Advanced Studies in Pure Mathematics 8, 1986
Complex Analytic Singularities
pp. 229–240

On π_2 of the Complements to Hypersurfaces which are Generic Projections

Anatoly Libgober

§ 1. Introduction

Let V denote a non-singular projective variety of dimension n embedded into a projective space of dimension N. Let L be a generic linear subspace of CP^N of dimension $N-n-2$. The projection with center at L defines a map $p: V \to CP^{n+1}$. The purpose of this note is to prove the following:

Main Theorem. *If V is a simply-connected non-singular variety of dimension greater than one then $\pi_2(CP^{n+1}-p(V)) \otimes Q$ is trivial.*

Several comments are in order to explain our interest in such kind of a result. The variety $p(V)$ is a singular hypersurface having singularities which are fairly well understood (at least in the case of small dimensions) [R1]. For example if $n=2$ then $p(V)$ has singularities along certain curve (double curve) near which $p(V)$ given by $xy=0$. Moreover it has finitely many triple points given locally by equation $xyz=0$ and finite number of pinch points locally given by $x^2-yz^2=0$. In particular this implies that if H is a generic plane in CP^{n+1} then $H \cap p(V)$ is an irreducible plane curve which has as singularities only nodes. Therefore $\pi_1(H-H \cap p(V))$ is isomorphic to Z/dZ where d is the degree of V ([D]). According to Zariski's theorem [Z], this implies that $\pi_1(CP^{n+1}-p(V))=Z/dZ$. Similarly if H is a generic linear 3-space, then $\pi_2(H-H \cap p(V))$ is isomorphic to $\pi_2(CP^{n+1}-p(V))$ and hence it is enough to prove our theorem only in the case $n=2$.

For a surface in CP^3 with isolated singularities the second homotopy group of the complement has properties similar to the properties of the Alexander modules attached to the fundamental groups of the complements to plane algebraic curves ([L1], [L2]). If S is a non-singular surface in CP^3 then $\pi_2(CP^3-S)$ is trivial, but in general $\pi_2(CP^3-S)$ is affected by the degree of the surface, by the type and by the position in CP^3 of the

Received February 28, 1985.
* Supported in part by NSF.

singular points of S. Our theorem says that the singularities of the hypersurfaces which are generic projections are too mild to produce a non-trivial $\pi_2(CP^{n+1}-p(V))\otimes Q$. In this sense it can be viewed as a higher dimensional analog of the theorem of Fulton-Deligne [D]. Our arguments are inspired by Nori's proof of a generalization of aforementioned Deligne-Fulton theorem to the case of nodal curves on arbitrary surfaces.

Let us explain now the main points in the proof comparing it with the case of plane curves. The isomorphism between the fundamental group of the complement to a plane irreducible nodal curve C with Z/dZ, where d is the degree of the curve, is usually deduced in two steps. The first one (which is rather trivial computation) shows that $H_1(CP^2-C, Z)=Z/dZ$ and the second (the major one) shows that $\pi_1(CP^2-C)$ is abelian. Similarly the major step in the proof of the theorem stated above is the following

Theorem′. *The action of $\pi_1(CP^3-p(V))=Z/dZ$ on $\pi_2(CP^3-p(V))\otimes Q$ is trivial.*

The action of π_1 here is the usual action of the fundamental group on higher homotopy groups (which on fundamental group itself amounts to conjugation). For our purposes the action on π_2 can be thought of as the action of the fundamental group on H_2 of the universal cover (the latter group of course can be identified with π_2).

Actually, for technical reason we work in affine situation and we prove the following

Theorem″. *If H is a generic hyperplane in CP^3 then the action of $\pi_1(CP^3-(p(V)\cup H))=Z$ on $\pi_2(CP^3-(p(V)\cup H))\otimes Q$ is trivial.*

The relationship between the "affine" and the "projective" one is given by the following lemma, proof of which will be given in [L2].

Lemma. *Let S be a surface in CP^3 such that $\pi_1(CP^3-S)$ is a cyclic group of order equal to the degree of S. Let H be a generic hyperplane in CP^3. Then $\pi_1(CP^3-(S\cup H))=Z$ and $\pi_2(CP^3-S)$ is isomorphic as $\pi_1(CP^3-S)$-module to the submodule of $\pi_2(CP^3-S\cup H)$ of elements invariant under the action of* $\operatorname{Ker}(\pi_1(CP^3-(S\cup H))\to\pi_1(CP^3-S))$, *which is also considered as $\pi_1(CP^3-S)$-module.*

Clearly the theorem″ implies the theorem′. The rest of this introduction explains how our main theorem follows from the theorem′. Recall (cf. for example [B]) that for any topological space X, one has the following exact sequence.

$$H_2(\pi_1(X), Z)\longrightarrow\pi_2(X)_{\pi_1}\longrightarrow H_2(X, Z)\longrightarrow H_1(\pi_1(X), Z)\longrightarrow 0$$

where $\pi_2(X)_{\pi_1(X)}$ denotes the factor group of $\pi_2(X)$ by the subgroup generated by the elements of the form $m-mg$ where $m \in \pi_2(X)$ and $g \in \pi_1(X)$. Now let us apply this to $X=CP^3-p(V)$. The theorem′ then implies that

$$\pi_2(CP^3-p(V))\otimes Q=H_2(CP^3-p(V), Q).$$

The latter group is zero as is shown in Lemma 2.3 of Section 2. This proves the main theorem.

