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Complex Analytic Singularities
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On the Singular Subspace of a Complex-Analytic Variety

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§ 1. Introduction

Theme of this paper is the following question:

"How does a germ of a complex-analytic variety relate to its singular subspace?"

The main point herein is to consider the singular locus of a variety not just as a set but provided with a suitable analytic structure. It thus becomes an in general non reduced subvariety of the initial variety, the singular subspace. There are several choices of analytic structures. We shall specify them in a moment. For each of those one can look at the information it contains about the variety one started with. It turns out that certain of them offer a precise insight into the local geometry of the variety around the singularity. We shall describe a number of related results and formulate some open questions.

The presentation avoids technical details and concentrates on the intuitive approach to the problem. Thus the article should be rather understood as a conceptual overview on the situation than a detailed discussion. For the latter, we refer the reader to [G-H].

§ 2. The singular subspace of a hypersurface singularity

Let $(X, 0)$ denote the germ of a hypersurface in $(\boldsymbol{C}^n, 0)$ defined by the analytic equation $f(x)=0$. If $(X, 0)$ is reduced, the set L of singular points of $(X, 0)$ is given by the vanishing of all first order partial derivatives $\partial_1 f, \cdots, \partial_n f$ of f. Denote by $j(f)=(\partial_1 f, \cdots, \partial_n f)\subset \mathcal{O}_n$ the jacobian ideal of f. There are essentially four different ways to make L into an analytic variety:

(1) $L(X, 0)$ the reduced analytic variety of local ring $\mathcal{O}_{L(X,0)}=\mathcal{O}_n/\sqrt{j(f)}$.

(2) $\mathrm{M}(X, 0)$ the variety whose local ring is the Milnor algebra $\mathcal{O}_{M(X,0)}=\mathcal{O}_n/j(f)$.

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(3) $\mathrm{Sing}(X, 0)$ defined by $\mathcal{O}_{\mathrm{Sing}(X,0)}=\mathcal{O}_n/(f)+j(f)$.

(4) $\mathrm{Sing}^*(X, 0)$ defined by $\mathcal{O}_{\mathrm{Sing}^*(X,0)}=\mathcal{O}_n/(f)+m\cdot j(f)$, $m\subset\mathcal{O}_n$ the maximal ideal.

Further possibilities are indicated in [G-H]. With this notation we have the following inclusions:

$$\begin{array}{ccccc} M(X,0) & \subset & L(X,0) & \subset & (X,0) \\ \cup & & & & \\ \mathrm{Sing}(X,0) & \subset & \mathrm{Sing}^*(X,0) & \subset & (X,0) \end{array}$$

All four varieties have the same underlying set which equals the set L of singular points of $(X, 0)$ if $(X, 0)$ is reduced. Analytically, they are quite different:

The first, $L(X, 0)$, contains very little information about the variety itself, so we can disregard it relative to the questions we shall consider. The other three, on the contrary, provide interesting information on the behaviour of the variety nearby its singular locus. Note that if f is quasi-homogeneous, $f\in j(f)$ and $M(X, 0)$ and $\mathrm{Sing}(X, 0)$ coincide. If not, $M(X, 0)$ depends on the choice of f. The isomorphism classes of $\mathrm{Sing}(X, 0)$ and $\mathrm{Sing}^*(X, 0)$ are analytic invariants of the variety; the local ring $\mathcal{O}_n/(f)+j(f)$ of the first is the base space of the semi-universal deformation of $(X, 0)$ (cf. [Ha 1] for the non isolated case), whereas the second $\mathcal{O}_n/(f)+m\cdot j(f)$ can be interpreted as the normal space to the orbit of f in $\mathcal{O}_n$ under the action of the group $K=\mathcal{O}_n^*\times\mathrm{Aut}(\boldsymbol{C}^n, 0)$ inducing analytic isomorphism on the level of defining functions. Let $S(X, 0)$ denote one of the last three singular subspaces of $(X, 0)$. Then we can pose the following problems:

(1) Does $S(X, 0)$ determine $(X, 0)$, i.e., does $S(X, 0)\simeq S(Y, 0)$ imply $(X, 0)\simeq(Y, 0)$ for any hypersurfaces $(X, 0)$ and $(Y, 0)$ in $(\boldsymbol{C}^n, 0)$?

