

Configurations Related to Maximal Rational Elliptic Surfaces

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Introduction

The purpose of this paper is to present a method to attack the problem of finding good configurations of plane curves. Our viewpoint is described as follows: Take two cubic curves intersecting properly on the complex projective plane $P_2(\mathbb{C})$ and then form the pencil generated by them. When regarded as a one-parameter family of elliptic curves, the pencil defines an elliptic surface in the sense of [Kd] which is obviously blown down to $P_2(\mathbb{C})$. The singular fibers then correspond to exactly the singular member of the pencil. A global section is mapped by the blowing down to either a base point of the pencil or a curve which, outside the base points, intersects the generic member of the pencil at exactly one point. In many cases the singular members and the images of global sections might already form an interesting configuration of plane curves. But, by fixing a global section to give the group structure to fibers, we can further observe the locus of m -torsion points of fibers for every $m > 0$ ($m \in \mathbb{Z}$), which is also regarded as a plane curve. Adding some of these loci might in general enrich the original configuration.

The motivation to the present work comes from the following fact: Hirzebruch [H2] uses some arrangements of lines on $P_2(\mathbb{C})$ for the purpose of constructing compact quotients of the unit ball in \mathbb{C}^2 in algebro-geometric way. A little earlier Inoue [In] and Livné [L] used the elliptic modular surfaces associated with some congruence subgroups of $SL(2, \mathbb{Z})$ for the same purpose (see [Sh] for this notion). But, some of these examples can actually be treated in a unified way from the viewpoint described above.

Now note that such quotients can not be deformed continuously while the moduli space of rational elliptic surfaces has eight parameters. Thus we have to impose the maximality assumption on our elliptic surface to make it rigid; this roughly amounts to assuming that the surface has only finitely many global sections.

The geometry of rational elliptic surfaces is equivalent to that of del Pezzo surfaces of degree one, the relation of which to the root system of

E_8 is classically well known [Du]. From the types of singular fibers one can at once associate a root subsystem of E_8 with the elliptic surface. In this way the classification of maximal rational elliptic surfaces is almost equivalent to that of root subsystems of E_8 with the maximal rank eight. (Only $2D_4$ gets out of the list.) Although this subject is classical, we will detail about it as a preliminary in Section 1. In Section 2 we construct the surfaces explicitly from pencils of cubic curves, we also describe their automorphisms in terms of Cremona transformations of $P_2(\mathbf{C})$. In Section 3 we explain how to produce more elliptic surfaces such as used in [In], [L] from those given in Section 2; the method is naturally covering formation by base change. In Section 4 we give remarks about examples from [H2], [In], [L] to explain the importance of the above viewpoints; we include also some ball-quotients which can be compactified by adding cusps, and discuss some relationship among them. This last standpoint is due to Hirzebruch; one example is in fact from a recent work of him. There are constructed by our method two more examples which are closely related to Hirzebruch's one.

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§ 1. Maximal rational elliptic surfaces

To open this section we make a remark about a deep relationship between the geometry of rational elliptic surfaces and the root system of E_8 . Suppose that we are given an elliptic surface S which is smooth, and rational. (Throughout this article any elliptic surface is assumed to have no exceptional curve of the 1st kind contained in a fiber.) Further let E and K be the class of fibers and the canonical class of S . Then, by the adjunction formula, one has the identities:

$$E^2=0 \quad E \cdot K=0.$$

Here the second one follows from the much stronger statement in the general theory [Kd], Theorem 12.1 that K is an integral multiple of E ; so we write

$$K=mE \quad (m \in \mathbf{Z}).$$

In particular we have $K^2=0$ and thus by Noether's formula [H1] the Euler

number of S is twelve i.e. we have the following Chern numbers for S :

$$c_1^2=0 \quad c_2=12.$$

Since $c_2=3$ or 4 for the (relatively) minimal rational surfaces, S should have at least one exceptional curve of the 1st kind. Let G be such an exceptional class. Again by the adjunction formula we have $-1=G \cdot K=m(G \cdot E)$, but we have $G \cdot E \geq 0$ since G and E are irreducible effective divisors. Thus we obtain the identities:

$$K=-E$$

$$G \cdot E=1.$$

From the second one we see also that any exceptional curve of the 1st kind is a global section. The converse is also true. Incidentally we have proved that the base curve of S is rational.

Now let b_2^+ (resp. b_2^-) be the maximal \mathbf{R} -dimension of subspaces of $H_2(S, \mathbf{R})$ on which the intersection form is positive (negative) definite and set $b_2=b_2^++b_2^-$ (the second Betti number). Since S is rational we have $b_2=c_2-2=10$. On the other hand, by Hirzebruch's index theorem [H1], we have $b_2^+-b_2^-(c_1^2-2c_2)/3=-8$. It follows thus

$$b_2^+=1 \quad b_2^-=9.$$

Now, observe that, since $E^2=0$, E is contained in its orthogonal complement $E^\perp(=K^\perp)=\{C \in H_2(S, \mathbf{Z}); C \cdot E=0\}$. Thus, the intersection form is restricted to the quotient module

$$L:=E^\perp/Z E=K^\perp/Z K.$$

By the second identity this is an even lattice. (A lattice means here a torsion-free \mathbf{Z} -module with a given \mathbf{Z} -inner product on it. The torsion-freeness of L follows from $G \cdot E=1$ above.) By Linear Algebra, L is of index $(0, 8)$ i.e. L is negative definite of rank 8. Thus, if one proves that L is unimodular, then it is isometric to $L(E_8)$ (=the lattice generated by roots of the simple group E_8 , the inner product of which is the Killing form). But, to see this, it suffices to observe the natural map $[G, E]^\perp \rightarrow L^\perp=E/Z E$ which is obviously an injective isometry and to note that $[G, E]^\perp(=Z G+Z E)$ and $[G, E]^\perp$ are simultaneously unimodular by the Poincaré duality and that $[G, E]$ is in fact unimodular since $E^2=0$, $G \cdot E=1$. We have thus proved:

Proposition 1.1. *The lattice L is isometric to $L(E_8)$. For any global section G of S we have the natural isomorphism $L \cong [G, E]^\perp$.*

According to this proposition it is reasonable to call the classes $x \in L$ with $(x, x) = x \cdot x = -2$ the *roots*. Let R denote the set of roots in L . Now we want to point out that there are some special roots which are of particular geometric importance. Suppose that we are given a (-2) -curve C on S . (For simplicity we mean by an m -curve a *smooth rational* curve of self-intersection number m .) By the adjunction formula we have $C \cdot E = 0$ ($\because E = -K$) i.e. $C \in E^\perp$, so C has its class in L which is obviously a root in the above sense. $C \cdot E = 0$ implies also that C is an irreducible component of some singular fiber of S . The roots in L coming from the (-2) -curves are now called the *distinguished* roots in L . According to [Kd] almost all singular fibers consist of (-2) -curves. (Since S has a global section, S does not have any multiple fibers). In fact, if one exclude some pathological singular fibers, then observing the distinguished roots in L is equivalent to observing the configuration of irreducible components of singular fibers of S . It is exactly for achieving this that we impose the following assumption on S .

Assumption I. S is free from the fibers of types II, III and IV.

II is a cuspidal rational curve, III is the union of two smooth rational curves being tangent at one point and IV is the union of three smooth rational curves meeting transversally at one point. Note that III, IV also consist of (-2) -curves. But we are not able to distinguish between I_2 and III or between I_3 and IV by their corresponding distinguished roots. This is the reason why we have also excluded III, IV. Now we want to identify any (-2) -curve on S with its class in L which is a distinguished root. We denote by $\mathcal{A}(S)$ the set of distinguished roots. As we have seen above, from the algebraic object $\mathcal{A}(S)$, we can immediately recover all singular fibers of S except for the type I_1 which is fiber being a nodal rational curve on S . This last class of fibers plays however an important geometric role in later discussions. To count the number ν of fibers of S of type I_1 , we introduce the concept of the Euler number $e(\mathcal{A}(S))$ of $\mathcal{A}(S)$ so that $\nu + e(\mathcal{A}(S)) = 12$. From the classification [Kd] $\mathcal{A}(S)$ is a union of mutually orthogonal *extended* Dynkin diagrams of types \tilde{A}_k ($k=1, 2, \dots$) or \tilde{D}_k ($k=4, 5, \dots$) or $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. We define now their Euler numbers by setting

$$e(\tilde{A}_k) = k + 1, \quad e(\tilde{D}_k) = k + 2, \quad e(\tilde{E}_k) = k + 2$$

and we define $e(\mathcal{A}(S))$ to be the sum of the Euler numbers of the diagrams constituting $\mathcal{A}(S)$. Note that for each constituent diagram, its Euler number is exactly that of the corresponding singular fiber. (\tilde{A}_k correspond to I_{k+1} in the notation of [Kd], \tilde{D}_k correspond to I_{k-4}^* , $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ correspond to IV^*, III^*, II^* .) Since the Euler number of S is the sum of Euler numbers

of singular fibers of S , we have the desired identity:

Proposition 1.2. *With ν being the number of fibers of type I_1 , $e(\mathcal{A}(S)) + \nu = 12$. In particular $e(\mathcal{A}(S)) \leq 12$.*

Now let $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \dots$ be the extended Dynkin diagrams in $\mathcal{A}(S)$. The vertices of $\tilde{\Gamma}_i$, which are roots in L , generate a sublattice \tilde{L}_i (with respect to the restriction of the inner product); if $\tilde{\Gamma}_i$ is of type \tilde{A}_k, \tilde{D}_k or \tilde{E}_k , then L_i is isomorphic to $L(A_k), L(D_k)$ or $L(E_k)$ and we let Γ_i stand for A_k, D_k or E_k . L_1, L_2, \dots are mutually orthogonal and the direct sum $L_1 \oplus L_2 \oplus \dots$ defines a root subsystem of $L \cong L(E_8)$ whose type is expressed by $\Gamma_1 \oplus \Gamma_2 \oplus \dots$. We set

$$\Gamma(S) := \Gamma_1 \oplus \Gamma_2 \oplus \dots$$

Since the type of $\mathcal{A}(S)$ is completely expressed by $\Gamma(S)$, we may observe this $\Gamma(S)$ as a root subsystem in E_8 . We can obviously define the Euler number for an arbitrary root subsystem of E_8 in such a way that the Euler numbers of $\mathcal{A}(S)$ and $\Gamma(S)$ coincide. We should remark here that not all root subsystems of E_8 come from rational elliptic surfaces. For example, the Euler number exceeds over 12 already for $7A_1$, which is prohibited by Proposition 1.2. But what is a miracle is the following:

Proposition 1.3. *Every root subsystem of E_8 whose Euler number is not greater than 12 can be realized as $\Gamma(S)$ for some rational elliptic surface S satisfying Assumption I above.*

Definition. The rational elliptic surface S is called *maximal* if the following conditions are fulfilled:

- 1) S satisfies Assumption I.
- 2) S is not of constant modulus.
- 3) $\Gamma(S)$ is a root system of maximal rank 8.

