-basis. Then $\mathscr{P}^l_{N/k'}$ is isomorphic to the truncated polynomial ring over $\mathcal{O}_N$ in three sets of variables $d\xi = (d\xi_1, \cdots, d\xi_\tau)$, $d\eta = (d\eta_1, \cdots, d\eta_\sigma)$ and $dc = (dc_1, \cdots, dc_t)$ of the form

$$\mathscr{P}^l_{N/k'} = \mathcal{O}_N[d\xi, d\eta, dc]/(d\xi, d\eta, dc)^{l+1},$$

which is a free $\mathcal{O}_N$-module with a basis consisting of

$$(d\xi)^\lambda (d\eta)^\mu (dc)^\nu := \textstyle\prod_{1 \leq i \leq \tau} (d\xi_i)^{\lambda_i} \prod_{1 \leq j \leq \sigma} (d\eta_j)^{\mu_j} \prod_{1 \leq n \leq t} (dc_n)^{\nu_n},$$

where $\lambda := (\lambda_1, \cdots, \lambda_\tau)$, $\mu := (\mu_1, \cdots, \mu_\sigma)$ and $\nu := (\nu_1, \cdots, \nu_t)$ run through τ-tuples, σ-tuples and t-tuples of nonnegative integers satisfying

$$l \geq |\lambda| + |\mu| + |\nu| := \textstyle\sum_{1 \leq i \leq \tau} \lambda_i + \sum_{1 \leq j \leq \sigma} \mu_j + \sum_{1 \leq n \leq t} \nu_n.$$

Correspondingly, we have the Taylor coefficient differential operators $\partial'_\lambda \partial''_\mu \partial_\nu$ from $\mathcal{O}_N$ to itself over k' so that for $f \in \mathcal{O}_N$ we have

$$j^l_{N/k'}(f) = \textstyle\sum_{|\lambda| + |\mu| + |\nu| \leq l} (\partial'_\lambda \partial''_\mu \partial_\nu f)(d\xi)^\lambda (d\eta)^\mu (dc)^\nu.$$

As we remarked immediately after Theorem 2.6, we may choose the subfield k' in such a way that $\kappa(\mathfrak{p}) = K$ is also separable over k'. Then by the Jacobian criterion, we have $\mathrm{ord}_M(f_j) \geq \deg f_j$ if and only if $\partial'_\lambda \partial''_\mu \partial_\nu f_j$ is in M for all $l < \deg f_j$, λ, μ and ν satisfying $|\lambda| + |\mu| + |\nu| \leq l$, that is, the image of $j^l_{N/k'}(f_j)$ under the canonical surjection

$$\mathscr{P}^l_{N/k'} \longrightarrow \kappa(\mathfrak{p}) \otimes_{\mathcal{O}_N} \mathscr{P}^l_{N/k'}$$

vanishes for all $l < \deg f_j$. To see that this latter condition is satisfied, consider the commutative diagram

$$\begin{array}{ccccc} \mathscr{P}^l_{N/k'} & \longrightarrow & \mathcal{O}_D \otimes_{\mathcal{O}_N} \mathscr{P}^l_{N/k'} & \longrightarrow & \kappa(\mathfrak{p}) \otimes_{\mathcal{O}_N} \mathscr{P}^l_{N/k'} \\ \rho \downarrow & & \downarrow & & \\ \mathscr{P}^l_{C/k'} & \longrightarrow & \mathcal{O}_D \otimes_{\mathcal{O}_N} \mathscr{P}^l_{C/k'} & & \end{array}$$

and the direct summand $\mathcal{O}_N$-submodule $\mathscr{H}^l$ of $\mathscr{P}^l_{N/k'}$ defined by

$$\mathscr{H}^l := \bigoplus \mathcal{O}_N (d\xi)^\lambda (d\eta)^\mu (dc)^\nu,$$

with λ, μ, ν running through τ-tuples, σ-tuples and t-tuples of nonnegative integers, respectively, satisfying

$$|\lambda|+|\mu|+|\nu|\leq l \text{ with } (\lambda, \mu) \text{ not in the exponent } \exp(I;\xi,\eta),$$

where $\exp(I; \xi, \eta)$ is the exponent of the ideal I with respect to the variables (ξ, η) of S as we defined in Section 1.

