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Regular Holonomic D-modules and Distributions on Complex Manifolds

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§ 0. Introduction

Let $(X, \mathcal{O}_X)$ be a complex manifold and $\mathcal{D}_X$ the sheaf of differential operators on X. The de Rham functor $\mathcal{DR}_X = \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, *)$ gives an equivalence of the category $\mathbf{RH}(\mathcal{D}_X)$ of regular holonomic $\mathcal{D}_X$-modules and the category $\mathbf{Perv}(\boldsymbol{C}_X)$ of perverse sheaves of $\boldsymbol{C}$-vector spaces on X ([K], [M], [B-B-D]).

To a perverse sheaf $F^{\cdot}$ on X we can associate its complex conjugate $\bar{F}^{\cdot}$. Then it is easily checked that $\bar{F}^{\cdot}$ is also perverse. We shall discuss here how to construct the corresponding functor $c: \mathbf{RH}(\mathcal{D}_X) \to \mathbf{RH}(\mathcal{D}_X)$ given by $\overline{\mathcal{DR}_X(\mathcal{M})} = \mathcal{DR}_X(\mathcal{M}^c)$.

The solution to this problem is given as follows. Let $\bar{X}$ be the complex conjugate of X and $\bar{\mathcal{M}}$ the complex conjugate of $\mathcal{M}$ (See § 1). Denoting by $\mathcal{D}b_{X_{\boldsymbol{R}}}$ the sheaf of distribution on the underlying real manifold $X_{\boldsymbol{R}}$ of X, $\mathcal{M}^c$ is given by

$$\mathcal{T}or_n^{\mathcal{D}_{\bar{X}}}(\Omega_{\bar{X}}^n \underset{\mathcal{O}_{\bar{X}}}{\otimes} \mathcal{D}b_{X_{\boldsymbol{R}}}, \bar{\mathcal{M}})$$

where $n = \dim X$ and $\Omega_{\bar{X}}^n$ denotes the sheaf of the highest degree differential forms on $\bar{X}$.

I would like to thank D. Barlet for helpful conversation.

§ 1. The complex conjugate

Let $\bar{X}$ be the complex conjugate of a complex manifold X. Hence $(\bar{X}, \mathcal{O}_{\bar{X}})$ is isomorphic to $(X, \mathcal{O}_X)$ as an $\boldsymbol{R}$-ringed space but the isomorphism $-: \mathcal{O}_X \to \mathcal{O}_{\bar{X}}$ is $\boldsymbol{C}$-anti-linear, i.e. $\overline{af} = \bar{a}\bar{f}$ for $a \in \boldsymbol{C}$ and $f \in \mathcal{O}_X$.

Let $\mathcal{D}_X$ and $\mathcal{D}_{\bar{X}}$ denote the sheaves of differential operators on X and $\bar{X}$, respectively. Then they are isomorphic as a sheaf of $\boldsymbol{R}$-rings. This isomorphism is also denoted by $-$. Through this isomorphism, we can associate the $\mathcal{D}_{\bar{X}}$ module $\bar{\mathcal{M}}$ to a $\mathcal{D}_X$-module $\mathcal{M}$. We call it the complex conjugate of $\mathcal{M}$. The $\mathcal{D}_{\bar{X}}$-module $\bar{\mathcal{M}}$ is isomorphic to $\mathcal{M}$ as a

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sheaf on X and if we denote this isomorphism by $-: \mathcal{M} \to \overline{\mathcal{M}}$ then we have $\overline{pu} = \bar{p}\bar{u}$ for $p \in \mathcal{D}_X$ and $u \in \mathcal{M}$. By this terminology, we have $\bar{\mathcal{O}}_X = \mathcal{O}_{\bar{X}}$ and $\bar{\mathcal{D}}_X = \mathcal{D}_{\bar{X}}$.

We can see easily that

$$(1.1) \qquad \mathcal{DR}_{\bar{X}}(\overline{\mathcal{M}}) \cong \overline{\mathcal{DR}_X(\mathcal{M})}$$

in the derived category of complexes of sheaves of $\boldsymbol{C}$-vector spaces on X.