Table I

Γ	$e(\Gamma)$	Γ	$e(\Gamma)$
E_8	10	A_8	9
$E_7 \oplus A_1$	11	$A_7 \oplus A_1$	10
$E_6 \oplus A_2$	11	$A_5 \oplus A_2 \oplus A_1$	11
D_8	10	$2A_4$	10
$D_6 \oplus 2A_1$	12	$2A_3 \oplus 2A_1$	12
$D_5 \oplus A_3$	11	$4A_2$	12
$2D_4$	12		

Above is the table for root subsystems of E_8 which are of rank 8 and for which the Euler number is not greater than 12 (see for example [Dy]):

There is a 1-parameter family of rational elliptic surfaces for which $\Gamma(S)=2D_4$. But these are not maximal in the sense above, since the functional invariant j is constant for them. (In fact j the parameter.) For the other maximal root subsystems the corresponding surfaces are unique up to isomorphisms; the maximality can be regarded as a kind of rigidity assumption, as is mentioned in the introduction.

Now we note that the global sections are obviously important data for describing the structure of S . For example we can introduce some natural groups of automorphisms of S by using them. We will see that the structures of such groups can also be described essentially by the diagram $\Lambda(S)$ at least in the case when S is maximal. We begin with the following observation: Given two global sections G_1 and G_2 on S , there is an automorphism of S which acts on every regular fiber as a translation (with respect to the group structure of the fibre) and which sends G_2 to G_1 . This is called the translation associated with the ordered pair (G_1, G_2) and denoted by $\text{tr}[G_1 - G_2]$. We denote by $T(S)$ the (abelian) group generated by translations. A theorem of Shioda [Sh] allows us to describe the structure of $T(S)$ by the homological invariants L, L_0 where L_0 is the submodule of L generated by the distinguished roots. We can state the result in the following form:

Proposition 1.4. *The natural map*

$$T(S) \ni \text{tr}[G_1 - G_2] \rightarrow [G_1 - G_2] \in L/L_0$$

is an isomorphism of the two abelian groups, where $[G_1 - G_2]$ is the class in L/L_0 of the algebraic cycle $G_1 - G_2$. In particular, if S is maximal, then $T(S)$ is finite and the order $|T(S)|$ is the square root of the absolute value of the determinant of the root system $\Gamma(S)$.

Note that $E \cdot (G_1 - G_2) = 0$ i.e. $G_1 - G_2 \in E^\perp$ so that one can consider its class in L and hence also in L/L_0 .

To any global section G of S we can also assign an involutive automorphism of S called the symmetry with center G : The regular part of any fiber has a unique analytic abelian group structure such that the intersection point with G is the zero, provided that each fiber is given the analytic structure through the projection. By $\text{sym}(G)$ we denote the involution $x \leftrightarrow -x$ which naturally extends to an automorphism of S . We denote by $G(S)$ the group generated by $T(S)$ and the symmetries. We have obviously the identity:

$$\text{sym}(G_1) \circ \text{sym}(G_2) = 2 \text{tr}[G_1 - G_2].$$

Thus we obtain the exact sequence:

$$1 \rightarrow T(S) \rightarrow G(S) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1$$

It is an almost obvious fact that $G(S)$ is the group of automorphisms which induce the identity transformation on the base curve. Thus $G(S)$ is a normal subgroup of the whole automorphism group $\text{Aut}(S)$ and the factor group $\text{Aut}(S)/G(S)$ acts effectively on the base. We denote this last group by $P(S)$. If S is maximal, then $P(S)$ can be represented as a permutation group over the set of singular fibers since the base is $P_1(\mathbb{C})$ and since there are at least three singular fibers for S according to Table I above.

§ 2. Explicit constructions

Our method of construction is described as follows: Take two cubics on $P_2(\mathbb{C})$: (x, y, z) in such a position that they intersect properly and that there is no common singular point of them, and consider the pencil generated by them. The pencil itself is already a fiber space over the parameter space $P_1(\mathbb{C})$, and the generic fibers are elliptic curves. But it might have some singular points. This is in fact the case if (and only if) the cubics have multiple intersection points; more precisely, if p is an intersection point of multiplicity k , then there exist exactly one A_{k-1} -singular point of the pencil over p (note that we have the natural projection of the pencil onto $P_2(\mathbb{C})$). One can see this as follows. Observe that the generic members are tangent at p with multiplicity k . This means that one needs to blow up over p exactly k -times to separate all members of the pencil there. By this process we obtain $k-1$ (-2) -curves which form an A_{k-1} -configuration and one (-1) -curve. By doing this for all base points of the pencil (=the intersection points of the original two cubics), we get a non-singular rational elliptic surface which naturally dominates the pencil. The above A_{k-1} -configuration contracts to the A_{k-1} -singular point over p under the blowing down to the pencil, while the (-1) -curve survives as a global section of the pencil. The A_{k-1} -configuration should be contained in a singular fiber of the elliptic surface if $k > 1$. The other irreducible components of this fiber should come from the strict transform of the cubic in the pencil which has singularity at p . The other type of singular fibers are the strict transforms of the singular members which are non-singular at any base point. Anyway the singular fibers come from the singular cubics in the pencil. We have also seen that the minimal desingularization of the pencil is a nine-points-blowing-up of $P_2(\mathbb{C})$. Conversely any rational elliptic surface

is obtained in this way.

In fact we can prove that the surface S in Section 1 is a nine-points-blowing up of $P_2(C)$. For, since $K = -E$, we have the following inequality for any non-singular rational curve C on S :

$$C^2 = -(C \cdot K + 2) = C \cdot E - 2 \geq -2.$$

The same obviously holds for any blowing down of S ; and this implies that S is blown down to either $P_2(C)$, or $\Sigma_0 = P_1(C) \times P_1(C)$ or Σ_2 . Since $c_2(S) = 12$, S is a nine-points-blowing-up for the first case and an eight-points-blowing-up for the latter two cases. The above inequality implies also that for the last case the center of the blowing-up is disjoint from the (-2) -curve of Σ_2 . Such a blowing-up can be blown down to $P_2(C)$. We see more easily that every one-point-blowing-up of Σ_0 is a two-points-blowing-up of $P_2(C)$. Thus, at any rate, S can be blown down to $P_2(C)$. The images of fibers by the blowing down must be cubic curves since $E = -K$ and $-K_{P_2(C)} = \mathcal{O}(3)$; that they form a pencil is also evident. In general it makes a difficulty if one wants to describe all global sections of the elliptic surface constructed in the way above, since there might be infinitely many of them in general. As we have seen in Section 1, we have only finitely many global sections if the surface is maximal. But, even in this special case, the (-1) -curves over the base points of the pencil do not always sweep out all global sections, for example, if a line passes through two base points and it is not entirely contained in any member of the pencil, then its strict transform should be a global section etc. Anyway, in the following constructions of maximal rational elliptic surfaces, everything will be given explicitly.

2.1. Case of E_8 . Take a nodal cubic on $P_2(C)$: (x, y, z) , say, the one defined by

$$C_1: 3(x^2 - y^2)z + 2x^3 = 0.$$

The line $z = 0$ is obviously an inflexion line of C_1 . The pencil P generated by C_1 and $z^3 = 0$ has the unique base point $p_0: x = z = 0$ of multiplicity 9. To desingularize P in the way described above we needed to blow up nine times over p_0 which produces an A_8 -configuration and one (-1) -curve on the minimal resolution S of P . Since the line $z = 0$ is an inflexion line for the generic members, its strict transform intersects the third exceptional curve of the blowing up which is in the A_8 -configuration. Thus we obtain the desired singular fiber of type \tilde{E}_8 . The strict transform of C_1 is one of the two singular fibers of type $\tilde{A}_0 (= I_1)$ which the surface should have. The other fiber comes from the following nodal cubic in the pencil:

$$C_2: 3(x^2 - y^2)z + 2x^3 - z^3 = (x + z)^2(2x - z) - 3y^2z = 0.$$

The node is given by $x + z = y = 0$. According to Proposition 1.4 there is only one global section of S which is nothing other than the (-1) -curve over p_0 . Thus $G(S) \simeq \mathbb{Z}/2\mathbb{Z}$ and its generator is induced by the involution $x \leftrightarrow -x, z \leftrightarrow -z$. The whole automorphism group of S is generated by this and the involution induced by the projective transformation: $(x, y, z) \rightarrow (\bar{x}, \bar{y}, \bar{z})$ where

$$\begin{aligned} \bar{x} &= x + z \\ \bar{y} &= y \\ \bar{z} &= -z. \end{aligned}$$

2.2. Case of $E_7 \oplus A_1$. First we take a (non-degenerate) conic Q on $P_2(\mathbb{C})$ and a point p_1 on Q . Draw further the tangent line T at p_1 for Q and take a line L which intersects Q transversally and which does not pass through p_1 . By p_2 we denote the intersection point of T and L . Explicitly we can for example set as follows:

$$\begin{aligned} Q: y^2 + x(z - 2x) &= 0 \\ T: x &= 0 & p_1: x = y = 0 \\ L: z &= 0 & p_2: x = z = 0. \end{aligned}$$

Denote by P the pencil generated by the two cubic curves

$$Q + L: z\{y^2 + x(z - 2x)\} = 0$$

and $3T: x^3 = 0$, and by S the minimal resolution of P . The base points of P are exactly p_1 and p_2 which have multiplicities 6 and 3. S is obtained by separating the members of P by blowing up six times over p_1 and three times over p_2 , and we get then an A_5 -configuration, an A_2 -configuration on S and two global sections of S . Together with these A_5, A_2 the strict transform of the line T forms the singular fiber of type \tilde{E}_7 . The strict transform of $Q + L$ is the fiber of type \tilde{A}_1 . By Proposition 1.2 there must be just one further singular fiber of type I_1 , which actually comes from the nodal cubic in the pencil:

$$C: z(y^2 + x(z - 2x)) + x^3 = x(x - z)^2 + y^2z = 0.$$

The node of C is obviously $(1, 0, 1)$. By the construction, T must be tangent to C at $p_1: x = y = 0$. The other intersection point, which is $p_2: x = z = 0$, is an inflexion point of C ; the corresponding inflexion line is $L: z = 0$, which is also obvious by the construction. Note further that Q

is tangent to C at p_1 with multiplicity 6. Thus we have incidentally proved the following proposition in the elementary geometry:

For such a point p on a nodal cubic C that the tangent line at p meets C again at an inflexion point, there is a (unique) conic Q which is tangent to C at p with the full multiplicity 6.

