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Splicing Algebraic Links

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§ 1. Introduction

In this paper we give an introduction to the terminology of splicing (see "Three-dimensional link theory and invariants of plane curve singularities" by Eisenbud and Neumann, [EN]) and then describe how to compute a normal form representation of the real monodromy and Seifert form for the link of a plane curve singularity from this point of view (it was done via a resolution diagram for the singularity in [N3]). It has been conjectured that this might be a complete invariant for the topology of an isolated complex hypersurface singularity in any dimension; the originator now denies responsibility and will remain unnamed, but the conjecture is still unresolved. Many of the required invariants are computed in [EN] and we just review these computations. The first four sections and Theorem 5.1 are survey and review; the main new result is the computation of the equivariant signatures of the monodromy via splicing in Theorem 5.3. This computation applies also to general graph links.

A *link* for us is a pair (Σ, K) where Σ is an oriented homology 3-sphere and K is a disjoint union of oriented circles in Σ. Let (V, p) be a germ of a normal complex surface at a $\boldsymbol{Z}$-homology manifold point, that is $H^*(V, V-p; \boldsymbol{Z}) = H^*(\boldsymbol{C}^2, \boldsymbol{C}^2-0; \boldsymbol{Z})$. Let $f\colon (V, p) \to (\boldsymbol{C}, 0)$ be the germ of an analytic map. We may assume (V, p) embedded in some ambient $(\boldsymbol{C}^n, 0)$ and then by intersecting $(V, f^{-1}(0))$ with a sufficiently small sphere about $0 \in \boldsymbol{C}^n$, we obtain the *link* $(\Sigma, K(f))$ of f. We call such a link an *algebraic graph link*; if $(V, p) = (\boldsymbol{C}^2, 0)$, it is just the link of a plane curve singularity. We make no reducedness assumption on f; thus each branch of $f^{-1}(0)$, and correspondingly each component of $K(f)$, carries a positive integer multiplicity; in the terminology of [EN], $(\Sigma, K(f))$ is a *multilink*. A link is the special case of a multilink with all multiplicities equal to 1.

The invariants we are interested in are invariants of the Milnor

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fibration

$$f/|f|: \Sigma - K(f) \longrightarrow S^1.$$

Namely, let F be a fiber of this fibration and $h: F \to F$ the geometric monodromy. We will compute the decomposition of the non-symmetric isometric structure $(H_1(F; \boldsymbol{C}), h_*, L)$, where L is the sesquilinearized Seifert linking form, as a sum of irreducibles. This is equivalent to computing the corresponding decomposition over $\boldsymbol{R}$: a real irreducible isometric structure is determined by its complexification, which is either irreducible and isomorphic to its conjugate or is the sum of two mutually conjugate irreducibles.

§ 2. Splicing

Given links (Σ', K') and (Σ'', K'') and components $S' \subset K'$ and $S'' \subset K''$, the *splice* $(\Sigma, K) = (\Sigma', K')_{\overline{S'S''}}(\Sigma'', K'')$ is constructed as follows. Σ is obtained by pasting together complements of open tubular neighborhoods $\Sigma'_0 = \Sigma' - N(S')$ and $\Sigma''_0 = \Sigma'' - N(S'')$ of S' and S'':

$$\Sigma = \Sigma'_0 \cup \Sigma''_0,$$

matching meridian of S' to longitude of S'' and vice versa.

$$K = (K' - S') \cup (K'' - S'')$$

is the union of the components of K' and K'' other than S' and S''.

