Advanced Studies in Pure Mathematics 8, 1986
Complex Analytic Singularities
pp. 101–124

On the Homotopy Theory of Arrangements

Michael Falk* and Richard Randell**

In this paper an arrangement $\mathscr{A}$ is a finite collection of hyperplanes $\{H_1, \cdots, H_n\}$ through the origin in $\boldsymbol{C}^l$. We wish to examine the complementary space $M = \boldsymbol{C}^l - \bigcup_{i=1}^n H_i$ from a topological point of view. More specifically, we will discuss the homotopy properties of M, and how these properties relate to various other well-known properties of arrangements. As a focal point we will consider the question:

Precisely when is M a $K(\pi, 1)$ space?

Arrangements arise in many contexts. For example, one may refer to papers by Orlik, Sommese, and Terao in this volume. The question of when M is a $K(\pi, 1)$ was first considered by Fadell and Neuwirth [6], who gave an affirmative answer for the arrangements of type A_k (see (2.1) for definitions). Such questions burst upon the singularities scene with the work of Arnol'd and Brieskorn reported on in [3], and the lovely result of Deligne [5] that real simplicial (hence real reflection) arrangements (see (2.4)) yield $K(\pi, 1)$ spaces. In the time since that work a number of other properties of arrangements have been defined, some with the $K(\pi, 1)$ property in mind, and some in other contexts. We intend here to mention those properties which seem relevant and to try to sort out their interrelationships.

Since many of our readers will be familiar with most of these properties, we will defer precise definitions and examples until Section 2. We start in Section 1 with a broad overview of the field. Then after giving the relevant definitions we will consider each possible implication in a systematic fashion in Section 3. The section may be treated as a reference section, though it begins with a discussion of some major positive results and their proofs. Where counterexamples are required, we have tried to manage with as few as possible. All this information is assembled in a chart at the end of Section 2. A quick glance at this chart shows quite a number of question marks. In the final section we construct a commuta-

Received January 1, 1985.
Revised February 14, 1986.

* This paper is an expanded version of part of the first author's thesis [7], and is partly based on the talk given by the second author at the U.S.-Japan Seminar.

** Supported by NSF.

tive diagram to help give some order to these questions, and we formulate several optimistic conjectures.

A few comments on notation are in order. We will consistently abuse notation by referring to the "fundamental group of the arrangement" rather than the "fundamental group of the complement of the arrangement" and so forth (unless of course there is danger of confusion). Also, when we have occasion to refer to specific examples, we will attempt to let the context, not the notation, make it clear precisely which arrangement we mean.

We are pleased to thank those with whom we have had discussion concerning this material, particularly Mutsuo Oka, Peter Orlik, Andrew Sommese, and Hiroaki Terao. Our thanks also go to the organizers of the conference, Tatsuo Suwa and Kyoji Saito in Japan, and Phil Wagreich and Peter Orlik, in the United States, for the opportunity they helped provide to meet and exchange ideas.

§ 1. Survey

We consider homotopy properties of the space $M=\boldsymbol{C}^l-\bigcup_{i=1}^n H_i$, where H_i is a complex linear subspace of complex codimension one. Thus M is the complement of a real codimension two embedding, and has a rich fundamental group structure. One may see that

(a) $H_1(M;\boldsymbol{Z})\cong\boldsymbol{Z}^n$, with generators loops meridional to the individual H_i.

(b) $\pi_1(M)\cong\pi_1(M\cap P)$, where P is a generic 3-space through the origin in $\boldsymbol{C}^l$ [12].

(c) $\pi_1(M\cap P)$ may be computed for a given arrangement with the standard "pencil" technique [4].

(d) For certain arrangements $\pi_1(M)$ may be written down in terms of generators and relations. For example, in [23] as corrected, a presentation is given for the complexification of a real arrangement.

For the higher homotopy groups much less is known. One does know certain conditions for which M is not a $K(\pi, 1)$. For example, Hattori [13] observed that the arrangement of 4 planes in $\boldsymbol{C}^3$ in general position does not yield a $K(\pi, 1)$ since in this case $\pi_2(M)$ is the free $\boldsymbol{Z}[\pi_1(M^*)]$-module of rank 1 ($M^*=M/\boldsymbol{C}^*$, $\pi_1(M^*)\cong\boldsymbol{Z}^3$). More generally (see Theorem 3.2), if $\pi_1(M)$ contains a subgroup isomorphic to $\boldsymbol{Z}^4$, then M cannot be a $K(\pi, 1)$ (for $M\subset\boldsymbol{C}^3$).

There are, however, a number of conditions under which M is a $K(\pi, 1)$. This happens, for example, if the arrangement comes from a real reflection group (2.3), or is the complexification of a real simplicial arrangement (2.4). Also, if M is the total space of a fiber bundle in which the base and fiber are $K(\pi, 1)$ spaces, then the long exact homotopy sequence of the bundle shows that M is a $K(\pi, 1)$ also. This property holds for a number of arrangements, including the type A_k considered by Fadell-Neuwirth. Recently, Jambu and Terao [15] have introduced the property of supersolvability (2.7.6) of an arrangement. It turns out (2.7.7) that a supersolvable arrangement has the fibering property mentioned above, so that supersolvable arrangements are $K(\pi, 1)$ arrangements.

On the other hand, there are several properties which seem to have something to do with the homotopy of M, but do not (or are not known to) imply that M is a $K(\pi, 1)$. Several of these properties are combinatorial in nature. That is, they depend only on the pattern of intersection of the hyperplanes, i.e., on the lattice associated to the arrangement (see 2.3). Some such properties, such as supersolvability, do imply M a $K(\pi, 1)$, while others, such as formality (2.7.11) do not. Since it is not known to us whether $K(\pi, 1)$-ness is combinatorial, the effects of these properties are of interest.

We should mention also two particular problems of special concern, since they have actually been stated in print.