Since the standard base $\{f_1, \cdots, f_r\}$ is chosen to be normalized with respect to (ξ, η), we see by Proposition 1.2 that $\partial'_\lambda \partial''_\mu f_j = 0$ if (λ, μ) is in $\exp(I; \xi, \eta)$ and $|\lambda|+|\mu|<\deg f_j$. Hence

$$j^l_{N/k'}(f_j) \in \mathscr{H}^l \qquad \text{for } l<\deg f_j \text{ and } 1\leq j\leq r.$$

The kernel of the left hand column $\rho: \mathscr{P}^l_{N/k'} \to \mathscr{P}^l_{C/k'}$ of the above commutative diagram is the ideal generated by $j^l_{N/k'}(f_1), \cdots, j^l_{N/k'}(f_r)$. The restriction of ρ to $\mathscr{H}^l$ induces an isomorphism

$$\kappa(O)\otimes_{\mathcal{O}_N}\mathscr{H}^l \xrightarrow{\sim} \kappa(O)\otimes_{\mathcal{O}_N}\mathscr{P}^l_{C/k'}.$$

Moreover, if $l<\deg f_j$, then the image of $j^l_{N/k'}(f_j) \in \mathscr{H}^l$ in $\mathscr{P}^l_{C/k'}$, hence that in $\mathcal{O}_D\otimes_{\mathcal{O}_C}\mathscr{P}^l_{C/k'}$, vanish. The middle column $\mathcal{O}_D\otimes_{\mathcal{O}_N}\mathscr{P}^l_{N/k'} \to \mathcal{O}_D\otimes_{\mathcal{O}_N}\mathscr{P}^l_{C/k'}$, however, induces an isomorphism

$$\mathcal{O}_D\otimes_{\mathcal{O}_N}\mathscr{H}^l \xrightarrow{\sim} \mathcal{O}_D\otimes_{\mathcal{O}_N}\mathscr{P}^l_{C/k'},$$

since both sides are $\mathcal{O}_D$-flat of finite rank as we saw above and since it induces an isomorphism when tensored with $\kappa(O)$ over $\mathcal{O}_N$. Consequently, if $l<\deg f_j$, then the image of $j^l_{N/k'}(f_j)$ in $\mathcal{O}_D\otimes_{\mathcal{O}_N}\mathscr{H}^l$, hence that in $\kappa(\mathfrak{p})\otimes_{\mathcal{O}_N}\mathscr{P}^l_{N/k'}$, vanish. q.e.d.

Lemma 4.8 (cf. [G_2, Corollaire 5.3]). *Suppose we have the equalities*

$$\operatorname{ord}_M(f_j)=\nu(j):=\deg f_j, \qquad 1\leq j\leq r,$$
$$\operatorname{ord}_M(h_i)=q(i):=\deg h_i, \qquad 1\leq i\leq \tau.$$

Then the map sending a homogeneous polynomial h in S to its M-adic initial form $\operatorname{in}_M(h)$ *gives rise to a homomorphism of graded rings*

$$\operatorname{in}_M: \mathfrak{A}_O(C, N) \longrightarrow \operatorname{gr}_\mathfrak{p}(N) := \operatorname{gr}_M(R).$$

Moreover, the canonical injection $K^\# \to K$ *of the residue fields at* $\mathfrak{p}^\#$ *and* $\mathfrak{p}$ *is an isomorphism. Furthermore, if we denote by* N' *and* N'' *the fibers of* $N \to N^\#$ *over* $O^\#$ *and* $\mathfrak{p}^\#$, *respectively, then there exists a unique point* $\mathfrak{p}''$ *of* N'' *lynig above* $\mathfrak{p}$ *in* N *and we have an isomorphism*

$$N'\times_k K^{\#}\xrightarrow{\sim} N'',$$

from the base extension of N' from k to $K^{\#}$, which sends the origin O to $\mathfrak{p}''$ and which also induces the following isomorphisms for the fibers C', A' (resp. C'', A'') *of C and A over $O^{\#}$* (resp. $\mathfrak{p}^{\#}$):

$$C'\times_k K^{\#}\xrightarrow{\sim} C'',\qquad A'\times_k K^{\#}\xrightarrow{\sim} A''.$$