Any algebraic graph link can be represented as the result of splicing together certain simple buidling blocks. The basic building block is the *Seifert link* $(\Sigma(\alpha_1, \cdots, \alpha_n), S_1 \cup \cdots \cup S_k)$. Here $1 \le k \le n$ and $\alpha_1, \cdots, \alpha_n$ are pairwise coprime positive integers. $\Sigma(\alpha_1, \cdots, \alpha_n)$ is the unique 3-dimensional Seifert fibered homology sphere having fibers $S_1, \cdots, S_n$ of degrees $\alpha_1, \cdots, \alpha_n$ and no other exceptional fibers (an exceptional fiber is one of degree > 1). $\Sigma(\alpha_1, \cdots, \alpha_n)$ can also be described as the link of the complete intersection surface singularity $(V(\alpha_1, \cdots, \alpha_n), 0)$, where

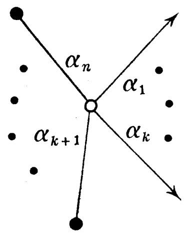
$$V(\alpha_1, \cdots, \alpha_n) = \{z \in \boldsymbol{C}^n \mid a_{i1} z_1^{\alpha_1} + \cdots + a_{in} z_n^{\alpha_n} = 0,\ i = 1, \cdots, n-2\},$$

(a_{ij}) being any sufficiently general coefficient matrix. That is:

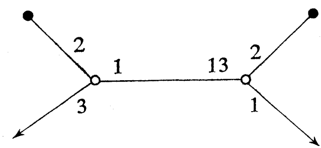
$$\Sigma(\alpha_1, \cdots, \alpha_n) = V(\alpha_1, \cdots, \alpha_n) \cap S^{2n-1}.$$

S_i is the intersection of $\Sigma(\alpha_1, \cdots, \alpha_n)$ with the hyperplane $z_i = 0$.

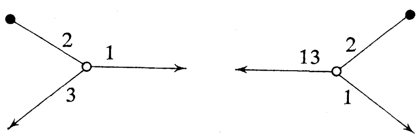
We symbolize the link $(\Sigma(\alpha_1, \cdots, \alpha_n), S_1 \cup \cdots \cup S_k)$ by the *splice diagram*



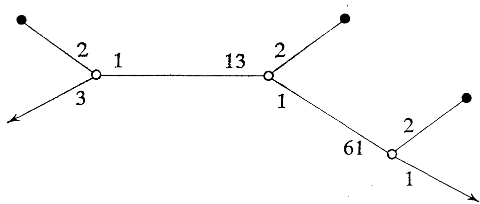
A diagram such as



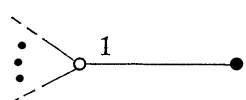
symbolizes the result of splicing the two Seifert links represented by the diagrams



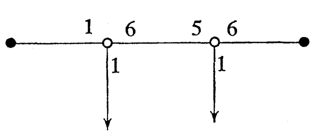
in the obvious way. One may iterate; for instance, splicing on an additional Seifert link could give

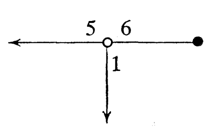


Note that edges of the form



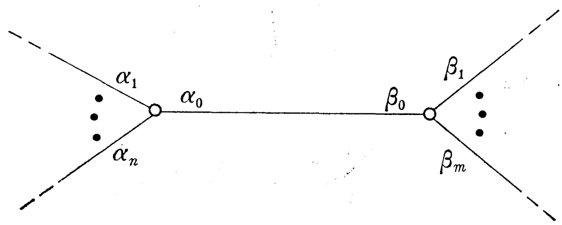
are redundant in a splice diagram and should be omitted; for example, the following two splice diagrams mean the same thing.





In [EN] it is shown that a link is an algebraic graph link if and only if it can be represented by a splice diagram satisfying the following condition; moreover, the diagram is then unique.

For any edge



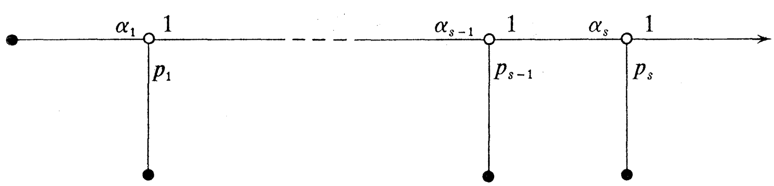
one has $\alpha_0\beta_0>\alpha_1\cdots\alpha_n\beta_1\cdots\beta_m$.