§ 2. Distribution solutions

Let us denote by $X_{\boldsymbol{R}}$ the underlying real analytic manifold of a complex manifold X. Then, by the diagonal map $X_{\boldsymbol{R}} \hookrightarrow X \times \bar{X}$, we can regard $X \times \bar{X}$ as the complexification of $X_{\boldsymbol{R}}$. Hence we have $\mathcal{D}_{X_{\boldsymbol{R}}} = \mathcal{D}_{X \times \bar{X}}|_{X_{\boldsymbol{R}}}$, $\mathcal{A}_{X_{\boldsymbol{R}}} = \mathcal{O}_{X \times \bar{X}}|_{X_{\boldsymbol{R}}}$. Let $\mathcal{D}b_{X_{\boldsymbol{R}}}$ denote the sheaf of distributions on $X_{\boldsymbol{R}}$ in Schwartz's sense. Then $\mathcal{D}b_{X_{\boldsymbol{R}}}$ is a $\mathcal{D}_{X_{\boldsymbol{R}}}$-module, and, in particular, this is endowed with the structure of a left $(\mathcal{D}_X, \mathcal{D}_{\bar{X}})$-bi-module.

Let us denote by $\mathbf{Mod}(\mathcal{D}_X)$ the category of left $\mathcal{D}_X$-modules, and by $\mathbf{D}(\mathcal{D}_X)$ its derived category. We denote by $\mathbf{D}^b_{\mathrm{rh}}(\mathcal{D}_X)$ the full subcategory of $\mathbf{D}(\mathcal{D}_X)$ consisting of bounded complexes with regular holonomic cohomology groups. We denote by $\mathbf{RH}(\mathcal{D}_X)$ the category of regular holonomic $\mathcal{D}_X$-modules.

Theorem 1. (i) *$C_X(*) \underset{\mathrm{def}}{=} \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(*, \mathcal{D}b_{X_{\boldsymbol{R}}})$ is the functor from $\mathbf{D}^b_{\mathrm{rh}}(\mathcal{D}_X)$ into $\mathbf{D}^b_{\mathrm{rh}}(\mathcal{D}_{\bar{X}})^{\circ}$. Here $\circ$ denotes the opposite category.*

(ii) *C_X is an equivalence of categories and $C_{\bar{X}} \circ C_X \cong \mathrm{id}$.*

(iii) *$\mathcal{DR}_{\bar{X}} \circ C_X \cong \mathcal{S}ol_X$, where*

$$\mathcal{S}ol_X(*) = \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(*, \mathcal{O}_X) \cong \boldsymbol{R}\mathcal{H}om_{\boldsymbol{C}}(\mathcal{DR}_X(*), \boldsymbol{C}_X).$$

The statement (iii) is easily derived from Dolbeaut's lemma for distributions

$$(2.1) \qquad \mathcal{DR}_{\bar{X}}(\mathcal{D}b_{X_{\boldsymbol{R}}}) \cong \mathcal{O}_X.$$

The property (ii) follows from (i), (iii) and the solution to Riemann-Hilbert problem ([K], [M]) for regular holonomic modules. In fact, in order to see (ii), it is enough to check

$$\mathcal{DR}_X(C_{\bar{X}} \circ C_X(\mathcal{M})) \cong \mathcal{DR}_X(\mathcal{M}).$$

The left-hand-side is isomorphic to

$$\begin{aligned}\mathcal{S}ol_{\bar{X}}(C_X(\mathcal{M})) &= \boldsymbol{R}\mathcal{H}om_{\boldsymbol{C}}(\mathcal{DR}_{\bar{X}}(C_X(\mathcal{M})), \boldsymbol{C}_X) = \boldsymbol{R}\mathcal{H}om_{\boldsymbol{C}}(\mathcal{S}ol_X(\mathcal{M}), \boldsymbol{C}_X) \\ &= \mathcal{DR}_X(\mathcal{M}).\end{aligned}$$

The proof of the assertion (i) is given in Sections 4–5.