Proof. The first assertion is obvious, since $\mathfrak{A}_O(C, N)=k[h_1, \cdots, h_\tau]$ and $\mathrm{ord}_M(h_i)=q(i)=\deg h_i$. As for the second, $A=\mathrm{Spec}\,(S/(h_1, \cdots, h_\tau))$ is finite and flat over $N^{\#}=\mathrm{Spec}\,(S^{\#})$ of degree $\prod_{1\le i\le\tau} q(i)$, since $S=k[\xi, \eta]$, $S^{\#}=k[\eta]$ and $h_1, \cdots, h_\tau$ are additive forms of degrees $q(1), \cdots, q(\tau)$ in triangular shape. At the point $\mathfrak{p}$, however, the length of $R/(h_1, \cdots, h_\tau)$ is already $\prod_{1\le i\le\tau} q(i)$, since $\mathrm{ord}_M(h_i)=q(i)$ for $1\le i\le\tau$. Hence $\mathfrak{p}$ is necessarily the unique point of A lying over $\mathfrak{p}^{\#}$ with the trivial residue field extension. Thus for each $1\le i\le\tau$ there exists $\alpha_i\in R^{\#}$ such that $\xi_i^*:=\xi_i-\alpha_i$ is in M. We denote $\xi^*:=(\xi_1^*, \cdots, \xi_\tau^*)$ and $\alpha:=(\alpha_1, \cdots, \alpha_\tau)$. Since h_i is an additive form, we have $h_i(\xi, \eta)=h_i(\xi^*+\alpha, \eta)=h_i(\xi^*, 0)+h_i(\alpha, \eta)$. Obviously, $h_i(\xi^*, 0)$ is in $M^{q(i)}$, while $h_i(\xi, \eta)$ is in $M^{q(i)}$ by assumption. Hence $h_i(\alpha, \eta)$ is in $M^{q(i)}\cap R^{\#}\subset M^{\#}$.

We know that there exist polynomials $\psi_1, \cdots, \psi_r$ in τ variables such that

$$f_j(\xi, \eta)=\psi_j(h_1(\xi, \eta), \cdots, h_\tau(\xi, \eta)),\qquad 1\le j\le r.$$

Thus their images in the ring $S'':=S\otimes_{S^{\#}}K^{\#}$ are

$$f_j(\xi'', 0)=\psi_j(h_1(\xi'', 0), \cdots, h_\tau(\xi'', 0)),\qquad 1\le j\le r,$$

where $\xi'':=(\xi_1'', \cdots, \xi_\tau'')$ with the image ξ_i'' of ξ_i^* in S''. Obviously, $S''=K^{\#}[\xi_1'', \cdots, \xi_\tau'']$ is a polynomial ring and we have an isomorphism of $K^{\#}$-algebras

$$S'\otimes_k K^{\#}=K^{\#}[\xi_1', \cdots, \xi_\tau']\xrightarrow{\sim} S''$$

sending ξ_i' to ξ_i''. We have $C'=\mathrm{Spec}\,(S'/(f_1(\xi', 0), \cdots, f_r(\xi', 0)))$ and $A'=\mathrm{Spec}\,(S'/(h_1(\xi', 0), \cdots, h_\tau(\xi', 0)))$, while $C''=\mathrm{Spec}\,(S''/(f_1(\xi'', 0), \cdots, f_r(\xi'', 0)))$ and $A''=\mathrm{Spec}\,(S''/(h_1(\xi'', 0), \cdots, h_\tau(\xi'', 0)))$. q.e.d.