§ 3. Plane curve singularities and Puiseux data

For a plane curve singularity the splice diagram is a quite direct coding of the Puiseux data for the singularity. If we have a single branch $f(x, y)=0$ whose Puiseux expansion (written in Newton form) is

$$y=x^{q_1/p_1}(a_1+x^{q_2/p_1p_2}(a_2+\cdots(a_{s-1}+a_sx^{q_s/p_1\cdots p_s})\cdots),$$

the corresponding splice diagram is



with

$$\alpha_1=q_1$$

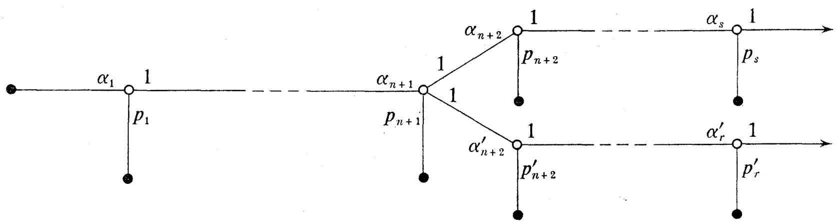
and, for $i\geq 1$,

$$\alpha_{i+1}=q_{i+1}+p_ip_{i+1}\alpha_i$$

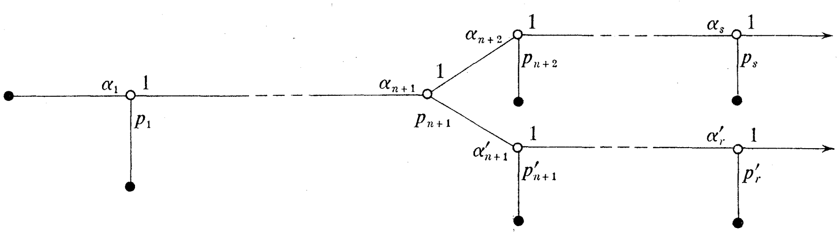
The case of two branches will suffice to describe the situation for more than one branch. Suppose the branches have expansions

$$y=x^{q_1/p_1}(a_1+x^{q_2/p_1p_2}(a_2+\cdots(a_{s-1}+a_sx^{q_s/p_1\cdots p_s})\cdots),$$
$$y=x^{q'_1/p'_1}(a'_1+x^{q'_2/p'_1p'_2}(a'_2+\cdots(a'_{r-1}+a'_rx^{q'_r/p'_1\cdots p_r})\cdots),$$

with exactly n common terms; that is, $p_i = p'_i$, $q_i = q'_i$, and $a_i = a'_i$ for $i = 1, \cdots, n$ but not for $i = n+1$. If $q_{n+1}/p_{n+1} = q'_{n+1}/p'_{n+1}$ the splice diagram is

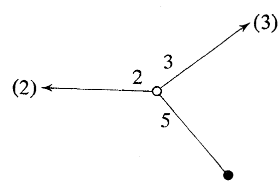


Otherwise, by exchanging the branches if necessary, we may assume $r = n$ or $q_{n+1}/p_{n+1} < q'_{n+1}/p'_{n+1}$ and the splice diagram is then



§ 4. Linking numbers and multilinks

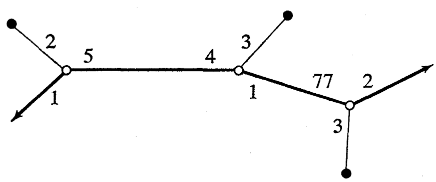
As mentioned in the introduction, we wish to allow link components of a link (Σ, K) to carry integer multiplicities. We write such multiplicities as labels at the arrowheads of the corresponding splice diagram. For example



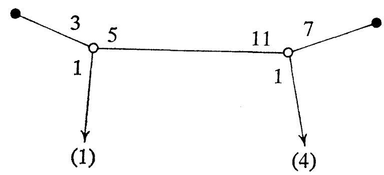
symbolizes the link of the map f: $(V(2, 3, 5), 0) \to (\boldsymbol{C}, 0)$, $f(z_1, z_2, z_3) = z_1^2 z_2^3$.

