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Canonical Cyclic Quotient Singularities of Dimension Three

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Introduction

The notion of canonical singularity of an algebraic variety defined over $\boldsymbol{C}$ was introduced by Reid [6] as a generalization in higher dimension of that of Du Val singularity which appears in the canonical model of a surface of general type. In this paper, we classify toric canonical singularities of dimension three. In particular, we can explicitly classify cyclic quotient singularities which are canonical of dimension three, since they are toric. Although the necessary and sufficient condition for a toric singularity to be canonical was already given in [6], explicit classification had some combinatorial difficulties. Terminal singularities [7] form an important subclass of the class of canonical singularities. Morrison and Stevens [4] gave a classification of terminal cyclic quotient singularities. As they mentioned in [4], White [8] had already proved the combinatorial part of their result.

Iwashita gave a classification of 3-dimensional canonical cyclic quotient singularities in her Master's thesis [2] by using the result of White. Independently, Morrison [3] also obtained the same result under a weak restriction by using the results on Fermat varieties by Aoki and Shioda.

In order to simplify the proof of [2], we change the method slightly. The classification of 3-dimensional cyclic quotient singularities is almost equivalent to that of 3-dimensional affine canonical toric varieties defined by trigonal cones. In Section 2, we determine those defined by g-gonal cones with $g \geqq 4$. The classification for the trigonal case is given in Section 3. These results are summarized as Theorem 3.6. In Section 4, we determine canonical cyclic quotient singularities of dimension three explicitly using the results in Section 3.

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Notation. For integers $n>0$ and m we denote by $[m]_n$ the integer which satisfies $0 \leqq [m]_n < n$ and $m \equiv [m]_n \pmod{n}$.

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For integers $a_1, \cdots, a_n$, we denote by g.c.d. $(a_1, \cdots, a_n)$ the greatest common divisor of $a_1, \cdots, a_n$.

Matrices represent the linear homomorphisms obtained by applying to column vectors from the left.

§ 1. Integral convex polygons

We consider the real plane $\boldsymbol{R}^2$ with the lattice $\boldsymbol{Z}^2$ and the fixed origin 0.

Definition. For two subsets S, S' of $\boldsymbol{R}^2$ and a positive integer k, we say S and S' are *k-equivalent* if $S'=S+u$ for an element $u \in k\boldsymbol{Z}^2$.

Definition. A subset P of $\boldsymbol{R}^2$ is said to be an *integral convex polygon* if P is equal to the convex hull of a finite subset of $\boldsymbol{Z}^2$ which is not contained in a line.

For an integral convex polygon P, we denote by $g(P)$ the least cardinality of finite subsets which span P, i.e., P is a $g(P)$-gon. We denote by $P(v_1, \cdots, v_g)$ the integral convex polygon minimally spanned by $\{v_1, \cdots, v_g\} \subset \boldsymbol{Z}^2$.

Lemma 1.1. *Let $P \subset \boldsymbol{R}^2$ be an integral convex polygon with $g(P) \geqq 4$. Then, for a coordinate system of $\boldsymbol{Z}^2$, the polygon P contains a quadrangle 1-equivalent to either*

(1) $D_1 = P((0, 0), (1, 0), (1, 1), (0, 1))$, *or*

(2) $D_2 = P((1, 0), (0, 1), (-1, 0), (0, -1))$.

Proof. Suppose there exists an integral convex g-gon P with $g \geqq 4$ such that P contains no square 1-equivalent to D_1 or D_2. Let P be one of such polygons with the smallest area. Note that this is possible since the area of an integral polygon is a half integer.

Claim: P is a quadrangle. Indeed, if P were a g-gon ($P(v_1, \cdots, v_g)$ with $g \geqq 5$, then $P(v_1, \cdots, v_4)$ would contain no quadrangle which is 1-equivalent to D_1 or D_2, a contradiction to the minimality of P. Hence P is a quadrangle.

Let $P=P(v_1, \cdots, v_4)$ and $\overline{v_1v_2}$, $\overline{v_2v_3}$, $\overline{v_3v_4}$ and $\overline{v_4v_1}$ be the edges of P. The quadrangle P has two diagonals $\overline{v_1v_3}$ and $\overline{v_2v_4}$. Let Q be the intersection of these two diagonals.

Claim: If P has a lattice point R other than the vertices, then $R=Q$. Indeed, if $R \neq Q$, then by renumbering $v_1, \cdots, v_4$, if necessary, we may assume R is in the triangle $P(v_3, v_4, v_1)$ and $R \notin \overline{v_1v_3}$. Then $P(v_1, v_2, v_3, R)$

would contain no quadrangle 1-equivalent to D_1 or D_2, again a contradiction.

If the point Q is a lattice point, then we may translate P so that Q is the origin. Since the triangle $P(0, v_1, v_2)$ contains no lattice point other than the vertices, $\{v_1, v_2\}$ is a basis of the lattice. Since Q is the unique lattice point in the relative interior of the diagonals, we have $v_3=-v_1$ and $v_4=-v_2$. Hence P is equal to the quadrangle D_2, a contradiction. If Q is not a lattice point, then the quadrangle P contains no lattice point other than the vertices. We translate P so that v_4 is the origin. Then $\{v_1, v_3\}$ is a basis of the lattice. Thus $v_2=xv_1+yv_3$ for integers x and y. By the convexity of P, we see that x, y are positive integers. Since the triangle $P(v_1, v_2, v_3)$ contains no lattice point other than the vertices, its area $(x+y-1)/2$ is equal to $1/2$. Thus $x=y=1$, and P is equal to D_1, a contradiction. q.e.d.

Definition. Let $P=P(v_1, \cdots, v_g)$ be an integral convex polygon and let $v_1=(a_1, b_1), \cdots, v_g=(a_g, b_g)$ for a coordinate system of $\boldsymbol{Z}^2$. Then, for a positive integer k, the polygon P is said to be *k-canonical* if the closed convex cone $C(P\times\{k\}):=\{c(a, b, k);\ c\geqq 0,\ (a, b)\in P\}$ in $\boldsymbol{R}^2\times\boldsymbol{R}=\boldsymbol{R}^3$ generated by $\{(a_1, b_1, k), \cdots, (a_g, b_g, k)\}$ has no lattice point (x_1, x_2, x_3) with $0<x_3<k$.

Remark 1.2. If two integral convex polygons are k-equivalent, then the cone $C(P\times\{k\})$ can be transformed to $C(P'\times\{k\})$ by a linear transformation in $GL_3(\boldsymbol{Z})$. Hence P' is k-canonical if and only if so is P.

The following lemma is obvious from the definition.

