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Strong Simultaneous Resolution for Surface Singularities

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Let $\lambda\colon \mathscr{V}\to T$ be (the germ of) a (flat) deformation of the two-dimensional isolated hypersurface singularity (V, p). We take T to be reduced. In [9], Teissier introduced, for all dimensions, various notions of simultaneous resolution for λ. Namely, let V_t denote $\lambda^{-1}(t)$, the fiber above t in T.

Definition 1. The map germ $\Pi\colon \mathscr{M}\to\mathscr{V}$ is *very weak* simultaneous resolution of λ if for all sufficiently small representatives of λ, the germ Π has a representative, also denoted Π, such that

(0) Π is a proper modification map.
(i) $\lambda\circ\Pi\colon \mathscr{M}\to T$ is a flat map.
(ii) $\Pi_t\colon M_t\to V_t$ is a resolution of V_t for all t.

Take V to have dimension two.

Let $\mathscr{A}$ denote the exceptional set in $\mathscr{M}$.

(W) Π is a *weak* simultaneous resolution if additionally the map induced by restriction $\widetilde{\lambda\circ\Pi}\colon \mathscr{A}\to T$ is simple, i.e. a locally trivial deformation.

Let $\mathscr{S}$ denote the singular locus of $\mathscr{V}$. Consider $\Pi^{-1}(\mathscr{S})$ as a non-reduced analytic space (with $\mathscr{A}$ as its underlying reduced space).

(S) Π is a *strong* simultaneous resolution if in addition to (0), (i) and (ii), the map induced by restriction $\widetilde{\lambda\circ\Pi}\colon \Pi^{-1}(\mathscr{S})\to T$ is simple.

(F) Π is a *flat* simultaneous resolution if in addition to (0), (i), and (ii), the map induced by restriction $\widetilde{\lambda\circ\Pi}\colon \Pi^{-1}(\mathscr{S})\to T$ is flat.

In [4] (see also [7]), very weak simultaneous resolution (after base change) and weak simultaneous resolution were each shown to be equivalent to the constancy as a function of t of suitable numerical invariants of the fibers. In this paper, it is shown, Theorem 1, that $\mu^*(V_t)$ constant implies strong simultaneous resolution for λ. It is known, [9], in all dimensions, that strong simultaneous resolution implies the Whitney conditions and, [8] [2], that the Whitney conditions are equivalent to $\mu^*(V_t)$ constant. So we complete an affirmative answer in dimension two to

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Teissier's question [9, 4.10.1b, p. 115] of whether $\mu^*(V_t)$ constant is equivalent to strong simultaneous resolution. Note that, in dimension two, Perron's work [6] and [4] give an affirmative answer to [9, 4.10. 1a, p. 115], whether $\mu(V_t)$ constant is equivalent to weak simultaneous resolution.

Also given in this paper is an example of a family with a weak simultaneous resolution which has no flat simultaneous resolution. This example should be compared to that of Briançon-Speder [1], where $\mu^{(3)}(V_t) := \mu(V_t)$ is constant, $\mu^{(2)}(V_t)$ is not constant, but there *is* a flat simultaneous resolution.

Theorem 1. *Let $\lambda: \mathscr{V} \to T$, with T reduced, be a (flat) family of isolated hypersurface two-dimensional singularities. Suppose that $\mu^*(V_t)$ is constant as a function of t. Then λ has a strong simultaneous resolution.*

We start with an outline of the proof. A complete proof is given after the outline.