Given a multilink (Σ, K), there is an associated cohomology class $m \in H^1(\Sigma - K; \boldsymbol{Z})$ whose value on any homology class C is the linking number of C with the link K, taking multiplicities into account. The class m determines the multiplicities: the multiplicity of a link component S is $m(M)$, where M is a meridian of S. If (Σ, K) was just a link rather than a multilink, we consider all multiplicities to be 1, so the multiplicity cohomology class m is still defined.

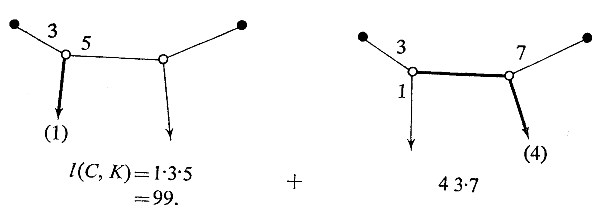
There is a simple method to compute the linking numbers of components of a graph link: join the corresponding arrowheads in the splice diagram by a simple path and take the product of all weights adjacent to, but not on, this path. For example, the linking number of the two components of the link given by the following diagram is $18 = 2 \cdot 3 \cdot 3$.



Non-arrowhead vertices of a splice diagram can be thought to correspond to fibers of the Seifert fibered structures of the splice component pieces of $\Sigma - K$, and mutual linking numbers can be computed in a similar way for them. For example, if (Σ, K) is the multilink (link with multiplicities) with diagram:



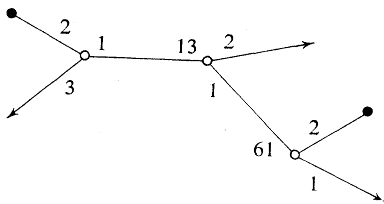
and C is a nonsingular fiber in the Seifert structure for the left hand piece, then the total linking number $m(C)$ of G with K is computed as follows:



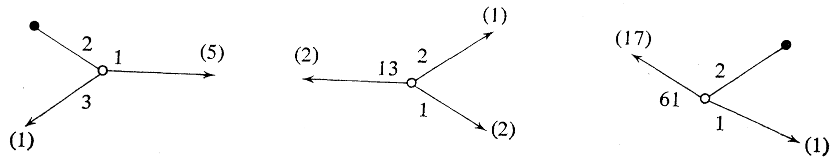
This total linking number, which is defined at any vertex v of the diagram, will be called the *multiplicity* l_v *at the vertex* v, and will be important in what follows.

If the link or multilink (Σ, K) is the result of splicing, $(\Sigma, K) = (\Sigma', K')_{\overline{S'S''}}(\Sigma'', K'')$, then the multiplicity class m for (Σ, K) restricts to cohomology classes m' and m'' on $\Sigma' - K'$ and $\Sigma'' - K''$ which give (Σ', K') and (Σ'', K'') the structure of multilinks. How to compute the relevant multiplicities for these "splice summands" is best illustrated in an example.

The plane curve link with diagram



has multilink splice components



where the multiplicity 5, for example, was computed as follows:

5=1 + 4.

We see that to any interior edge e of the splice diagram (edge connecting two nodes) can be associated two numbers (e.g. 2 and 5 for the left interior edge of the above example) which are the multiplicities for the link components used to splice at that edge. Denote by d_e the g.c.d. of these two numbers associated to the edge e. For a node v, denote by d_v the g.c.d. of all link component multiplicities for the Seifert multilink splice component corresponding to the node v (this can also be computed as the g.c.d. of the d_e's at all adjacent interior edges and the link component multiplicities at all adjacent arrowhead vertices to v). Finally, denote by d the g.c.d. of all link component multiplicities of (Σ, K) (this is the number of components of the Milnor fiber F). We shall need these numbers below.

§ 5. Invariants

Let (Σ, K) be an algebraic graph multilink with Milnor fibration $\mu\colon \Sigma - K \to S^1$. Let F be the fiber and $h\colon F \to F$ be the monodromy. The algebraic monodromy $h_*\colon H_1(F) \to H_1(F)$ has only 1×1 and 2×2 blocks in its Jordan normal form and the eigenvalues are roots of unity. Let N be a common multiple of the orders of the eigenvalues, so $(h_*^N - 1)^2 = 0$.