Lemma 1.3. *Let P and P' be integral convex polygons with $P'\subset P$. If P is k-canonical for a positive integer k, then so is P'.*

Let N be a free $\boldsymbol{Z}$-module of rank 3, and let M be its dual $\boldsymbol{Z}$-module. For a closed convex cone π of dimension 3 in $N_{\boldsymbol{R}}=N\otimes_{\boldsymbol{Z}}\boldsymbol{R}$ which is generated by a finite number of elements $n_1, \cdots, n_s$ of N, we consider the subsemigroup $M\cap\pi^\vee=\{m\in M;\ \langle m, n\rangle\geqq 0,\ n\in\pi\}$ and its semigroup ring $\boldsymbol{C}[M\cap\pi^\vee]=\bigoplus_{m\in M\cap\pi^\vee}\boldsymbol{C}e(m)$, where $\{e(m);\ m\in M\cap\pi^\vee\}$ is a $\boldsymbol{C}$-basis with multiplication defined by $e(m)\cdot e(m')=e(m+m')$ for $m, m'\in M\cap\pi^\vee$. By Reid [6, Theorem 3.9 and the footnote on p. 294], the point of the affine toric variety $X_\pi=\mathrm{Spec}\,(\boldsymbol{C}[M\cap\pi^\vee])$ which corresponds to the maximal ideal $\boldsymbol{C}[M\cap\pi^\vee\backslash\{0\}]$ is canonical of index k, if and only if there exists a coordinate system (x_1, x_2, x_3) of N such that the intersection of π with the plane $\{x_3=k\}\subset N_{\boldsymbol{R}}$ is a polygon spanned by lattice points and there is no lattice point (x_1, x_2, x_3) in π with $0<x_3<k$. Hence the classification of 3-dimensional toric canonical singularities of index k is reduced to that of

k-canonical integral convex polygons.

Remark 1.4. Any integral convex polygon is 1-canonical. Since this is exactly the case where the ring $C[M\cap\pi^{\vee}]$ is Gorenstein by Ishida [1, Theorem 7.7], we conclude that every Gorenstein toric singularity is canonical. This holds in any dimension.

§ 2. Canonical g-gon with $g\geqq 4$

The purpose of this section is to show the following:

Theorem 2.1. *Let P be an integral convex polygon with $g(P)\geqq 4$. If P is k-canonical for an integer $k\geqq 2$, then we have $k=2$ and P is 2-equivalent to $P((1,0),(1+m,1),(1,2),(1-n,1))$ for some positive integers m,n and for a coordinate system of Z^2.*

We need the following two lemmas.

Lemma 2.2. *Let a and b be integers. Then the square $P=P((a,b),(a+1,b),(a+1,b+1),(a,b+1))$ is not k-canonical for $k\geqq 2$.*

Proof. Set $v_0=(a,b,k)$, $v_1=(a+1,b,k)$, $v_2=(a,b+1,k)$ and $v_3=(a+1,b+1,k)$ for the integer $k\geqq 2$. Let r and s be the integers which satisfy $a\equiv r$, $b\equiv s$ (mod k) and $0\leqq r,s\leqq k-1$. Then the point

$$\begin{aligned}&\{(k-1-r)(k-1-s)/k(k-1)\}v_0+\{r(k-1-s)/k(k-1)\}v_1\\&\quad+\{(k-1-r)s/k(k-1)\}v_2+\{rs/k(k-1)\}v_3\\&\qquad=(\{(k-1)a+r\}/k,\ \{(k-1)b+s\}/k,\ k-1)\end{aligned}$$

is a lattice point in the cone $C(P\times\{k\})$, since the coefficients are non-negative and $(k-1)a+r\equiv -a+r\equiv 0$, $(k-1)b+s\equiv -b+s\equiv 0$ (mod k). Hence the square is not k-canonical by definition. q.e.d.

Lemma 2.3. *Let (a,b) be a point of Z^2. If the quadrangle $P=P((a+1,b),(a,b+1),(a-1,b),(a,b-1))$ is k-canonical for an integer $k\geqq 2$, then $k=2$ and a and b are odd.*

Proof. Let r and s be the integers such that $-a\equiv r$, $-b\equiv s$ (mod k) and $0\leqq r,s\leqq k-1$. By changing the signs of the coordinates of Z^2, which means a or b is replaced by $-a$ or $-b$, respectively, if necessary, we may assume $0\leqq r,s\leqq k/2$. Assume r or s is not equal to $k/2$. Then we have $0\leqq r+s\leqq k-1$. We set $v_0=(a+1,b,k)$, $v_1=(a,b+1,k)$, $v_2=(a-1,b,k)$ and $v_3=(a,b-1,k)$. Then the point

$$\{(k-1-r-s)r/2k(r+s)\}v_0+\{(k-1-r-s)s/2k(r+s)\}v_1$$
$$+\{(k-1+r+s)r/2k(r+s)\}v_2+\{(k-1+r+s)s/2k(r+s)\}v_3$$
$$=(\{(k-1)a-r\}/k,\ \{(k-1)b-s\}/k,\ k-1)$$

is in $\boldsymbol{Z}^3 \cap C(P\times\{k\})$, since the coefficients are nonnegative and since $(k-1)a-r\equiv -a-r\equiv 0$ and $(k-1)b-s\equiv -b-s\equiv 0 \pmod k$. Hence, if the quadrangle is k-canonical, then we have $r=s=k/2$. In particular, k is even. Furthermore, since $-2a\equiv 2r\equiv 0, -2b\equiv 2s\equiv 0 \pmod k$, we have $(k-2)a\equiv(k-2)b\equiv 0 \pmod k$. Hence

$$\{(k-2)/k\}(a, b, k)=((k-2)a/k,\ (k-2)b/k,\ k-2) \in \boldsymbol{Z}^3.$$

Hence, if $k>2$, the quadrangle P is not canonical. Hence $k=2$, $r=s=1$ and $a\equiv b\equiv 1 \pmod 2$. q.e.d.

Let P be a k-canonical integral convex polygon for an integer $k\geqq 2$. Then by Lemmas 2.2 and 1.3, P does not contain the square $P((a, b), (a+1, b), (a+1, b+1), (a, b+1))$. Hence, if $g(P)\geqq 4$, we know by Lemma 1.1 that P contains a quadrangle 1-equivalent to $D_2=P((1, 0), (0, 1), (-1, 0), (0, -1))$ for a coordinate system of $\boldsymbol{Z}^2$.

The proof of the theorem is thus reduced to the following:

Proposition 2.4. *Let P be a k-canonical convex polygon with $k\geqq 2$ which contains a quadrangle 1-equivalent to D_2. Then $k=2$ and P is 2-equivalent to a quadrangle $P((1, 0), (1+m, 1), (1, 2), (1-n, 1))$ for some positive integers m, n and for a coordinate system of $\boldsymbol{Z}^2$.*

Proof. We have $k=2$ by Lemma 2.3. We take P' which is 1-equivalent to P and contains D_2. Let (a, b) be a lattice point of P' which is not in D_2.

Claim: $a=0$ or $b=0$. Indeed, suppose $a\neq 0$ and $b\neq 0$. Then by changing the signs of the coordinates of $\boldsymbol{Z}^2$, if necessary, we may assume $a, b>0$. Then P' contains D_1 and P is not k-canonical by Lemmas 1.3 and 2.2.