We will see later that this fact can be used for the construction of the elliptic surfaces corresponding to D_8 and $A_7 + A_1$. Proposition 1.4 says that there are exactly 2 global sections for S , which are nothing other than the (-1) -curves over the base points p_1, p_2 . The symmetries corresponding to p_1, p_2 are identical ($T(S) \cong \mathbf{Z}/2\mathbf{Z}$) and are induced by the involution $y \leftrightarrow -y$. If one denotes the global sections over p_1, p_2 by G_1, G_2 , then the translation $\text{tr}[G_2 - G_1]$ is induced by the involutive Cremona transformation $(x, y, z) \rightarrow (\bar{x}, \bar{y}, \bar{z})$:

$$\begin{cases} \bar{x} = x^2 \\ \bar{y} = -xy \\ \bar{z} = -(y^2 - x(2x - z)). \end{cases}$$

The whole automorphism group of S coincides with the group $G(S)$ since the three singular fibers are of different types. $G(S) \cong (\mathbf{Z}/2\mathbf{Z})^2$.

2.3. Case of $E_6 \oplus A_2$. The desired surface should have the three singular fibers of types $\tilde{E}_6, \tilde{A}_2, \tilde{A}_0 (=I_1)$ this time. We start with a triangle Δ and a line L which does not pass through any vertex of Δ . For the explicitness we set

$$\begin{aligned} \Delta: & \quad xyz = 0 \\ L: & \quad x + y + z = 0. \end{aligned}$$

We show that the minimal resolution S of the pencil P generated by $\Delta: xyz = 0$ and $3L: (x + y + z)^3 = 0$ is the required elliptic surface. The base points are the three intersection points $p_1: (0, 1, -1), p_2: (1, 0, -1), p_3: (1, -1, 0)$, each being counted with multiplicity 3. To obtain S we should blow up three times over each base point in order that the strict transforms of members of the pencil do not intersect each other. The resulting three A_2 -configurations on S , together with the strict transform of L form the fiber of type \tilde{E}_6 . The strict transform of Δ is the fiber of type \tilde{A}_2 . The fiber of type \tilde{A}_0 comes from the following nodal cubic in the pencil:

$$C: (x + y + z)^3 - 27xyz = 0.$$

(The node of C is $(1, 1, 1)$.) By Proposition 1.4 the global sections are the (-1) -curves over the base points. The permutations of (x, y, z) can obviously be lifted to automorphisms of the surface S . This is just $G(S)$. For the same reason as in the previous case $G(S)$ is the whole automorphism group.

2.4. Case of D_8 . Let Q, T, C be the same conic and line and cubic as in Section 2.2. Recall that Q and C are tangent at exactly one point $p_1: x=y=0$ with full multiplicity 6, and that T is the common tangent of them. T meets C transversally in the further intersection point $p_2: x=z=0$. For the later convenience we choose another coordinate system of $P_2(C)$, in which Q, T, C given as follows:

$$\begin{aligned} Q: q &= 0 \\ T: x &= 0 \\ C: (z + 2x) \cdot q + x^3 &= 0 \quad (\text{node: } (1, 0, -1)) \end{aligned}$$

where we have put

$$q = xz + y^2.$$

This time we denote by P the pencil generated by C and the degenerate cubic $Q+T: xq=0$, and by S the minimal resolution of P . The points p_1, p_2 are the base points of multiplicity 8 and 1, and S is obtained by blowing up $P_2(C)$ 8-times over p_1 and once over p_2 ; so, on S , we have one A_7 -configuration and the two global sections. The strict transform of Q (resp. T) intersects the sixth (resp. second) exceptional divisor in the A_7 -configuration since it is tangent to the generic member of P with multiplicity 6 (resp. 2). We have thus described the fiber of S of type \tilde{D}_8 . Besides the one corresponding to C , there must be just one more fiber of type I_1 , which in fact comes from the following nodal cubic in P :

$$C': (z + 2x)q + x^3 - 4xq = y^2(z - 2x) + x(x - z)^2 = 0$$

whose node is obviously $(1, 0, 1)$. Proposition 1.4 implies that the global sections of S are only those lying over p_1, p_2 , which we denote by G_1, G_2 . The symmetries associated with G_1, G_2 are induced by the involution $y \leftrightarrow -y$; the translation $\text{tr}[G_1 - G_2]$ is induced by the following involutive Cremona transformation:

$$\begin{cases} \bar{x} = -x^2q \\ \bar{y} = xyq \\ \bar{z} = q(q - xz) - x^4. \end{cases}$$

Thus $G(S) \simeq (\mathbb{Z}/2\mathbb{Z})^2$; the index of $G(S)$ in the whole automorphism group is 2 since the following projective transformation induces an automorphism of order 4 of the surface S whose square is equal to the symmetry:

$$\begin{cases} \bar{x} = x \\ \bar{y} = iy \\ \bar{z} = -z. \end{cases}$$

The extension $1 \rightarrow G(S) \rightarrow \text{Aut}(S) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1$ is thus nontrivial.

2.5. Case of $D_6 \oplus 2A_1$. Take a conic Q on $P_2(\mathbb{C})$, two tangent lines T_1, T_2 of Q and a line L passing through the intersection point of T_1 and T_2 . Explicitly they are given by the equations:

$$\begin{aligned} Q: z^2 + xy &= 0 \\ T_1: x &= 0 \\ T_2: y &= 0 \\ L: x + y &= 0. \end{aligned}$$

In this case we consider the pencil P generated by $Q + L: (x + y)(z^2 + xy) = 0$ and $2T_1 + T_2: x^2y = 0$. Then we have the base points $p_1: y = z = 0, p_2: x = y = 0, p_3: x = z = 0$ with multiplicities 2, 3, 4. As before we blow up $P_2(\mathbb{C})$ over p_1, p_2, p_3 as many times as their multiplicities to obtain the minimal resolution S of P . On S there are therefore one A_1 -, one A_2 - and one A_3 -configurations and three global sections which lie over p_1, p_2, p_3 . One can easily see that these A_2, A_3 and the strict transforms of T_1, T_2 form the fiber of S of type \tilde{D}_6 . Now note that the cubic in the pencil

$$C: xy^2 + (x + y)z^2 = (x + y)(xy + z^2) - x^2y = 0$$

has a node at $p_1: y = z = 0$. Thus the strict transform of C and the above A_1 -configuration form a fiber of type \tilde{A}_1 . The other fiber of type \tilde{A}_1 is obviously the strict transform of $Q \cup L$. According to Proposition 1.4 the surface S should have four global sections of which we have already found three. The remaining one is the strict transform of the line

$$M: z = 0$$

which in fact intersects each member of P at most one point outside the base points on it. By G_1, G_2, G_3, G_4 we denote the global sections corresponding to p_1, p_2, p_3, M . Since $\text{tr}[G_i - G_j]$ are all of order 2, the symmetries with centers G_1, G_2, G_3, G_4 are identical, and in fact with the automorphism induced by the involution $z \leftrightarrow -z$. The translation $\text{tr}[G_1 - G_2]$

is induced by the Cremona transformation:

$$\begin{cases} \bar{x} = -xyz^2 \\ \bar{y} = y(xy^2 + xz^2 + yz^2) \\ \bar{z} = -z(xy^2 + xz^2 + yz^2) \end{cases}$$

The translation $\text{tr}[G_3 - G_2]$ is induced by the Cremona transformation:

$$\begin{cases} \bar{x} = xy \\ \bar{y} = z^2 \\ \bar{z} = -yz \end{cases}$$

and $\text{tr}[G_4 - G_2]$ is induced by the Cremona transformation:

$$\begin{cases} \bar{x} = -xy^2z \\ \bar{y} = z\{xy^2 + (x+y)z^2\} \\ \bar{z} = -y\{xy^2 + (x+y)z^2\}. \end{cases}$$

The whole automorphism group of S is generated by $G(S)$ and the automorphism induced by the Cremona transformation:

$$\begin{cases} \bar{x} = xy \\ \bar{y} = -y(x+y) \\ \bar{z} = i(x+y)z \end{cases}$$

which actually interchanges the two fibers of type \tilde{A}_1 .

2.6. Case of $D_5 \oplus A_3$. We begin with a conic Q , two tangent T_1, T_2 of Q and a general line L which passes through the contact point of T_1 and Q . In suitable coordinates they are defined by the following equations:

$$Q: (x+y)z + y^2 = 0$$

$$T_1: x + y = 0$$

$$T_2: z = 0$$

$$L: x = 0.$$

P denotes the pencil generated by the cubic curves $Q + L: x\{(x+y)z + y^2\}$ and $T_1 + 2T_2: (x+y)z^2 = 0$. The base points of the pencil are $p_1: x = z = 0$, $p_2: z = y = 0$, $p_3: y = z = 0$ which have multiplicities 2, 3, 4. As before let S be the minimal resolution of P obtained by blowing up $P_2(\mathbb{C})$ over the points p_1, p_2, p_3 as many times as their multiplicities. We obtain thus an