Remark 2.1. Similarly we have $\mathcal{D}_{\bar{X}}^{\infty}\otimes_{\mathcal{D}_{\bar{X}}}C_X(\mathcal{M})=\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_{X_{\boldsymbol{R}}})$. Here $\mathcal{D}_{\bar{X}}^{\infty}$ denotes the sheaf of differential operators of infinite order on $\bar{X}$ ([K-K]) and $\mathcal{B}_{X_{\boldsymbol{R}}}$ denotes the sheaf of hyperfunctions on $X_{\boldsymbol{R}}$. In fact, since $\mathcal{B}_{X_{\boldsymbol{R}}}=\boldsymbol{R}\Gamma_{X_{\boldsymbol{R}}}(\mathcal{O}_{X\times\bar{X}})[2n]$, $n=\dim X$, we have

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_{X_{\boldsymbol{R}}})[2n]=\boldsymbol{R}\Gamma_{X_{\boldsymbol{R}}}(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_{X\times\bar{X}}))[2n].$$

By Proposition 1.4.3 [K-K], we have $\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_{X\times\bar{X}})=\mathcal{S}ol_X(\mathcal{M})\hat{\otimes}\mathcal{O}_{\bar{X}}$, where $\hat{\otimes}$ denotes the external tensor product. Hence Proposition 1.4.2 [K-K] implies

$$\begin{aligned}&\boldsymbol{R}\Gamma_{X_{\boldsymbol{R}}}(\boldsymbol{R}\mathcal{H}om_{\boldsymbol{C}}(\mathcal{DR}_X(\mathcal{M}),\boldsymbol{C}_X)\hat{\otimes}\mathcal{O}_{\bar{X}})[2n]\\&\quad=\boldsymbol{R}\mathcal{H}om_{\boldsymbol{C}}(\mathcal{DR}_X(\mathcal{M}),\mathcal{O}_{\bar{X}})=\boldsymbol{R}\mathcal{H}om_{\boldsymbol{C}_X}(\mathcal{S}ol_{\bar{X}}(C_X(\mathcal{M})),\mathcal{O}_{\bar{X}})\\&\quad=\mathcal{D}_{\bar{X}}^{\infty}\underset{\mathcal{D}_{\bar{X}}}{\otimes}C_X(\mathcal{M}).\end{aligned}$$

Here the last identity follows from Theorem 1.4.9 [K-K].

Theorem 2. (i) *For any regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$, we have*

$$\mathcal{E}xt_{\mathcal{D}_X}^{j}(\mathcal{M},\mathcal{D}b_{X_{\boldsymbol{R}}})=0\qquad \text{for } j\neq 0$$

and $C_X(\mathcal{M})=\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}b_{X_{\boldsymbol{R}}})$ is a regular holonomic $\mathcal{D}_{\bar{X}}$-module.

(ii) $\mathcal{DR}_{\bar{X}}\circ C_X=\mathcal{S}ol_X$.

(iii) *C_X gives an equivalence of the categories $\mathbf{RH}(\mathcal{D}_X)$ and $\mathbf{RH}(\mathcal{D}_{\bar{X}})^{\circ}$. Here $\mathbf{RH}(\mathcal{D}_X)$ denotes the category of regular holonomic $\mathcal{D}_X$-modules.*

Proof. By Theorem 1, if $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module, then $C_X(\mathcal{M})=\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}b_{X_{\boldsymbol{R}}})$ belongs to $\mathbf{D}^b_{\mathrm{rh}}(\mathcal{D}_{\bar{X}})$. In order to see $H^j(C_X(\mathcal{M}))=0$ for $j\neq 0$, it is necessary and sufficient to show that $\mathcal{DR}_{\bar{X}}(C_X(\mathcal{M}))=\mathcal{S}ol_X(\mathcal{M})$ is perverse on $\bar{X}$. Since the set of analytic subsets of $\bar{X}$ is equal to that of X, the perversity on X is equivalent to the perversity on $\bar{X}$. Hence the perversity of $\mathcal{DR}_{\bar{X}}(C_X(\mathcal{M}))$ on $\bar{X}$ follows from the perversity of $\mathcal{S}ol_X(\mathcal{M})$ on X. The other statements follow immediately from (i) and the preceding theorem. Q.E.D.

Together with (1.1), we have

Corollary 3. *The functor c:*

$$\mathcal{M}\mapsto\mathcal{H}om_{\mathcal{D}_{\bar{X}}}(\bar{\mathcal{M}}^{*},\mathcal{D}b_{X_{\boldsymbol{R}}})=\mathcal{T}or_n^{\mathcal{D}_{\bar{X}}}(\Omega_{\bar{X}}^{n}\underset{\mathcal{O}_{\bar{X}}}{\otimes}\bar{\mathcal{M}},\mathcal{D}b_{X_{\boldsymbol{R}}})$$

is an automorphism of $\mathbf{RH}(\mathcal{D}_X)$, which satisfies $\mathcal{DR}_X(\mathcal{M}^c)=\overline{DR_X(\mathcal{M})}$. Here $n=\dim X$ and $*$ denotes the dual system.