§ 2. Preliminaries

2.1. Lemma. *Let f: $X\to Y$ be a branched covering of a complex manifold Y of finite degree. Then the induced map $H_i(X, Q)\to H_i(Y, Q)$ is surjective.*

Proof. Let $B_X\subset X$ and $B_Y\subset Y$ be closed subsets of X and Y such that $X-B_X$ is an unbranched covering of $Y-B_Y$. Then $f_*(\pi_1(X-B_X))$ is a subgroup of finite index of $\pi_1(Y-B_Y)$. Let us consider a subgroup S of $f_*(\pi_1(X-B_X))$ which is a normal subgroup of finite index of $\pi_1(Y-B_Y)$ (e.g. the intersection of conjugates of $f_*(\pi_1(X-B_X))$ in $\pi_1(Y-B_Y)$). Let $\tilde{X}$ be the branched covering of Y corresponding to the subgroup S (cf. [F]). Then the quotient space X/S can be identified with Y and the projection map π: $\tilde{X}\to Y$ can be factored as $\tilde{X}\xrightarrow{\varphi}X\xrightarrow{f}Y$. The map $(f\varphi)^*: H^i(Y, Q)\to H^i(\tilde{X}, Q)$ is injective because its composition with the transfer $H^i(X, Q)\to H^i(Y, Q)$ is the multiplication by the index of S. Therefore $f^*H^i(Y, Q)\to H^i(X, Q)$ is injective. But this fact is just dual to our lemma.

2.2. Let $X\to C$ be a dominant map with connected generic fibre F. Then the map $\pi_1(F)\to\pi_1(X)$ induced by inclusion of F into X is surjective.

Proof. This follows from the exact sequence

$$\pi_1(F)\longrightarrow\pi_1(X)\longrightarrow\pi_1(C)$$

(cf. [N], Lemma 1.5).

2.3. Lemma. *Let V be a simply connected algebraic surface in CP^N and p: $V\to p(V)\subset CP^3$ be a generic projection. Let H be a generic hyperplane in CP^3. Then $H_2(CP^3-p(V)-H, Q)=0$.*

Proof. First let us fix sufficiently small regular neighbourhood of $p(V)$ and H, which we shall denote by T_V and T_H respectively. Let ∂_∞ be the intersection of the boundary of T_H with T_V and let ∂ denotes the complement to T_H in the boundary of T_V. Next note that the following isomorphism $H_2(CP^3p(V)-H,Q)=H^3(p(V),p(V)\cap H,Q)$ takes place. Indeed it follows from the exact sequence

$$H_3(CP^3-H,Q)\longrightarrow H_3(CP^3-H,CP^3-p(V)-H,Q)\\ \longrightarrow H_2(CP^3-p(V)-H,Q)\longrightarrow H_2(CP^3-H,Q)$$

combined with excision isomorphism $H_3(CP^3-H,CP^3-p(V)-H,Q)=H_3(T_V-T_H\cap T_V,\partial,Q)$, the Lefshetz duality $H_3(T_V-T_H,\partial,Q)=H^3(T_V-T_H,\partial_\infty,Q)$ and the retraction isomorphism $H^3(T_V-T_H,\partial_\infty,Q)=H^3(p(V),p(V)\cap H,Q)$.

Now let $C\subset V$ denotes the double curve of projection $p\colon V\to CP^3$. In the following diagram

$$\begin{array}{ccccccc} H^2(C,Q) & \longrightarrow & H^3(V,C,Q) & \longrightarrow & H^3(V,Q) & \longrightarrow & 0\\ \uparrow\alpha & & \uparrow\beta & & \uparrow\gamma & & \\ H^2(p(C),Q) & \longrightarrow & H^3(p(V),p(C),Q) & \longrightarrow & H^3(p(V),Q) & \longrightarrow & 0 \end{array}$$

the maps α and β are isomorphisms. Indeed $H^3(V,C,Q)=H^3(V,T(C),Q)=H_1(V-C,Q)$, where $T(C)$ is a regular neighbourhood of C; on the other hand similarly $H^3(p(V),p(C),Q)=H_1(p(V)-p(C),Q)$. Hence β is an isomorphism. We will see below that α is an isomorphism and therefore γ is an isomorphism as well. Because V is simply-connected this implies that $H^3(p(V),Q)=0$. Clearly the map $H^2(p(V),Q)\to H^2(H\cap p(V),Q)$ is surjective. Hence we can infer from the exact sequence