(2) In the cases where (1) admits an affirmative answer, is there an intrinsic description of $(X, 0)$ in terms of $S(X, 0)$?

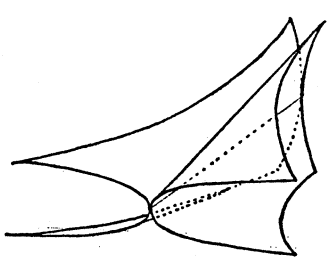
(3) Characterize the classes of varieties in $(\boldsymbol{C}^n, 0)$ which can occur as the singular subspace $S(X, 0)$ of a hypersurface.

(4) Formulate and answer the analogous questions for the non hypersurface case.

In this paper we shall describe answers to question (1) and (4). Question (2) consists in reconstructing the ideal (f) from the knowledge of the ideals $j(f)$, $(f)+j(f)$ or $(f)+m\cdot j(f)$. This can be done algorithmically by solving a system of linear equations in the unknown coefficients of the expansion of f. However, this is not what one would like to consider as a valuable reconstruction of the variety $(X, 0)$. At the moment we don't know of any satisfying answer to question (2) and to the probably much harder third one. Therefore we shall restrict ourselves to questions (1) and (4):

N. Shoshitaishvili [Sh] and M. Benson [Be 2] showed independently that isolated hypersurface singularities which are defined by a quasihomogeneous (resp. homogeneous) equation are determined by their singular subspace $\mathrm{Sing}(X, 0)=M(X, 0)$. Then J. Mather and S.S.-T. Yau [M-Y 2] proved that $\mathrm{Sing}(X, 0)$ as well as $\mathrm{Sing}^*(X, 0)$ always determine the analytic type of an isolated hypersurface singularity. Benson constructed in his proof the required isomorphism more or less explicitly, whereas the other authors used that isolated singularities are finitely determined thus reducing the problem to finite dimensional jet-spaces. Once this is done, Mather's results [M] of the action of a Lie group on manifolds allow to prove the assertion. Note that the statement is false over the reals or a field of characteristic $p>0$, and for the space $M(X, 0)$ in the non quasihomogeneous case. However there are similar results by J. Scherk [Sch] and S.S.-T. Yau [Y2] for $M(X, 0)$ provided one considers $\mathcal{O}_{M(X, 0)}=\mathcal{O}_n/j(f)$ with the additional structure of a $\boldsymbol{C}\{t\}/(t^n)$-algebra.

In [G-H], T. Gaffney and H. Hauser extended the results on $\mathrm{Sing}(X, 0)$ and $\mathrm{Sing}^*(X, 0)$ to the case of a non isolated singularity. The answer obtained was somewhat surprising: $\mathrm{Sing}^*(X, 0)$ always determines the analytic type of $(X, 0)$, but this is not true for $\mathrm{Sing}(X, 0)$, even though the difference between these two singular subspaces is just the finite dimensional $\boldsymbol{C}$-algebra $(f)+j(f)/(f)+m\cdot j(f)$. Here is the counter-example: Take an element h of $\mathcal{O}_n$ which does not belong to its jacobian ideal, $h \notin j(h)$. Define a family of functions $f_t \in \mathcal{O}_{2n+1}$, $t \in \boldsymbol{C}$, by $f_t(x, y, z)=h(x)+(1+t+z)\cdot h(y)$ and let $(X_t, 0)\subset(\boldsymbol{C}^{2n+1}, 0)$ be the corresponding family of varieties. Then $\mathrm{Sing}(X_t, 0)=\mathrm{Sing}(X_0, 0)$ but $(X_t, 0)\not\cong(X_0, 0)$ for $t \in \boldsymbol{C}$ close to 0 (cf. also the example in [Te 1] p. 271). Note that the example of a family of four lines through 0 with varying crossratio does not work, as the analytic type of $\mathrm{Sing}(X, x)$ varies along the singular locus. One might visualize the phenomena of the counter-example as follows:



This leads to the following

Definition. A variety $(X, 0)$ is said to have isolated singularity type (I.S.T.) at 0, if $\mathrm{Sing}(X, x) \not\simeq \mathrm{Sing}(X, 0)$ for all $x \in X$ close to 0.