By [4], λ has a weak simultaneous resolution. So this theorem is about the pull-back, for each t, of the maximal ideal $\mathfrak{m}_t$ of the singularity p_t in V_t. This pull-back will be controlled via the following theorems of Neumann [5]. Fix t, so that we may omit the subscript. Take $p := p_t$ to be the origin 0. Let K, a 3-manifold, be the usual intersection of V with a small sphere about 0. Let $\pi: M \to V$ be the minimal resolution of $(V, 0)$ with normal crossings in the exceptional set A. Then [5, Theorem 2], the topology of K uniquely determines the weighted dual graph Γ of A. Let (x, y, z) be coordinates for the ambient space for V such that $H := \{z=0\}$ is a generic (with respect to V) hyperplane. Let B be the boundary of a tubular neighborhood of $H \cap K$ in K. $H \cap K$ is of course a link, so B is the union of real two-dimensional tori. The map

$$\phi: B \longrightarrow S^1, \qquad \phi(x, y, z) = z/|z|$$

specifies the meridians on the tori of B. Then, by a variant of Neumann [5, Theorem, Appendix Sec. 2 and Theorem 8.2], the topological pair (K, ϕ) determines the topological nature near A of the divisor $(z \circ \pi)$ of $z \circ \pi$.

Now replace the subscript t's. The $\mu^*(V_t)$ is constant condition is equivalent to the Whitney conditions along all of the p_t. Via Hironaka's strict Whitney conditions [3, p. 129, Lemma 5.2], it follows that the topology of (K_t, ϕ_t) for each p_t is independent of t. Let $(\mathfrak{m}_t \circ \pi_t)$ be the non-reduced locus of the pull-back to M_t of the maximal ideal $\mathfrak{m}_t$. It is shown that near A_t, $(z \circ \pi_t)$ and $(\mathfrak{m}_t \circ \pi_t)$ difier in a simple manner. Then the hypothesis of constant multiplicity implies that $(\mathfrak{m}_t \circ \pi_t)$ is in fact topologically constant.

Here is our minor modification of Neumnn's work.

Consider a holomorphic function g near a two-dimensional singularity (V, p), $g(p)=0$. On $V-p$, g should be square-free. Start with the minimal resolution π: $M \to V$ of (V, p). Let $\tilde{g}=g \circ \pi$ be the pull-back of g to M. We want to perform (canonically) quadratic transformations on M, as follows, until reaching the minimum configuration which falls into Neumann's calculus of graph manifolds with meridians. In Neumann's calculus, the only singularities allowed on the exceptional set are ordinary double points. But, in contrast with other current definitions of "good resolution", these double points may occur within an irreducible component A_i of the exceptional set A. We consider $X=\{\tilde{g}=0\}$, near A. Think of X as reduced. Blow up all singular points of X which are not double points. Repeat the process until only double point singularities remain. (Since plane curve singularities are resolved by successive quadratic transformations, this process does terminate.) Let X' denote this new locus for this new $\tilde{g}$. Then X' determines a decorated plumbing graph Γ as follows. See [5, Appendix]. The irreducible components $\{A_i\}$ of A' are the vertices of Γ; a double point in an A_i is denoted by a loop in Γ. The other, non-compact, components X'_j of X', are the decorations. Each such X'_j is denoted by an arrow on Γ, with tail at the A_i which meets X'_j. In Γ, we think of each decoration as an edge. Each vertex receives its usual (geometric) genus and weight given by the topological self-intersection. Then the minimality condition on the construction of Γ is: All rational -1 vertices in Γ have degree at least 3 (where, again, decorations are edges which count towards the degree). The graph manifold with meridians is formed, using g, as follows. Construct K, the usual 3-manifold which is the link of p, by plumbing using Γ without the decorations. Delete from K an open tubular neighborhood of $K \cap X'$, forming the 3-manifold K' with boundary B. Then the needed fibration ϕ: $B \to S^1$ is given by

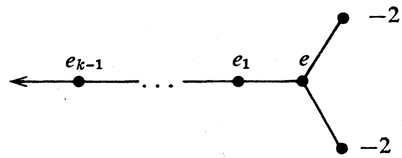
$$\phi(z)=g(z)/|g(z)|.$$

Since we have taken g to be square-free on $V-p$, K' with this meridian structure is homeomorphic to the graph manifold with meridian structure given by Γ.