(a) Is every free arrangement a $K(\pi, 1)$?

(b) Is every complex reflection arrangement a $K(\pi, 1)$?

The first of these has been suggested by K. Saito [26, p. 295], the second by P. Orlik and L. Solomon [21]. (Actually, they ask if such arrangements are of fiber-type.) Since complex reflection arrangements are free [27], an answer of "yes" to (a) implies a similar answer to (b). The answer to (b) is "yes" if the arrangement is associated to the symmetry group of a complex polytope [Orlik and Solomon, private communication].

Thus there are several conditions under which an arrangement is a $K(\pi, 1)$. If we assume that M is a $K(\pi, 1)$, however, much less can be said. The only positive result we know of is that a $K(\pi, 1)$ arrangement must be "formal." (The precise definition of formal will be given later (2.7.11).) For now, suffice it to say that formal means "as much general position as possible, given the intersection pattern in codimension one and two."

In summary, we have several conditions which force M to be a $K(\pi, 1)$. All but one of these (supersolvability) are clearly more geometrical than combinatorial. Also, examples are known which are $K(\pi, 1)$ but not supersolvable.

On the other hand we have a combinatorial condition (formal) which is necessary for M to be a $K(\pi, 1)$. What seems still possible is the existence of a fairly weak topological condition $(*)$ so that M is a $K(\pi, 1)$ if and only if M is formal and $(*)$.

§ 2. Definitions and Examples

Now we take up the task of presenting precise definitions of the relevant properties and of presenting several classes of examples. We first consider

(2.1) **Real reflection arrangements**

Let W be a finite irreducible group generated by reflections in $\boldsymbol{R}^l$. Then W operates also on $\boldsymbol{C}^l$, and we have the arrangement $\mathscr{A}$ of reflecting hyperplanes. The list [2] of such groups is comprised of

$$A_l,\ C_l,\ D_l,\ E_6,\ E_7,\ E_8,\ F_4,\ G_2,\ H_3,\ H_4,\ I_2(p).$$

We consider two such classes in more detail. For A_l, we have $M\times \boldsymbol{C}\cong Z_{l+1}$, where

$$Z_{l+1}=\{(z_1,\cdots,z_{l+1})\in \boldsymbol{C}^{l+1}\,|\,z_i\neq z_j, \text{ for all } i\neq j\}.$$

Thus the number n of hyperplanes is given by $n=(l+1)l/2$. Notice that projection onto the first l coordinates gives a fiber bundle $Z_{l+1}\to Z_l$, with fiber $\boldsymbol{C}-\{z_1,\cdots,z_l\}$, so that M is a $K(\pi, 1)$.

For D_l, we have $M=\{(y_1,\cdots,y_l)\in \boldsymbol{C}^l\,|\,y_i\neq \pm y_j, \text{ for all } i\neq j\}$. Following [3], set $Z=\{(z_1,\cdots,z_{l-1})\in \boldsymbol{C}^{l-1}\,|\,z_i\neq 0 \text{ and } z_i\neq z_j \text{ for all } i\neq j\}$. Then Z is a $K(\pi, 1)$ by projection as above, while setting $z_i=y_l^2-y_i^2$ gives a fiber bundle $M\to Z$, showing that M is a $K(\pi, 1)$. Notice that this is not a projection or even linear mapping.

(2.2) **Pictures and real arrangements**

For certain arrangements it is easy to draw a very nice picture. These are the real arrangements in $\boldsymbol{C}^3$. Suppose the hyperplanes H_i of an arrangement are defined by linear forms $f_i(z_1,\cdots,z_l)=0$. We say the arrangement is *real* if all the coefficients of all the f_i may be taken as real, not just complex, numbers. The real reflection arrangements are examples.

Suppose then that we have a real arrangement in $\boldsymbol{C}^3$, and let us use (x, y, z) as coordinates. Then we may linearly change coordinates so that $f_1(x, y, z)=z$, and for all other f_i, we may set $z=1$ to obtain linear equations which we may graph on an xy coordinate system. Thus, if $f_1=z$, $f_2=x, f_3=y$, we would draw the picture

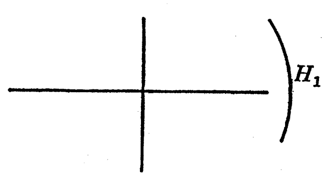


Figure (2.2.1)

Of course, we are really drawing the real part of the projectivized arrangement in $\boldsymbol{CP}^2 = \boldsymbol{C}^3 - \{0\}/\boldsymbol{C} - \{0\}$. Thus, for D_3, which has 6 lines, we have

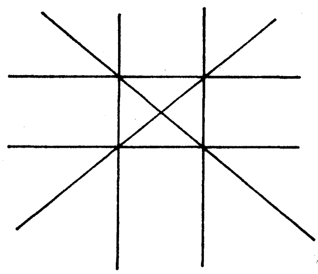


Figure (2.2.2)

If we wish a picture with the line at infinity as one line of our arrangement we may take a coordinate change

$$z' = y + z$$
$$y' = y$$
$$x' = x$$

which yields the 6 lines

$$x' = \pm y'$$
$$x' = \pm (z' - y')$$
$$y' = \pm (z' - y'), \quad \text{or } 2y' = z' \text{ and } 0 = z'.$$

This gives

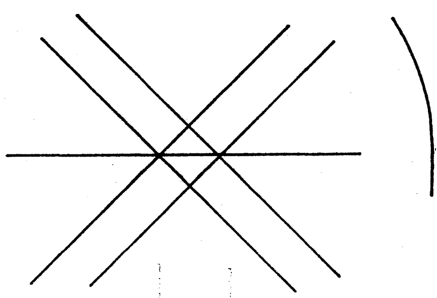


Figure (2.2.3)