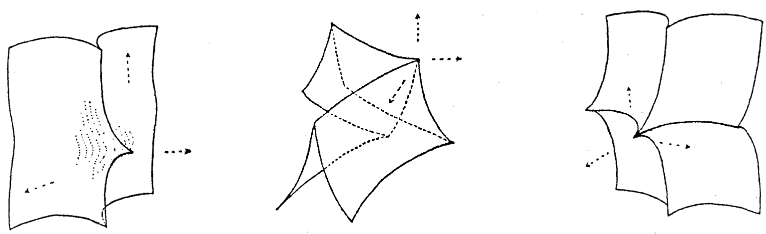
We then have the

Theorem. (cf. [G-H]). *Let $(X, 0)$ and $(Y, 0)$ be two hypersurfaces in $(\boldsymbol{C}^n, 0)$. Then $(X, 0) \simeq (Y, 0)$ if and only if* $\mathrm{Sing}^*(X, 0) \simeq \mathrm{Sing}^*(Y, 0)$ *for arbitrary $(X, 0)$ and $(Y, 0)$ and $(X, 0) \simeq (Y, 0)$ if and only if* $\mathrm{Sing}(X, 0) \simeq \mathrm{Sing}(Y, 0)$ *for $(X, 0)$ and $(Y, 0)$ of isolated singularity type.*

Examples of I.S.T. (1) Isolated singularities.

(2) The Whitney umbrella, the product of two cusps, or more generally, varieties whose singular subspace $\mathrm{Sing}(X, 0)$ has an isolated singularity at 0.

(3) Hypersurfaces given by homogeneous equations. It is an open question whether all quasihomogeneous singularities are I.S.T. Problem: Describe all singularities which are not of isolated singularity type.



Outline of proof. If $\mathrm{Sing}(X, 0) \simeq \mathrm{Sing}(Y, 0)$, a change of coordinates in $(\boldsymbol{C}^n, 0)$ allows us to assume $\mathrm{Sing}(X, 0) = \mathrm{Sing}(Y, 0)$. Define a deformation $(X_t, 0)$ from $(X, 0)$ to $(Y, 0)$ via $f_t = (1-t) \cdot f + t \cdot g$, $t \in \boldsymbol{C}$, for equations f and g of $(X, 0)$ and $(Y, 0)$. It can be shown that $\mathrm{Sing}(X_t, 0) = \mathrm{Sing}(X_0, 0)$ for all $t \in \boldsymbol{C}$ except a finite number. Let $T \subset \boldsymbol{C}$ be the complement of those. Then T is connected (this fails over $\boldsymbol{R}$ or char $p > 0$) and therefore the assertion will follow if we show the triviality of the local family $((X_t, 0)_{t \in (T, t_0)})$ for any point t_0 in T. Using a standard result of deformation theory, the triviality of this family is equivalent to saying that $\partial_t f_t \in (f_t) + m \cdot j(f_t)$, the membership depending analytically on t. This holds in our case if we start off with $\mathrm{Sing}^*(X, 0)$, thus proving the first part of the theorem. For $\mathrm{Sing}(X, 0)$, the constancy of $\mathrm{Sing}(X_t, 0)$ only implies that $\partial_t f_t \in (f_t) + j(f_t)$, which allows the trivializing isomorphism ϕ_t to remove the origin, thus proving $(X_{t_0}, 0) \simeq (X_t, \phi_t(0))$ instead of $(X_{t_0}, 0) \simeq (X_t, 0)$. Now, if $(X, 0)$ and $(Y, 0)$ are supposed to be of isolated singularity type, it follows that the fibers $(X_t, 0)$ are of isolated singularity type too. Using that $(X_{t_0}, 0) \simeq (X_t, \phi_t(0))$ implies $\mathrm{Sing}(X_t, 0) \simeq \mathrm{Sing}(X_t, \phi_t(0))$ for all $t \in (T, t_0)$ and that $\phi_t(0) \to 0$ for $t \to t_0$ one can then show that ϕ_t must fix the origin, $\phi_t(0) = 0$, and therefore $(X_{t_0}, 0) \simeq (X_t, 0)$ for all $t_0 \in T$ and $t \in (T, t_0)$, whence the assertion.