Corollary 4.9. *If we have the equality $H_O^{(0)}(C)=H_{\mathfrak{p}}^{(c)}(C)$ in (2_H) with $c=\mathrm{tr.\,deg}_k K$, then we have*

$$\sigma=c+\mathrm{height}\,(\mathfrak{p}^{\#}),\qquad H_O^{(\sigma)}(C')=H_{\mathfrak{p}''}^{(\sigma)}(C'')$$

$$\nu_O^*(C', N')=\nu_{\mathfrak{p}''}^*(C'', N''),\qquad \tau_O(C', N')=\tau_{\mathfrak{p}''}(C'', N'')$$

and an isomorphism of graded $K^{\#}$-algebras

$$\mathfrak{A}_O(C', N')\otimes_k K^{\#} \xrightarrow{\sim} \mathfrak{A}_{\mathfrak{p}''}(C'', N'')$$

which sends additive forms to additive forms and which sends

$$\{\mathrm{in}_O(C', N')\cap \mathfrak{A}_O(C', N')\}\otimes_k K^{\#} \quad to \quad \mathrm{in}_{\mathfrak{p}''}(C'', N'')\cap \mathfrak{A}_{\mathfrak{p}''}(C'', N'').$$

Proof. By Lemma 4.7, the assumption here implies that of Lemma 4.8. Thus by Lemma 4.8, we have $c=\mathrm{tr.deg}_k K=\mathrm{tr.deg}_k K^{\#}$, which is well known to equal $\sigma-\mathrm{height}(\mathfrak{p}^{\#})$, since $\sigma=\dim S^{\#}$. The remaining assertions are clear by Lemma 4.8. q.e.d.

Proof of Proposition 4.5. (i) follows from the upper-semicontinuity of the Hilbert-Samuel functions in Theorem 2.5, (i). Note that Theorem 2.6 guarantees the inequality (2_H) only for $l\geq \dim_k \Omega^1_{k/k'}$ for the k' appearing in the proof of Lemma 4.7.

To show (ii), assume that the equality $H^{(0)}_O(C)=H^{(c)}_{\mathfrak{p}}(C)$ in (2_H) is satisfied. Then by Lemma 4.6 and Corollary 4.9, we have

$$H^{(c)}_{\mathfrak{p}}(C)=H^{(0)}_O(C)=H^{(\sigma)}_O(C')=H^{(\sigma)}_{\mathfrak{p}''}(C'')$$

and $\sigma-c=\mathrm{height}(\mathfrak{p}^{\#})=:\rho$. Here $\mathfrak{p}''$ is the unique point of N'' lying above $\mathfrak{p}\in N$. Clearly, the local ring $R/RM^{\#}$ of N'' at $\mathfrak{p}''$ coincides with the local ring at $\mathfrak{p}$ of the closed subscheme $W:=\mathrm{Spec}(S/S\mathfrak{p}^{\#})$ of N. The morphism $N\to N^{\#}$ is smooth, hence W is regular at $\mathfrak{p}$ of codimension ρ in N and $RM^{\#}$ is generated by a regular system of parameters $\{\theta_1, \cdots, \theta_\rho\}$ of $M^{\#}$. We have

$$T_{\mathfrak{p}}(N)\supset T_{\mathfrak{p}}(W)\xleftarrow{\sim} T_{\mathfrak{p}''}(N'')$$

for the tangent spaces, as well as

$$C_{\mathfrak{p}}(C)\supset C_{\mathfrak{p}}(C\cap W)\xleftarrow{\sim} C_{\mathfrak{p}''}(C'')$$

for the tangent cones.