§ 3. Applications

Proposition 4. *Let u be a distribution on $X_{\boldsymbol{R}}$. Then the following conditions are equivalent.*

(a) *$\mathscr{D}_X u$ is a regular holonomic $\mathscr{D}_X$-module.*

($\bar{\text{a}}$) *$\mathscr{D}_{\bar{X}} u$ is a regular holonomic $\mathscr{D}_{\bar{X}}$-module.*

(b) *Any point of X has a neighborhood U and a coherent left $(\mathscr{D}_X|_U)$-ideal $\mathscr{J}$ such that $\mathscr{J}u=0$ on U and that $(\mathscr{D}_X|_U)/\mathscr{J}$ is regular holonomic (we say shortly that u solves locally a regular holonomic system on X).*

($\bar{\text{b}}$) *u solves locally a regular holonomic system on $\bar{X}$.*

Proof. The implication (a)$\Rightarrow$(b), ($\bar{\text{a}}$)$\Rightarrow$($\bar{\text{b}}$) is evident. Hence it is sufficient to show (b)$\Rightarrow$($\bar{\text{a}}$). Let $\mathscr{M}=\mathscr{D}_X/\mathscr{J}$ be a regular holonomic $\mathscr{D}_X$-module with $\mathscr{J}u=0$. Then u gives the $\mathscr{D}_X$-linear homomorphism $\varphi\colon \mathscr{M}\mapsto \mathscr{D}b_{X_{\boldsymbol{R}}}$, and hence φ is a section of $C_X(\mathscr{M})=\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}b_{X_{\boldsymbol{R}}})$. Since $\mathscr{D}_{\bar{X}}u$ is isomorphic to the sub-$\mathscr{D}_{\bar{X}}$-module of $C_X(\mathscr{M})$ generated by φ, $\mathscr{D}_{\bar{X}}u$ is a regular holonomic $\mathscr{D}_{\bar{X}}$-module. Q.E.D.

Remark 3.1. The conditions (a)$\sim$($\bar{\text{b}}$) for u are also equivalent to the conditions (a)$\sim$($\bar{\text{b}}$) for the complex conjugate $\bar{u}$ of u.

Proposition 5. *Let $\mathscr{M}$ be a regular holonomic $\mathscr{D}_X$-module and φ a section of $C_X(\mathscr{M})$. Then the following conditions are equivalent.*

(i) *φ is an injective sheaf homomorphism from $\mathscr{M}$ to $\mathscr{D}b_{X_{\boldsymbol{R}}}$.*

(ii) *φ generates $C_X(\mathscr{M})$ as $\mathscr{D}_{\bar{X}}$-module.*

Proof. Set $\mathscr{N}=\mathscr{D}_{\bar{X}}\varphi\subset C_X(\mathscr{M})$. Applying the functor $C_{\bar{X}}$ to the exact sequence $0\to\mathscr{N}\xrightarrow{\alpha}C_X(\mathscr{M})$, we obtain the exact sequence $0\leftarrow C_{\bar{X}}(\mathscr{N})\xleftarrow{\beta}\mathscr{M}$. The homomorphism β is given by $\beta(u)\colon \mathscr{N}\ni P\varphi\mapsto P(\varphi(u))\in\mathscr{D}b_{X_{\boldsymbol{R}}}$ for $u\in\mathscr{M}$, $P\in\mathscr{D}_{\bar{X}}$. Hence $\operatorname{Ker}\beta=\operatorname{Ker}(\varphi\colon \mathscr{M}\to\mathscr{D}b_{X_{\boldsymbol{R}}})$. The equivalence of $C_{\bar{X}}$ implies: φ is injective $\Leftrightarrow\beta$ is injective $\Leftrightarrow\alpha$ is surjective. Q.E.D.

Remark 3.2. If u satisfies (a)$\sim$($\bar{\text{b}}$) of Proposition 4, then $C_X(\mathscr{D}_X u)=\mathscr{D}_{\bar{X}}u$. In particular, any distribution v which satisfies $Pv=0$ for any $P\in\mathscr{D}_X$ with $Pu=0$, can be written locally in the form Qu with $Q\in\mathscr{D}_{\bar{X}}$.