A_1 -configuration ((-2) -curve), an A_2 -configuration and an A_3 -configuration over p_1, p_2 and p_3 . We have also three global sections over p_1, p_2, p_3 , which we denote by G_1, G_2, G_3 . These A_1 and A_3 , together with the strict transforms of T_1 and T_2 , form the fiber of type \tilde{D}_5 while the A_2 and the strict transforms of Q, L form the fiber of type \tilde{A}_3 . The fiber of type I_1 comes from the nodal cubic in P

$$C: (x+y)z^2 - 16x\{(x+y)z + y^2\} = (x+y)(8x-z)^2 - 16x(2x+y)^2 = 0$$

whose node is $(1, -2, 8)$. According to Proposition 1.4 we know that, besides G_1, G_2, G_3 above, there is one more global section of S , which is in fact the strict transform of the line:

$$M: y=0.$$

We denote this global section by G_4 . Since the singular fibers are of different types, $G(S)$ is the whole automorphism group. We can as before write down $G(S)$ as Cremona-transformation, but we do not prefer this way; we instead describe S as the (minimal) desingularization of a pencil of $(2, 2)$ -curves on $P_1(\mathbb{C}) \times P_1(\mathbb{C})$, which is more natural for the purpose. We write $(\zeta_1, \zeta_2; \eta_1, \eta_2)$ for the double homogeneous coordinates of $P_1(\mathbb{C}) \times P_1(\mathbb{C})$. As the two curves, which are to generate the desired pencil P , we introduce the following:

$$C_1: (\zeta_1 - \zeta_2)^2(\eta_1 - \eta_2)^2 = 0$$

$$C_2: \zeta_1\zeta_2\eta_1\eta_2 = 0.$$

We get now the four base points of the pencil $p_1: (1, 1; 0, 1), p_2: (1, 1; 1, 0), p_3: (0, 1; 1, 1), p_4: (1, 0; 1, 1)$ which are all of multiplicity 2. To obtain the desingularization S of the pencil we blow up twice over each of these points in such a way that all members of the pencil are separated from each other on S . Thus we have the four (-2) -curves, which together with the strict transform of C_1 , form the singular fiber of type \tilde{D}_5 , and the four (-1) -curves, which are the global sections of S . The fiber of type \tilde{A}_3 is just the strict transform of C_2 . The fiber of type I_1 is the strict transform of the following singular member of P

$$\begin{aligned} 0 &= (\zeta_1 - \zeta_2)^2(\eta_1 - \eta_2)^2 - 16\zeta_1\zeta_2\eta_1\eta_2 \\ &= (\zeta_1 + \zeta_2)^2(\eta_1 + \eta_2)^2 - 4\{\zeta_1\zeta_2(\eta_1 + \eta_2)^2 + \eta_1\eta_2(\zeta_1 + \zeta_2)^2\} \end{aligned}$$

which has actually node at $(1, -1; 1, -1)$. $G(S)$ is now generated by the automorphisms induced by the involutions:

$$(\zeta_1, \zeta_2; \eta_1, \eta_2) \longleftrightarrow (\zeta_2, \zeta_1; \eta_1, \eta_2)$$

$$\begin{aligned}
 (\zeta_1, \zeta_2; \eta_1, \eta_2) &\longleftrightarrow (\zeta_1, \zeta_2; \eta_2, \eta_1) \\
 (\zeta_1, \zeta_2; \eta_1, \eta_2) &\longleftrightarrow (\eta_1, \eta_2; \zeta_1, \zeta_2).
 \end{aligned}$$

It is left to the reader to make explicit the relation between these two constructions.

2.7. Case of A_8 . We begin by giving cubic curve with a node on $P_2(\mathbb{C})$: (x, y, z) , say

$$C: x^2y + y^2z + z^2x - 3xyz = 0 \quad (1, 1; 1): \text{ the node of } C.$$

The regular part $C^* = C - \{(1, 1, 1)\}$ has the natural group structure isomorphic to \mathbb{C}^* if one fixes an inflexion point as the identity. Starting with a point p_1 on C we draw the tangent line from p_1 to C , the further intersection point with C of which we denote by p_2 . Regarding this p_2 as the starting point, we can repeat the same process, from which we obtain the further intersection point p_3 . Now, in order that p_1 is on the tangent line at p_3 , it is necessary and sufficient that p_1 is a point of order exactly 9 of the group $C^* \cong \mathbb{C}^*$, which we assume now. (Note that the 9-torsion points are independent from the choice of the inflexion point.) In this way we obtain a “tangent triangle” of C , and since we have six 9-torsion points on C , there are exactly two of them. Now we note that the coordinates are so conveniently chosen that one of them is given simply by

$$\begin{aligned}
 \Delta: xyz = 0 \\
 (p_1 = (1, 0, 0), p_2 = (0, 1, 0), p_3 = (0, 0, 1)).
 \end{aligned}$$

We now consider the pencil P generated by the two cubic curves C and Δ and denote by S the minimal desingularization of P , which is to be the elliptic surface of the desired type. To get S , we blow up $P_2(\mathbb{C})$ three times over each of the base points p_1, p_2, p_3 ; they are of multiplicity 3 by the construction. On S we have thus the three A_2 -configurations, which together with the strict transform of Δ form the singular fiber of type \bar{A}_8 , and the three (-1) -curves, which are the global sections of S . S should have the three further singular fibers of type I_1 , one of which is the strict transform of C . The other two come from the following nodal cubics in the pencil:

$$x^2y + y^2z + z^2x - 3\omega^v xyz = 0 \quad (v = 1, 2, \omega^2 + \omega + 1 = 0)$$

The group $T(S)$ of translations is generated by the projective transformation of order 3 $(x, y, z) \rightarrow (y, z, x)$. $G(S)$ is generated by $T(S)$ and the involutive Cremona transformation $(x, y, z) \rightarrow (yz, xy, zx)$. The whole

automorphism group is generated by $G(S)$ and the projective transformation $(x, y, z) \rightarrow (x, \omega y, \omega^2 z)$ which permutes the three fibers of type I_1 evenly.

2.8. Case of $A_7 \oplus A_1$. In Section 2.2 we have shown that there exist a nodal cubic C and a conic Q which intersect at exactly one point p i.e. they are tangent at p with multiplicity 6. We can of course take a line L which intersects C (and Q) transversally at p and which is tangent to C at another point, say q . Now we consider the pencil P generated by the cubic curves C and $Q \cup L$; p is then the base point of C with multiplicity 7 and q is the base point multiplicity 2. To get the minimal desingularization S of P we should blow up seven times over p and twice over q . The A_6 -configuration on S and the strict transform of $Q \cup L$ form the singular fiber of type \tilde{A}_7 ; the singular fiber of type \tilde{A}_1 is formed by the A_1 -configuration over q and the strict transform of the member of P for which q is a singular point (a node). The remaining singular fibers of S are the two fibers of type I_1 , of which one is the strict transform of C ; the other comes from the nodal cubic in the pencil whose node is neither q nor the node of C . Explicitly L, Q, C are given by

$$\begin{aligned} L: y + 2z &= 0 \\ Q: xz - y^2 + z^2 &= 0 \\ C: x(xz - y^2) + y^2z &= 0 \\ p &= (1, 0, 0) \quad q = (2, -2, 1). \end{aligned}$$

The member of the pencil P which has a node at q is therefore defined by

$$x(xz - y^2) + y^2z + 4(y + 2z)(xz - y^2 + z^2) = 0.$$

The other nodal cubics in P are the original C and the cubic:

$$x(xz - y^2) + y^2z + 8(y + 2z)(xz - y^2 + z^2) = 0$$

whose node is actually $(16, -4, 1)$. Note now that we obtained only two (-1) -curves on S by the blowing up, while S should have 4 global sections. The other two global sections are the strict transforms of the following two curves:

$$\begin{aligned} z &= 0 \\ xz - y^2 + 2z^2 &= 0 \end{aligned}$$

(Check that each of these intersects the generic member of the pencil at exactly one point outside the base points.) Instead of writing down the automorphisms of S as Cremona transformations of $P_2(C)$ we introduce a

birational correspondence between $P_2(\mathbb{C}) : (x, y, z)$ and a 2-dimensional variety under which the Cremona transformations correspond to holomorphic automorphisms of the variety. This variety is given in $\mathbb{C}^5 : (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta)$ by the equations:

$$\begin{aligned} \zeta_1 + \zeta_2 &= 1 \\ \zeta_3 + \zeta_4 &= 1 \\ \zeta^2 &= \zeta_1 \zeta_2 \zeta_3 \zeta_4 \end{aligned}$$

The birational correspondence is given by

$$\begin{aligned} \zeta_1 &= \frac{xz - y^2 + z^2}{xz - y^2 + 2z^2} \\ \zeta_2 &= \frac{z^2}{xz - y^2 + 2z^2} \\ \zeta_3 &= \frac{(xz - y^2 + 2z^2)^2}{z\{x(xz - y^2) + y^2z + 4(y + 2z)(xz - y^2 + z^2)\}} \\ \zeta_4 &= \frac{(y + 2z)^2(xz - y^2 + z^2)}{z\{x(xz - y^2) + y^2z + 4(y + 2z)(xz - y^2 + z^2)\}} \\ \zeta &= \frac{(y + 2z)(xz - y^2 + z^2)}{\{x(xz - y^2) + y^2z + 4(y + 2z)(xz - y^2 + z^2)\}} \end{aligned}$$

Note also that the inverse mapping $(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta) \rightarrow (x, y, z)$ is given by

$$\begin{aligned} \frac{x}{z} &= \left(\frac{\zeta_4}{\zeta} - 2\right)^2 + \frac{\zeta_1}{\zeta_2} - 1 \\ \frac{y}{z} &= \frac{\zeta_4}{\zeta} - 2. \end{aligned}$$

The fibers of S correspond to the family of curves on the variety defined by

$$\zeta = \text{const.}$$

The group $T(S)$ of translations corresponds to the cyclic group generated by the automorphism $(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta) \rightarrow (\zeta_3, \zeta_4, \zeta_2, \zeta_1, \zeta)$; $G(S)$ is then generated by $T(S)$ and the involution induced by $(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta) \leftrightarrow (\zeta_2, \zeta_1, \zeta_3, \zeta_4, \zeta)$ (or $(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta) \leftrightarrow (\zeta_1, \zeta_2, \zeta_4, \zeta_3, \zeta)$). The whole automorphism group $\text{Aut}(S)$ is generated by $G(S)$ and the involution of S corresponding to $(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta) \leftrightarrow (\zeta_1, \zeta_2, \zeta_3, \zeta_4, -\zeta)$. Now, it becomes an obvious translation to write

down $\text{Aut}(S)$ in the form of Cremona transformations of $P_2(\mathbf{C})$.