Denote by $\Delta(t)$ and $\Delta^1(t)$ the characteristic polynomials of h_* and $h_* \mid \mathrm{Ker}\,(h_*^N - 1)$ respectively, so the roots of $\Delta(t)$ are the eigenvalues of h and the roots of $\Delta^1(t)$ are the eigenvalues belonging to 2×2 blocks of the Jordan normal form. The following combines special cases of Theorems 11.3 and 14.1 of [EN].

Theorem 5.1. *Let δ_v be the degree of vertex v in the splice diagram (number of incident edges) and let d and the l_v, d_v and d_e be as in Section 4. Then*

$$\Delta(t) = (t^d - 1) \prod (t^{l_v} - 1)^{\delta_v - 2},$$

product over all non arrowhead vertices, and

$$\Delta^1(t) = (t^d - 1) \prod_e (t^{d_e} - 1) / \prod_v (t^{d_v} - 1),$$

products respectively over all interior edges and all nodes of the splice diagram.

Now let $H = H_1(F: \boldsymbol{C})$ and let $H = \bigoplus_\lambda H_\lambda$ be the splitting of H according to the eigenvalues of $h_*: H \to H$. Let L be the sesquilinearized Seifert form on H. Then $S = L - L^*$ is the skew hermitian intersection form on H, so iS is an hermitian form. Define

$$\sigma_\lambda^- = \mathrm{sign}\,(iS \mid H_\lambda).$$

We shall describe how to compute σ_λ^- in Theorem 5.3 below.

Denote by m_λ and m_λ^1 the multiplicity of λ as a root of $\Delta(t)$ and $\Delta^1(t)$ respectively, so $m_\lambda - 2m_\lambda^1$ and m_λ^1 are the number of 1×1 and 2×2 Jordan blocks for the eigenvalue λ respectively.

Denote the components of K by S_i, $i = 1, \cdots, n$. For each S_i, denote by m_i its multiplicity and by l_i its linking number with the rest of K (taking multiplicities of the other components of K into account). Then (m_i, l_i) represents the homology class of the intersection $F \cap T_i$ of F with the boundary T_i of a tubular neighborhood $N(S_i)$, so $d_i = \gcd\,(m_i, l_i)$ is the number of components of $F \cap T_i$. It follows easily that if $H' = \mathrm{Im}\,(H_1(\partial F; \boldsymbol{C}) \to H_1(F: \boldsymbol{C}))$ then the characteristic polynomial of $h_* \mid H'$ is

$$\Delta'(t) = (t^d - 1)^{-1} \prod_{i=1}^{n} (t^{d_i} - 1)$$

Let m'_λ be the multiplicity of λ as a root of $\Delta'(t)$.

The following result is proved, in slightly different formulation, in [N3] (there was a misprint in the relevant Table 1 of the paper; the bottom right entry should read "1 for $\lambda = -1$ and 0 else")

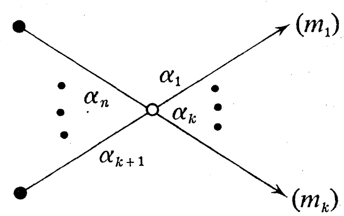
Theorem 5.2. *The indecomposable summands of the above* (H, h_*, L), *with their multiplicities, are all given in the following list.*

Summand,	*multiplicity,*	*comment*
$\Gamma_\lambda := (\boldsymbol{C}, (\lambda), (0))$,	m'_λ,	$\lambda \neq 1$
$-\Lambda_1^1 := (\boldsymbol{C}, (1), (-1))$,	$n-1\ (=m'_1)$	
$\Lambda_\lambda^1 := (\boldsymbol{C}, (\lambda), i(\bar\lambda - 1))$,	$(m_\lambda - m'_\lambda - 2m_\lambda^1 + \sigma_\lambda^-)/2$,	$\lambda \neq 1$
$-\Lambda_\lambda^1 := (\boldsymbol{C}, (\lambda), i(1-\bar\lambda))$,	$(m_\lambda - m'_\lambda - 2m_\lambda^1 - \sigma_\lambda^-)/2$,	$\lambda \neq 1$
$\Lambda_\lambda^2 := \left(\boldsymbol{C}^2, \begin{pmatrix}\lambda & 0\\ \lambda & \lambda\end{pmatrix}, i\begin{pmatrix}1 & \bar\lambda - 1\\ 1-\bar\lambda & 0\end{pmatrix}\right)$,	m_λ^1	$\lambda \neq 1$