We now use Neumann's M-calculus to reduce Γ to M-normal form. [5, Theorem, Appendix Sec. 2] guarantees the uniqueness of M-normal form. In drawing weighted dual graphs, we follow the convention that vertices without a genus lable are understood to have genus 0. Other genera are enclosed in square brackets. Then, mimicking [5, Theorem 8.2], we have

Theorem 2. *The M-normal form Γ_n of the minimal Γ from above may be obtained as follows:*

(1) *If Γ is of the form*

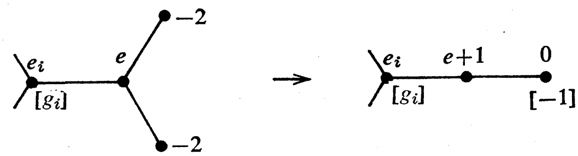


$k \geq 1$

where all of the genera are 0 *and the e are the weights, then* $\Gamma_n = \Gamma$.

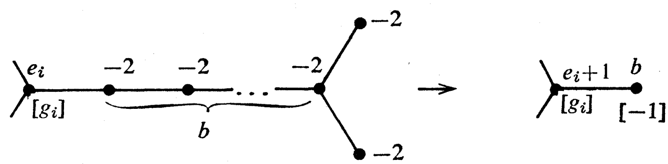
(2) *In all other cases,* Γ_n *is obtained from* Γ *by applying the following operations to* Γ *whenever possible*:

a.



$e_i \leq -3$

b.



$b \geq 1$, $\quad b$ maximal

Moreover, Γ_n *has the following properties*:

(i) *All edge signs are* $+$.

(ii) *Any genus weights satisfy* $g_i \geq -1$.

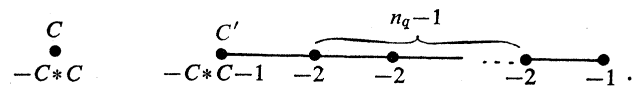
(iii) *Any vertex i with* $g_i = -1$ *has degree* 1; *moreover, it has* (*euler number*) *weight* $e_i \geq 0$, *and if* $e_i = 0$, *then the maximal chain ending at i has length* ≥ 1.

(iv) Γ *is uniquely determined by* Γ_n.

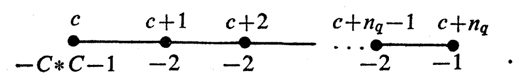
Proof. As in [5, Theorem 8.2], Theorem 2 is an immediate consequence of Neumann's work [5]. One remark may still be appropriate: Case 1 of Neumann's [5, Theorem 8.2] does not occur in Theorem 2 because the graph in [5, Theorem 8.2, 1] has no decorations and so cannot arise as the decorated graph of the *function f*. That is, after performing 2a, especially in the case $e = -3$, the vertex with weight $e+1$ cannot be changed by any additional operation and so always is part of a chain. [This is needed to prove (iii) and hence to verify condition N4 for normal form.]

Here is the needed analysis of $(z \circ \pi)$. The function z of the outline becomes the function g of Lemma 3.

Lemma 3. *Let (V, p) be an isolated two-dimensional singularity. Let π' be the minimal resolution of (V, p) such that the pullback of the maximal ideal $\mathfrak{m}$ of p is locally principal. Let g be a generic element of $\mathfrak{m}$, i.e. such that $(g \circ \pi') = (\mathfrak{m} \circ \pi') + W'_g$ with W'_g meeting the exceptional set A' transversely. Let π be the minimal resolution of (V, p) such that writing $(g \circ \pi) = D + W_g$ with D supported on A has W_g meeting the exceptional set A transversely. Let h be a second generic element of $\mathfrak{m}$ such that $g \circ \pi'$ and $h \circ \pi'$ generate $\mathfrak{m} \circ \pi'$. For h, as for g above, we write $(h \circ \pi) = D + W_h$. Consider a point q in $A \cap W_g$, i.e. a point where $g \circ \pi$ does not generate $\mathfrak{m} \circ \pi$. Let n_q be the intersection multiplicity near q of W_g and W_h. Let C denote the irreducible component of A which contains q. Let Γ and Γ' denote the weighted dual graphs for A and A' respectively. Then near q, with the possibly non-zero genus of C omitted, Γ and Γ' are given by*



Let c be the coefficient of C in D. Then, repeating the second diagram above, near the quadratic transformations, $(\mathfrak{m} \circ \pi')$ is given by



The multiplicity of p is given by

$$-D*D+\sum n_q$$

where the sum is over all embedded points q.