Remark 3.3. The subsheaf of $\mathscr{D}b_{X_{\boldsymbol{R}}}$ consisting of distributions satisfying (a)$\sim$($\bar{\text{b}}$) of Proposition 4 is not a $\mathscr{D}_{X_{\boldsymbol{R}}}$-module. In fact, $e^{z\bar{z}}$, nor $1/(1+z\bar{z})$, does not satisfy any holomorphic linear differential equation on $\boldsymbol{C}$; that is, $\mathscr{D}_{\boldsymbol{C}}\ni P\mapsto Pe^{z\bar{z}}\in\mathscr{D}b_{X_{\boldsymbol{R}}}$ is injective.

Corollary 6. *Any regular holonomic $\mathscr{D}_X$-module is locally embedded into $\mathscr{D}b_{X_{\boldsymbol{R}}}$.*

In fact, for a regular holonomic $\mathscr{D}_X$-module $\mathscr{M}$, $C_X(\mathscr{M})$ is locally generated by one section ([B]).

Example 3.4. (1) $X=\boldsymbol{C}$, $u=(z+\bar{z})^n$. Then $\mathscr{D}_X u\cong\mathscr{D}_X/\mathscr{D}_X\partial^{n+1}$ and $\mathscr{D}_{\bar{X}}u\cong\mathscr{D}_{\bar{X}}/\mathscr{D}_{\bar{X}}\bar{\partial}^{n+1}$.

(2) $X=\boldsymbol{C}$, $u=1/z$. Then $\mathscr{D}_X u\cong\mathscr{D}_X/\mathscr{D}_X(z\partial+1)$ and $\mathscr{D}_{\bar{X}}u\cong\mathscr{D}_{\bar{X}}/\mathscr{D}_{\bar{X}}\bar{z}\bar{\partial}$. Remark that $\bar{\partial}u=\pi^{-1}\delta(\operatorname{Re} z)\delta(\operatorname{Im} z)$.

Remark 3.5. We conjecture that Theorem 2 is still true for arbitrary holonomic $\mathscr{D}_X$-modules.

Example (D. Barlet). When $X=\boldsymbol{C}$, $\mathscr{M}=\mathscr{D}_X/\mathscr{D}_X(z^2\partial_z+1)$, let us take $u=e^{1/z-1/\bar{z}}$. Since u is a bounded function outside the origin, u can be considered as a distribution on X. Then, we have

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}b_{X_{\boldsymbol{R}}})\cong\mathscr{D}_{\bar{X}}u=\mathscr{D}_{\bar{X}}/\mathscr{D}_{\bar{X}}(\bar{z}^2\partial_{\bar{z}}-1).$$

The vanishing $\mathscr{E}xt^1_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}b_{X_{\boldsymbol{R}}})=0$ follows from the solvability of the constant-coefficient differential operator $-\partial_t+1$ $(=z^2\partial_z+1,\ t=1/z)$ on the space of tempered distributions on $\boldsymbol{C}_t$.

§ 4. Proof of Theorem 1

We shall prove Theorem 1 (i) by reducing it to a simple case (Lemma 7) by using Hironaka's desingularization theorem ([H]).

Lemma 7. *Let $X=\boldsymbol{C}^n$, $f=x_1\cdots x_l$ $(l\leq n)$ and let $\mathscr{M}$ be a regular holonomic right $\mathscr{D}_X$-module such that $\mathscr{M}_f=\mathscr{M}$ and that the characteristic variety of $\mathscr{M}$ is contained in the zero section outside $f^{-1}(0)$. Here $*_f$ denotes the localization by f. Then we have $\mathscr{T}or_j^{\mathscr{D}_X}(\mathscr{M},\mathscr{D}b_{X_{\boldsymbol{R}}})=0$ for $j\neq n$ and $\mathscr{T}or_n^{\mathscr{D}_X}(\mathscr{M},\mathscr{D}b_{X_{\boldsymbol{R}}})$ is a regular holonomic $\mathscr{D}_{\bar{X}}$-module.*