By these results, we see that $H^{(c)}_{\mathfrak{p}}(C)=H^{(c+\rho)}_{\mathfrak{p}}(C\cap W)$ and Corollary 2.4, (ii) is applicable. Thus $C_{\mathfrak{p}}(C\cap W)=C_{\mathfrak{p}}(C)\cap T_{\mathfrak{p}}(W)$ and $\nu^*_{\mathfrak{p}}(C, N)=\nu^*_{\mathfrak{p}}(C\cap W, W)=\nu^*_{\mathfrak{p}''}(C'', N'')$, which equals $\nu^*_O(C', N')=\nu^*_O(C, N)$ by Corollary 4.9 and Lemma 4.6. Hence we get the equality (2_ν).

Let $\xi^*:=(\xi^*_1, \cdots, \xi^*_\tau)$ be as in the proof of Lemma 4.8 and let $\theta:=(\theta_1, \cdots, \theta_\rho)$ be as above. Then (ξ^*, θ) is a regular system of parameters for M. Thus $\mathrm{gr}_M(R)$ is the polynomial ring over K with variables $\Xi:=(\Xi_1, \cdots, \Xi_\tau)$ and $\Theta:=(\Theta_1, \cdots, \Theta_\rho)$, where

$$\Xi_i := \mathrm{in}_M(\xi_i^*) \qquad \text{for } 1 \leq i \leq \tau \quad \text{and}$$
$$\Theta_j := \mathrm{in}_M(\theta_j) \qquad \text{for } 1 \leq j \leq \rho.$$

As in the proof of Lemm 4.8, we have $h_i(\xi, \eta) = h_i(\xi^*, 0) + h_i(\alpha, \eta)$ with $\mathrm{ord}_M(h_i(\xi^*, 0)) = \mathrm{ord}_M(h_i(\alpha, \eta)) = q(i)$ for $1 \leq i \leq \tau$. We see that $a_i' := \mathrm{in}_M(h_i(\xi^*, 0)) = h_i(\Xi, 0)$ is an additive form in Ξ of degree $q(i)$ in triangular shape, while $b_i' := \mathrm{in}_M(h_i(\alpha, \eta))$ is a homogeneous polynomial in Θ of degree $q(i)$. Since $f_j = \psi_j(h_1, \cdots, h_\tau)$, we see that

$$\mathrm{in}_M(f_j) = \psi_j(a_1' + b_1', \cdots, a_\tau' + b_\tau') \qquad \text{for } 1 \leq j \leq r.$$

Thus for each τ-tuple m of nonnegative integers, we have

$$\mathrm{in}_M(\partial_m' f_j) = \partial_m'(\mathrm{in}_M(f_j)) \qquad \text{for } 1 \leq j \leq r,$$

where ∂_m' on the left (resp. right) hand side is the Taylor coefficient differential operator with respect to ξ^m (resp. Ξ^m). We also have

$$\mathrm{in}_M(h_i) = \varphi_i(\cdots, \partial_m'(\mathrm{in}_M(f_j)), \cdots) \qquad \text{for } 1 \leq i \leq \tau.$$

Moreover, $K^\# = K$ by Lemma 4.8, and the canonical surjection $\mathrm{gr}_M(R) = K[\Xi, \Theta] \to S'' = K[\xi'']$ sends $\mathrm{in}_M(f_j)$ to $f_j(\xi'', 0)$ for $1 \leq j \leq r$. By Lemma 4.8, $\{f_j(\xi'', 0); 1 \leq j \leq r\}$ is a standard base of $\mathrm{in}_{\mathfrak{p}''}(C'', N'')$ quasi-normalized with respect to ξ'' and $\mathfrak{A}_{\mathfrak{p}''}(C'', N'') = K[h_1(\xi'', 0), \cdots, h_\tau(\xi'', 0)]$.

Even though b_{ij}''s may not be additive forms in Θ, the proof of Corollary 1.3 works here, and we conclude that $\{\mathrm{in}_M(f_j); 1 \leq j \leq r\}$ is a standard base of $\mathrm{in}_\mathfrak{p}(C, N)$ quasi-normalized with respect to (Ξ, Θ). In particular, $\mathfrak{A}_\mathfrak{p}(C, N)$ contains $\mathrm{in}_M(f_j)$ for $1 \leq j \leq r$, hence contains $\mathrm{in}_M(h_i) = \varphi_i(\cdots, \partial_m'(\mathrm{in}_M(f_j)), \cdots)$ for $1 \leq i \leq \tau$, since $\mathfrak{A}_\mathfrak{p}(C, N)$ is stable under ∂_m'.