It remains to compute the σ_λ^-. For $x \in \boldsymbol{R}$ let $\{x\}$ be the fractional part of x and

$$((x)) = \begin{cases} \frac{1}{2} - \{x\}, & x \notin \boldsymbol{Z}, \\ 0, & x \in \boldsymbol{Z}. \end{cases}$$

Theorem 5.3. σ_λ^- *is the sum of the values of* σ_λ^- *over the Seifert multilink splice components of* (Σ, K). *For the Seifert multilink with diagram*



put $m_i = 0$ *for* $i = k+1, \cdots, n$, *so* $m = \sum_j \alpha_1 \cdots \hat\alpha_j \cdots \alpha_n m_j$ *is the multiplicity of the central node. Choose integers* β_j, $j = 1, \cdots, n$, *with* $\beta_j \alpha_1 \cdots \hat\alpha_j \cdots \alpha_n \equiv 1$ (modulo α_j) *for each* j *and put* $s_j = (m_j - \beta_j m)/\alpha_j$. *If* $\lambda = \exp(2\pi i p/q)$, *with* p/q *in lowest terms, then*

$$\sigma_\lambda^- = \begin{cases} 0 & \text{if } q \text{ does not divide } m, \\ 2\sum_{j=1}^{n} ((s_j p/q)) & \text{if } q \text{ divides } m. \end{cases}$$

Proof. The signatures σ_λ^- are the equivariant signatures of $h: F \to F$. Such signatures are discussed in [N2] for example; they are defined for any orientation preserving self homeomorphism of an even dimensional manifold and they satisfy Novikov additivity (additivity with respect to pasting along boundary components). In [EN] it is shown that the monodromy $h: F \to F$ can be obtained by pasting along boundary circles the monodromy maps on the Milnor fibers of the splice components. The first statement of Theorem 5.3 thus follows.

Let (Σ, K) be the Seifert multilink described in the theorem. We will use the analytic description of it from Section 2: $\Sigma = V \cap S^{2n-1}$, where

$$V = \{z \in \boldsymbol{C}^n \mid a_{i1} z_1^{\alpha_1} + \cdots + a_{in} z_n^{\alpha_n} = 0,\ i = 1, \cdots, n-2\},$$

for some coefficient matrix (a_{ij}), and K is the link for the map $f: V \to \boldsymbol{C}$ given by

$$f(z_1, \cdots, z_n) = z_1^{m_1} \cdots z_n^{m_n}.$$

The Milnor fibration is therefore $\mu = f/|f|: \Sigma - K \to S^1$. Σ has an S^1-action given by $t(z_1, \cdots, z_n) = (t^{\alpha/\alpha_1} z_1, \cdots, t^{\alpha/\alpha_n} z_n)$, where $\alpha = \alpha_1 \cdots \alpha_n$. μ is equivariant with respect to this S^1-action on $\Sigma - K$ and the non-effective S^1-action on S^1 given by $t.s = t^m s$. In particular, the orbits of the S^1-action on Σ are transverse to the Milnor fiber F and a general orbit intersects F in m points. Also, h can be described as the exp $(2\pi i/m)$-map of the S^1-action, so it has order m. σ_λ^- is thus zero if λ is not a m-th root of unity. Assume now that $\lambda = \exp(2\pi i p/q)$ and q divides m.