(2.3) The lattice and cohomology of an arrangement

Following Orlik-Solomon [20], we define the *lattice* $L = L(\mathscr{A})$ of an arrangement. The set $L(\mathscr{A})$ is the set of nonempty intersections of subsets of $\mathscr{A}$, partially ordered by reverse inclusion. The Möbius function of L

is defined recursively by $\mu(C^l)=1$, and $\mu(X)=-\sum_{\substack{Y\in L\\ Y<X}}\mu(Y)$. For any $X\in L$, let the rank of X, $r(X)$, be defined by $r(X)=l-\dim_C X$. Finally, let the Poincaré polynomial of M be defined by

$$P_M(t)=\sum_{i=0}^{\infty} b_i(M)t^i,$$

where $b_i(M)$ is the i-th betti number, $b_i(M)=\dim H^i(M;C)$. Then [20] one has

$$P_M(t)=\sum_{Y\in L}[\mu(Y)(-t)^{r(Y)}].$$

Thus, for D_3 one has the lattice

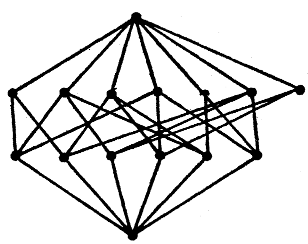


Figure (2.3.1)

and $P_M(t)=1+6t+11t^2+6t^3=(1+t)(1+2t)(1+3t)$.

This is a very special case of a result of Brieskorn [3]: For real reflection arrangements, $P_M(t)$ factors into a product of terms $(1+m_it)$, where the m_i are the exponents of the corresponding reflection group.

(2.4) Simplicial arrangements

Let $\mathscr{A}$ be a real arrangement. Then we consider the intersection of the real hyperplanes in $\boldsymbol{R}^l$ with the unit $l-1$ sphere S^{l-1}. If the resulting subdivision is simplicial, i.e., yields in the obvious way the structure of a simplicial complex on S^{l-1}, we say the arrangement is a *simplicial* one. Thus D_3 is simplicial (interpret the second picture as a picture on S^2), but the arrangement below is not:

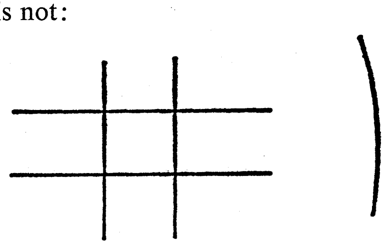


Figure (2.4.1)

This notion is relevant because of Deligne's result [5] that the complexification of a real simplicial arrangement yields a $K(\pi, 1)$ space, together with the fact [2] that real reflection arrangements are simplicial.

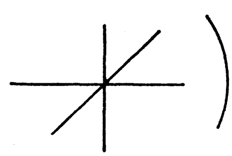
(2.5) Unitary reflection arrangements

Another natural class to consider is that of the unitary reflection arrangements, defined analogously to the real reflection arrangements. The list here (see [24]) includes the real reflection groups, certain monomial groups $G(p, q, r)$, and a list of exceptional groups G_i.

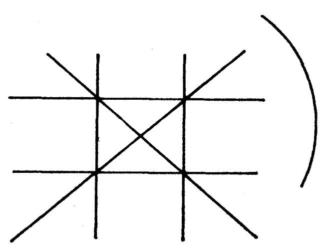
(2.6) Certain exceptional arrangements

In this section we will define an interesting infinite collection of arrangements J_k, $k \geq 1$; along with certain other arrangements which will be useful as counterexamples.

Consider the family of all nonsingular Fermat curves $ax^k + by^k + cz^k = 0$, subject to $a+b+c=0$. A routine computation shows that nonsingularity holds unless $(a:b:c) \in \boldsymbol{P}^2$ is one of $(1:0:-1)$, $(0:1:-1)$, or $(1:-1:0)$. Further, each $(x:y:z) \in \boldsymbol{P}^2$ not lying on a singular curve is on exactly one nonsingular curve. Thus we have a well-defined mapping $p: \boldsymbol{P}^2 - \bigcup H_i \to \boldsymbol{P}^1$ defined by $p(x:y:z) = (a:b:c)$, where the H_i are the $3k+1$ projective lines $z=0$, $x^k = y^k$, $y^k = z^k$, $x^k = z^k$. Let J_k be the corresponding arrangement of $(3k+1)$ lines in $\boldsymbol{C}^3$. Then J_1 is



and J_2 is



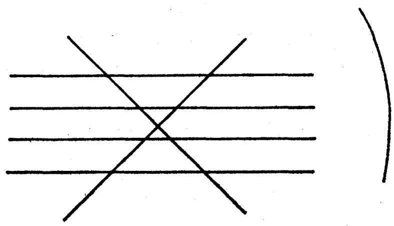
If $k > 2$, the arrangement is not real. Using the map p above, one may show that $M^* = M/\boldsymbol{C}^*$ fibers over a three-times punctured 2-sphere, with fiber a surface of genus $(k-1)(k-2)/2$ with $k^2 + k$ punctures. Thus M_k is a $K(\pi, 1)$ space. One may compute

$$
\begin{aligned}
P_{M_k}(t) &= 1+(3k+1)t+(3k^2+k+1)t^2+(3k^2-2k+1)t^3 \\
&= (1+t)(1+3kt+(3k^2-2k+1)t^2)
\end{aligned}
$$

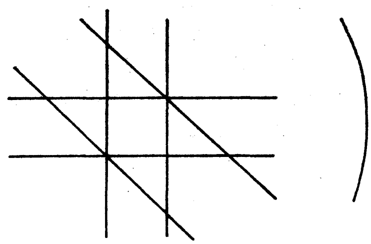
so that $P_{M_k}(t)$ factors over the integers if and only if $k=1$ or $k=2$.

Finally, we list some particular real arrangements with interesting properties.