2.9. Case of $A_5 \oplus A_2 \oplus A_1$. For this case we begin with a nodal cubic C and a conic Q which intersect only at the node p of C (with multiplicity 6). (Q touches one of the two branches of C at p with multiplicity 5.) Further we take an inflexion point q of C and the corresponding inflexion line L . Explicitly we can introduced L, Q, C as follows:

$$L: z=0$$

$$Q: x^2 + xy + xz + y^2 = 0$$

$$C: -x^2z + xyz + y^3 = 0$$

$$p = (0, 0, 1)$$

$$q = (1, 0, 0)$$

As before we consider the pencil P generated by the two cubic curves $L \cup Q$ and C . The member of P which has a node at q is given by

$$C_1: f_{(1)}(x, y, z) = 0$$

where we have set

$$f_{(a)}(x, y, z) = z(x^2 + xy + xz + y^2) + a(-x^2z + xyz + y^3).$$

Since p, q are the base points of P with multiplicity 6, 3, we blow up $P_2(\mathbf{C})$ six times over p and three times over q to obtain the minimal desingularization S of P . The A_5 -configuration over p , together with the strict transform of C , forms the singular fiber of type \tilde{A}_5 ; the A_2 -configuration over q , together with the strict transform of C_1 , forms then the fiber of type \tilde{A}_2 . The strict transform of $Q \cup L$ itself is the fiber of type \tilde{A}_1 . Since S should have just one singular fiber of type I_1 , there must exactly be one further nodal cubic in P which is regular at p and q . This is in fact given by

$$C_2: f_{(0)}(x, y, z) = 0$$

and its node is $(1, 4, -24)$ as is checked directly. So far we have only two global sections on S which are the (-1) -curves arising from the blowing up. The other four global sections are the strict transforms of the following curves:

$$x=0$$

$$y=0$$

$$g_1=0 \quad (g_1 := xz + y^2)$$

$$g_2=0 \quad (g_2 := xy + xz + y^2)$$

each of which actually intersects the generic member $f_{(a)}=0$ at $(0, 1, -a)$, $(1, 0, a-1)$, $(-a^2, a(a-1), (a-1)^2)$, $(a^2, -(a-1), a-1)$ outside the base points p, q . (The expressions g_1, g_2 will be used later; we also make a convention: $f_{(\infty)}=z(x^2+xy+xz+y^2)$.) As in the previous section we now introduce a birational correspondence between $P_2(C)$; (x, y, z) and a 2-dimensional affine variety in C : $(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6)$ in such a way that the automorphism group of S can be written in a particularly simple form. The variety, denoted by V , is given by the equations

$$\begin{cases} \zeta_1 + \zeta_4 = 0 \\ \zeta_2 + \zeta_5 = 0 \\ \zeta_3 + \zeta_6 = 0 \\ \zeta_1\zeta_3\zeta_5 - \zeta_4\zeta_5\zeta_6 = 0 \end{cases}$$

and the correspondence is defined by

$$\begin{aligned} \zeta_1 &= (xy)/g_1 \\ \zeta_2 &= (-xg_1)/(yg_2) \\ \zeta_3 &= -g_1g_2/(xf_1) \\ \zeta_4 &= -g_2/g_1 \\ \zeta_5 &= -f_{(\infty)}/(yg_2) \\ \zeta_6 &= yf_{(\infty)}/(xf_{(1)}). \end{aligned}$$

(To see that this is birational, one needs only to check that the inverse is given by $x/z = \zeta_1\zeta_2^2\zeta_4^2/(1-\zeta_4\zeta_5)$, $y/z = \zeta_1\zeta_2\zeta_4/(1-\zeta_4\zeta_5)$.) Note now that the whole automorphism group coincides with $G(S)$ since the four singular fibers are of different types in this case. Note also that the fibration on S corresponds to the family of curves on V given by

$$\zeta_1\zeta_3\zeta_5 (= \zeta_2\zeta_4\zeta_6) = \text{const.}$$

Thus, in particular, the cyclic permutation of the coordinates $\zeta_1 \rightarrow \zeta_2 \rightarrow \zeta_3 \rightarrow \zeta_4 \rightarrow \zeta_5 \rightarrow \zeta_6 \rightarrow \zeta_1$ induces an automorphism of S . In fact this generates the group of translations which should be of order 6. The group $G(S)$ is generated for example by the involution $\zeta_2 \leftrightarrow \zeta_6, \zeta_3 \leftrightarrow \zeta_5$ (ζ_1, ζ_4 : unchanged) and this cyclic permutation.

2.10. Case of $2A_4$. First we take the four points on $P_2(C)$ in general position which are well known to be unique projectively: We set

$$p_1 = (1, -1, 0)$$

$$\begin{aligned}
 p_2 &= (1, 0, -1) \\
 q_1 &= (0, 1, 0) \\
 q_2 &= (0, 0, 1).
 \end{aligned}$$

Then we observe two triangles $\Delta_1 = \overline{p_1q_1} \cup \overline{p_2q_2} \cup \overline{p_1p_2}$ and $\Delta_2 = \overline{p_1q_2} \cup \overline{p_2q_1} \cup \overline{q_1q_2}$. By the equations they are given as follows:

$$\begin{aligned}
 \Delta_1: & \quad yz(x+y+z)=0 \\
 \Delta_2: & \quad x(x+y)(x+z)=0
 \end{aligned}$$

Now we form the pencil P generated by the two cubic curves Δ_1 and Δ_2 and denote by S the minimal desingularization of P obtained by blowing up $P_2(\mathbb{C})$ over the base points of P ; p_1, p_2, q_1, q_2 are the base points of multiplicity 2 and $r=(0, 1, -1)$ the simple base point. Thus we have already obtained all the five global sections of S . The (-2) -curves over p_1, p_2 and the strict transform of Δ_1 form a singular fiber of type \tilde{A}_4 while the other fiber of the same type is obtained from Δ_2 and the (-2) -curves over q_1, q_2 . But there must be still two singular fibers of type I_1 which we have to find. They are nodal cubics in P ; if we set

$$C_{(a)}: yz(x+y+z) - ax(x+y)(x+z) = 0$$

then these correspond to the values $a = (-11 \pm 5\sqrt{5})/2$ of the parameter. One can of course directly check this fact. But one can also determine the above values if one knows the position of the nodes of them which should be the fixed point of the action of $T(S)$ (or even $G(S)$). We will now introduce a 2-dimensional affine variety V of $\mathbb{C}^5: (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5)$ and a birational correspondence between $P_2(\mathbb{C})$ and V so that the action of $T(S)$ is particularly simply described by passing to V : The equations for V are

$$\begin{cases}
 \zeta_3\zeta_4 + \zeta_1 - 1 = 0 \\
 \zeta_4\zeta_5 + \zeta_2 - 1 = 0 \\
 \zeta_3\zeta_1 + \zeta_3 - 1 = 0 \\
 \zeta_1\zeta_2 + \zeta_4 - 1 = 0 \\
 \zeta_2\zeta_3 + \zeta_5 - 1 = 0
 \end{cases}$$

and the correspondence is given by

$$\begin{aligned}
 \zeta_1 &= (x+y)/x \\
 \zeta_2 &= x(x+y+z)/((x+y)(x+z))
 \end{aligned}$$

$$\begin{aligned} \zeta_3 &= (x+z)/x \\ \zeta_4 &= -y/(x+z) \\ \zeta_5 &= -z/(x+y). \end{aligned}$$

One sees now that the generic member $C_{(a)}$ of the pencil P corresponds to the curve on V defined by

$$(*) \quad \zeta_1 \zeta_2 \zeta_3 \zeta_4 \zeta_5 = a.$$

Thus the cyclic automorphism $\zeta_1 \rightarrow \zeta_2 \rightarrow \zeta_3 \rightarrow \zeta_4 \rightarrow \zeta_5 \rightarrow \zeta_1$ of V induces the automorphism of S which preserves each fiber and therefore belongs to $G(S)$. In fact this generates the group $T(S)$. The five symmetries associated with the global sections are $\zeta_2 \leftrightarrow \zeta_5, \zeta_3 \leftrightarrow \zeta_4, \zeta_1 \leftrightarrow \zeta_3, \zeta_4 \leftrightarrow \zeta_5$ etc. Thus $G(S)$ is a dihedral group of order 10. The fixed points of $G(S)$ are obviously

$$\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \zeta_5 = (-1 \pm \sqrt{5})/2.$$

The curves in (*) on which either of them lies are therefore

$$\zeta_1 \zeta_2 \zeta_3 \zeta_4 \zeta_5 = \{(-1 \pm \sqrt{5})/2\}^5$$

which just correspond to the nodal cubics in P mentioned above.

Now, since the values $a = (-11 \pm 5\sqrt{5})/2$ are not interchanged by the map $a \leftrightarrow -a$, $G(S)$ can not be of index 4 in the whole automorphism group of S . The index is exactly 2 since the birational automorphism of V which maps $(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5)$ to $(-\zeta_1/(\zeta_3 \zeta_4), -\zeta_3/(\zeta_1 \zeta_5), -\zeta_5/(\zeta_2 \zeta_3), -\zeta_2/(\zeta_4 \zeta_5), -\zeta_4/(\zeta_1 \zeta_2))$ induces the transformation of the parameter $a \leftrightarrow -1/a$ which actually transposes the two fibers of type \tilde{A}_4 and those of type I_1 .

2.11. Case of $2A_3 \oplus 2A_1$. We begin with the configuration of a conic Q and the three (different) tangents L_1, L_2, L_3 of Q , which are explicitly given by the equations

$$\begin{aligned} Q: \quad & xy - z^2 = 0 \\ L_1: \quad & x = 0 \\ L_2: \quad & y = 0 \\ L_3: \quad & x + y - 2z = 0. \end{aligned}$$

The three contact points p_1, p_2, p_3 on L_1, L_2, L_3 have the coordinates;

$$\begin{aligned} p_1 &= (0, 1, 0) \\ p_2 &= (1, 0, 0) \end{aligned}$$

$$p_3 = (1, 1, 1).$$

We denote the intersection point of L_1 and L_2 by q and the line passing through q, p_3 by L :

$$q = (0, 0, 1)$$

$$L: x = y$$

Now we form the pencil P generated by the two curves $Q \cup L$ and $L_1 \cup L_2 \cup L_3$; the base points are then q, p_1, p_2, p_3 , which have the multiplicities 2, 2, 2, 3. Thus, by the blowing up, we obtain the three A_1 -configurations and the A_2 -configuration and the four (-1) -curves on the minimal desingularization S of P . Besides the given ones, there are exactly two singular members of P , which are nodal cubics:

$$\begin{aligned} C_1: & 2x^2y - 2xyz - xz^2 + yz^2 \\ & = -(x-y)(xy - z^2) + xy(x+y-2z) = 0 \end{aligned}$$

$$\begin{aligned} C_2: & 2xy^2 - 2xyz + xz^2 - yz^2 \\ & = (x-y)(xy - z^2) + xy(x+y-2z) = 0 \end{aligned}$$