$$H^2(p(V),Q)\longrightarrow H^2(H\cap p(V),Q)\longrightarrow H^3(p(V),p(V)\cap H,Q)\\ \longrightarrow H^3(p(V),Q)$$

that $H^3(p(V),p(V)\cap H,Q)=0$ and the lemma follows. We will deduce that α is an isomorphism by showing that preimage of each irreducible component D_i of the double curve $p(C)$ is an irreducible. This in turn will follow from the fact that each irreducible component of double curve of $p(C)$ contains pinch points. Indeed if $p^{-1}(D_i)$ is reducible then p is an isomorphism on each irreducible component and $p|_{p^{-1}(D_i)}$ does not have ramification points i.e. D_i does not contain pinch points. Existence of pinch points on each irreducible component follows from the following argument due to Lawrence Ein. Let Sec V be the subvariety of the Grassmannian Gr$(N-3,N)$ consisting of $(N-3)$-subspaces of CP^N containing at least two points of V. Let $I\subset\mathrm{Sec}\,V\times\mathrm{Gr}(N-4,N)$ be the incidence correspondence consisting of pairs (L_1,L_0) where L_1 is a secant $(N-3)$-subspace of CP^N containing a $(N-4)$-subspace L_0. Denote by p_2 projection of I on the second factor. Clearly I is an irreducible variety and the fibre of I over $L_0 \in \mathrm{Gr}(N-4, N)$ is the double curve of projection of V from L_0. Over a Zariski open set U the map p_2 is a locally trivial fibration. Irreducibility of I implies that $\pi_1(U)$ acts transitively on irreducible components of a double curve of projection from a generic center. Therefore all irreducible components D_i contain the same number of pinch points and our claim follows. Note that if $N \geq 5$ then according to Severi's Theorem ([M], p. 72) the double curve is irreducible with single exception of Veronese surface of degree 4 in CP^5. In this case the double curve consists of 3 lines each containing 2 pinch points (cf. [R]).

2.4. Let $\varphi: X \to Y$ be a continuous map of topological spaces. Let G and H be subgroups of $\pi_1(X, x)$ and $\pi_1(Y, \varphi(x))$ respectively. Denote by $\tilde{X}_G$ and $\tilde{Y}_H$ the covering spaces of X and Y respectively corresponding to the subgroups G and H. Assume that $\varphi_*(G) \subset H$. Then φ induces the map $\varphi_{G,H}: \tilde{X}_G \to \tilde{Y}_H$.

This can be easily seen for example by interpreting $\tilde{X}_G$ (resp. $\tilde{Y}_H$) as the space of equivalence classes of paths starting at the base point, where two paths considered to be equivalent if and only if they have the same end point and the class of the loop generated by union of those paths belongs to G (resp. H).

2.5. Let Y be a subspace of a topological space X and let $\varphi: Z \to X$ be an unbranched covering of X. Assume that $\pi_1(Y, a)$ acts trivially on $\varphi^{-1}(a)$. Then $\varphi^{-1}(Y)$ is a disjoint union of copies of Y.

2.6. Lemma. *Let U be a complex manifold. Let S be a non-singular hypersurface in U and $T(S)$ be a tubular neighbourhood of S in U. Assume that S is simply connected. Then $\pi_1(T(S)-S)$ acts trivially on $\pi_i(T(S)-S)$ for $i>1$.*

Proof. The exact homotopy sequence of fibration $p_s: T(S)-S \to S$ identifies $\pi_i(T(S)-S)$ with a subgroup of $\pi_i(S)$ for $i>1$. For $\alpha \in \pi_i(T(S)-S)$ and $\beta \in \pi_1(T(S)-S)$ one has $p_{s*}(\beta\alpha) = p_{s*}(\beta)\, p_{s*}(\alpha) = p_{s*}(\alpha)$ because $p_{s*}(\beta)=1$. Hence $\beta\alpha=\alpha$. Q.E.D.

§ 3. Proof of the Main Theorem

Let us consider the action $\psi: C \times CP^3 \to CP^3$ of the additive group of C on CP^3 given by $\psi(a)\cdot(z_0, \cdots, z_3) = (z_0 + az_3, z_1, z_2, z_3)$. The orbits of this action are affine lines in $C^3 = CP^3 - H$ where H is given by equation $z_3 = 0$, such that they pass through the point $(1, 0, 0, 0)$. The points of hyperplane H are fixed by all elements in C. Denote also by $\tilde{H}$ the hyperplane in CP^N which is taken by projection $p: CP^N \to CP^3$ into H.