By exchanging the first and the second coordinates of $\boldsymbol{Z}^2$, if necessary, we may assume $b=0$ and $a\geqq 2$. Let (a', b') be another lattice point of $P'\backslash D_2$. Then we have $a'=0$ or $b'=0$. If $a'=0$, then we may assume $b'\geqq 2$. Hence $D_1\subset P'$, which is impossible by Lemma 2.2. Thus every lattice point of P' other than $(0, 1)$ and $(0, -1)$ is on the line $\{(x, 0); x \in \boldsymbol{R}\}$. Let m and n be the largest integers with $(m, 0) \in P'$ and $(-n, 0) \in P'$, respectively. Since P' is an integral convex polygon, we know $P'=$

$P((0,-1),(m,0),(0,1),(-n,0))$. Hence $P=P((a,b-1),(a+m,b),(a,b+1),(a-n,b))$ for some integers a, b. By Lemmas 1.3 and 2.3, we know a, b are odd, and hence P is 2-equivalent to $P((1,0),(1+m,1),(1,2),(1-n,1))$. q.e.d.

Proposition 2.4 also implies the following.

Corollary 2.5. *Let P be a k-canonical integral triangle for an integer $k\geqq 2$. Then P does not contain any integral convex quadrangle.*

Proof. Suppose P contains an integral convex quadrangle P'. Then by Lemmas 1.3 and 2.2, P' does not contain any quadrangle 1-equivalent to D_1 for any coordinate system of $\boldsymbol{Z}^2$. Hence, by Lemma 1.1, P' contains a quadrangle 1-equivalent to D_2 for a coordinate system of $\boldsymbol{Z}^2$. Then by Proposition 2.4, P must be a quadrangle. q.e.d.

§ 3. Canonical triangles

In this section we classify k-canonical integral convex triangles. Since every integral triangle is 1-canonical, we consider the case $k\geqq 2$. By Corollary 2.5, they contain no integral convex quadrangle in case $k\geqq 2$.

Proposition 3.1. *Let $P=P(v_1, v_2, v_3)$ be an integral convex triangle which contains no integral convex quadrangle. Then the triangle P is 1-equivalent to either*

(1) $P_1=P((0,0),(1,0),(0,1))$,
(2) $P_{2,m}=P((0,0),(1,0),(0,m))$; $(m\geqq 2)$,
(3) $P_{3,m}=P((0,0),(1,0),(-1,2m))$; $(m\geqq 2)$,
(4) $P_{4,m}=P((0,0),(1,0),(-1,2m+1))$; $(m\geqq 1)$, *or*
(5) $P_5=P((0,-1),(2,0),(-1,2))$,

for a coordinate system of $\boldsymbol{Z}^2$.

For distinct points Q, Q' on $\boldsymbol{R}^2$, we denote by $L(Q,Q')$ the line which goes through Q and Q'. We need the following lemma.

Lemma 3.2. *In the situation of Proposition* 3.1, *suppose the triangle P contains two lattice points S, S' in the interior. Then the line $L(S,S')$ goes through one of the vertices of P.*

Proof. It is sufficient to show that S' is on the union $\bigcup_{i=1}^3 L(v_i, S)$. Otherwise, we may assume S' is in the interior of the area K in Figure 1 by renumbering v_i's, if necessary. Then $P(v_1, S, S', v_3)$ is an integral convex quadrangle. q.e.d.

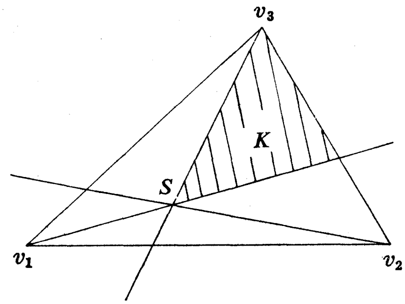

Figure 1.

Proof of Proposition 3.1. (i) When the triangle P contains no lattice point other than the vertices, (v_2-v_1, v_3-v_1) is a basis of $\boldsymbol{Z}^2$. Hence P is 1-equivalent to P_1 for the coordinate system associated to this basis.

(ii) We consider the case where P contains a lattice point other than the vertices on an edge but there is no lattice point in the interior. Let $\overline{v_1v_2}$, $\overline{v_2v_3}$, $\overline{v_1v_3}$ be the edges of P. Suppose there exist lattice points $u_1 \in \overline{v_1v_2}$, $u_2 \in \overline{v_1v_3}$ other than the vertices. Then P contains the integral convex quadrangle $P(v_2, v_3, u_1, u_2)$. This implies that the lattice points on the boundary of P other than v_1, v_2, v_3 are on a common edge. Hence we may assume that they are contained in $\overline{v_1v_3}$. Let w be the lattice point on $\overline{v_1v_3}$ closest to v_1. Since the triangle $P(v_1, v_2, w)$ contains no lattice point other than the vertices, it is 1-equivalent to P_1 for the coordinate system associated with the basis $(v_2-v_1, w-v_1)$. Hence P is 1-equivalent to $P_{2,m}$ with respect to the integer m with $v_3=v_1+m(w-v_1)$.

(iii) Now suppose the triangle P contains a lattice point in the interior of P. There are the following two cases:

Case (a). The triangle P has a lattice point on an edge other than the vertices.

Case (b). The triangle P has no lattice point on the edges except for vertices.

We first consider Case (a). As we saw in the proof of (ii), such an edge is unique for P. We may assume that a lattice point u is on $\overline{v_2v_3}\backslash\{v_2, v_3\}$ by renumbering v_i's, if necessary. Let w be a lattice point in the interior of P. If $w \notin \overline{v_1u}$, we may assume that w is in the interior of the triangle $P(v_1, u, v_3)$. Then P contains the quadrangle $P(v_1, v_2, u, w)$, a contradiction. Hence $w \in \overline{v_1u}$. Moreover if there exists another lattice point u' on $\overline{v_2v_3}\backslash\{v_2, v_3\}$, then we have $w \notin \overline{v_1u'}$, again a contradiction. Hence u is a unique lattice point on $\overline{v_2v_3}\backslash\{v_2, v_3\}$ and any lattice point in the interior of P is on the segment $\overline{v_1 u}$. Let w' be the lattice point on $\overline{v_1 u}$ closest to v_1. Since $(v_2-v_1, w'-v_1)$ is a basis of $\boldsymbol{Z}^2$, we see that $P(v_1, v_2, v_3)$ is 1-equivalent to $P_{3,m}$ for the coordinate system of $\boldsymbol{Z}^2$ associated with this basis and with respect to the integer m with $u=v_1+m(w'-v_1)$.

We divide Case (b) into two subcases according as whether all the lattice points in the interior of P are on a common line L or not.

In the first subcase we may assume that the line L goes through the vertex v_1 by renumbering the indices of v_1, v_2, v_3. Indeed, this follows from Lemma 3.2, when there are at least two lattice points, while it is trivially true if there exists only one lattice point in the interior of P. Let S be the lattice point on L closest to v_1. Then $(v_2-v_1, S-v_1)$ is a basis of $\boldsymbol{Z}^2$. For the coordinate system of $\boldsymbol{Z}^2$ associated with this basis, P is 1-equivalent to $P_{4,m}$ where m is the number of lattice points in the interior.