Remark. The basis of the proof is to construct the right deformation $(X_t, 0)$ joining $(X, 0)$ and $(Y, 0)$. Choosing an arbitrary trivial deformation $(Z_t, 0)$ joining $\mathrm{Sing}(X, 0)$ and $\mathrm{Sing}(Y, 0)$ is not sufficient, as one has to find a family of varieties $(X_t, 0)$ such that $\mathrm{Sing}(X_t, 0) = (Z_t, 0)$ and joining $(X, 0)$ and $(Y, 0)$. This problem is a very special case of question (2) of the beginning of this section.

§ 3. The non hypersurface case

There is a certain ambiguity on what should be thought of as the singular subspace of a variety $(X, 0)$ in $(\boldsymbol{C}^n, 0)$ defined by p functions $f_1, \cdots, f_p$. Set-theoretically, it is given for reduced $(X, 0)$ as the set of points where the jacobian matrix $\partial f = (\partial_i f_j)$ of the vector $f = (f_1, \cdots, f_p) \in \mathcal{O}_n^p$ does not have maximal rank k. One possibility therefore is the analytic structure $Z(X, 0)$ defined by the ideal of $\mathcal{O}_n$ which is generated by the f_j's and all $k \times k$-minors of ∂f, say $I + J_k(f)$. We have the following result of A. Dimca:

Theorem ([D 1]). *The analytic type of any zero-dimensional or homogeneous complete intersection* $(X, 0)$ *is determined by its singular subspace* $Z(X, 0)$.

The proof consists in constructing explicitly the isomorphism between two varieties $(X, 0)$ and $(Y, 0)$ with $Z(X, 0) \simeq Z(Y, 0)$. One might wonder whether the proof of the theorem of the last section could not be used to prove the above result in more generality. The serious problem one encounters is to prove for the constructed deformation $(X_t, 0)$ that $Z(X_t, 0)$ is a trivial family (cf. the proof of the analogous statement for functions and A-equivalence, [Ha 2]). However, one can prove a local version:

Theorem. *Let* $((X_t, 0)_{t \in (T, 0)})$ *be an analytic family of arbitrary varieties in* $(\boldsymbol{C}^n, 0)$ *and* $Z^*(X_t, 0)$ *be defined by the ideal* $I_t + m \cdot J_k(f_t)$ *(with the obvious notation). Then the family* $((X_t, 0)_{t \in (T, 0)})$ *is trivial if and only if* $(Z^*(X_t, 0)_{t \in (T, 0)})$ *is trivial.*

We would like to mention also a related result of K. Wirthmüller [Wi] saying that any complete intersection with isolated singularity is determined up to analytic isomorphism by the discriminant of its semi-universal deformation (cf. also [Ga]). One has to exclude the hypersurface case, where a slightly different characterization holds.

There is a second possibility of defining the singular subspace of an arbitrary variety. For this purpose we shall introduce the concept of a generalized analytic variety:

§ 4. The category G of generalized analytic varieties

We propose an extension of the category of germs of analytic varieties to germs of varieties whose analytic structure is given by an analytic module instead of a local ring. As a justification we mention that many analytic sets are defined by rank conditions on some matrices (e.g. singular or critical loci, direct images) and usually provided with the analytic structure of the corresponding Fitting ideal. We shall take instead the module generated by the rows (or columns) of the matrix, thus preserving more information by an object easier to work with. The concept we suggest here is a slightly modified version of the outline given in [G-H]. We hope to provide soon a detailed description and study of generalized analytic varieties. Here we shall just present the basic definitions, which will enable us to formulate the extension of the theorem of the second section.

Definition. Set $A=\mathcal{O}_n$. For a finitely generated A-module M we define the core $\bar{M}$ of M as the A-module $\bar{M}$ for which there exists a free A-module L of maximal dimension such that M is isomorphic to $\bar{M}\oplus L$. The ring A being local, the core $\bar{M}$ of M is well defined, i.e. unique up to A-isomorphism ([Sw], Prop. 11.7.). We say M is bare, if it equals its score, say, has no free factors.