C_1 has the node at the base point p_1 and C_2 at p_2 . The strict transform of C_1 and the A_1 -configuration over p_1 form a singular fiber of type \tilde{A}_1 . In the same way we obtain the other fiber of type \tilde{A}_1 from C_2 and the (-2) -curve over p_2 . One singular fiber of type \tilde{A}_3 is formed by the (-2) -curve over q and the strict transform of the triangle $L_1 \cup L_2 \cup L_3$. The other one is the union of the A_2 -configuration and the strict transform of $Q \cup L$. Note that, by the blowing up, we have obtained only four among the eight global sections of the surface S . The other four are the strict transforms of the following curves:

$$G_1: x = z$$

$$G_2: y = z$$

$$G_3: z = 0$$

$$G_4: 2xy - (x+y)z = 0$$

Now we blow down the eight global sections to get $P_1(\mathbf{C}) \times P_1(\mathbf{C})$. In fact we can explicitly write down the birational mapping of $P_2(\mathbf{C})$: (x, y, z) as follows.

$$\zeta = (x-z)(y-z)/(xy-z^2)$$

$$\eta = \{(x+y)z - 2xy\}/\{(x-y)z\}$$

where (ζ, η) are the coordinates of the open set $C \times C$ of $P_1(C) \times P_1(C)$. One can directly check that the fibration of S corresponds to the pencil of $(2, 2)$ -curves on $P_1(C) \times P_1(C)$ given by

$$(\zeta_1^2 - \zeta_2^2)(\eta_1^2 - \eta_2^2) - 4a\zeta_1\zeta_2\eta_1\eta_2 = 0$$

where $(\zeta_1, \zeta_2; \eta_1, \eta_2)$ are the double homogeneous coordinates $(\zeta = \zeta_1/\zeta_2, \eta = \eta_1/\eta_2)$ and a the parameter ranging over $P_1(C) = C \cup \{\infty\}$. One immediately sees that the fibers of type \tilde{A}_3 correspond to the quadrangles

$$(\zeta_1^2 - \zeta_2^2)(\eta_1^2 - \eta_2^2) = 0, \quad \zeta_1\zeta_2\eta_1\eta_2 = 0$$

and that the global sections correspond to the base points $(1, 0; 1, \pm 1)$, $(0, 1; 1, \pm 1)$, $(1, \pm 1; 1, 0)$, $(1, \pm 1; 0, 1)$. This clearly shows that in this case the construction by using $P_1(C) \times P_1(C)$ is more transparent than that by using $P_2(C)$. In fact the group $T(S)$ of translations, which is isomorphic to $(\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, corresponds to the group generated by the transformation $(\zeta, \eta) \rightarrow (\eta, -1/\zeta)$ of order 4 and the involution $(\zeta, \eta) \leftrightarrow (-\zeta, -\eta)$ which commute with each other. There are exactly the two symmetries associated with the global sections, which are induced by $(\zeta, \eta) \leftrightarrow (-\zeta, 1/\eta)$, $(\zeta, \eta) \leftrightarrow (1/\zeta, -\eta)$. The whole automorphism group is generated by $G(S)$ and the two involutions induced by $(\zeta, \eta) \leftrightarrow (1/\zeta, \eta)$, $(\zeta, \eta) \leftrightarrow ((\zeta + 1)/(\zeta - 1), (\eta + 1)/(\eta - 1))$. The former involution permutes the fibers of type \tilde{A}_1 since it changes the sign of the parameter a while the latter permutes the fiber of type \tilde{A}_2 and brings a to its inverse. Finally we remark that the fibers of type \tilde{A}_1 correspond to the values $a = \pm 1$ for which the $(2, 2)$ -curve of the pencil decomposes to two $(1, 1)$ -curves:

$$\begin{aligned} \{(\zeta\eta - 1) + (\zeta + \eta)\} \{(\zeta\eta - 1) - (\zeta + \eta)\} &= 0 \\ \{(\zeta\eta + 1) + (\zeta - \eta)\} \{(\zeta\eta + 1) - (\zeta - \eta)\} &= 0 \end{aligned}$$

2.12. Case of $4A_2$. This case is the most classical one. We observe for example the Fermat cubic in $P_2(C)$: (x, y, z)

$$C_0: x^3 + y^3 + z^3 = 0.$$

Since the Hessian of this is $216 \times xyz$, the nine inflexion points are three and three on the edges of the triangle

$$\Delta: xzy = 0.$$

But there are three more triangles with this property; in fact the pencil P

generated by C_0, A , which is represented as the parametrized curve

$$C_a: x^3 + y^3 + z^3 - 3axyz = 0,$$

degenerates to a triangle for each value $a = 1, \omega, \omega^2$ ($\omega^2 + \omega + 1 = 0$). As an elliptic surface, P is already non-singular, since the base points are just the nine inflexion points and are thus of multiplicity 1. P is in fact the blowing up of $P_2(\mathbb{C})$ with these points as center. As is seen above, P has four singular fibers of type \tilde{A}_2 (and no other singular fiber). The automorphisms of P are all induced by the projective transformations which map the nine points onto themselves: $T(S)$ is isomorphic to the abelian group $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$ and is generated by the two transformations of order 3, $x \rightarrow y \rightarrow z \rightarrow x$, $(x, y, z) \rightarrow (x, \omega y, \omega^2 z)$, which obviously commute with each other. $G(S)$ is obtained by adding a transposition $y \leftrightarrow z$ to $T(S)$. (By the conjugation we get the nine symmetries from this which correspond to the nine inflexion points.) Note now that the transformation $(x, y, z) \rightarrow (\omega x, y, z)$ induces the automorphism of the pencil P which induces the non-trivial mapping $a \rightarrow \omega^2 a$ of the parameter space. We have also the following involutive automorphism $(x, y, z; a) \rightarrow (\bar{x}, \bar{y}, \bar{z}; \bar{a})$ of the pencil:

$$\bar{x} = x + y + z$$

$$\bar{y} = x + \omega y + \omega^2 z$$

$$\bar{z} = x + \omega^2 y + \omega z$$

$$\bar{a} = (a + 2)/(a - 1)$$

These two automorphisms generate a group of order 24 which permutes evenly the four points $a = 1, \omega, \omega^2, \infty$ over which the singular fibers lie. Since this permutation group is the alternating group of four letters and the four points are the vertices of a tetrahedron in the number sphere $\mathbb{C} \cup \{\infty\}$, we can thus obtain all automorphisms of P by composing the ones given above.

§ 3. Covering formation

In this section we will remark that some important elliptic surfaces can be described as coverings of maximal rational elliptic surfaces. Such coverings should usually be obtained by base change; more precisely we take a union of singular fibers to be the ramification locus and observe only a cyclic covering branched over it. The covering is in general a *singular* elliptic surface (with respect to the induced fibration) i.e. it can have (isolated) singular points. We can of course resolve them; but, after the minimal desingularization, it might still have (-1) -curves contained in

some fibers. By a successive blowing down of such (-1) -curves we get the elliptic surface which has no such curves anymore. This is independent of the process of the blowing down as is guaranteed by the theory [Kd]. This last surface is called the *reduced* cyclic covering associated with the original union of fibers which was the ramification locus; its degree should of course be that of the cyclic covering which appeared by the way of this construction. (We use this terminology even if there occurs no blowing down of (-1) -curves, in which case the reduced covering means simply the minimal resolution of the cyclic covering.)

Now we apply this formation to some surfaces constructed in Section 2. We begin with the elliptic surface S associated with \tilde{E}_8 . Besides the one of type \tilde{E}_8 , this has two singular fibers of type I_1 . The reduced double covering, denoted by \tilde{S} , of S branched over these two fibers is now an elliptic surface of which two singular fibers are of type \tilde{E}_8 and the other two of type \tilde{A}_1 . The unique global section of S is lifted to a global section of \tilde{S} which is a (-2) -curve. We have thus $c_2(\tilde{S})=24$, $c_1(\tilde{S})=0$ and \tilde{S} is simply connected. This implies that \tilde{S} is a $K3$ surface. The configuration of singular fibers and the global section gives us an imbedding of $A_3 \oplus 2E_8$ into the $K3$ lattice (use the global section!). From the surface associated with $E_7 \oplus A_1$, we can construct exactly in the same way an elliptic $K3$ surface over $P_1(\mathbb{C})$ with four singular fibers of which two are of type \tilde{E}_7 and one is of type \tilde{A}_3 and the rest of type \tilde{A}_1 . In this case there are 2 global sections which are (-2) -curves; we obtain an imbedding of $D_5 \oplus 2E_7$ into the $K3$ lattice.

For the surface associated with $E_6 \oplus A_2$ we consider the reduced covering of degree 3 ramified over the singular fibers of types \tilde{E}_6 and I_1 . The fiber of the covering over this \tilde{E}_6 is however regular; over the fiber of type I_1 there lies a singular fiber of type \tilde{A}_2 . Over the fiber of type \tilde{A}_2 , which is outside the ramification locus, there lie three singular fibers of the same type. Thus we have an elliptic surface over $P_1(\mathbb{C})$ and with four singular fibers of type \tilde{A}_2 . (Note that the self-intersection number of the lifted global sections, which was equal to -3 , is increased to -1 by the blowing down for getting the reduced covering.) This is in fact the maximal rational elliptic surface associated with $4A_2$. We mention here also another strange covering relation: Starting with the same maximal elliptic surface, we consider the reduced covering of degree 3 branched over the fibers of types \tilde{E}_6 and \tilde{A}_2 . We have then one fiber of type \tilde{A}_6 and three fibers of type I_1 on the covering, which shows that it is nothing other than the maximal rational elliptic surface associated with A_6 . (These two covering formations are communicated by K. Ueno.)

For the surfaces constructed in Sections 2.4~2.6 we note only that we can construct from them the elliptic $K3$ surfaces over $P_1(\mathbb{C})$ whose

combinations of singular fibers are $(2\tilde{D}_8, 2\tilde{A}_1)$, $(2\tilde{D}_8, 2\tilde{A}_3)$, $(2\tilde{D}_5, \tilde{A}_7, \tilde{A}_1)$. These give imbeddings of $2D_8 \oplus A_3$, $2D_8 \oplus A_7$, $2D_5 \oplus A_7 \oplus A_2$ into the $K3$ lattice. The details are left to the reader.