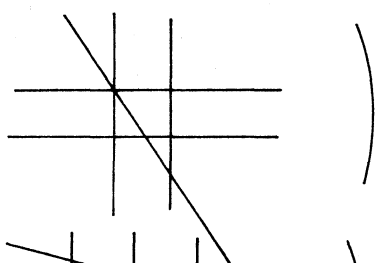
X_1:



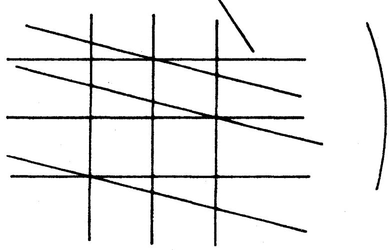
X_2:



X_3:



X_4:



(2.7) **Definitions**

For the convenience of the reader we gather a number of definitions in this section. Several new properties (fibered, fiber-type, LCS, formal, and simple) are defined.

Definition (2.7.1). An arrangement $\mathscr{A}$ is *fibered* if M is the total space of a fiber bundle $F \to M \to M'$, where F is a punctured surface, and M' is a $K(\pi, 1)$ space.

Definition (2.7.2). An arrangement $\mathscr{A}$ *factors* if $P_M(t)$ factors into linear terms over $\boldsymbol{Z}[t]$.

In [28], Terao showed that a "free" arrangement factors. To define freeness, we think of $D = \bigcup H_i$ as a divisor in $\boldsymbol{C}^l$, and let Q be a local defining equation at 0. Define $\Omega^q(\log D) = \{$rational q-forms $w \mid Qw$ and Qdw are regular$\}$. The divisor D is *free* if $\Omega^1(\log D)$ is a free $\mathcal{O}_{C^l,0}$ module.

Definition (2.7.3). An arrangement $\mathscr{A}$ is *free* if and only if the associated divisor D is free.

This concept was introduced by K. Saito, who conjectured that an arrangement is free if and only if M is a $K(\pi, 1)$-space. Terao [26] found examples for which M is a $K(\pi, 1)$ but which are not free. The other half of the problem is open, as far as we know.

Next we formalize terminology we have been using throughout.

Definition (2.7.4). An arrangement $\mathscr{A}$ is a $K(\pi, 1)$-*arrangement* if and only if M is a $K(\pi, 1)$ space.

The following three properties are closely connected. We consider them in order of decreasing strength.

Definition (2.7.5). Let L be the lattice associated to an arrangement $\mathscr{A}$. An element $x \in L$ is called a modular element if it forms a modular pair with every $y \in L$; i.e., if $y < z$, then $y \vee (x \wedge z) = (y \vee x) \wedge z$.

Here $a \vee b = a \cap b$, and $a \wedge b = \bigcap V_i$, the intersection over all $V_i \in \mathscr{A}$ such that $a \cup b \subset V_i$.

Definition (2.7.6). $\mathscr{A}$ is *supersolvable* if there exists a maximal modular chain

$$0 = x_0 < x_1 < \cdots < x_l = 1$$

in the lattice; i.e., each x_i is a modular element.

Supersolvable arrangements are free [15]. In [30] H. Terao has shown that an arrangement is supersolvable if and only if it is "fiber-type," which we now define recursively.

Definition (2.7.7). (i) The arrangement $\{0\}$ in $\boldsymbol{C}^1$ is a *fiber-type* arrangement.

(ii) Suppose that, after suitable linear coordinate change, projection to the first $(l-1)$ coordinates is a fiber bundle projection $M \to M'$, where M' is the complement of a fiber-type arrangement in C^{l-1}. Then $\mathscr{A}$ is a *fiber-type arrangement.*

For example, A_l is a fiber-type arrangement for all l.

For the next definition we set $G=\pi_1(M)$, $G_1=G$, and $G_{n+1}=[G_n, G]$, where $[A, B]$ denotes the commutator subgroup of A and B. That is, $[A, B]$ is the subgroup generated by $a^{-1}b^{-1}ab$, $a \in A$, $b \in B$. Then the G_n are just the terms of the lower central series of G. We further set $G(n)=G_n/G_{n+1}$. By [18, Theorem 5.4], $G(n)$ is a finitely generated abelian group. We set $\varphi_n = \operatorname{rank} G(n)$.

Definition (2.7.8). We will say *the lower central series of $\mathscr{A}$ is cohomologically determined* provided that

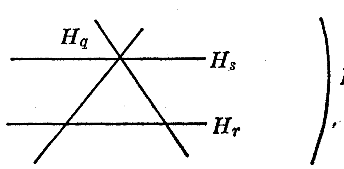
$$\prod_{j=1}^{\infty} (1-t^j)^{-\varphi_j} = \frac{1}{P_M(-t)} \quad (\text{in } Z[[t]]).$$

We will say that $\mathscr{A}$ is LCS for short. Kohno showed this for A_l in [17], and T. Oda [private communication] proved LCS for the C_l arrangements. We have shown in [9] that this holds for all fiber-type arrangements, but not for all $K(\pi, 1)$ arrangements.

The next two definitions concern the rational homotopy theory and minimal model of M. Let $\mathscr{M}$ denote this minimal model and S denote the 1-minimal model [10].

Definition (2.7.9) [16]. M is a *rational* $K(\pi, 1)$ if its minimal model is generated by elements of degree ≤ 1, i.e., if $S=\mathscr{M}$.

Definition (2.7.10). An arrangement $\mathscr{A}$ in C^3 is said to be *parallel* (with respect to $H \in \mathscr{A}$) if for any three H_p, H_q, H_r in general position there is a fourth H_s such that $H_s \cap H_r \subset H$ and $H_s \supset H_p \cap H_q$. Here is a picture:



In [16], Kohno shows that parallel arrangements are rational $K(\pi, 1)$'s.

We close this section of examples and definitions with two new properties designed to be related to the (topological) $K(\pi, 1)$ property.