Denote by N_i a small S^1-invariant tubular neighborhood of $S_i = \Sigma \cap \{z_i = 0\}$ and define $\Sigma_0 = \Sigma - \operatorname{int}(N_1 \cup \cdots \cup N_n)$ and $F_0 = F \cap \Sigma_0$. F_0 results from F by the removal of some disks and annuli, which support no signature. Thus the σ_λ^- are the equivariant signatures of $h_0 = h \mid F_0$. h_0 generates a free $\boldsymbol{Z}/m$-action on F_0, so h_0 is a covering transformation for some m-fold cyclic cover $F_0 \to F'$. Let $\varphi: \pi_1(F') \to \boldsymbol{Z}/m$ be the classifying map for this covering. Let $\rho_\lambda: \boldsymbol{Z}/m \to U(1)$ be the representation which takes the generator to $(\lambda) \in U(1)$ and put $\rho = \rho_\lambda \circ \varphi$. In [APS] and [N1] it was shown that σ_λ^- only depends on $\rho \mid \partial F'$ and that a circle $S^1 \subseteq \partial F'$ on which ρ takes $1 \in \pi_1(S^1)$ to $\exp(2\pi i s/q)$ contributes $2((s/q))$ to this signature. Thus, if we show that φ takes the class of the j-th boundary component of F' to $s_j \in \boldsymbol{Z}/m$, with s_j as in the Theorem, we will have completed the proof.

We shall take $j = 1$ for simplicity of notation. A small $\boldsymbol{Z}/\alpha_1$-invariant transverse disk to the orbit $S_1 = \Sigma \cap \{z_1 = 0\}$ can be parametrized in the form

$$\{(\varepsilon z, z_2(z), \cdots, z_n(z)) \mid z \leq 1\}$$

with ε small and $z_2(z), \cdots, z_n(z)$ approximately constant. The tubular neighborhood N_1 can then be chosen as

$$N_1=\{(t^{\alpha/\alpha_1}\varepsilon z,\ t^{\alpha/\alpha_2}z_2(z),\ \cdots,\ t^{\alpha/\alpha_n}z_n(z)\,|\,z\leq 1,\ t\in S^1\}.$$

We can trivialize the S^1-action on N_1 by the map $g\colon S^1\times S^1\to N_1$ given by

$$g\colon (s,t)\longmapsto (s^{-1/\alpha_1+\beta_1/\alpha_1\cdot\alpha/\alpha_1}t^{\alpha/\alpha_1}\varepsilon,\ s^{\beta_1/\alpha_1\cdot\alpha/\alpha_2}t^{\alpha/\alpha_2}z_2(s^{-1/\alpha_1}),\ \cdots,\ s^{\beta_1/\alpha_1\cdot\alpha/\alpha_n}t^{\alpha/\alpha_n}z_n(s^{-1/\alpha_1})).$$

Indeed, it is an elementary computation to check that this map is bijective and it clearly takes rotation of the second factor of $S^1\times S^1$ to our given S^1 action on ∂N_1. The composition $\mu\circ g\colon S^1\times S^1\to S^1$ is, up to an almost constant factor, $(s,t)\to s^b t^a$ with

$$a=\sum_{i=1}^{n} m_i\cdot\frac{\alpha}{\alpha_i}=m$$

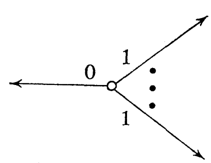
and

$$b=m_1\cdot\frac{-1}{\alpha_1}+\sum_{i=1}^{n} m_i\cdot\frac{\beta_1}{\alpha_1}\cdot\frac{\alpha}{\alpha_i}=-s_1.$$

Thus $g^{-1}(F\cap\partial N_1)=(\mu\circ g)^{-1}(1)$ is equivariantly the pull back of the standard Z/m cover of S^1 by the degree s_1 map $S^1\to S^1$. This is what we needed to prove.