Let us consider the map ϕ: $V\times\boldsymbol{C}\to\boldsymbol{CP}^3$ given by $\phi(v, a)=\psi(a)\cdot p(v)$ where $v \in V$ and $a \in C$. This map is a branched covering of $\boldsymbol{CP}^3$ of degree equal to the degree of V. If $P \in \boldsymbol{C}^3=\boldsymbol{CP}^3-H$ then $\phi^{-1}(P)$ consists of the pairs (v_i, a_i) $i=1, \cdots, \deg V$, where $p(v_i)$ $(i=1, \cdots, \deg V)$ form the intersection of $p(V)$ and the line passing through P and $(1, 0, 0, 0)$. The branching locus $B\subset \boldsymbol{C}^3$ consists of the union of the cone over the double curve of $p(V)$ and the cone over the branching curve of the projection of $p(V)$ from the point $(1, 0, 0, 0)$.

Lemma 1. *The inverse image $\phi^{-1}(p(V))$ has two irreducible components one of which is $V\times 0\subset V\times \boldsymbol{C}$.*

Proof. Let us denote the union of irreducible components of $\phi^{-1}(p(V))$ different from V by V'. We claim that $V'-\phi^{-1}(B)$ is connected. Because this space is non-singular (it is an unbranched covering of a manifold) the lemma follows.

In order to show the connectivity of $V'-\phi^{-1}(B)$ for any triple of distinct points v_0, v_1, v_2 belonging to $p(V)$ and lying on a line of projection (i.e. on a line having $(1, 0, 0, 0)$ in its closure) we construct a path α_1 in $p(V)$ connecting v_1 and v_2 and a closed path α_0 with ends in v_0 such that projection of $\boldsymbol{CP}^3$ from the point $(1, 0, 0, 0)$ on $\boldsymbol{CP}^2$ takes α_1 and α_0 into the same loop.

Then the path $\gamma(t)=(\alpha_1(t), a(t))$ where $a(t)$ is defined from $\alpha_0(t)=\psi(a(t))\alpha_1(t)$ defines a path connecting $(v_0, a(1))$ and $(v_0, a(2))$. Both $(v_0, a(1))$ and $(v_0, a(2))$ belong to $\phi^{-1}(p(v))$. Clearly the connectivity of V implies that any pair of points in V' can be deformed into a pair of points of such type and this gives the connectivity of $V'-\varphi^{-1}(B)$.

The existence of the paths α_0 and α_1 in turn follows from the fact that the Galois group of the covering defined by generic projection (and we do assume that the projection from $(1, 0, 0, 0)$ is a generic projection of $p(V)$) is the full symmetric group. This group has the property that the stabilizer of any point of the covering acts transitively on other elements of the fibre, which implies the existence of α_0 and α_1.

Corollary 1. *Let k be the largest integer which divides the homology class $h \in H_2(V, Z)$ of hyperplane section $\tilde{H}$. Denote by V_{aff} the affine part of V i.e. $V_{\mathrm{aff}}=V-\tilde{H}$. Then*

$$H_1(V_{\mathrm{aff}}\times \boldsymbol{C}-\phi^{-1}(p(V)))=\boldsymbol{Z}+\boldsymbol{Z}+\boldsymbol{Z}/k\boldsymbol{Z}.$$

Proof. Let us consider the cycles $V\times 0$, $V\times\infty$, $\tilde{H}\times \boldsymbol{CP}^1$, V', where ∞ denotes the point $\boldsymbol{CP}^1-\boldsymbol{C}$ and by abuse of notation V' is the closure of the space V' considered in lemma 1 above. Let $\ell \in H_2(V\times \boldsymbol{CP}^1, Z)$ be the class of $a\times CP^1(a \in V)$. Denote the homology class defined by a cycle by the same symbol as the cycle itself. Note that we have the following intersection indices

$$(V\times 0, \ell)=1; \quad (V\times\infty, \ell)=1; \quad (\ell, h\times CP^1)=0; \quad (\ell, V')=d-1$$
$$(d=\deg V);$$

and for any $\alpha \in H_2(V, Z)$ considered as homology class in $H_2(V\times CP^1, Z)$ via embedding $V\to V\times\alpha\subset V\times CP^1$ for sufficiently generic $a\neq 0$

$$(\alpha, V\times 0)=0, \quad (\alpha, V\times\infty)=0, \quad (\alpha, h\times CP^1)=(\alpha, h)_V,$$
$$(\alpha, V')=\sharp((v, a)\mid \psi(a)p(V)\in p(\alpha\times\alpha))=(a, p(V)\cap\psi(\alpha)p(V))=d(h, \alpha)_V.$$

Next the group $H_2(V_{\text{aff}}\times C-\phi^{-1}(p(V)), Z)$ by Lefshetz duality is isomorphic to $H^4(V\times 0\cup V\times\infty\cup h\times CP^1\cup V', Z)$ and hence is just $Z+Z+Z+Z$ (use irreducibility of V'). The image of the group $H_2(V\times CP^1, Z)$ in $H_2(V\times CP^1, V_{\text{aff}}\times C-\phi^{-1}(p(V)), Z)$ is the subgroup of Z^4 generated by the vectors $(1, 1, 0, d-1)$ and $(0, 0, k, dk)$ where k is the largest integer dividing h in $H_2(V, Z)$. Indeed if $h=kh'$ and h' is indivisible then by Poincaré duality there is $\bar{h}$ such that $(h', \bar{h})=1$. Hence (α, h) is a multiple of k. Clearly the factor of Z^4 by the subgroup generated by this two vectors is isomorphic to $Z+Z+Z/kZ$. On the other hand $H_1(V\times CP^1, Z)=0$ and the corollary follows.