Now consider the subcase where the lattice points in the interior of P are not on a common line. Obviously, there are three such points S, S', S'' not on a line. By Lemma 3.2, we may assume the line $L(S, S')$ is equal to $L(v_1, S)$. We may assume v_1, S, S', are on $L(S, S')$ in this order. Let S_3 and S_4 be the intersection $L(v_2, S)\cap L(v_3, S')$ and $L(v_3, S)\cap L(v_2, S')$, respectively (see Figure 2). By Lemma 3.2, S'' is on both $\bigcup_{i=1}^{3} L(v_i, S)$ and $\bigcup_{i=1}^{3} L(v_i, S')$. Since $S''\notin L(v_1, S)$, this implies S'' is either S_3 or S_4. It does not happen that there exist two lattice points not on $L(v_1, S)$, because otherwise they are S_3 and S_4 and P would contain the integral quadrangle $P(S, S_3, S', S_4)$. We may assume $S''=S_3$ by exchanging the indices of v_2, v_3, if necessary. Then by a similar argument, we see that S (resp. S') is the unique lattice point in the interior of P which is not on $L(v_3, S')$ (resp. $L(v_2, S)$). Therefore $\{S, S', S''\}$ is the complete set of lattice points in the interior of P and $v_1-S=S-S'$, $v_2-S''=S''-S$ and $v_3-S'=S'-S''$. Since the triangle $P(S, S', S'')$ contains no lattice point other than the vertices, $(S''-S, S'-S)$ is a basis of $\boldsymbol{Z}^2$. Thus for the coordinate of $\boldsymbol{Z}^2$

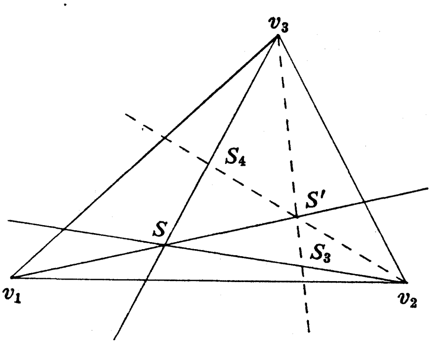

Figure 2.

associated with this basis, it is easy to see that $P(v_1, v_2, v_3)$ is equal to P_5+S. q.e.d.

Proposition 3.2. *Let $k\geqq 2$, $m\geqq 1$, a, b be integers. Then, for $v_1=(a, b)$, $v_2=(a+1, b)$, $v_3=(a, b+m) \in \boldsymbol{Z}^2$, the triangle $P=P(v_1, v_2, v_3)$ is k-canonical if and only if*

(i) $a\equiv -1 \pmod{k}$ *and* g.c.d. $(b, k)=1$ *in case* $m\geqq 2$, *and*

(ii) g.c.d. $(a, k)=$g.c.d. $(b, k)=1$ *and either* $a\equiv -1$, $b\equiv -1$ *or* $a+b \equiv 0 \pmod{k}$ *in case* $m=1$.

Our proof of this proposition depends on the following result:

Lemma 3.3 (White [8, Theorem 1 and the remark after it]). *Let T be a tetrahedron in $\boldsymbol{R}^3$ with the vertices in the lattice $\boldsymbol{Z}^3$, and let $\{\sigma_i, \sigma_i'\}$, $i=1, 2, 3$ be the three pairs of edges of T with $\sigma_i \cap \sigma_i'=\varnothing$. Then the following conditions are equivalent.*

(1) *For an i in $\{1, 2, 3\}$, every lattice point on the tetrahedron T is contained in $\sigma_i \cup \sigma_i'$.*

(2) *There exist adjacent lattice planes π and π' with $\sigma_j \subset \pi$ and $\sigma_j' \subset \pi'$ for some $j \in \{1, 2, 3\}$.*

Furthermore, if $\sigma_i \cup \sigma_i'$ in (1) *contains a lattice point of T other than the vertices, then j in* (2) *is equal to i.*

Remark. Two lattice planes in $\boldsymbol{R}^3$ are said to be *adjacent* if they are disjoint and there is no lattice point between them. This is equivalent to saying that there exist a surjective homomorphism $\phi\colon \boldsymbol{Z}^3 \to \boldsymbol{Z}$ and an integer a such that these two planes are spanned by $\phi^{-1}(a)$ and $\phi^{-1}(a+1)$, respectively.

We set $v_i'=(v_i, k) \in \boldsymbol{Z}^3$, for $i=1, 2, 3$ and let T be the convex hull of $\{0, v_1', v_2', v_3'\}$.

Lemma 3.4. *In the situation of Proposition* 3.2, *the triangle P is k-canonical and the condition* (2) *of Lemma* 3.3 *is satisfied for the pair $(\overline{0v_2'}, \overline{v_1'v_3'})$ if and only if $a\equiv -1 \pmod{k}$ and* g.c.d. $(b, k)=1$.

Proof of Lemma 3.4. By the remark above, (2) of Lemma 3.3 is satisfied for $(\overline{0v_2'}, \overline{v_1'v_3'})$ if and only if there exist integers x, y, z which satisfy

$$
\text{(A1)}\qquad
\begin{aligned}
&x(a+1)+yb+zk=0\\
&xa+yb+zk=1\\
&xa+y(b+m)+zk=1.
\end{aligned}
$$

From these equations, we have $x=-1$ and $y=0$. Hence it is equivalent to the existence of an integer z with $-a+zk=1$, i.e., $a\equiv-1 \pmod{k}$. Under these equivalent conditions P is k-canonical if and only if $\overline{0v_2'}$ contains no lattice point other than the vertices. This is equivalent to the condition g.c.d. $(b, k)=1$, since $v_2'=(a+1, b, k)$ and g.c.d. $(a+1, k)=k$.

q.e.d.

Proof of Proposition 3.2. Suppose P is k-canonical. Then, for $m\geqq 2$, we know that (2) of Lemma 3.3 is satisfied for the pair $(\overline{0v_2'}, \overline{v_1'v_3'})$. By applying Lemma 3.4, we get (i) of Proposition 3.2. Now assume $m=1$. Since the lattice points of T are necessarily its vertices, every pair of disjoint edges of T satisfies (1) of Lemma 3.3. Hence (2) of Lemma 3.3 is satisfied for one of the pairs $E_1=\{\overline{0v_1'}, \overline{v_2'v_3'}\}$, $E_2=\{\overline{0v_2'}, \overline{v_1'v_3'}\}$, $E_3=\{\overline{0v_3'}, \overline{v_1'v_2'}\}$. Let f_1, f_2, f_3 be automorphisms of $\boldsymbol{Z}^2$ defined by matrices

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \text{ and } \begin{pmatrix}-1 & -1\\ 0 & 1\end{pmatrix},$$

respectively. We denote their scalar extension to $\boldsymbol{R}$ by the same letter f_i. Applying f_i to the triangle P for each $i=1, 2, 3$, we come to the situation of Lemma 3.4. For $i=1$, we have $f_1(P(v_1, v_2, v_3))=P_1+(a, b)$. Hence we have $a\equiv-1 \pmod{k}$ and g.c.d. $(b, k)=1$. For $i=2$, we have $f_2(P(v_1, v_2, v_3))\equiv P_1+(b, a)$ and hence $b\equiv-1 \pmod{k}$ and g.c.d. $(a, k)=1$. For $i=3$, we have $f_3(P(v_1, v_2, v_3))=P_1+(-a-b-1, b)$ and hence $a+b\equiv 0 \pmod{k}$ and g.c.d. $(b, k)=1$. Since we have g.c.d. $(a, k)=$ g.c.d. $(b, k)=1$ in every case, the logical sum of these conditions is equal to the condition (ii) of Proposition 3.2.