§ 6. General graph links

As described in [EN], a splice component of a general (possibly non-algebraic) graph multilink may be one of our standard Seifert multilinks but with the orientation of the ambient homology sphere Σ reversed, indicated by weighting the corresponding vertex of the splice diagram with a -1; it may have some non-positive link component multiplicities; and it may be an additional type of splice component—an unknotted circle in S^3 plus several disjoint meridians, represented by the splice diagram

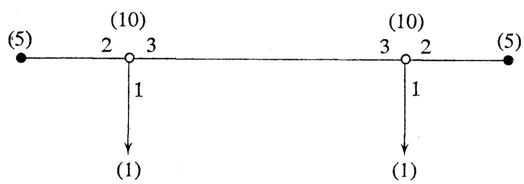


Such a multilink may not be fibered, but the signatures σ_λ^- are still defined (see for instance [G] for a survey of various equivalent definitions in the literature) and Theorem 5.3 still applies to compute them; the only change is that the s_j must be multiplied by -1 if m is negative and the σ_λ^- are zero if m is zero. The proof is an easy extension of the above proof. Note however that Theorem 5.2 and the formula for $\Delta^1(t)$ of Theorem

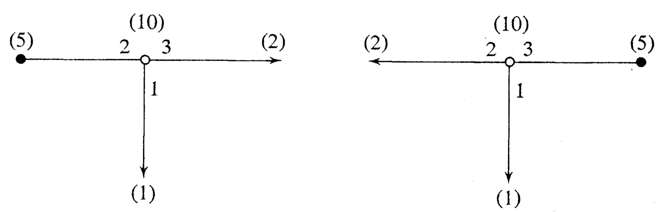
5.1 fail in general for non-algebraic multilinks, although the formula for $\Delta(t)$ is still valid (it is the Alexander polynomial in the non-fibered case), see [EN] for details.

§ 7. Examples

We compute the example of two transverse cusps: $(x^2+y^3)(x^3+y^2)=0$. This has splice diagram (numbers in parentheses are multiplicities):



The splice components are:



Thus, by Theorem 5.1,

$$\Delta=(t-1)(t^{10}-1)^2/(t^5-1)^2=(t-1)(t^5+1)^2$$
$$\Delta^1=(t-1)(t^2-1)/(t-1)^2=(t+1).$$

The two splice components are isomorphic, so they contribute equally to the equivariant signatures. We may take $\alpha_1=1$, $\alpha_2=2$, $\alpha_3=3$, $\beta_1=0$, $\beta_2=-1$, $\beta_3=-1$; then $s_1=1$, $s_2=5$, $s_3=4$. Theorem 5.3 thus gives that each splice component contributes as follows to the signature σ_λ^- for $\lambda=\zeta^k$, $\zeta=\exp(2\pi i/10)$:

$k=1$	2	3	4	5	6	7	8	9
$\sigma_\lambda^-=1$	0	1	0	0	0	-1	0	-1

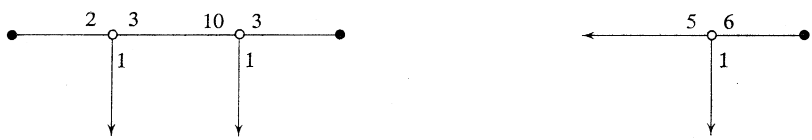
and we see by Theorem 5.2 that

$$(H, h_*, L)=-\Lambda_1^1\oplus 2(\Lambda_\zeta^1)\oplus 2(\Lambda_{\zeta^3}^1)\oplus 2(-\Lambda_{\zeta^7}^1)\oplus 2(-\Lambda_{\zeta^9}^1)\oplus \Lambda_{-1}^2.$$

A similar type of example is the family found by Marie-Claire Grima: if $p+r=q+s$ and $ps<qr$ and $\gcd(pe, qf)=\gcd(re, sf)=\gcd(pf, qe)=\gcd(rf, se)=1$, then the plane curve singularity links

$$(x^{pe}+y^{qf})(x^{re}+y^{sf})=0,$$
$$(x^{pf}+y^{qe})(x^{rf}+y^{se})=0,$$

have the same $\Delta(t)$ and $\Delta^1(t)$, as Theorem 5.1 shows. But computer experiments indicate that they are always distinguished by their equivariant signatures, for example if $(p, q, r, s, e, f)=(1, 3, 5, 3, 2, 1)$ then the two links have splice diagrams



By Theorem 5.3 their signatures differ at $\exp(2\pi ik/36)$ for $k=11, 13, 17, 19, 23, 25$.

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