Proof. The assertion being closed under extension of $\mathscr{D}_X$-modules, we may assume from the beginning that

$$\mathscr{M}=\mathscr{D}_X\Big/\sum_{j=1}^{l}(x_j\partial_j-\lambda_j)\mathscr{D}_X+\sum_{j=l+1}^{n}\mathscr{D}_X\partial_j$$

with $\lambda_j\in\boldsymbol{C}\backslash\{-1,-2,\cdots\}$. We have $\mathscr{M}\otimes^L_{\mathscr{D}_X}\mathscr{D}b_{X_{\boldsymbol{R}}}=\mathscr{M}\otimes^L_{\mathscr{D}_X}(\mathscr{D}b_{X_{\boldsymbol{R}}})_f$. Hence this is isomorphic to the Koszul complex

$$(4.1)\quad (\mathscr{D}b_{X_{\boldsymbol{R}}})_f\xrightarrow[\begin{bmatrix}P_1\\ \vdots\\ P_n\end{bmatrix}]{}(\mathscr{D}b_{X_{\boldsymbol{R}}})_f^n\to\cdots\to(\mathscr{D}b_{X_{\boldsymbol{R}}})_f^n\xrightarrow[(P_1,\cdots,P_n)]{}(\mathscr{D}b_{X_{\boldsymbol{R}}})_f\longrightarrow 0$$

where $P_j=\begin{cases} x_j\partial_j-\lambda_j & (j\leqq l) \\ \partial_j & (j>l). \end{cases}$

The map φ: $u\mapsto|x_1^{\lambda_1}\cdots x_n^{\lambda}|^2u$ gives an isomorphism of $(\mathcal{D}b_{X_{\boldsymbol{R}}})_f$ and φ transforms (4.1) to the Koszul complex

$$(4.2)\qquad (\mathcal{D}b_{X_{\boldsymbol{R}}})_f\xrightarrow[\begin{bmatrix}Q_1\\ \vdots\\ Q_n\end{bmatrix}]{}(\mathcal{D}b_{X_{\boldsymbol{R}}})_f^n\to\cdots\to(\mathcal{D}b_{X_{\boldsymbol{R}}})_f\longrightarrow 0$$

where $Q_j=\begin{cases} x_j\partial_j & (j\leqq l) \\ \partial_j & (j>l). \end{cases}$

By Dolbeaut's lemma, its homology group is concentrated at the degree n and the n-th homology group is isomorphic to $(\mathcal{O}_{\bar{X}})_{\bar{f}}$. Here we used the following

$$(4.3)\qquad (\mathcal{D}b_{X_{\boldsymbol{R}}})_f=(\mathcal{D}b_{X_{\boldsymbol{R}}})_{\bar{f}}.$$

Hence $\mathcal{T}or_j^{\mathcal{D}_X}(\mathcal{M},\mathcal{D}b_{X_{\boldsymbol{R}}})=0$ for $j\neq n$ and $\mathcal{T}or_n^{\mathcal{D}_X}(\mathcal{M},\mathcal{D}b_{X_{\boldsymbol{R}}})$ is isomorphic to $(\mathcal{O}_{\bar{X}})_{\bar{f}}$ with the structure of $\mathcal{D}_{\bar{X}}$-module by

$$(4.4)\qquad \bar{\partial}_j\circ u=\bar{x}_j^{\lambda_j}\bar{\partial}_j\bar{x}_j^{-\lambda_j}u=(\bar{\partial}_j-\lambda_j\bar{x}_j^{-1})u.$$

Hence this is a regular holonomic $\mathcal{D}_{\bar{X}}$-module. Q.E.D.

Lemma 8. *Let Y be a smooth submanifold of a complex manifold X, H a normally crossing hypersurface of Y and $\mathcal{M}$ a regular holonomic $\mathcal{D}_X$-module satisfying*

$$(4.5)\qquad \operatorname{Supp}\mathcal{M}\subset Y,$$

$$(4.6)\qquad \mathcal{H}^j_{[H]}(\mathcal{M})=0\quad \textit{for any } j,$$

(4.7) $\operatorname{Ch}\mathcal{M}\subset T_Y^*X\cup\pi^{-1}H$, *where* Ch *denotes the characteristic variety and π is the projection from T^*X to X. Then $\mathcal{T}or_j^{\mathcal{D}_X}(\mathcal{M},\mathcal{D}b_{X_{\boldsymbol{R}}})$ is a regular holonomic $\mathcal{D}_{\bar{X}}$-module for any j.*

Proof. There exists a regular holonomic $\mathcal{D}_Y$-module $\mathcal{N}$ such that $\mathcal{M}=\mathcal{N}\otimes_{\mathcal{D}_Y}\mathcal{D}_{Y\hookrightarrow X}$. We can easily show the following