Next we consider the maximal rational elliptic surfaces with only singular fibers of A types. To begin with, we suppose that S is the surface associated with A_8 (Section 2.7). Besides the one of type \tilde{A}_8 , S has three singular fibers of type I_1 , the union of which we regard as the ramification locus and form the reduced cyclic covering S_1 of degree 3. S_1 has three \tilde{A}_8 and three \tilde{A}_2 and the liftings to S_1 of the three global sections of S are elliptic curves of self-intersection number -3 . Now we take the union of three singular fibers of type \tilde{A}_2 and form the reduced cyclic covering S_2 of degree 3 with this as the ramification locus. Then all singular fibers of S_2 are of the same type \tilde{A}_8 and we have just 12 of them. We have at least three global sections of S_2 which are of self-intersection number -9 and of genus 4. Choose one of them as the zero section and give the group structure to (the regular part of) every fiber. Then we see that we find exactly $72 (= 9 \times 9 - 3 \times 3)$ 9-torsion points on each fiber, including singular fibers. This implies that these points form a smooth curve in S_2 and that the projection to the base curve induces an unramified covering, when restricted to this locus of 9-torsion points. This covering is of degree 3 if it is restricted to a connected component of the locus. Now we note that we can further form the base change by this restriction and that we thus obtain an elliptic surface with 36 singular fibers of type \tilde{A}_8 for which the 9-torsion points form exactly 72 global sections; the genus of global sections is 10. (The 3 global sections of S_2 are lifted to 9 global sections.) This is nothing other than the elliptic modular surface in the sense of [Sh] associated with the principal congruence subgroup of $SL(2, \mathbf{Z})$ of level 9. (Check the coincidence of numbers of singular fibers and global sections!)

Next we assume that S is the maximal rational elliptic surface associated with $A_7 \oplus A_1$ (Section 2.8). First we consider the reduced double covering S_1 of S branched over the fibers of type I_1 , which is an elliptic $K3$ surface with two \tilde{A}_7 and four \tilde{A}_1 . On S_1 we have four global sections lifted from S which are (-2) -curves. We further form the reduced cyclic covering S_2 of degree 4 of S_1 branched over the four fibers of type \tilde{A}_1 . S_2 has now 12 \tilde{A}_7 as its singular fibers and we have at least four global sections of genus 3 and of self-intersection number -8 . As before we choose one global section as the zero to give the group structure to fibers of S_2 and we see that the locus of 8-torsion points is a smooth curve and that the projection, restricted to a connected component of the locus, defines an unramified double covering of the base curve of the elliptic surface S_2 . Now, by making the base change by this, we obtain an elliptic surface with 24 singular fibers of type \tilde{A}_7 and the 48 $(= 8 \times 8 - 4 \times 4)$ global sections

which correspond to the 8-torsion points of fibers. This is the elliptic modular surface associated with the principal congruence subgroup of $SL(2, \mathbb{Z})$ of level 8.

Now suppose that S is the maximal rational elliptic surface associated with $A_5 \oplus A_2 \oplus A_1$ (Section 2.9). We then denote by S_1 the reduced cyclic covering of degree 3 of S ramified over the two singular fibers of types \tilde{A}_1 and I_1 . S_1 has four \tilde{A}_5 and four \tilde{A}_2 as singular fibers, and the global sections of S_1 are (-3) -curves. We then form the reduced double covering S_2 of S_1 branched over the four fibers of type \tilde{A}_2 . Thus we obtain 12 singular fibers of type \tilde{A}_5 and the global sections are elliptic curves of self-intersection number -6 . The group structure of fibers of S_2 is given as before. This time the locus of 6-torsion points of fibers decomposes into 24 $(= 6 \times 6 - 3 \times 3 - 2 \times 2 + 1)$ global sections. Thus S_2 itself is the elliptic modular surface associated with the principal congruence subgroup of $SL(2, \mathbb{Z})$ of level 6. We note here that the elliptic modular surfaces obtained so far are successive cyclic coverings of the surfaces constructed in Section 2.7~2.9, which however makes a distinction from the following relatively simple constructions.

In fact the elliptic modular surface associated with the principal congruence subgroup of $SL(2, \mathbb{Z})$ of level 5 is obtained as a reduced cyclic covering of degree 5 of the surface S associated with $2A_4$ (Section 2.10). Recall that S has four singular fibers of which two are of type \tilde{A}_4 and the other two of type I_1 . We then denote by S_1 the reduced cyclic covering of order 5 ramified over the fibers of type I_1 . On S_1 we have thus only 12 singular fibers of type \tilde{A}_4 and the global sections are (-5) -curves. As before we define the group structure of fibers and we see that the projection induces the unramified covering between the locus of 5-torsion points and the base curve which is rational. The last fact implies that the locus decomposes into 24 $(= 5 \times 5 - 1 \times 1)$ global sections, which shows that S_1 itself is the desired elliptic modular surface. Together with the zero section there are thus 25 global sections, among which only five are those lifted from S . Now the natural question arises: What kind of curves are the images of the other 20 global sections under the morphism S_1 onto S ? Since S can further be blown down to $P_2(\mathbb{C})$, the image curves should be given by their explicit equations. The same question can naturally be asked also for the other elliptic modular surfaces constructed above. The reason for why we ask it here is that the answer to it shows in a way how natural the (birational) coordinates $(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5)$ in Section 2.10 are given to S . Without any reasoning we state that the image curves are the following four curves:

$$\varepsilon \zeta_1 + \varepsilon^2 \zeta_2 + \varepsilon^3 \zeta_3 + \varepsilon^4 \zeta_4 + \zeta_5 = 0$$

where ε ranges over the set of primitive 5-th roots of unity. (When lifted to S_1 , each of these curves decomposes into five global sections. Note also that each of these is mapped onto itself by the action of $T(S)$.) The blowing down $S \rightarrow P_2(\mathbb{C})$ maps further the four curves to the four cubic curves passing through four base points p_1, p_2, q_1, q_2 each of which has such a cusp at one of the nodes of $C_{(a)}$ ($a = (-11 \pm 5\sqrt{5})/2$) that the tangent of the cusp contacts one of the branches of the curve at the node. The equations of these cubics are a little cumbersome, when compared with the expressions above.

The similar construction for the rational elliptic surface associated with $2A_3 \oplus 2A_1$ proceeds much easier: We assume that S is this surface. Then we denote by S_1 the reduced double covering of S branched over the two fibers of type \tilde{A}_1 . S_1 has thus an elliptic $K3$ surface with six singular fibers of type \tilde{A}_3 . Since the base curve of S_1 is $P_1(\mathbb{C})$, the 4-torsion points form exactly 12 ($= 4 \times 4 - 2 \times 2$) global sections. (The 2-torsion points give 3 global sections since they are obtained from the 4-torsion points by the duplication.) S_1 is the elliptic modular surface associated with the principal congruence subgroup of level 4. Among the 16 global sections only eight are the lifted ones. The other eight are mapped by the projection $S_1 \rightarrow S$ onto the four curves;

$$\zeta = \pm i$$

$$\eta = \pm i$$

where ζ, η are the birational coordinates introduced in Section 2.11. These do not seem so interesting as in the previous case.

The rational elliptic surface associated with $4A_2$ is itself the elliptic modular surface associated with the principal congruence subgroup of level 3.

More sophisticated covering formations are of course possible and might be used for many purposes. The images of global sections of such coverings might enrich the original configurations on $P_2(\mathbb{C})$ which are used for the construction of maximal rational elliptic surfaces. But this is anyway included in the principle of observing the loci of torsion points of cubic curves in the pencils, which is our viewpoint described in the introduction.

It remains to be a strange fact however that we could not construct the elliptic modular surface for the level 7 in such a way as described above.

§ 4. Some quotients of unit ball

In this section we will shed a light on the importance of our viewpoint by showing that some (volume-finite) quotients of the 2-dimensional unit

ball are obtained by using the elliptic surfaces constructed above. Most of them are included in the works of Hirzebruch [H2], Inoue [In], Livné [L] and others. We begin with the result of [L]: Let $E(m)$ denote the elliptic modular surface associated with the principal congruence subgroup of level m . Livné shows that, for positive number d which divides the numerator of $m/2$, there exist cyclic coverings of $E(m)$ of degree d branched over the union of the m^2 global sections. (He shows also that, if d divides the numerator of $m/6$, then there exists uniquely such a covering to which all automorphisms of $E(m)$ can be lifted.) The main result of [L] states that the universal covering of such a covering is (isomorphic to) the unit ball in C^2 if and only if (m, d) is either of $(5, 5), (7, 7), (8, 4), (9, 3), (12, 2)$. (Note that d is exactly the numerator of $m/6$ for these pairs.) The same result is also obtained independently by Inoue [In]. As is shown in Section 3, the surfaces $E(5), E(8), E(9)$ are described as certain coverings of some maximal rational elliptic surfaces; $E(12)$ is also an unramified base change of degree 4 of the (reduced) double covering of $E(6)$ branched over the union of singular fibers. (As is remarked at the end of Section 3, the surface $E(7)$ is unfortunately missing in our construction.) To obtain the above result Inoue and Livné used the Yau-Miyaoaka theory [Y], [M1] concerning the inequality $c_1^2 \leq 3c_2$. For later examples we need however a generalized form of this: Suppose that X is a non-singular compact surface and that on X there are given non-singular elliptic curves C_1, C_2, \dots, C_m and rational double configurations F_1, F_2, \dots, F_n . (By a rational double configuration we mean a configuration of (-2) -curves which can be blown down to a rational double point, so it is either of A_k ($k=1, 2, \dots$), D_k ($k=4, 5, \dots$) or E_6, E_7, E_8). Suppose further that $C_1, \dots, C_m; F_1, \dots, F_n$ are mutually disjoint. Now, for each F_i $1 \leq i \leq n$, we introduce a rational number called its *correction invariant*:

$$\mu_i = 3 \left(e(F_i) - \frac{1}{|G_i|} \right)$$

where $e(F_i)$ is the Euler number of F_i and G_i is the finite subgroup of $SL(2, C)$ such that the quotient singularity C^2/G_i is (locally) equivalent to the singularity obtained by blowing down F_i . (Note that $e(A_k) = k + 1$, $e(D_k) = k + 1$, $e(E_k) = k + 1$.) Now, under the additional assumption that X is a minimal surface of general type (or more generally that $K_X + \sum_{i=1}^m C_i$ is numerically effective and $\kappa(K_X + \sum_{i=1}^m C_i + \sum_{i=1}^n F_i) = 2$) we have the inequality:

$$3c_2(X) - c_1(X)^2 + \sum_{i=1}^m C_i \cdot C_i - \sum_{i=1}^n \mu_i \geq 0.$$

For this see Sakai [S], Miyaoka [M2], Kobayashi [Ko]. [Ko] proves further that, if the equality holds above, then $\check{X} - \cup_{i=1}^n C_i$ is the quotient of the unit ball in C^2 by some automorphism group of the ball acting properly discontinuously, where \check{X} denotes the surface obtained by blowing down F_i separately. Such a quotient is in general compactified by adding a finite number of cusps and, when the cusps are (minimally) resolved, we recover X ; C_1, C_2, \dots, C_m are then the exceptional sets of resolution.