Proof. Near q, W_h and W_g are submanifolds with contact of order n_q. So n_q successive quadratic transformations are required for their proper transforms not to intersect. The last statement of Lemma 3 follows from [10, p. 420].

Proof of Theorem 1. It suffices to prove Theorem 1 under the restriction that T is one-dimensional and smooth: For suppose that the one-dimensional case has been proved and consider the general case. Look at the simultaneous minimal resolution. By considering one-dimensional subspaces of T, it follows that the order of vanishing of the pull-back of the maximal ideal is constant on the irreducible components of the exceptional sets. Moreover, the embedded points of the pull-back are topologically constant and their location varies holomorphically with t. Then successive simultaneous blow-ups at the embedded points gives the general result.

Let (x, y, z) be ambient coordinates for V_0 in V, $p_0=(0, 0, 0)=0$. For suitable constants a, b, c, which we may take to be independent of t, let

$$\tau\colon H=\{(x, y, z, t)\,|\,ax+by+cz=0\}$$

be a family of hyperplanes such that near $t=0$, the function $g=ax+by+cz$ is generic with respect to V_t as in Lemma **3.** [The existence of such a τ may be verified as follows: Consider a weak simultaneous resolution of λ. Then an open dense set of (a, b, c) give generic g for $t=0$ and another open dense set of (a, b, c) give generic g for $t\neq 0$.] Via the projection map onto T, $V\cap H$ is an equisingular family of plane curve singularities. Let $S_{t,e}$ be the sphere in $\boldsymbol{C}^3$ about p_t of radius e. Then, by [3, Lemma 5.2], there is a $d>0$ and an $e>0$ such that for $|t|<d$, $V_t\cap H_t$ has p_t as its only singularity inside S_t, and $V_t\cap H_t$ meets $S_{t,r}$ transversely for all $0<r<e$. The condition that $(V_t-p_t)\cap H_t$ is non-singular is equivalent to the condition that V_t-p_t and H_t meet transversely. Also by [3, Lemma 5.2], we may further restrict d and e from above so that V_t meets $S_{t,r}$ transversely for all $0<r<e$. Then for all $|t|<d$ and all $0<r<e$, V_t, H_t and $S_{t,r}$ meet with normal crossings. Then for each fixed t, the graph manifold with meridians $(K_{t,r}, \phi_{t,r})$ is topologically independent of r, $0<r<e$. Now fix r, $0<r<e$. By [3, Lemma 5.2], $(K_{t,r}, \phi_{t,r})$ is independent of t for small t. Hence for small t, $(K_{t,r}, \phi_{t,r})$ is independent of both t and r. Let $g_t\,(x, y, z)=ax+by+cz$. By [5, Theorem, Appendix, Sec. 2] and Theorem 2 (iv), the decorated plumbing graph Γ_t for the (K_t, ϕ_t) is independent of t. Γ_t describes the topology of the exceptional set A_t as well as the topology of the reduced locus of $\{g_t\circ\pi_t=0\}$.

As in Lemma 3, write

$$(g_t\circ\pi_t)=D_t+W_{t,g}.$$

Then

$$(D_t+W_{t,g})*A_{t,i}=0 \tag{1}$$

for every irreducible component $A_{t,i}$ of A_t. Moreover, the intersection matrix $(A_{t,i}*A_{t,j})$ is negative definite and in particular non-singular. $W_{t,g}*A_{t,i}$ is determined by (K_t, ϕ_t). So (1) yields a system of simultaneous linear equations in the coefficients of D_t which has a unique solution. So (K_t, ϕ_t) determines the topology of $D_t+W_{t,g}$.