Next we shall consider the infinite abelian covering of $V_{\text{aff}}\times C-\phi^{-1}(p(V))$ corresponding to the kernel of the composition of homomorphisms

$$\pi_1(V_{\text{aff}}\times C-\phi^{-1}(p(V)))\longrightarrow H_1(V_{\text{aff}}\times C-\phi^{-1}(p(V)))\longrightarrow Z+Z$$

where the last homomorphism is taking quotient by the torsion subgroup. Denote it by $(V_{\text{aff}}\times \widetilde{C-\phi^{-1}}(p(V)))_{\text{ab}}$ and the corresponding covering map denote by c. Aforementioned kernel consists of elements in $\pi_1(V_{\text{aff}}\times C-\phi^{-1}(p(V)))$ for which the linking number with $V\times 0\cup V'$ is zero. The infinite cyclic covering corresponding to the subgroup of $\pi_1(V_{\text{aff}}\times C-\phi^{-1}p(V))$ consisting of paths having zero linking number with V' we shall denote by $(V_{\text{aff}}\times \widetilde{C-\phi^{-1}}(p(V)))$ and the infinite cyclic cover of C^3-V will be denoted by $\widetilde{C^3-V}$. According to 2.4 the map ϕ: $V_{\text{aff}}\times C-\phi^1(p(V))\to C^3-p(V)$ induces the map $\tilde{\phi}$: $(V_{\text{aff}}\times \widetilde{C-\phi^{-1}}(p(V)))_{\text{ab}}\to \widetilde{C^3-p}(V)$. Let F_a denote $V\times a-\phi^{-1}(p(V))$ and $\tilde{F}_a$ be the infinite cyclic cover of F_a corresponding to subgroup of $\pi_1(F_a)$ of paths having zero linking number with V'.

Lemma 2. *The map*

$$(\tilde{\phi})_*\colon H_2(V_{\text{aff}}\times \widetilde{C-\phi^{-1}(p(V))})_{\text{ab}}, Q)\to H_2(\widetilde{C^3-p(V)}, Q)$$

induced by $\tilde{\phi}$ *is surjective.*

The proof of this lemmas requires the following

Proposition. *The fundamental group of* $\tilde{F}_a$ *is finite.*

Proof of Lemma 2. The map $(V_{\text{aff}}\times \widetilde{C-\phi^{-1}(p(V))})\to \widetilde{C^3-p(V)}$ is a map of finite degree. Hence according to 2.1 it is enough to show that the map $H_2((V_{\text{aff}}\times \widetilde{C-\phi^{-1}(p(V))})_{\text{ab}})\to H_2(V_{\text{aff}}\times \widetilde{C-\phi^{-1}(p(V))})$ is surjective.

Note that $(V_{\text{aff}}\times C-\phi^{-1}(p(V)))_{\text{ab}}$ is an infinite cyclic cover of $(V_{\text{aff}}\times \widetilde{C-\phi^{-1}(p(V))})$. Denote by $C_i(X)$ the group of i-chains of a topological space X with Q-coefficients. Then we have the following exact sequence

$$\begin{aligned}&C_2((V_{\text{aff}}\times \widetilde{C-\phi^{-1}(p(V))})_{\text{ab}}\longrightarrow C_2((V_{\text{aff}}\times \widetilde{C-\phi^{-1}(p(V))})_{\text{ab}})\\&\longrightarrow C_2(V_{\text{aff}}\times \widetilde{C-\phi^{-1}(p(V))})\longrightarrow C_1((V_{\text{aff}}\times \widetilde{C-\phi^{-1}(p(V))})_{\text{ab}})\\&\longrightarrow C_1((V_{\text{aff}}\times \widetilde{C-\phi^{-1}(p(V))})_{\text{ab}}).\end{aligned}$$

This gives the exact homology sequence

$$\begin{aligned}&H_2((V_{\text{aff}}\times \widetilde{C-\phi^{-1}(p(V))})_{\text{ab}}, Q)\longrightarrow H_2(V_{\text{aff}}\times \widetilde{C-\phi^{-1}(p(V))}, Q)\\&\longrightarrow H_1((V_{\text{aff}}\times \widetilde{C-\phi^{-1}(p(V))})_{\text{ab}}, Q)\end{aligned}$$

Now 2.2 and the proposition imply that the last group in this sequence is trivial and therefore our lemma follows.