We are now ready to produce more examples: First we observe the reduced double covering $E^{(2)}(3)$ of $E(3)$ branched over the union of singular fibers. We take this surface as X and the nine global sections lifted from $E(3)$ as C_i 's. Note that they are elliptic curves of self-intersection number -2 . As F_i 's we take the $12(=3 \times 4)$ (-2) -curves in the singular fibers which are disjoint from C_i 's. Obviously $c_1^2(X)=0$ and $c_2(X)=6 \times 4=24$ since X has four singular fibers of type \tilde{A}_5 . Note also that $\mu_i=9/2$ for each F_i . We can now calculate the left hand side of the inequality:

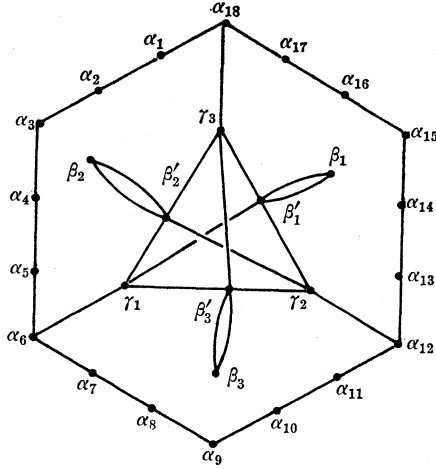
$$3 \times 24 + 9 \times (-2) - 12 \times \left(\frac{9}{2}\right) = 0$$

Thus $\check{X} - \cup_{i=1}^9 C_i$ is a quotient of the unit ball. This example is due to Hirzebruch.

By using the maximal rational elliptic surface associated with the root system A_8 , we can also construct another example of this kind: We denote this surface by $S(A_8)$. Since $S(A_8)$ has exactly four singular fibers, we can form the (reduced) double covering $S^{(2)}(A_8)$ branched over all singular fibers. $S^{(2)}(A_8)$ has then one fiber of type \tilde{A}_{17} and three fibers of type \tilde{A}_1 and we have the three global sections lifted from $S(A_8)$. By observing the (-2) -curves which are disjoint from the lifted global sections, we obtain three A_5 -configurations from the fiber \tilde{A}_{17} and three A_1 -configurations from the three \tilde{A}_1 . Let F_1, F_2, \dots, F_6 denote these six configurations and C_1, C_2, C_3 the global sections. For $X=S^{(2)}(A_8)$ we have $c_2(X)=18+3 \times 2=24$ (and $c_1^2(X)=0$). Thus the left hand side of the inequality is:

$$3 \times 24 + 3 \times (-2) - 3 \times 3 \cdot \left(6 - \frac{1}{6}\right) - 3 \times 3 \left(2 - \frac{1}{2}\right) = 0.$$

Thus we are also done in this case. We denote by Γ the discrete subgroup of $PSU(2, 1)$; $S^{(2)}(A_8) - (C_1 \cup C_2 \cup C_3) \cong B/\Gamma$ where B is the unit ball on which $PSU(2, 1)$ acts holomorphically. Now we want to show that Γ has (twelve) subgroups of index 2. For this purpose we give names to the three global sections and the (-2) -curves in the singular fibers of $S^{(2)}(A_8)$ as follows:



The \tilde{A}_{17} consists of $\alpha_1, \alpha_2, \dots, \alpha_{18}$; the three \tilde{A}_1 are $(\beta_1, \beta'_1), (\beta_2, \beta'_2), (\beta_3, \beta'_3)$; the three lifted global sections are $\gamma_1, \gamma_2, \gamma_3$. Check that in the homology group we have the identity:

$$\begin{aligned} & \gamma_1 + \gamma_2 + \alpha_1 + \alpha_3 + \alpha_5 + \alpha_{13} + \alpha_{15} + \alpha_{17} \\ &= 2 \times \left\{ \gamma_3 + 2 \cdot (\alpha_1 + \alpha_{18} + \alpha_{17}) + \alpha_2 + \alpha_3 + \alpha_{15} + \alpha_{16} - \sum_{i=6}^{12} \alpha_i \right\}. \end{aligned}$$

(By a theorem of [Sh] we have to check only that the intersection numbers of the both hand sides with any curve in the configuration coincide with each other). This shows that there are (four) double coverings of $S^{(2)}(A_8)$ branched over the above effective divisor. Take one of them and blow down the inverse images of $\alpha_1, \alpha_3, \alpha_5, \alpha_{13}, \alpha_{15}, \alpha_{17}$ which are (-1) -curves on the covering. We denote the resulting surface by \tilde{S} . Now, on \tilde{S} , we have two A_2 -configurations coming from $(\alpha_2, \alpha_4), (\alpha_{14}, \alpha_{16})$ and two A_5 -configurations lying over $\alpha_7 - \alpha_8 - \alpha_9 - \alpha_{10} - \alpha_{11}$ and six A_1 -configuration lying over $\beta_1, \beta_2, \beta_3$ and two elliptic curves of self-intersection number -2 lying over γ_3 and two elliptic curves of self-intersection number -1 lying over γ_1, γ_2 . We let these be denoted by $F_1, F_2, \dots, F_{10}; C_1, C_2, C_3, C_4$. We have further $c_2(X) = 30, c_1^2(X) = 6$ for $X = \tilde{S}$. Now the left hand side of the inequality is:

$$\begin{aligned} & 3 \times 30 - 6 + 2 \times (-2) + 2 \times (-1) - 2 \times 3 \cdot \left(3 - \frac{1}{3} \right) \\ & - 2 \times 3 \cdot \left(6 - \frac{1}{6} \right) - 6 \times 3 \cdot \left(2 - \frac{1}{2} \right) = 0. \end{aligned}$$

Recall that we have seen in Section 3 that the surfaces $S(A_8)$ and $E(3)$ are closely related, i.e. they are both cyclic coverings of degree 3 of the elliptic surface associated with the root system E_6 . In fact, by using $E(3)$, Masahiko Saito constructed the following example: Take one global section of the double covering $E^{(2)}(3)$ and observe the (-2) -curves in the singular fibers which are disjoint from this section. Then they form four disjoint A_5 -configurations, which we further set to be F_1, F_2, F_3, F_4 . We denote the fixed global section by C_1 and calculate the left hand side of the inequality for $X = E^{(2)}(3)$ and the configurations $C_1; F_1, F_2, F_3, F_4$. That is:

$$3 \times 24 + (-2) + 4 \times 3 \cdot \left(6 - \frac{1}{6}\right) = 0$$

Thus we obtain another quotient of the ball. But we want to relate this to $S^{(2)}(A_8)$; We denote by τ the translation sending, say γ_1 to γ_2 , which is an automorphism of order 3. Then the quotient $S^{(2)}(A_8)/\tau$ has six singular points of type A_2 coming from the six singular points of the three fibers of type \tilde{A}_2 which are the fixed points of τ . If we resolve these, then we obtain three more singular fibers of type \tilde{A}_5 in addition to that coming from the fiber of type \tilde{A}_{17} . Thus this is nothing other than the surface $E^{(2)}(3)$. The configurations $C_1, C_2, C_3; F_1, F_2, F_3, F_4, F_5, F_6$ on $S^{(2)}(A_8)$ pass to $C_1; F_1, F_2, F_3$ on $E^{(2)}(3)$, some being reduced by the action of τ and some being increased by the exceptional sets of A_2 -singularities. Thus we have shown that the discrete subgroup corresponding to Saito's example contains Γ as a normal subgroup of index 3. We have to note further that also $S^{(2)}(A_8)$ is obtained in a similar way by dividing $E^{(2)}(3)$ by a (lifted) translation of order 3 and that Hirzebruch's configuration induces exactly the one given on $S^{(2)}(A_8)$. This implies that the three examples above are essentially the same. It is thus clear that this kind of subtle covering relations between surfaces for which the inequality above is equality should be understood from the inclusion relations between the corresponding discrete subgroups of $PSU(2, 1)$. But it is in general difficult problem to determine such subgroups explicitly, or to describe the structure of them by a reasonable way.

To close this section, we make only a remark about the result of Ishida [Is] which states that there is a natural covering relation between Inoue-Livne's example for $(m, d) = (5, 5)$ and the following example of Hirzebruch [H2]: Let $l_1 = 0, l_2 = 0, \dots, l_6 = 0$ be the equations of six lines on $P_2(\mathbb{C})$ intersecting three and three at four points. [H2] shows that we have $c_1^2 = 3c_2$ for the minimal desingularization $S(H)$ of the abelian covering of $P(\mathbb{C})$ which arises by the adjunction of 5-th roots of ratios l_i/l_j i.e. that the universal covering of this is the unit ball. By $\Gamma(H)$ we denote the fundamental group of $S(H)$, which can be regarded as a subgroup of

$PSU(2, 1)$. By $S(L)$ we denote the 5-fold cyclic covering of $E(5)$ branched over the global sections to which the automorphisms of $E(5)$ are liftable, and by $\Gamma(L)$ the fundamental group of $S(L)$ imbedded into $PSU(2, 1)$. We have $N/\Gamma(L) \simeq \text{Aut}(S(L))$ where N is the normalizer of $\Gamma(L)$ in $PSU(2, 1)$. [Is] shows that $S(H)$ is naturally a 25-fold unramified covering of $S(L)$. Thus we obtain in particular the inclusion $\Gamma(H) \subseteq \Gamma(L) \subseteq N$. Furthermore [L] proves that N can be lifted isomorphically to $SU(2, 1)$. Now, by modifying the representation of [L] slightly, one can show that N can be represented as a matrix group defined over the ring \mathcal{O} of integers of $\mathcal{Q}(\zeta)$ where $\zeta = \exp(2\pi i/5)$. One can further show that $\Gamma(L)$ is a (normal) subgroup of index 5 of $N(\mathfrak{p})$ and that $\Gamma(H) = N(\mathfrak{p}^2)$, where \mathfrak{p} is the principal ideal $(\zeta - 1)$ and $N(\alpha)$ denotes the principal congruence subgroup of N of level α for any ideal α of \mathcal{O} .

Hirzebruch also constructs interesting free quotients of the ball by using configurations related to $E(3)$; we refer the reader to [H2] for the details.

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