By Lemma 1.4, we have $\tau_\mathfrak{p}(C, N) \geq \tau_\mathfrak{p}(C \cap W, W) = \tau_{\mathfrak{p}''}(C'', N'')$, which equals $\tau_O(C', N') = \tau_O(C, N)$ again by Corollary 4.9 and Lemma 4.6, hence we get the inequality (2_τ). Moreover, we have homomorphisms of graded K-algebras $\mathfrak{A}_\mathfrak{p}(C, N) \to \mathrm{gr}_\mathfrak{p}(W) \supset \mathfrak{A}_\mathfrak{p}(C \cap W, W) \rightleftarrows \mathfrak{A}_{\mathfrak{p}''}(C'', N'') = K[h_1(\xi'', 0), \cdots, h_\tau(\xi'', 0)]$, as well as isomorphisms of graded k-algebras $\mathfrak{A}_O(C, N) = k[h_1(\xi, \eta), \cdots, h_\tau(\xi, \eta)] \rightrightarrows \mathfrak{A}_O(C', N') = k[h_1(\xi', 0), \cdots, h_\tau(\xi', 0)]$. By Lemma 1.4, the equality in (2_τ) is equivalent to the equality $\tau_\mathfrak{p}(C, N) = \tau_\mathfrak{p}(C \cap W, W)$, which holds if and only if the above homomorphisms induce a composite isomorphism $\mathfrak{A}_\mathfrak{p}(C, N) \rightrightarrows \mathfrak{A}_\mathfrak{p}(C \cap W, W) \rightleftarrows \mathfrak{A}_{\mathfrak{p}''}(C'', N'')$ which sends additive forms to additive forms and which sends $\mathrm{in}_\mathfrak{p}(C, N) \cap \mathfrak{A}_\mathfrak{p}(C, N)$ to $\mathrm{in}_{\mathfrak{p}''}(C'', N'') \cap \mathfrak{A}_{\mathfrak{p}''}(C'', N'')$. This is certainly the case if and only if $\mathrm{in}_M(h_i)$, which we have already shown to be in $\mathfrak{A}_\mathfrak{p}(C, N)$, is an additive form for $1 \leq i \leq \tau$, since its image $h_i(\xi'', 0)$ is an additive form and since

$$\mathfrak{A}_{\mathfrak{p}''}(C'', N'')=K[h_1(\xi'', 0), \cdots, h_t(\xi'', 0)]. \qquad \text{q.e.d.}$$

We now combine the results obtained so far in this section to prove most of the assertions in Section 3.

Proof of Theorem 3.1. Since $d:=\text{tr. deg}_k k'$ and $c:=\text{tr. deg}_k K$ satisfy $c-1=d$, we have the chain of equalities and inequalities for all $j\geq 0$

$$\begin{aligned} H_x^{(j)}(X)=H_O^{(j)}(C_x(X))=H_O^{(j+s)}(C_x(X, Y))\geq H_{\mathfrak{p}}^{(j+s+c)}(C_x(X, Y)) \\ =H_{x'}^{(j+s+c)}(X'\cap \Pi^{-1}(x))\geq H_{x'}^{(j+d)}(X') \end{aligned}$$

successively by Proposition 4.1, (0_H), by Proposition 4.2, (1_H), by Proposition 4.5, (2_H), by Proposition 4.3, (3_H) and by Proposition 4.4, (4_H), where $s=\dim Y$. q.e.d.

Remark. The above proof shows that x' is an *infinitely near point* of x if and only if the equalities in (2_H) and (4_H) are satisfied.