Proof of the proposition. Let C_a denote $\phi^{-1}(p(V)\cap\psi(a)\cdot p(V))$. C_a is an irreducible curve on V which for generic a has as singularities only nodes. F_a can be identified with $V-(C_a\cup H)$ where H is the chosen earlier hyperplane section. From the exact homology sequence

$$H_2(V, Z)\longrightarrow H_2(V, V-C_a\cup H, Z)\longrightarrow H_1(V-C_a\cup H, Z)\longrightarrow H_1(V, Z)$$

combined with the fact that the last group is trivial and the Lefshetz Duality isomorphism we obtain $H_2(V, V-C_a\cup H, Z)=H^2(C_a\cup H, Z)=Z+Z$. The homology classes of C_a and H are linearly dependent. Therefore from the fact that the subgroup of Z consisting of integers (h, α), $\alpha\in H_2(V, Z)$ is generated by k we infer that $H_1(V-C_a\cup H, Z)=Z+Z/kZ$ (k is defined in Corollary 1).

It follows easily from Nori's theorem [N] that $\pi_1(V-H)$ is abelian and homology computations similar to above show that it is in fact just Z/kZ. The five term sequence of low dimensional homologies coming from Leray spectral sequence of extention (K denotes the kernel of the map of fundamental groups)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \pi_1(F_a) & \longrightarrow & \pi_1(V-H) & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & Z/k & & \end{array}$$

gives $H_1(K, Z)=Z\oplus Z/sZ$ where s is a divisor of k. Our lemma now follows from the fact that K is abelian. Indeed if so then $K=Z+Z/sZ$ and the kernel of the map of K onto Z defined by linking with C is finite. Therefore $\pi_1(\tilde{F}_a)=\mathrm{Ker}\,(\pi_1(F_a)\rightarrow Z)$ is finite.

The commutativity of K we shall deduce from Nori's theorem. The number of nodes of C_a is equal to $2\delta \deg V$ where δ is the degree of double curve C of $p(V)$. On the other hand the self-intersection of C_a is equal to $2(\deg V)^3$. Clearly the degree of C is equal to $(\deg V-1)(\deg V-2)/2-g$ where g is the genus of image of generic hyperplane section of V under chosen projection p. Now the inequality

$$2(\deg V)^3 > 4\left(\deg V\cdot\left(\frac{(\deg V-1)(\deg V-2)}{2}-g\right)\right)$$

according to [N] guaranties that $\mathrm{Ker}\,(\pi_1(F)\rightarrow Z/kZ)$ is abelian.

Lemma 3. *For generic a the map*

$$H_2(\tilde{F}_a, Q)\rightarrow H_2((V_{\mathrm{aff}}\times \widetilde{C-\phi^{-1}}(p(V)))_{\mathrm{ab}}, Q)$$

is surjective.

Proof. First note that the map in the lemma is induced by the lifting $\tilde{i}\colon \tilde{F}_a\rightarrow(V_{\mathrm{aff}}\times\widetilde{C-\phi^{-1}}(p(V)))_{\mathrm{ab}}$ of the natural inclusion map $i\colon F_a\hookrightarrow V_{\mathrm{aff}}-\phi^{-1}(p(V))$ which existence follows from 2.4 because $i_*(\mathrm{Ker}\,(\pi_1(F_a)\xrightarrow{\ell} Z))$ belongs to $\mathrm{Ker}\,(\pi_1(V_{\mathrm{aff}}\times C-\phi^{-1}(p(V)))\xrightarrow{\ell} Z+Z)$ (ℓ denotes the linking coefficient with $\phi^{-1}(p(V))$).

Next we lift the canonical projection $V_{\mathrm{aff}}\times C-\phi^{-1}(p(V))\rightarrow C^*$ to the map $s\colon (V_{\mathrm{aff}}\times\widetilde{C-\phi^{-1}}(p(V)))_{\mathrm{ab}}\rightarrow C$ which can be done according to 2.4. For sufficiently generic a the fibre of s can be identified with F_a as can be deduced from 2.5. The relevant maps fit in the following diagram

$$\begin{array}{ccc} & (V_{\mathrm{aff}}\times \widetilde{C-\phi^{-1}(p(V))})_{\mathrm{ab}} & \\ \swarrow \varphi & & \searrow s \\ (V_{\mathrm{aff}}\times \widetilde{C-\phi^{-1}(p(V))})_{\mathrm{c}} & \text{———} & C \\ \downarrow \overline{\exp} & & \downarrow \exp \\ V_{\mathrm{aff}}\times C-\phi^{-1}(p(V)) & \xrightarrow{\ \mathrm{pr}\ } & C^* \end{array}$$

where $(V_{\mathrm{aff}}\times \widetilde{C-\phi^{-1}(p(V))})_{\mathrm{c}}$ is an infinite cyclic cover of $V_{\mathrm{aff}}\times C-\phi^{-1}(p(V))$ corresponding to the kernel of the homomorphism $\pi_1(V_{\mathrm{aff}}\times C-\phi^{-1}(p(V)))\to Z$ defined by linking with $V_{\mathrm{aff}}\times 0\subset V_{\mathrm{aff}}\times C$ or alternatively $(V_{\mathrm{aff}}\times C-\phi^{-1}(p(V)))_{\mathrm{c}}$ is a pull back of the exponential covering along the canonical projection pr. In what follows we prove the dual statement to our lemma, namely that the map $H^2(V_{\mathrm{aff}}\times \widetilde{C-\phi^{-1}(p(V))}_{\mathrm{ab}}, Q)\to H^1(\tilde{F}_a, Q)$ is injective.