Lemma 9. $\mathcal{T}or_j^{\mathcal{D}_X}(\mathcal{D}_{Y\hookrightarrow X},\mathcal{D}b_{X_{\boldsymbol{R}}})=0$ *for*

$$j\neq\operatorname{codim} Y\quad \textit{and}\quad \cong\mathcal{D}_{\bar{X}\leftarrow\bar{Y}}\underset{\mathcal{D}_{\bar{Y}}}{\otimes}\mathcal{D}b_{Y_{\boldsymbol{R}}}\quad \textit{for } j=\operatorname{codim} Y.$$

By this lemma, we have, denoting $c=\operatorname{codim} Y$,

$$\mathcal{T}or_j^{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_{X_{\boldsymbol{R}}}) \cong \mathcal{D}_{\bar{X} \leftarrow \bar{Y}} \underset{\mathcal{D}_{\boldsymbol{R}}}{\otimes} \mathcal{T}or_{j-c}^{\mathcal{D}_Y}(\mathcal{N}, \mathcal{D}b_{Y_{\boldsymbol{R}}}).$$

On the other hand, Lemma 7 implies the regular holonomicity of $\mathcal{T}or_j^{\mathcal{D}_Y}(\mathcal{N}, \mathcal{D}b_{Y_{\boldsymbol{R}}})$ and the lemma follows. Q.E.D.

§ 5. End of Proof of Theorem 1

We shall show the following statement (5.1) by the induction of dim Supp $\mathcal{M}$;

(5.1) $\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{D}b_{X_{\boldsymbol{R}}} \in \mathbf{D}^b_{\mathrm{rh}}(\mathcal{D}_{\bar{X}})$ *for any bounded complex* $\mathcal{M}$ *of right* $\mathcal{D}_X$-*modules with regular holonomic cohomology groups.*

Here $S = \mathrm{Supp}\, \mathcal{M}$ is, by definition, the union of Supp $\mathcal{H}^j(\mathcal{M})$.

There exists a nowhere dense closed analytic subset S_0 of S which satisfies the following two conditions.

(5.2) S_0 contains the singular locus of S.

(5.3) $\mathrm{Ch}\, \mathcal{H}^j(\mathcal{M}) \subset T^*_S X \cup \pi^{-1}(S_0)$.

The question being local, we may assume further

$$S_0 = S \cap \varphi^{-1}(0) \quad \text{for a } \varphi \in \Gamma(X; \mathcal{O}_X). \tag{5.4}$$

Let us consider a distinguished triangle $\mathcal{M} \to \mathcal{M}_\varphi \to \mathcal{N} \xrightarrow{+1} \mathcal{M}$. Since $\mathrm{Supp}\, \mathcal{N} \subset S_0$, $\mathcal{N} \otimes^L_{\mathcal{D}_X} \mathcal{D}b_{X_{\boldsymbol{R}}}$ belongs to $\mathbf{D}^b_{\mathrm{rh}}(\mathcal{D}_{\bar{X}})$ by the hypothesis of induction. Hence in order to see $\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{D}b_{X_{\boldsymbol{R}}} \in \mathbf{D}^b_{\mathrm{rh}}(\mathcal{D}_{\bar{X}})$, it is enough to show $\mathcal{M}_\varphi \otimes^L_{\mathcal{D}_X} \mathcal{D}b_{X_{\boldsymbol{R}}} \in \mathbf{D}^b_{\mathrm{rh}}(\mathcal{D}_{\bar{X}})$. Hence, replacing $\mathcal{M}$ with $\mathcal{M}_\varphi$, we may assume further

$$\mathcal{M} = \mathcal{M}_\varphi. \tag{5.5}$$

Now, by Hironaka's desingularization theorem ([H]), there exists a smooth manifold X' a projective morphism f: $X' \to X$ and a non-singular submanifold S' of X' which satisfy the following properties:

(5.6) f gives an isomorphism from $X' \backslash f^{-1}(S_0)$ onto $X \backslash S_0$ and $f(S') = S$.

(5.7) $S'_0 = S' \cap f^{-1}(S_0)$ is normally crossing hypersurface of S'.