Conversely, since two conditions in Lemma 3.4 are equivalent, P satisfying (i) is k-canonical. For P satisfying (ii), we can apply Lemma 3.4 after the application of f_i as above.

q.e.d.

Proposition 3.5. *Let P be one of the triangles P_1, $P_{2,m}$, $P_{3,m}$, $P_{4,m}$, P_5 in Proposition* 3.1. *Then, for integers $k\geqq 2$, a, b, the triangle $P+(a, b)$ is k-canonical if and only if the integers satisfy the following respective condition for each of the cases in Proposition* 3.1.

(1) *If $P=P_1$, then* g.c.d. $(a, k)=$g.c.d. $(b, k)=1$ *and either* $a\equiv-1$, $b\equiv-1$ *or* $a+b\equiv 0 \pmod{k}$.

(2) *If $P=P_{2,m}$, then* $a\equiv-1 \pmod{k}$ *and* g.c.d. $(b, k)=1$.

(3) *If $P=P_{3,m}$, then* $k=2$ *and* $a\equiv b\equiv 1 \pmod{2}$.

(4) *If $P=P_{4,m}$, then* $m=1$, $k=3$, g.c.d. $(a, 3)=1$ *and* $b\equiv-1 \pmod{3}$.

(5) *If $P=P_5$, then* $k=2$ *and* $a\equiv b\equiv 1 \pmod{2}$.

Proof. The cases $P=P_1$ and $P=P_{2,m}$ are already proved as Proposition 3.2.

Let $P=P_{3,m}$. Then the triangle $P+(a, b)$ is equal to $P(v_1, v_2, v_3)$ for $v_1=(a, b)$, $v_2=(a+1, b)$, $v_3=(a-1, b+2m)$. Let $v=(a, b+m)$. Then $P(v_1, v_2, v_3)$ is the union of triangles $P(v_1, v_2, v)$ and $P(v_1, v_3, v)$. Since $P(v_1, v_2, v)=P_{2,m}+(a, b)$, it is k-canonical if and only if

(B1) $a\equiv -1 \pmod{k}$ and g.c.d. $(b, k)=1$,

by Proposition 3.2. Let f be the linear automorphism of $\boldsymbol{Z}^2$ given by $f((1, 0))=(-1, 2m)$ and $f((0, 1))=(0, 1)$. Then we have $f(P(v_1, v_3, v))=P_{2,m}+(-a, 2ma+b)$. Since $P(v_1, v_3, v)$ is k-canonical if and only if so is $f(P(v_1, v_3, v))$, we know it is k-canonical if and only if

(B2) $-a\equiv -1 \pmod{k}$ and g.c.d. $(2ma+b, k)=1$,

by Proposition 3.2. It is clear that the logical product of (B1) and (B2) is equivalent to the condition (3) of the proposition, since $k\geqq 2$.

Now assume $P=P_{4,m}$. Then we have $P+(a, b)=P(v_1, v_2, v_3)$ with $v_1=(a, b)$, $v_2=(a+1, b)$ and $v_3=(a-1, b+2m+1)$. Let v be the point $(a, b+m)=(v_1+mv_2+mv_3)/(2m+1)$ in the interior of $P(v_1, v_2, v_3)$. Since the triangle $P(v_1, v_2, v_3)$ is the union of three triangles $P(v_1, v_2, v)$, $P(v_1, v_3, v)$ and $P(v_2, v_3, v)$, it is k-canonical if and only if so are these three triangles. We now show that it is not the case if $m>1$. Since $P(v_1, v_2, v)=P_{2,m}+(a, b)$, we have

(B3) $a\equiv -1 \pmod{k}$ and g.c.d. $(b, k)=1$,

by Proposition 3.2. Let g be the linear automorphism of $\boldsymbol{Z}^2$ defined by $g((1, 0))=(-1, 2m+1)$ and $g((0, 1))=(0, 1)$. Then we have $g(P(v_1, v_3, v))=P_{2,m}+(-a, (2m+1)a+b)$. Hence we have

(B4) $-a\equiv -1 \pmod{k}$ and g.c.d. $((2m+1)a+b, k)=1$.

Clearly, (B3) and (B4) imply $k=2$ and a is odd. This is a contradiction, since (B3) implies b is odd and (B4) implies b is even. Hence only the case $m=1$ is possible. Then each of the three triangles has no lattice point other than the vertices. Since $P(v_1, v_2, v_3)=P_1+(a, b)$, we have

(C1) g.c.d. $(a, k)=$g.c.d. $(b, k)=1$ and either (C1–1) $a\equiv -1$, (C1–2) $b\equiv -1$ or (C1–3) $a+b\equiv 0 \pmod{k}$.

Let h_1 be the linear automorphism of $\boldsymbol{Z}^2$ defined by $h_1((1, 0))=(-1, 3)$ and $h_1((0, 1))=(0, 1)$. Since $h_1(P(v_1, v_3, v))=P_1+(-a, b+3a)$, we have

(C2) g.c.d. $(a, k)=$g.c.d. $(b+3a, k)=1$ and either (C2–1) $-a\equiv -1$, (C2–2) $b+3a\equiv -1$ or (C2–3) $2a+b\equiv 0 \pmod{k}$.

Let h_2 be the linear automorphism of $\boldsymbol{Z}^2$ defined by $h_2((1, 0))=(2, 1)$ and $h_2((0, 1))=(1, 1)$. Since $h_2(P(v_2, v_3, v))=P_1+(2a+b+1, a+b+1)$, we have

(C3) g.c.d. $(2a+b+1, k)=$g.c.d. $(a+b+1, k)=1$ and either (C3–1) $2a+b+1\equiv -1$, (C3–2) $a+b+1\equiv -1$ or (C3–3) $3a+2b+2\equiv 0$ (mod k).

Thus the integers k, a, b must satisfy the above conditions (C1), (C2) and (C3). It is easy to check that these conditions are satisfied if $k=3$, g.c.d. $(a, k)=1$ and $b\equiv -1$ (mod k).