Proof of Proposition 3.4. Since x' is assumed to be infinitely near to x, we have the equality in (2_H), hence we have a homomorphism in_M of graded rings from

$$\mathfrak{A}_x(X, Z)=\mathfrak{A}_O(C_x(X), T_x(Z))=\mathfrak{A}_O(C_x(X, Y), N_x(Z, Y))$$

to $\text{gr}_M(R)$ by Proposition 4.1, (0_τ), by Proposition 4.2, (1_τ), by the remark above and by Lemmas 4.7 and 4.8. q.e.d.

Proof of Theorem 3.3. Since x' is assumed to be infinitely near to x, we have the equalities in (2_H) and (4_H) by the above remark. Hence we have the chain of equalities and inequalities

$$\begin{aligned} \tau_x(X, Z)=\tau_O(C_x(X), T_x(Z))=\tau_O(C_x(X, Y), N_x(Z, Y)) \\ \leq \tau_{\mathfrak{p}}(C_x(X, Y), N_x(Z, Y))=\tau_{x'}(X'\cap \Pi^{-1}(x), \Pi^{-1}(x))\leq \tau_{x'}(X', Z') \end{aligned}$$

successively by Proposition 4.1, (0_τ), by Proposition 4.2, (1_τ), by Proposition 4.5, (2_τ), by Proposition 4.3, (3_τ) and by Proposition 4.4, (4_τ). q.e.d.

Remark. The above proof and the remark immediately after the proof of Theorem 3.1 show that x' is an *infinitely very near point* of x if and only if the equalities in (2_H), (2_τ), (4_H) and (4_τ) are satisfied.

Proof of Theorem 3.5. As we saw above, x' is infinitely near to x if and only if the equalities in (2_H) and (4_H) are satisfied. Under the circumstances, x' is infinitely very near to x if and only if the equalities in

(2_τ) and (4_τ) hold. By Proposition 4.1 (0_τ) and by Proposition 4.2, (1_τ), we have

$$\mathfrak{A}_x(X, Z)=\mathfrak{A}_O(C_x(X),\ T_x(Z))=\mathfrak{A}_O(C_x(X, Y),\ N_x(Z, Y)).$$

On the one hand, Proposition 4.5, (ii) shows that under the equality in (2_H), the equality in (2_τ) holds if and only if in_M induces an isomorphism of graded K-algebras

$$K\otimes_k \mathfrak{A}_O(C_x(X, Y), N_x(Z, Y)) \xrightarrow{\sim} \mathfrak{A}_{\mathfrak{p}}(C_x(X, Y), N_x(Z, Y))$$

having the required properties. On the other hand, Proposition 4.4, (iv) shows that under the equality in (4_H), the equality in (4_τ) holds if and only if we have an isomorphism of graded k'-algebras

$$\mathfrak{A}_{x'}(X' \cap \Pi^{-1}(x), \Pi^{-1}(x)) \xleftarrow{\sim} \mathfrak{A}_{x'}(X', Z')$$

having the required properties. q.e.d.

Corollary 3.7 follows immediately from Theorem 3.6 and the proof of Propositions 4.4 and 4.5 as well as that of Theorem 3.5 above.

Proof of a part of Theorem 3.2. Hironaka [H_4, Proposition 21, Lemmas 23 and 24 and Corollary 23.2] showed the first assertion and the "only if" part of the second assertion simultaneously in the following form (see also Singh [S_3] and [HO]): If $\nu_x^*(X, Z) \le \nu_{x'}^*(X', Z')$, then $\nu_x^*(X, Z)=\nu_{x'}^*(X', Z')$ and the equalities in (2_H) and (4_H) are satisfied. Note that our lexicographic order for ν^* is a total order.

We do not reproduce the proof of this part here, since it is long and since we do not use it directly in this paper.