The first definition ("formal") is motivated by the following observations:

(a) If M is a $K(\pi,1)$, then $\pi=\pi_1(M)$ determines $H^*(M)$.

(b) By a Lefschetz-Zariski theorem [12], $\pi_1(M)\cong\pi_1(Q\cap M)$, where Q is a generic 3-space in $\boldsymbol{C}^l$.

Thus we single out certain arrangements among those which have identical 3-space sections. Loosely speaking, we ask for as much general position as possible, given the intersections in (complex) codimension one and two.

More precisely, $\mathscr{A}=\{H_1, \cdots, H_n\}$, where $H_i\subset \boldsymbol{C}^l$, $H_i=f_i^{-1}(0)$. Let $f_i=a_{i1}z_1+\cdots+a_{il}z_l$. We may set $f_i^*=a_{i1}z_1+\cdots a_{il}z_l+Oz_{l+1}+\cdots+Oz_n$ (assuming $n\geq l$), and obtain an arrangement $\mathscr{A}^*=\{H_1^*, \cdots, H_n^*\}$ in $\boldsymbol{C}^n$, with $L(\mathscr{A})=L(\mathscr{A}^*)$. Now consider a rank 2 element of L $(=L(\mathscr{A}))$. Such an element corresponds to an $(n-2)$ dimensional intersection of 2 or more hyperplanes, say $H_1^*, \cdots, H_j^*$. Thus

$$(*)\qquad f_m^*=b_{1m}f_1^*+b_{2m}f_2^*, \qquad \text{for } 3\leq m\leq j,$$

and we may consider the affine variety $V\subset \boldsymbol{C}^{n^2}$ consisting of those collections of n distinct hyperplanes in $\boldsymbol{C}^n$ satisfying conditions $(*)$ (for all rank 2 elements of the lattice). Since $(*)$ are linear conditions in the coefficients α_{ij} defining the hyperplanes, V is an open subvariety of a linear variety, hence V is irreducible. Also, $\mathscr{A}^*\in V$. Let V_Z be the Zariski open subset of V consisting of all points at which all possible minors of (α_{ij}) are nonzero. Then $\mathscr{A}^*$ may or may not be an element of V_Z. Note that any two points of V_Z have the same lattice L_Z.

Definition (2.7.11). $\mathscr{A}$ is a *formal* arrangement if $L(\mathscr{A})=L_Z$.

The arrangement $x=0$, $y=0$, $z=0$, $x+y+z=0$ in $\boldsymbol{C}^3$ is not formal, while J_2 is formal. Later we will see that $\mathscr{A}$ $K(\pi, 1)$ implies $\mathscr{A}$ formal, but not conversely.

To see that the first example is not formal, note that we have

$$\begin{aligned}
f_1^*&=1x+0y+0z+0w\\
f_2^*&=0x+1y+0z+0w\\
f_3^*&=0x+0y+1z+0w\\
f_4^*&=1x+1y+1z+0w.
\end{aligned}$$

The condition $(*)$ is vacuous in this case, since there are no rank two elements corresponding to intersections of more than two hyperplanes. Thus V_Z consists of all points of V at which all minors of the 4×4 matrix (α_{ij}) are nonzero. Thus the lattice L_Z is the boolean lattice on 4 elements, and in particular has an element of rank 4. Thus $L_Z\neq L(\mathscr{A})$, and the arrangement is not formal.

Next, we will show that J_2 is formal. The hyperplanes of J_2 (in $\boldsymbol{C}^3$) are $f_1=z=0$, $f_2=x+y=0$, $f_3=x-y=0$, $f_4=x+z=0$, $f_5=x-z=0$, $f_6=y+z=0$, $f_7=y-z=0$. There are some rank 2 elements of the lattice corresponding to 3 or more hyperplanes. These are

a) $z=0,\ y\pm z=0$
b) $z=0,\ x\pm z=0$
c) $x=z,\ y=z,\ x=y$
d) $x=z,\ y=-z,\ x=-y$
e) $x=-z,\ y=z,\ x=-y$
f) $x=-z,\ y=-z,\ x=y$.

Notice that these correspond to the six triple points in the picture of J_2 given earlier. The equations $(*)$ become

a) $f_7^*=y-z=(y+z)-2z=f_6^*-2f_1^*$
b) $f_5^*=x-z=(x+z)-2z=f_4^*-2f_1^*$
c) $f_7^*=y-z=-(x-y)+(x-z)=-f_3^*+f_5^*$
d) $f_6^*=y+z=(x+y)-(x-z)=f_2^*-f_5^*$
e) $f_7^*=y-z=(x+y)-(x+z)=f_2^*-f_4^*$
f) $f_6^*=(y+z)=-(x-y)+(x+z)=-f_3^*+f_4^*$.

Since we have seven hyperplanes, we consider $\boldsymbol{C}^7$, with coordinates $(x=w_1, y=w_2, z=w_3, w_4, \cdots, w_7)$. Then points of V consist of seven hyperplanes $g_i^*=\sum_{j=1}^7 \alpha_{ij}w_j$, satisfying $g_7^*=g_6^*-2g_1^*$, etc. We claim $L_Z=L(J_2)$ here. By construction, one always has $L_Z=L(\mathscr{A})$ in ranks less than three. Since $L(J_2)$ has no elements of rank more than three, it suffices to show that L_Z has a single rank three element. This amounts in this case to showing that there are three of the g_i^* which have the property that any g_i^* can be written as a linear combination of them. Referring to the picture of J_2 given before, we start at a triple point, say c), so we write $g_7^*=-g_3^*+g_5^*$. We next find another triple point on $f_5=0$, say the one corresponding to b), so $g_5^*=g_4^*-2g_1^*$. We see thus far that g_4^* and g_3^* are linear combinations of g_1^*, g_5^*, g_7^*. We next look for multiple points involving g_2^* and g_6^* in terms of these and note that the points corresponding to e) and a) respectively will work. Thus each g_i^* is a linear combination of g_1^*, g_5^*, g_7^*, and so L_Z has a unique rank three element and no elements of higher rank, as was to be shown.