The converse is proved as follows: We have to show $k=3$ and $b\equiv -1$ (mod 3). We have $k\neq 2$, since otherwise (C1) implies a and b are odd, while (C2) implies $b+3a$ is odd. (i) We assume (C1–1) $a\equiv -1$ (mod k). Then (C2–1) and (C3–1) do not occur, since $k\neq 2$ and g.c.d. $(b, k)=1$, respectively. In this case, (C2–2) and (C2–3) both mean $b\equiv 2$ (mod k), and either (C3–2) or (C3–3) is satisfied if and only if $k=3$. (ii) Next assume (C1–2) $b\equiv -1$ (mod k). Then (C2–1), (C2–2) and (C2–3) mean $a\equiv 1$, $3a\equiv 0$ and $2a\equiv 1$ (mod k), respectively, while (C3–1), (C3–2) and (C3–3) mean $2a\equiv -1$, $a\equiv -1$ and $3a\equiv 0$ (mod k), respectively. We exclude (C3–2), since we have already checked the case $a\equiv -1$ (mod k). Since g.c.d. $(a, k)=1$ and $3a\equiv 0$ (mod k) imply $k=3$, we also exclude (C2–2) and (C3–3). Hence we may assume (C3–1) $2a\equiv -1$ (mod k) holds. Then (C2–3) does not occur since $k\neq 2$. In the remaining case in which (C2–1) and (C3–1) hold, we have $k=3$, since $2a\equiv 2\equiv -1$ (mod k). (iii) Assume (C1–3) $a+b\equiv 0$ (mod k). Then (C2–3) and (C3–2) do not occur since g.c.d. $(a, k)=1$ and $k\neq 2$, respectively. Then both (C3–1) and (C3–3) mean $a\equiv -2$ (mod k), and (C2–1) and (C2–2) both imply $k=3$, for this a.

Finally we assume $P=P_5$. Then we have $P+(a, b)=P(v_1, v_2, v_3)$ with $v_1=(a, b-1)$, $v_2=(a+2, b)$ and $v_3=(a-1, b+2)$. Let u_1, u_2 and u_3 be the lattice points (a, b), $(a+1, b)$ and $(a, b+1)$ on $P(v_1, v_2, v_3)$, respectively. Since the triangle $P(v_1, v_2, v_3)$ is the union of four triangles $P(v_1, v_2, u_1)$, $P(v_2, v_3, u_2)$, $P(v_1, v_3, u_3)$ and $P(u_1, u_2, u_3)$, it is k-canonical if and only if so are these four triangles. Thus we consider each triangle. Let h_3 be the linear automorphism of $\boldsymbol{Z}^2$ defined by $h_3((1, 0))=(0, 1)$ and $h_3((0, 1))=(-1, 0)$. We have $h_3(P(v_1, v_2, u_1))=P_{2,2}+(-b, a)$. Hence by Proposition 3.2, we have

(D1) $-b\equiv -1$ (mod k) and g.c.d. $(a, k)=1$.

Let h_4 be the linear automorphism of $\boldsymbol{Z}^2$ defined by $h_4((1, 0))=(1, 0)$ and $h_4((0, 1))=(1, 1)$. We have $h_4(P(v_2, v_3, u_2))=P_{2,2}+(a+b+1, b)$. Hence

(D2) $a+b+1\equiv -1$ (mod k) and g.c.d. $(b, k)=1$.

For the linear automorphism h_5 of $\boldsymbol{Z}^2$ defined by $h_5((1, 0))=(-1, -1)$ and $h_5((0, 1))=(0, -1)$, we have $h_5(P(v_1, v_3, u_3))=P_{2,2}+(-a, -a-b-1)$. Hence

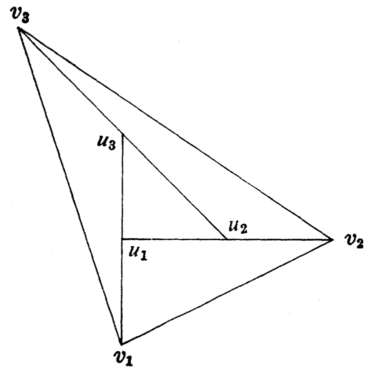

Figure 3.

(D3) $-a \equiv -1 \pmod{k}$ and g.c.d. $(-a-b-1, k)=1$.

By (D1), (D2) and (D3), we have $1=(-b)+(a+b+1)+(-a) \equiv -1-1-1 \equiv -3 \pmod{k}$. Thus k is 2 or 4 and $a \equiv b \equiv 1 \pmod{k}$. Moreover, $P(u_1, u_2, u_3)=P_1+(a, b)$. Hence by Proposition 3.2, (ii), we have

(D4) g.c.d. $(a, k)=$g.c.d. $(b, k)=1$ and either $a \equiv -1, b \equiv -1$ or $a+b \equiv 0 \pmod{k}$.

If $k=4$, (D4) is not compatible with the condition $a \equiv b \equiv 1 \pmod{k}$. Thus we have $k=2$ and g.c.d. $(a, 2)=$g.c.d. $(b, 2)=1$. This is the condition (5) of Proposition 3.4. It is easy to see that (5) implies (D1), (D2), (D3) and (D4). q.e.d.

By combining this proposition and the results in the previous sections, we get the classification of 3-dimensional toric canonical singularities as follows.

Theorem 3.6. *Let $r=3$, and let σ be a 3-dimensional cone. Then the associated affine toric variety X_σ is canonical of index $k>0$ if and only if, for a coordinate system $N \cong \boldsymbol{Z}^3$, the cone σ is equal to $C(P \times \{k\})$ for one of the following polygons $P \subset \boldsymbol{R}^2$:*

(i) *$k=1$ and any integral convex polygon P.*

(ii) *$k=2$ and $P=P((1, 0), (1+m, 1), (1, 2), (1-n, 1))$ for some positive integers m, n.*

(iii) *$k \geqq 2$ and $P=P((-1, b), (0, b), (-1, b+m))$ for integers $m \geqq 1$, b with* g.c.d. $(b, k)=1$.

(iv) *$k=2$ and $P=P((-1, 1), (0, 1), (-2, 1+2m))$ for an integer $m \geqq 2$.*

(v) *$k=3$ and $P=P((-1, -1), (0, -1), (-2, 2))$.*

(vi) *$k=2$ and $P=P((-1, -2), (1, -1), (-2, 1))$.*

Proof. We know these cones define canonical toric singularities by Remark 4.1, Theorem 2.1 and Proposition 3.5. We should prove the converse. If the index $k=1$, then we know by Remark 1.4 that the cone is of type (i). If σ is a g-gonal cone with $g\geqq 4$ and $k\geqq 2$, then it is of type (ii) by Theorem 2.1. If the cone σ is trigonal and $k\geqq 2$, then the result follows from Proposition 3.5. Here note that the triangle of (1) of Proposition 3.5 is k-equivalent to a triangle of type (iii) for $m=1$ for a suitable coordinate system of $\boldsymbol{Z}^2$. q.e.d.