We here show the "if" part of the second assertion. If x' is infinitely near to x, then the equalities in (2_H) and (4_H) hold by the remark immediately after the proof of Theorem 3.1. Then the equalities in (2_ν) and (4_ν) hold so that we have the chain of equalities

$$\begin{aligned}\nu_x^*(X, Z)&=\nu_O^*(C_x(X), T_x(Z))=\nu_O^*(C_x(X, Y), N_x(Z, Y))\\ &=\nu_{\mathfrak{p}}^*(C_x(X, Y), N_x(Z, Y))=\nu_{x'}^*(X' \cap \Pi^{-1}(x), \Pi^{-1}(x))=\nu_{x'}^*(X', Z')\end{aligned}$$

by Proposition 4.1, (0_ν), by Proposition 4.2, (1_ν), by Proposition 4.5, (ii), by Proposition 4.3, (3_ν) and by Proposition 4.4, (ii). q.e.d.

References

[B] B.-M. Bennett, On the characterization of a local ring, Ann. of Math., **91** (1970), 25–87.

[EGA] A. Grothendieck and J. Dieudonné, Élément de Géométrie Algébrique Chapter $0_{IV,1}$, Inst. Hautes Études Sci. Publ. Math., **20**, 1964.

[G_1] J. Giraud, Etude locale des singularités, Cours de 3ème Cycle, Univ. de Paris XI, U.E.R. Math. 91-Orsay, **26** (1971–72), (mimeographed).

[G_2] ——, Contact maximal en caractéristique positive, Ann. Sci. Ecole Norm. Sup., (4) **8** (1975), 201–234.

[H_1] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, I, II, Ann. of Math., **79** (1964), 109–326.

[H_2] ——, Desingularization of excellent surfaces (Notes by B.-M. Bennett), Advanced Science Seminar in Algebraic Geometry, Bowdoin College, Summer, 1967. Reproduced in Springer Lecture Notes, **1101** (1984).

[H_3] ——, Additive groups associated with points of a projective space, Ann. of Math., **92** (1970), 327–334.

[H_4] ——, Certain numerical characters of singularities, J. Math. Kyoto Univ., **10** (1970), 151–187.

[HO] M. Herrmann and U. Orbanz, Remark on a paper by B. Singh on certain numerical characters of singularities, J. Pure Appl. Algebra, **24** (1982), 151–156.

[O_1] T. Oda, Hironaka's additive group scheme, in Number Theory, Algebraic Geometry and Commutative Algebra in honor of Y. Akizuki (Y. Kusunoki et al., eds.) Kinokuniya, Tokyo, 1973, 181–219.

[O_2] ——, Hironaka's additive group scheme, II, Publ. Res. Inst. Math. Sci., Kyoto Univ., **19** (1983), 1163–1179.

[O_3] ——, Hironaka group schemes and resolution of singularities, in Algebraic Geometry, Proc. of Japan-France Conf., Tokyo and Kyoto 1982 (M. Raynaud and T. Shioda, eds.), Lecture Notes in Math. **1016**, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983, 295–312.

[O_4] ——, Additive forms having additive initial forms, to appear in a volume dedicated to the late Takehiko Miyata.

[S_1] B. Singh, Effect of a permissible blowing-up on the local Hilbert functions, Invent. Math., **26** (1974), 201–212.

[S_2] ——, A numerical criterion for the permissibility of a blowing-up, Compositio Math., **33** (1976), 15–28.

[S_3] ——, Relations between certain numerical characters of singularities, J. Pure Appl. Algebra, **16** (1980), 99–108.

[S_4] M. Spivakovsky, A counterexample to Hironaka's "hard" polyhedra game, Publ. Res. Inst. Math. Sci., Kyoto Univ., **18** (1982), 1009–1012.

[S_5] ——, A Solution to Hironaka's polyhedra game, in Arithmetic and Geometry, papers dedicated to I. R. Shafarevich on the occasion of his sixtieth birthday (M. Artin and J. Tate, eds.) vol. II, Geometry, Progress in Math. **36**, Birkhäuser, Boston, Basel, Stuttgart, 1983, 419–432.

[Z] O. Zariski, Reduction of singularities of algebraic three dimensional varieties, Ann. of Math., **45** (1944), 472–542; also in Collected Works, vol. I, 441–511.

Mathematical Institute
Tohoku University
Sendai, 980
Japan