Our final definition, that of a "simple" arrangement in $\boldsymbol{C}^3$, is motivated by work of A. Sommese [25, Question 4.2]. Let $\mathscr{A}=\{H_1, \cdots, H_n\}$ be an arrangement in $\boldsymbol{C}^3$, and let $\mathscr{A}_{\boldsymbol{P}}=\{L_1, \cdots, L_n\}$ be the corresponding arrangement in $\boldsymbol{P}^2$. For any point $p \in \boldsymbol{P}^2$, let r_p be the number of lines L_j with $p \in L_j$.

Definition (2.7.12). $\mathscr{A}$ is *simple* if and only if

(i) for every L_j, $|\{p \in L_j \mid r_p \geq 3\}| \geq 2$;
(ii) given p, q with $r_p \geq 3$, $r_q \geq 3$, there exist $L_{j_1}, \cdots, L_{j_t}$ with $p \in$

L_{j_1}, $q \in L_{j_t}$ and $L_{j_i} \cap L_{j_{i+1}} = \{z\}$, with $r_z \geq 3$.

For example, the arrangement X_4 is simple, but not formal or a $K(\pi, 1)$ (see 3.13). Sommese's question is thus whether the Hirzebruch surface $\mathscr{H}(\mathscr{A}, n)$ associated to a simple arrangement is a $K(\pi, 1)$ space.

(2.8) **Table of implications**

The table below gathers what we know about these properties. The rows correspond to hypotheses, the columns to conclusions. The footnotes refer to results we learned of shortly before publication. Section 3 will analyze this table in detail.

	fibered	free	factor	fiber type	SS	LCS	rat'l $K(\pi, 1)$	formal	$K(\pi, 1)$	comb.
R-Refl.	??	T	T	F	F	F′)	F[1]	T	T	NA
Simp.	?	F	F	F	F	F	F	T	T	T[2]
C-Refl.	??	T	T	F	F	F′)	F	?	??	NA
fibered		F	F	F	F	F	F	T	T	?
free	?		T	F	F	F	F	??	??	??
factor	F	F		F	F	F	F	F	F	T
fiber type	T	T	T		T	T	T	T	T	T
super-solvable	T	T	T	T		T	T	T	T	T
lower central series	?	F[*]	F[*]	F[1]	F[1]		F[1]	??	??	T
Rat'l $K(\pi, 1)$	?	F[*]	F[*]	F[*]	F[*]	T[3]		T	??	T
Parallel	?	F	F	F	F	T[*]	T[*]	T[*]	??	T
formal	F	F	F	F	F	F	F		F	?
simple	F	F	F	F	F	F	F	F	F	T
$K(\pi, 1)$	??	F	F	F	F	F	F	T		??

Key: rows are hypotheses
columns are conclusions
T = true
F = false
NA = not applicable
? = not known
?? = not known, of significant interest

*) Depends on the assertion that parallel implies rational $K(\pi, 1)$, proof of which has not appeared.

1) D_4 is not rational $K(\pi, 1)$ [8], but is LCS [T. Kohno, Poincaré series of the Malcev completion of generalized pure braid groups, preprint].

2) C. Toda, as communicated by H. Terao.

3) M. Falk [8] and T. Kohno [Rational $K(\pi, 1)$ arrangements satisfy the LCS formula, preprint].

′) Added in proof: The LCS formula fails for H_3.

§ 3. Remarks and References

In this section we discuss the various assertions in the table, proceeding row by row. We cite references or sketch proofs for the valid implications, and refer to arrangements discussed in Section 2 for specific counterexamples. We also summarize the state of current research into the open questions:

The following results will be useful in identifying counterexamples.

Theorem (3.1). *Suppose the arrangement* $\mathscr{A}$ *contains hyperplanes* H_1, H_2, H_3, *and* H_4 *with the following properties*:

(i) $\bigcap_{i=1}^4 H_i$ *has codimension* 3 *in* $\mathbf{C}^l$, *and*

(ii) *for* $1 \leq i < j \leq 3$, $H_i \cap H_j$ *is contained in no hyperplane of* $\mathscr{A}$ *other than* H_i *and* H_j. *Then* M *is not a rational* $K(\pi, 1)$. □

A proof will appear in [8]. The idea is that $\{H_1, \cdots, H_4\}$ gives rise to a relation in $H^*(M)$ which cannot be exact in the 1-minimal model of M (see § 4).

The following criteria generalize a result of A. Hattori [13].

Theorem (3.2). *Let* $\mathscr{A}$ *be an arrangement in* $\mathbf{C}^l$, *and suppose* $\pi_1(M)$ *contains a subgroup of cohomological dimension greater than* l. *Then* M *is not a* $K(\pi, 1)$.

Proof. The complement M deforms onto an l-complex K. Let $\tilde{K}$ be the covering space of K corresponding to the particular subgroup of $\pi_1(M) \cong \pi_1(K)$ mentioned in the theorem. Then $\tilde{K}$ is also an l-complex so $H^p(\tilde{K}) = 0$ for $p > l$. If $M \simeq K$ is aspherical, $\tilde{K}$ is also, and $H^*(\tilde{K}) \cong H^*(\pi_1(\tilde{K}))$. Then $\pi_1(\tilde{K})$ has cohomological dimension at most l, providing a contradiction.

Corollary (3.3). *Let* $\mathscr{A}$ *be an arrangement in* $\mathbf{C}^3$ *with real defining equations. Suppose* H_1, H_2, H_3, *and* H_4 *are hyperplanes of* $\mathscr{A}$ *which satisfy properties* (i) *and* (ii) *of Theorem* 3.1, *and the real parts of* H_1, H_2, *and* H_3 *bound a connected component of the real part of* M. *Then* M *is not a* $K(\pi, 1)$.