§ 4. Canonical cyclic quotient singularities

Let r be a positive integer and let μ_r be the cyclic group $\boldsymbol{Z}/r\boldsymbol{Z}$ of order r. Assume μ_r acts on the vector space $\boldsymbol{C}^3$ linearly and effectively. Since every representation of a finite abelian group in a $\boldsymbol{C}$-vector space is a direct sum of 1-dimensional representations, the action of μ_r is given by $\rho(z_1, z_2, z_3)=(\xi^{a_1}z_1, \xi^{a_2}z_2, \xi^{a_3}z_3)$ for a coordinate (z_1, z_2, z_3) of $\boldsymbol{C}^3$ and for some integers a_1, a_2, a_3 with $0\leqq a_1, a_2, a_3<r$, where ρ is a generator of μ_r and $\xi=\exp(2\pi i/r)$. We denote by $Q_r(a_1, a_2, a_3)$ the quotient of $\boldsymbol{C}^3$ by this normalized action of the cyclic group. If the three integers r, a_1, a_2 have a common divisor $d>1$, then we have $\rho^{r/d}(z_1, z_2, z_3)=(z_1, z_2, \xi^e z_3)$ where e is the integer with $0\leqq e<r$ and $e\equiv ra_3/d \pmod r$. By the effectivity of the action ξ^e is a primitive d-th root of 1. The quotient of $\boldsymbol{C}^3$ by the subgroup $\mu_d=(\rho^{r/d})$ is $\boldsymbol{C}^3$ with the coordinate $(w_1, w_2, w_3)=(z_1, z_2, z_3^d)$ and the action of the factor group $\mu_{r/d}=\mu_r/\mu_d$ on it is given by $\bar{\rho}(w_1, w_2, w_3)=(\zeta^{a_1/d}w_1, \zeta^{a_2/d}w_2, \zeta^c w_3)$ for $\zeta=\xi^d$ where $\bar{\rho}$ is the image of ρ in $\mu_{r/d}$ and c is the integer defined by $0\leqq c<r/d$ and $c\equiv a_3 \pmod{r/d}$. Thus we know $Q_r(a_1, a_2, a_3)$ and $Q_{r/d}(a_1/d, a_2/d, c)$ are isomorphic. By the same argument for the triples (r, a_2, a_3) and (r, a_1, a_3), we may restrict ourselves to considering the quadruple (r, a_1, a_2, a_3) such that each of these triples has no common divisor. Namely we set

$$\begin{aligned}\Lambda=\{(r, a_1, a_2, a_3);\ & r, a_1, a_2, a_3 \in \boldsymbol{Z},\ r>0,\ 0\leqq a_1, a_2, a_3<r \\ & \text{and g.c.d. } (r, a_i, a_j)=1 \text{ for any } i, j \text{ with } 1\leqq i<j\leqq 3\}\end{aligned}$$

and we classify $(r, a_1, a_2, a_3)\in\Lambda$ for which $Q_r(a_1, a_2, a_3)$ is canonical. The elements (r, a_1, a_2, a_3) and (r', a_1', a_2', a_3') in Λ are said to be conjugate if (i) $r=r'$, and (ii) there exists $x\in\boldsymbol{Z}$ with $1\leqq x\leqq r-1$ and g.c.d. $(x, r)=1$ such that (a_1', a_2', a_3') is equal to $([xa_1]_r, [xa_2]_r, [xa_3]_r)$ or its permutation. These two elements of Λ are conjugate to each other if and only if the subgroups of $GL_3(\boldsymbol{Z})$ generated by

$$\begin{pmatrix}\xi^{a_1} & & \\ & \xi^{a_2} & \\ & & \xi^{a_3}\end{pmatrix} \text{ and } \begin{pmatrix}\xi^{a_1'} & & \\ & \xi^{a_2'} & \\ & & \xi^{a_3'}\end{pmatrix},$$

respectively, are conjugate to each other. Hence cyclic quotients $Q_r(a_1, a_2, a_3)$ and $Q_{r'}(a_1', a_2', a_3')$ are isomorphic in this case.

Theorem 4.1. *Let (r, a_1, a_2, a_3) be a quadruple in Λ. Then the cyclic quotient $Q_r(a_1, a_2, a_3)$ is canonical if and only if it belongs to one of the following four classes.*

Class I. (r, a_1, a_2, a_3)'s with g.c.d. $(a_i, a_j)=1$ and $a_i+a_j\equiv 0 \pmod r$ for some $1\leqq i<j\leqq 3$.

Class II. (r, a_1, a_2, a_3)'s with $a_1+a_2+a_3\equiv 0 \pmod r$.

Class III. (r, a_1, a_2, a_3)'s which are conjugate to $(r, 1, r/2+1, r-2)$ for some $r\geqq 8$ with $r\equiv 0 \pmod 4$.

Class IV. (r, a_1, a_2, a_3)'s which are conjugate to either $(9, 1, 4, 7)$ or $(14, 1, 9, 11)$.

Remark 4.2. Morrison [3] gave the same result under the restriction that each of a_1, a_2, a_3 is prime to r. This is equivalent to assuming that the cyclic quotient singularity is isolated. Obviously, the quadruples of Class III do not occur in this case.

The cyclic quotient $Q_r(a_1, a_2, a_3)$ is described in terms of torus embeddings as follows.

Let N be a free $\boldsymbol{Z}$-module of rank 3, and let M be its dual $\boldsymbol{Z}$-module. Let $\{n_1, n_2, n_3\}$ be a basis of N, and let $\sigma\subset N_{\boldsymbol{R}}$ be the trigonal cone generated by $\{n_1, n_2, n_3\}$. We consider the semigroup ring $\boldsymbol{C}[M\cap\pi^\vee]=\bigoplus_{m\in M\cap\pi^\vee}\boldsymbol{C}e(m)$. Then the affine scheme $X_{\sigma,N}$ with the coordinate ring $\boldsymbol{C}[M\cap\pi^\vee]$ is equal to $\boldsymbol{C}^3$ with the coordinate $(z_1, z_2, z_3)=(e(m_1), e(m_2), e(m_3))$, where $\{m_1, m_2, m_3\}$ is the basis of M dual to $\{n_1, n_2, n_3\}$. Let T_N be the algebraic torus $\mathrm{Hom}_{\boldsymbol{Z}}(M, \boldsymbol{C}^\times)=N\otimes_{\boldsymbol{Z}}\boldsymbol{C}^\times$, where $\boldsymbol{C}^\times$ is the multiplicative group $\boldsymbol{C}\backslash\{0\}$. Then T_N acts on $X_{\sigma,N}$ by $(t_1, t_2, t_3)\colon (z_1, z_2, z_3)\mapsto(t_1z_1, t_2z_2, t_3z_3)$ for $(t_1, t_2, t_3)\in T_N$ and $(z_1, z_2, z_3)\in\boldsymbol{C}^3$.