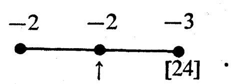
Now let $\{h_t\}$ be a second family of generic elements of $\mathfrak{m}_t$ such that $g_t\circ\pi_t$ and $h_t\circ\pi_t$ generate $\mathfrak{m}_t\circ\pi_t$. Let q_t in A_t be as in Lemma 3, with corresponding $n_{q,t}$. Since $n_{q,t}$ is the order of contact between submanifolds, it is upper semi-continuous as a function of t. The multiplicity of p_t, equal to $1+\mu^{(1)}(V_t)$, is independent of t. By Lemma 3, $n_{q,t}$ is independent of t for all q. By Lemma 3 again, the π_t' provide a strong simultaneous resolution.

As mentioned in the introduction, by [6] and [4], in dimension two, $\mu(V_t)$ constant is equivalent to weak simultaneous resolution. Consider, however, the following family:

$$z^7+x(y^9+x^{27})+tzy^8=0 \tag{2}$$

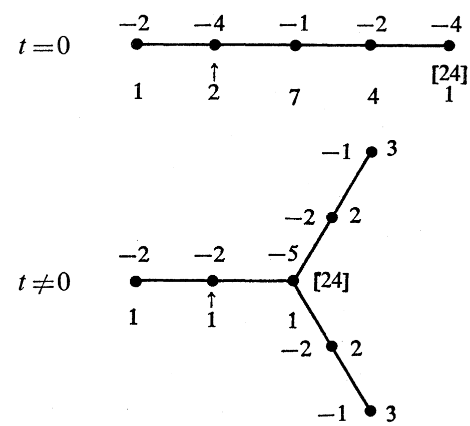
For all t, V_t has an isolated singularity with C^* action, and the weights are independent of t. $\mu^{(3)}(V_t)=\mu(V_t)=1350$ for all t. For all t, $\mu^{(2)}(V_t)$ may be computed using the hyperplane $H=\{x=y\}$. Then $\mu^{(2)}(V_0)=54$ and for $t\neq 0$, $\mu^{(2)}(V_t)=50$. For all t, the minimal resolution of V_t has the weighted dual graph

(3)



For $t=0$ and $t\neq 0$, consider the minimal resolution for which the pullback i_t of the maximal ideal $\mathfrak{m}_t$ is locally principal. Here are the weighted dual graphs along with the divisor d_t corresponding to the pullback i_t.

(4)



The curves marked ↑ by in (4) are the proper transform of the curve marked by ↑ in (3). Observe that the coefficient in d_t for these curves is different for $t=0$ and $t\neq 0$. In particular, the family (2), which has a weak simultaneous resolution, has no flat simultaneous resolution. See [9, p. 107].

References

[1] Briançon, J. and Speder, J., La trivialité toplogique n'implique pas les conditions de Whitney, C.R. Acad. Sci. Paris, ser A., **280** (1975), 365.

[2] ——, Les conditions de Whitney impliquent "$\mu^{(*)}$ constant," Ann. Inst. Fourier (Grenoble), **26** (1976), 153–164.

[3] Hironaka, H., Normal cones in analytic Whitney stratifications, Publ. Math. n° 31 I.H.E.S., (1970).

[4] Laufer, H., Weak simultaneous resolution for deformations of Gorenstein surface singularities, Proc. Symposia in Pure Math., Vol. 40, Part 2 (Amer. Math. Soc. 1983), 1–30.

[5] Neumann, W., A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves, Trans. Amer. Math. Soc., **268** (1982), 299–344.

[6] Perron, B., μ "constant" implique "type topologique constant" en dimension trois, Universite de Dijon, preprint.

[7] Shepherd-Barron, N., Some questions on singularities in 2 and 3 dimensions, Thesis, Mathematics Institute, University of Warwick, 1980.

[8] Teissier, B., Cycles evanescents, sections planes et conditions de Whitney, Asterisque, **7, 8** (1973), 285–362.

[9] Teissier, B., Resolution simultanée I, II, Seminaire sur les Singularites des Surfaces, ed. Demazure et al., Lecture Notes Mathematics, **777**, Springer-Verlag, Berlin 1980.

[10] Wagreich, P., Elliptic singularities of surfaces, Amer. J. Math., **92** (1970), 295–312.

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