A basic analytic module is the equivalence class of a finitely generated A-module, where two modules are equivalent if their cores are isomorphic as A-modules. Any basic analytic module can be represented by a bare module. This will be done tacitely in the sequel.

Let $M=M_1\times\cdots\times M_s$ be a cartesian product of basic analytic modules with presentations $A^{p_i}\xrightarrow{F_i}A^{m_i}\to M_i\to 0$. We may assume M_i bare and $p_i=p$, $m_i=m$ for all i. We say M is retracted, if no F_i can be written

$$F_i=P\circ F_j\circ Q$$

for some $j\neq i$ and homomorphisms $P\colon A^m\to A^m$ and $Q\colon A^p\to A^p$. Roughly speaking, M is retracted if for all i and j no M_j is a quotient module of M_i. A cartesian product $N=N_1\times\cdots\times N_t$ of basic analytic modules is a retraction of M, if N is retracted and for presentations $A^p \xrightarrow{G_j} A^m \to N_j \to 0$ (N_j bare) there exist homomorphisms

$$P_i,\ R_j\colon A^m \longrightarrow A^m \quad \text{and} \quad Q_i,\ S_j\colon A^p \longrightarrow A^p \quad (\text{all } i \text{ and } j)$$

such that

$$F_i = P_i \circ G_{j_i} \circ Q_i \quad \text{and} \quad G_j = R_j \circ F_{i_j} \circ S_j \quad \text{for some } j_i \text{ and } i_j.$$

We call N the retraction of M.

An analytic module is the equivalence class of a cartesian product $M = M_1 \times \cdots \times M_s$ of basic analytic modules, where $M \sim M'$ if their retractions coincide. It is thus the equivalence class of a couple consisting of a module M and a product decomposition of it.

In particular we have:

$$\begin{aligned} &A^m/I \sim A^m \times A^m/I \times I \simeq (A^m/I)^2, \\ &A^m \times A^m/I \times J \sim A^m/I && \text{if } J \subset I, \\ &A^m \times A^m/I \times J \nsim A^m/I + J && \text{in general}, \\ &A^m \times A^m/I \times J \nsim A^m \times A^m \times A^m/I \times J \times (I+J). \end{aligned}$$

A morphism $f\colon M \to M'$ of analytic modules is the equivalence class of an A-linear map $\tilde{f}\colon \tilde{M} \to \tilde{M}'$ of A-modules representing M and M'. Note that this passage to the equivalence class of a map is not functorial w.r.t. composition.

Definition. A basic generalized analytic variety is a couple $(X, 0) = ((X, 0), \mathcal{O}_{X,0})$ where $\mathcal{O}_{X,0}$ is a basic analytic module over $\mathcal{O}_n$ and $(X, 0) = ((X_k, 0)_{k \in N})$ is a collection of subsets of $(\boldsymbol{C}^n, 0)$ defined by

$$(X_k, 0) = \{x \in (\boldsymbol{C}^n, 0),\ \operatorname{rk} F(x) \leq m-k\} = \{x \in (\boldsymbol{C}^n, 0),\ \textstyle\bigwedge^{m-k+1} F(x) = 0\}$$

for some presentation F of a representative $M = \mathcal{O}_n^m/I$ of $\mathcal{O}_{X,0}$:

$$\mathcal{O}_n^p \xrightarrow{F} \mathcal{O}_n^m \longrightarrow \mathcal{O}_n^m/I \longrightarrow 0.$$

The sets $(X_k, 0)$ do not depend on the choice of the presentation F and the representative M of $\mathcal{O}_{X,0}$. We have the filtration:

$$(\boldsymbol{C}^n, 0) = (X_0, 0) \supset (X_1, 0) \supset \cdots \supset (X_m, 0) \supset \emptyset = (X_{m+i}, 0).$$

A generalized analytic variety is a finite intersection of simple ones, more precisely, a couple $(X, 0) = ((X, 0), \mathcal{O}_{X,0})$ where $\mathcal{O}_{X,0} = \mathcal{O}_{X(1),0} \times \cdots \times \mathcal{O}_{X(s),0}$ is an analytic module over $\mathcal{O}_n$, $(X(i), 0)$ basic, and $(X, 0)$ the collection of all possible intersections of the underlying sets of the $(X(i), 0)$.