Set $\varphi' = \varphi \circ f$, and

$$\mathcal{M}' = f^{-1}\mathcal{M} \underset{f^{-1}\mathcal{D}_X}{\otimes} \mathcal{D}_{X \leftarrow X'}. \tag{5.8}$$

Since $(\mathcal{D}_{X \leftarrow X'})_\varphi = (\mathcal{D}_{X'})_{\varphi'}$, we have

$$\mathcal{M}' \cong \mathcal{M} \underset{\mathcal{D}_X}{\otimes} (\mathcal{D}_{X'})_{\varphi'} \cong \mathcal{M}'_{\varphi'}. \tag{5.9}$$

Lemma 8 implies $\mathscr{M}' \otimes^{L}_{\mathscr{D}_{X'}} \mathscr{D}b_{X'_R} \in \mathbf{D}^b_{\mathrm{rh}}(\mathscr{D}_{\bar{X}'})$. Therefore its integration

$$\boldsymbol{R}f_*\left(\mathscr{D}_{\bar{X}\leftarrow\bar{X}'} \overset{L}{\underset{\mathscr{D}_{\bar{X}'}}{\otimes}} \left(\mathscr{M}' \overset{L}{\underset{\mathscr{D}_{\bar{X}'}}{\otimes}} \mathscr{D}b_{X'_R}\right)\right)$$

also belongs to $\mathbf{D}^b_{\mathrm{rh}}(\mathscr{D}_{\bar{X}})$ (Theorem 6.2.1 [K-K]).

On the other hand, we have

$$\begin{aligned}\mathscr{M}' \overset{L}{\underset{\mathscr{D}_{X'}}{\otimes}} \mathscr{D}b_{X'_R} &= \mathscr{M} \overset{L}{\underset{\mathscr{D}_X}{\otimes}} (\mathscr{D}_{X'})_{\varphi'} \overset{L}{\underset{\mathscr{D}_{X'}}{\otimes}} \mathscr{D}b_{X'_R}\\ &= \mathscr{M} \overset{L}{\underset{\mathscr{D}_X}{\otimes}} (\mathscr{D}b_{X'_R})_{\varphi'} = (\mathscr{D}_{\bar{X}'})_{\bar{\varphi}'} \overset{L}{\underset{\mathscr{D}_{\bar{X}'}}{\otimes}} \left(\mathscr{M} \overset{L}{\underset{\mathscr{D}_X}{\otimes}} (\mathscr{D}b_{X'_R})_{\varphi'}\right),\end{aligned}$$

and

$$\mathscr{D}_{\bar{X}\leftarrow\bar{X}'} \overset{L}{\underset{\mathscr{D}_{\bar{X}'}}{\otimes}} (\mathscr{D}_{\bar{X}'})_{\bar{\varphi}'} = (\mathscr{D}_{\bar{X}'})_{\bar{\varphi}'}.$$

Here we used $(\mathscr{D}b_{X'_R})_{\varphi'} = (\mathscr{D}b_{X'_R})_{\bar{\varphi}'}$. Hence, we obtain

$$\begin{aligned}&\boldsymbol{R}f_*\left(\mathscr{D}_{\bar{X}\leftarrow\bar{X}'} \overset{L}{\underset{\mathscr{D}_{\bar{X}'}}{\otimes}} \left(\mathscr{M}' \overset{L}{\underset{\mathscr{D}_{X'}}{\otimes}} \mathscr{D}b_{X'_R}\right)\right)\\ &\quad = \boldsymbol{R}f_*\left(\mathscr{M} \overset{L}{\underset{\mathscr{D}_X}{\otimes}} (\mathscr{D}b_{X'_R})_{\varphi'}\right) = \mathscr{M} \overset{L}{\underset{\mathscr{D}_X}{\otimes}} \boldsymbol{R}f_*((\mathscr{D}b_{X'_R})_{\varphi'}).\end{aligned}$$

Since $\boldsymbol{R}f_*((\mathscr{D}b_{X'_R})_{\varphi'}) = (\mathscr{D}b_{X_R})_{\varphi}$ we finally conclude

$$\mathscr{M} \overset{L}{\underset{\mathscr{D}_X}{\otimes}} (\mathscr{D}b_{X_R})_{\varphi} = \mathscr{M} \overset{L}{\underset{\mathscr{D}_X}{\otimes}} \mathscr{D}b_{X_R} \in \mathbf{D}^b_{\mathrm{rh}}(\mathscr{D}_{\bar{X}}).$$

This shows (5.1) and completes the proof of Theorem 1.

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