For a quadruple (r, a, b, c) in Λ, we set $n=(an_1+bn_2+cn_3)/r\in N_{\boldsymbol{R}}$. Let N' be the free $\boldsymbol{Z}$-module $N+\boldsymbol{Z}n\subset N_{\boldsymbol{R}}$. Then we have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N'/N & \longrightarrow & T_N & \longrightarrow & T_{N'} & \longrightarrow & 0. \\ & & & & \cap & & \cap & & \\ & & & \boldsymbol{C}^3= & X_{\sigma,N} & \longrightarrow & X_{\sigma,N'} & & \end{array}$$

Since the image of the class $n+N\in N'/N$ in T_N is $(\xi^{a_1}, \xi^{a_2}, \xi^{a_3})$, we know that $X_{\sigma,N'}$ is equal to the cyclic quotient $Q_r(a_1, a_2, a_3)$. By the definition of Λ, we know n_1, n_2, n_3 are primitive in N'. Hence by Reid [6], $Q_r(a_1, a_2, a_3)$ is canonical if and only if every lattice point in the tetrahedron spanned by $\{0, n_1, n_2, n_3\}$ is either equal to 0 or contained in the triangle spanned by $\{n_1, n_2, n_3\}$. By this observation, we get the following:

Lemma 4.3. *Let (r, a_1, a_2, a_3) be a quadruple in Λ and k be a positive integer. Then the cyclic quotient $Q_r(a_1, a_2, a_3)$ is canonical of index k if and only if there exists a k-canonical triangle $P(v_1, v_2, v_3)\subset \boldsymbol{R}^2$ such that $\boldsymbol{Z}^3=\boldsymbol{Z}v_1'+\boldsymbol{Z}v_2'+\boldsymbol{Z}v_3'+\boldsymbol{Z}(a_1v_1'+a_2v_2'+a_3v_3')/r$, where $v_1'=(v_1, k)$, $v_2'=(v_2, k)$, $v_3'=(v_3, k)\in \boldsymbol{Z}^3$.*

First, assume $Q_r(a_1, a_2, a_3)$ is canonical of index $k\geqq 2$. Then the triangle $P(v_1, v_2, v_3)$ in the above lemma is equal to one of $P+(a, b)$ in Proposition 3.5. By Remark 1.2, we may replace $P(v_1, v_2, v_3)$ by a k-equivalent one. Hence we assume $0\leqq a, b<k$. Now we look at each of the cases in Proposition 3.5.

Among the three subcases of the case (1) $P=P_1$, suppose $a=k-1$, g.c.d. $(b, k)=1$. Then the triangle $P_1+(a, b)$ is equal to $P(v_1, v_2, v_3)$ for $v_1=(k-1, b)$, $v_2=(k, b)$ and $v_3=(k-1, b+1)$. Let α, β be integers with $\beta k+b\alpha=1$. Then for the lattice point $v=(-\alpha, \beta, -\alpha)$, we know that $\{v, v_2', v_1'\}$ is a basis of $\boldsymbol{Z}^3$ and $v_3'=kv+\alpha v_2'+v_1'$. Thus we get $\boldsymbol{Z}^3=N+\boldsymbol{Z}n_1$ for $n_1=(v_3'+(k-1)v_1'+(k-\alpha)v_2')/k\in N_{\boldsymbol{R}}$ and α is relatively prime to k. By Lemma 4.3, $X_{\sigma,N'}$ is equal to $Q_k(1, k-1, k-\alpha)$. Hence the quadruple $(k, 1, k-1, k-\alpha)$ in Λ belongs to Class I in Theorem 4.1. Similarly, we get quadruples in Class I from the other cases.

In case (2) $P=P_{2,m}$, we know $a=k-1$ and g.c.d. $(b, k)=1$. Then the triangle $P_2+(a, b)$ is equal to $P(v_1, v_2, v_3)$ for $v_1=(k-1, b)$, $v_2=(k, b)$ and $v_3=(k-1, b+m)$. Let v be the same point v as in case (1). We have $v_3'=mkv+m\alpha v_2'+v_1'$. Thus we get $\boldsymbol{Z}^3=N+\boldsymbol{Z}n_2$ for

$$n_2=(v_3'+(mk-m\alpha)v_2'+(mk-1)v_1')/mk\in N_{\boldsymbol{R}}.$$

By Lemma 4.3, $X_{\sigma,N}$ is the cyclic quotient $Q_{mk}(1, mk-m\alpha, mk-1)$. This belongs to Class I, too.

In case (3) $P=P_{3,m}$, we know $k=2$, $a=1$ and $b=1$. Then the triangle $P+(a, b)$ is equal to $P(v_1, v_2, v_3)$ for $v_1=(1, 1)$, $v_2=(2, 1)$, $v_3=(0, 1+2m)$. For the lattice point $v=(-1, 0, -1)$, we know that $\{v, v_2', v_1'\}$ is a basis of $\boldsymbol{Z}^3$ and $v_3'=4mv+(2m-1)v_2'+2v_1'$. Thus we get $\boldsymbol{Z}^3=N+\boldsymbol{Z}n_3$ for

$$n_3=(v_3'+(2m+1)v_2'+(4m-2)v_1')/4m\in N_{\boldsymbol{R}}.$$

By Lemma 4.3, we have $X_{\sigma,N'}=Q_{4m}(1, 2m+1, 4m-2)$. This belongs to Class III.

In case (4) $P=P_{4,1}$, we know $k=3$, $a=1$ or 2 and $b=2$. Suppose $a=1$. Then the triangle $P+(a, b)$ is equal to $P(v_1, v_2, v_3)$ for $v_1=(1, 2)$, $v_2=(2, 2)$, $v_3=(0, 5)$. If we set $v=(0, -3, -2)$, then $\{v, v_3', v_1'\}$ is a basis of $\boldsymbol{Z}^3$ and $v_2'=9v+5v_3'+2v_1'$. Thus we get $\boldsymbol{Z}^3=N+\boldsymbol{Z}n_4$ for

$$n_4=(v_2'+4v_3'+7v_1')/9\in N_{\boldsymbol{R}}.$$

By Lemma 4.3, $X_{\sigma,N'}$ is the cyclic quotient $Q_9(1, 4, 7)$ which belongs to Class IV. Similarly, we get the same result for $a=2$.

In case (5) $P=P_5$, we know $k=2$, $a=1$ and $b=1$. Hence the triangle $P+(a, b)$ is equal to $P(v_1, v_2, v_3)$ for $v_1=(1, 0)$, $v_2=(3, 1)$ and $v_3=(0, 3)$. For $v=(0, -1, -1)$, we know that $\{v, v_3', v_1'\}$ is a basis of Z^3 and $v_2'=14v+5v_3'+3v_1'$. By Lemma 4.3, $X_{\sigma,N'}$ is equal to $Q_{14}(1, 9, 11)$ and it belongs to Class IV, too.

Finally, assume $Q_r(a_1, a_2, a_3)$ is canonical of index 1. Then since $(a_1v_1+a_2v_2+a_3v_3)/r$ is a lattice point, the third coordinate $(a_1+a_2+a_3)/r$ is an integer. Hence (r, a_1, a_2, a_3) is in Class II.

Conversely, it is clear from the above investigation that $Q_r(a_1, a_2, a_3)$ is canonical for (r, a_1, a_2, a_3) in Class III or Class IV. By Reid [6, Theorem 3.1], the quotient $Q_r(a_1, a_2, a_3)$ is canonical if and only if $[xa_1]_r+[xa_2]_r+[xa_3]_r\geqq r$ is satisfied for every integer x such that $1\leqq x\leqq r-1$. It is clear by this criterion that $Q_r(a_1, a_2, a_3)$ is canonical for (r, a_1, a_2, a_3) in Class I or Class II.

Thus we complete proof of Theorem 4.1.

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