Proof. Using Randell's presentation for $\pi_1(M)$ [23], one detects a free abelian subgroup of rank 4, and Theorem 3.2 applies. □

We now proceed to the table of implications. The paragraph numbers refer to the rows (hypotheses) in the table. A certain amount of crossreferencing is unavoidable.

(3.1) Real reflection arrangements

The properties of real reflection arrangements motivated many of the more well known theorems and conjectures in this table. Arnol'd [1] showed that the arrangements of type A_l factor, and Brieskorn [3] extended this result to all real reflection arrangements. The integer coefficients of the linear factors coincide with the exponents of the corresponding reflection group. Fadell and Neuwirth [6] showed that the arrangements of type A_l are fiber-type, hence have the $K(\pi, 1)$ property. Brieskorn [3] showed that the arrangements of type C_l are also fiber-type, and constructed nonlinear fiberings for types D_l and F_4. He conjectured that all real reflection arrangements are $K(\pi, 1)$. This conjecture is true but it is not known whether the arrangements of types H_3, H_4, E_6, E_7, and E_8 are fibered. The $K(\pi, 1)$ conjecture is confirmed by Deligne's result [5], since all irreducible real reflection arrangements are simplicial [2] (see 3.2). The formality of these arrangements follows (see 3.14). Terao [27] has shown that all these arrangements are free. The LCS and rational $K(\pi, 1)$ properties seem related to fiber-type and $K(\pi, 1)$ questions. But we have not yet been able to prove or disprove these properties for the smallest nontrivial example, D_4. Jambu and Terao [15] remark that this example is not super-solvable, and therefore not fiber-type (see 3.7).

(3.2) Simplicial arrangements

Deligne [5] showed that all simplicial arrangements are $K(\pi, 1)$, but his proof does not involve the fibration property. These arrangements must then be formal (see 3.14). The simplicial arrangement $A_4(13)$ of [11] has Poincaré polynomial $P_M(t)=(1+t)(1+12t+39t^2)$, which does not factor. Hence $A_4(13)$ is not free (see 3.5). The arrangement J_2 of Section 2 (labelled $A_1(7)$ in [11]) provides a counterexample for all the remaining assertions. In [9] we remark that the LCS formula does not hold for this arrangement. This may be verified using the method outlined in Section 4. Thus it cannot be fiber-type or supersolvable (see 3.7 and 3.8). An application of Theorem 3.1 above will show this arrangement is not a rational $K(\pi, 1)$.

(3.3) Unitary reflection arrangements

Orlik and Solomon [22] extended Brieskorn's factorization result to all unitary reflection arrangements, and subsequently Terao [27] showed that these arrangements are indeed free. All of these but a handful of exceptional arrangements are known to be $K(\pi, 1)$. Using Theorem 3.1 and the combinatorial information provided in [19], we can show that the reflection arrangement labelled G_{26} is not a rational $K(\pi, 1)$. It is therefore neither fiber-type nor supersolvable (see 3.7 and 3.8). One expects the

LCS formula to fail for this arrangement; this remains to be seen[*]. This arrangement comes from the symmetry group of a complex polytope, and is therefore $K(\pi, 1)$ by a recent result of Orlik and Solomon [private communication].

(3.4) **Fibered arrangements**

A fibered arrangement is automatically $K(\pi, 1)$, and formal (see 3.14). However, the fibered arrangements J_k of Section 2 provide counter-examples for all the other assertions. If $k \geq 3$, this arrangement does not factor, and is therefore not free (see 3.5), fiber-type, or supersolvable (see 3.7 and 3.8). As mentioned in (3.2), the arrangement J_2 is not a rational $K(\pi, 1)$, and the LCS formula does not hold.

(3.5) **Free arrangements**

Terao introduced free arrangements in [26], and showed that all free arrangements factor [28]. This class includes the reflection arrangements [27], so the $K(\pi, 1)$ conjecture for free arrangements, due to Saito, is a natural extension of Brieskorn's original question. The arrangement J_2 of Section 2 is seen to be free by applying Terao's Addition-Removal Theorem [29]. This arrangement fails to be a rational $K(\pi, 1)$, fiber-type, or supersolvable arrangement, and the LCS formula does not hold (see 3.2). Whether freeness is a combinatorial property or not remains an open problem [29]. Though defined algebraically, there is an inductive technique, the Addition-Removal Theorem [29], which is combinatorial and accounts for many, but not all, of the free arrangements.

(3.6) **Factored arrangements**

On the other hand the more general factorization property is combinatorial, as noted in Section 2. And though many classes of arrangements enjoy this property, it seems to have little significance per se, as the arrangement X_1 of Section 2 demonstrates. This arrangement factors, $P_M(t)=(1+t)(1+3t)^2$, but fails to have any of the other properties. Application of Theorem 3.1 and Corollary 3.3 shows that this arrangement is neither $K(\pi, 1)$ nor rational $K(\pi, 1)$. So X_1 cannot be fibered, or fiber-type, or supersolvable (see 3.8). The LCS formula fails, as may be checked by examining the one-minimal model (see § 4), and one also sees that the definition of formality is not satisfied. Finally, the simplicial arrangement $A_2(18)$ of [11] factors, with $P_M(t)=(1+t)(1+8t)(1+9t)$, but is not free [26].

(3.7) **Fiber-type arrangements**

Fiber-type arrangements are automatically fibered and $K(\pi, 1)$. Formality is a consequence of the $K(\pi, 1)$ property (see 3.14). In [9] we show

[*] Added in proof: The LCS formula fails for G_{26}.

that fiber-type arrangements factor, and that the LCS formula holds. Terao [30] has shown that fiber-type arrangements are super-solvable and therefore also free. The methods of [9] may be employed to show fiber-type arrangements are rational $K(\pi, 1)$ (see [8] for details).