Let us consider the Leray spectral sequence of the map s:

$$E_2^{p,q}=H^p(C, R^q s_* Q)\Longrightarrow H^{p+q}((V_{\mathrm{aff}}\times \widetilde{C-\phi^{-1}(p(V))})_{\mathrm{ab}}, Q).$$

$R^0 s_* Q$ is a constant sheaf, while $R^1 s_* Q$ has nonzero fibers only at isolated points because for generic a we have $H^1(\tilde{F}_a, Q)=0$ as follows from the proposition above. Therefore $E_2^{2,0}=H^2(C, R^0 s_* Q)=0$ and $E_2^{1,1}=H^1(C, R^1 s_* Q)=0$. Hence $H^2((V_{\mathrm{aff}}\times \widetilde{C-\phi^{-1}(p(V))})_{\mathrm{ab}}, Q)$ is isomorphic to $E_\infty^{0,2}$ and therefore injects into $H^0(C, R^2 s_* Q)$. Next we show that for generic a the map $i^*\colon H^0(C, R^2 s_* Q)\to H^2(\tilde{F}_a, Q)$ induced by embedding of the point a into C is injective. The sheaf $R^2 s_* Q$ is locally constant outside of a discrete set, say A. Therefore any section of $R^2 s_* Q$ vanishing at generic point a vanishes everywhere except possibly for points of A. Let D_ε be a small disk about one of the points, say a_i of A which does not contain any other point of A. Assume also that exp restricted on D_ε is a homeomorphism. The injectivity of i^* will follow from the injectivity of $H^2(s^{-1}(D_\varepsilon), Q)\to H^2(\tilde{F}_a, Q)$ because $H^0(D_\varepsilon, R^2 s_* Q)=H^2(s^{-1}(D_\varepsilon), Q)$. Note that $s^{-1}(D_\varepsilon)$ can be identified with the universal cyclic cover of $\mathrm{pr}^{-1}(\exp D_\varepsilon)$ because $\overline{\exp}^{-1}(\mathrm{pr}^{-1}(\exp D_\varepsilon))$ is a disjoint union of $\mathrm{pr}^{-1}(\exp D_\varepsilon)$ (cf. 2.5). Outside of a small ball B in $V_{\mathrm{aff}}\times C$ about a singular point of $V_{\mathrm{aff}}\times a_i\cap\phi^{-1}(p(V))$ the map pr: $\mathrm{pr}^{-1}(\exp D_\varepsilon)=B\to D_\varepsilon$ is a locally trivial fibration while $\partial B-\partial B\cap\phi^{-1}(p(V))$ is a deformation retract of $B-\phi^{-1}(p(V))$. The same is true for the cyclic covers i.e. $s\colon s^{-1}(D_\varepsilon)-\widetilde{B-\phi^{-1}(p(V))}\to D_\varepsilon$ is a locally trivial fibration and cyclic cover $\partial B-\phi^{-1}(p(V))$ is a retract of $B-\phi^{-1}(p(V))$. Here $B-\phi^{-1}(p(V))$ denotes the cyclic cover of $\widetilde{B-\phi^{-1}(p(V))}$. Now let us consider the Mayer-Vietories sequence of the union of the pairs $(s^{-1}(D_\varepsilon) - \widetilde{B-\phi^{-1}(p(V))}, \widetilde{F_a - B})$ and $(\widetilde{B-\phi^{-1}(p(V))}, \widetilde{F_a \cap B})$ intersecting along $(\widetilde{\partial B-\phi^{-1}(p(V))}, F_a \cap \partial B)$. We have

$$\begin{aligned} &H^1(s^{-1}(D_\varepsilon) - \widetilde{B-\phi^{-1}(p(V))}, \widetilde{F_a \cap B}) \oplus H^1(\widetilde{B-\phi^{-1}(p(V))}, \widetilde{F_a \cap B}) \\ &\longrightarrow H^1(\widetilde{\partial B-\phi^{-1}(p(V))}, \widetilde{F_a \cap \partial B - \phi^{-1}(p(V))}) \longrightarrow H^2(s^{-1}(D_\varepsilon), \tilde{F}_a) \\ &\longrightarrow H^2(s^{-1}(D_\varepsilon) - \widetilde{B-\phi^{-1}(p(V)}, \widetilde{F_a \cap B}) \oplus H^2(B-\phi^{-1}(p(V)), \widetilde{F_a \cap B}) \\ &\longrightarrow H^2(\widetilde{\partial B-\phi^{-1}(p(V))}, \widetilde{F_a \cap \partial B - \phi^{-1}(p(V))}). \end{aligned}$$