A morphism h: $(X, 0) \to (Y, 0)$ of generalized analytic varieties is a composed map $h = L \circ \phi^* : \mathcal{O}_{Y,0} \to \mathcal{O}_{X,0}$ where ϕ: $(\boldsymbol{C}^n, 0) \to (\boldsymbol{C}^n, 0)$ is an analytic map-germ and L: $\phi^*(\mathcal{O}_{Y,0}) \to \mathcal{O}_{X,0}$ a homomorphism of analytic modules. Note that if L is injective, h induces well defined maps on the collection of underlying sets.

A generalized analytic variety $(Y, 0)$ is a subvariety of $(X, 0)$, if for suitable choices of representatives, $\mathcal{O}_{Y,0}$ is a quotient module of $\mathcal{O}_{X,0}$.

We now come back to the notion of the singular subspace of a classical analytic variety:

Definition. Let $(X, 0) \subset (\boldsymbol{C}^n, 0)$ be an analytic variety defined by p functions $f_1, \cdots, f_p$ in $\mathcal{O}_n$ and let $\mathcal{O}_{X,0} = \mathcal{O}_n/I$. Let $f = (f_1, \cdots, f_p) \in \mathcal{O}_n^p$ denote the vector of components f_i and $j(f) \subset \mathcal{O}_n^p$ the submodule generated by its partial derivatives $\partial_1 f, \cdots, \partial_n f \in \mathcal{O}_n^p$. We define Sing $(X, 0)$ (resp. Sing* $(X, 0)$) as the generalized analytic variety of analytic module $\mathcal{O}_{\mathrm{Sing}(X,0)} = \mathcal{O}_n^p / I \cdot \mathcal{O}_n^p + j(f)$ (resp. $\mathcal{O}_{\mathrm{Sing}^*(X,0)} = \mathcal{O}_n^p / I \cdot \mathcal{O}_n^p + m \cdot j(f)$. Then we have

$$\mathrm{Sing}\,(X, 0) \subset \mathrm{Sing}^*\,(X, 0) \subset (X, 0).$$

Theorem. ([G-H]). *Let $(X, 0)$ and $(Y, 0)$ be (classical) analytic varieties in $(\boldsymbol{C}^n, 0)$. Then $(X, 0) \simeq (Y, 0)$ if and only if* Sing* $(X, 0) \simeq$ Sing* $(Y, 0)$ *in general and $(X, 0) \simeq (Y, 0)$ if and only if* Sing $(X, 0) \simeq$ Sing $(Y, 0)$ *for $(X, 0)$ and $(Y, 0)$ of isolated singularity type.*

The proof is a straightforward extension of the corresponding result for hypersurfaces.

Certainly we could have just considered modules and omitted the construction of generalized analytic varieties. We want to conclude by a simple example visualizing aside of the above theorem that generalized analytic varieties can be a useful tool in the study of singularities:

Example. Let $(X, 0) \subset (\boldsymbol{C}^4, 0)$ be the hypersurface of equation $x^2 y - z^2 w = 0$, i.e., the variety whose hyperplane sections with $y = \mathrm{const} \neq 0$ and $w = \mathrm{const} \neq 0$ are Whitney umbrellas. Then Sing $(X, 0)$ is defined by the ideal $(x^2, xy, z^2, zw) \subset \mathcal{O}_4$ and we can consider the generalized analytic variety $(Z, 0) = \mathrm{Sing}\,(\mathrm{Sing}\,(X, 0))$, the singular subspace of the singular subspace of $(X, 0)$. As the collection of underlying sets we obtain:

$(Z_3, 0) = (\boldsymbol{C}^4, 0)$ $(Z_2, 0) = yw$-plane $(Z_1, 0) = (y\text{-axis}) \cup (w\text{-axis})$
$(Z_0, 0) = 0,$

filtration which induces the Whitney stratification of $(X, 0)$. Note that, by the theorem, $(Z, 0)$ determines $\operatorname{Sing}(X, 0)$ and hence $(X, 0)$.

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