It follows that

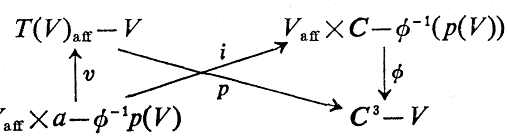
$$H^i(s^{-1}(D_\varepsilon) - \widetilde{B-\phi^{-1}(p(V))}, \widetilde{F_a - B}) = 0$$

and

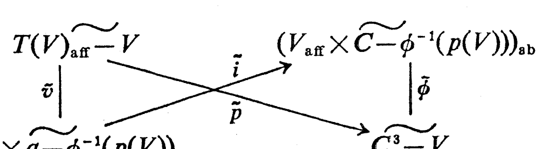
$$H^i(\widetilde{B-\phi^{-1}(p(V))}, \widetilde{F_a \cap B}) \longrightarrow H^i(\widetilde{\partial B-\phi^{-1}(p(V))}, F_a \cap B)$$

is an isomorphism. Hence $H^2(s^{-1}(D_\varepsilon), \tilde{F}_a) = 0$ and the lemma follows.

Now we are in position to conclude the proof of the theorem$''$. Let U be a 3-dimensional submanifold of CP^N which contains V and such that the cone of projection p onto CP^3 is transversal to U. Let $T(V)$ be a tubular neighbourhood of V in U. Let U_{aff} and $T(V)_{\text{aff}}$ be the affine portions of U and $T(V)$ respectively. For a sufficiently close to zero one can find a map $V \to T(V) - V$ such that the following diagram commutes.



Indeed $\phi^{-i}(V_{\text{aff}} \times a)$ is just translation of $p(V)$ by a. This translation of CP^3 is induced by a linear transformation L_a which is close to identity. For a sufficiently closed to zero the intersection of the image of projecting cone under L_a and U is transversal and belongs to $T(V)$. This intersection is a surface diffeomorphic to V. The vertical map v is restriction of this diffeomorphism. The points of $V_{\text{aff}} \times a$ which the map $\phi \cdot i$ takes into $p(V)$ are taken by the map v into $V \subset U$. The diagram above gives rise to the following diagram of the covering spaces.



According to Lemmas 3.2 and 3.3 the map $(\tilde{\phi}\tilde{i})_*: H_2(V_{\text{aff}} \times \widetilde{a-\phi^{-1}(p(V))}, Q) \to H_2(\widetilde{C^3-p(V)}, Q)$ is surjective. Therefore $H_2(\widetilde{T(V)_{\text{aff}}-V}, Q) \to H_2(\widetilde{C^3-p(V)}, Q)$ is surjective as well. It follows from 2.6 that the action of $\pi_1(\widetilde{T(V)_{\text{aff}}-V})$ on $H_2(\widetilde{T(V)_{\text{aff}}-V}) = \pi_2(T(V)_{\text{aff}}-V)$ is trivial which implies that the action of $\pi_1(C^3-p(V))$ on $H_2(\widetilde{C^3-p(V)}) = \pi_2(C^3-p(V)) \otimes Q$ is trivial as well. This concludes the proof of the theorem″ and therefore the proof of the main theorem.

References

[B] K. Brown, Cohomology of groups, Springer Verlag 1982.

[F] R. Fox, Covering spaces with singularities, Symposium in honor of S. Lefshetz, Princeton University Press, 1957, 243–257.

[D] P. Deligne, Le groupe fondamental du complement d'une courbe plane n'ayant que des point double ordinaires est abelien, Seminaire Bourbaki, No. 543, Nov. 1979.

[L1] A. Libgober, Homotopy groups of the complements to singular hypersurfaces, Bull. Amer. Math. Soc., **13**, No. 1 (1985), 49–51.

[L2] ——, Cyclic covers of CP and the topology of the complement to their branching loci, in preparation.

[M] B. Moishezon, Complex surfaces and connected sums of complex projective planes, Lecture Notes in Math. **603**, Springer Verlag 1977.

[N] M. Nori, Zariski's conjecture and related problems, Ann. Sci. Ec. Norm. Sup. 4 Serie, t **16** (1983), 305–344.

[R1] J. Roberts, Singularities subschemes and generic projections, Trans. Amer. Math. Soc., **212** (1975), 229–268.

[R2] ——, Hypersurfaces with nonsingular normalization and their double loci. J. of Algebra, **53**, No. 1, July 1978.

[S] E. Spanier, Algebraic topology, McGraw-Hill, 1966.

[Z] O. Zariski, On the Poincare group of algebraic hypersurface, Ann. of Math., **38** (1937), 131–141.

Department of Mathematics
University of Illinois at Chicago
Chicago, Illinois 60680
USA