(3.8) **Super-solvable arrangements**

Super-solvable arrangements were shown to be free by Jambu and Terao in [15]. In [30] Terao has shown all super-solvable arrangements are fiber-type. Referring to (3.7), we see that this purely combinatorial condition implies all the properties listed.

(3.9) **LCS arrangements**

Absent some other conditions, specifically fiber-type, the lower central series formula is virtually impossible to verify. This makes counter-examples difficult to identify. It is possible that this formula is implied by the rational $K(\pi, 1)$ property, in which case many of the blanks could be filled in.*) Since the formula depends only on the cohomology and one-minimal model, the LCS property is combinatorial.

(3.10) **Rational $K(\pi, 1)$ arrangements**

The rational $K(\pi, 1)$ property depends on the one-minimal model, which is determined by the cohomology. This follows from Brieskorn's work [3], and is also evident from Orlik and Solomon's presentation of $H^*(M)$ (see § 4). So the rational $K(\pi, 1)$ property is combinatorial. However, its relationship to the topological $K(\pi, 1)$ property remains unclear. The example X_2 of Section 2 is, according to Kohno's assertion, a rational $K(\pi, 1)$ (see 3.11), but does not factor ($P_M(t)=(1+t)(1+6t+10t^2)$). Thus X_2 cannot be free, fiber-type, or supersolvable (see 3.5, 3.7, 3.8). The LCS formula has been checked only to third order for this example, but it is now known to hold for all rational $K(\pi, 1)$ arrangements.') We have been able to show that all rational $K(\pi, 1)$ arrangements are formal—see Section 4.

(3.11) **Parallel arrangements**

This parallel postulate was introduced by T. Kohno in [16], where he claims that all parallel arrangements are rational $K(\pi, 1)$. Formality is then a consequence of Theorem 4.1. The example X_2 of Section 2 belies all the other assertions here—refer to 3.10.

(3.12) **Formal arrangements**

Formality is clearly related to the $K(\pi, 1)$ property, but a glance at the table shows that some additional hypothesis should be added. The

*) Added in proof: This is resolved; see Table 2.8.

') Added in proof: In fact, parallel implies LCS to order 3.

single example X_3 of Section 2 serves as a counterexample for all the assertions here. That X_3 is neither $K(\pi, 1)$ nor rational $K(\pi, 1)$ is seen by applying Theorem 3.1 and Corollary 3.3. The method discussed in Section 4 will show that the LCS formula fails. This arrangement has Poincaré polynomial $P_M(t)=1+6t+12t^2+7t^3$, which does not factor. This encompasses all the remaining assertions. The formality of X_3 must be checked directly. Formality is not a priori a combinatorial property.

(3.13) **Simple arrangements**

Simple arrangements were defined by A. Sommese in [25], and are included here because of the similarity with the definition of formal, and the relation to $K(\pi, 1)$ results. Specifically, Sommese [25] asks if the Hirzebruch surfaces [14] associated to simple arrangements are $K(\pi, 1)$'s. Simple arrangements are not all formal, though, as example X_4 of Section 2 shows. None of the other properties listed here hold for simple arrangements. The example X_4 just cited cannot be rational $K(\pi, 1)$ or $K(\pi, 1)$ since it is not formal. Thus it cannot be fibered, fiber-type, or supersolvable. The example J_2 of Section 2 is simple, but the LCS formula does not hold. And the arrangement X_4 of Section 2 is simple but does not factor ($P_M(t)=(1+t)(1+9t+23t^2)$) and is therefore not free.

(3.14) $K(\pi, 1)$ **arrangements**

Much of the current research is focused on the question of whether the $K(\pi, 1)$ property is combinatorial or not. However, the fact that all $K(\pi, 1)$ arrangements are formal (a proof is given in Section 4) gives a reasonable geometric criterion which is necessary for the $K(\pi, 1)$ property to hold. Formality is not sufficient, though, as remarked in 3.12. Perhaps the most surprising counterexample here is the arrangement J_2 of Section 2, which is $K(\pi, 1)$ (being fibered), but not a rational $K(\pi, 1)$, by Theorem 3.1. The LCS formula also fails for this arrangement. The simplicial arrangement $A_4(13)$ of [11] provides a counterexample for all the other assertions here. Being simplicial, it is a $K(\pi, 1)$ arrangement (see 3.2), but does not factor. Hence $A_4(13)$ is neither free, fiber-type, nor supersolvable.

§ 4. Minimal models, formality, open questions

In this section we discuss some of the newer results in the table and propose some specific conjectures. The basis for these ideas is a commutative diagram $(*)$ which relates the $K(\pi, 1)$ and rational $K(\pi, 1)$ properties to the formality condition introduced in Section 2. We deduce from the diagram that a $K(\pi, 1)$ or rational $K(\pi, 1)$ arrangement is necessarily formal (Theorems 4.1 and 4.2). The diagram suggests several reasonable conjectures consistent with the table, some of which we mention at the end of the section.

The construction of the diagram $(*)$ requires a preliminary discussion of formal arrangements and one-minimal models. This discussion depends on a combinatorial model for $H^*(M)$ discovered by P. Orlik and L. Solomon [20]. Let us describe their result.

Let $\mathscr{A}$ be an arrangement in $\boldsymbol{C}^l$. For each $H \in \mathscr{A}$, let φ_H be a linear form with $H = \ker \varphi_H$. Let $\omega_H = (1/2\pi i)(d\varphi_H/\varphi_H)$, representing the one-dimensional cohomology class meridional to the hyperplane H. Let R be the subcomplex of the De Rham complex $A^*(M)$ generated by $\{\omega_H \mid H \in \mathscr{A}\}$. Brieskorn [3, Lemma 5] showed that the natural map $R \to A^*(M)$ induces an isomorphism on cohomology. Note that the differential d is trivial on R.

Let E be the free exterior algebra with one-dimensional generators $\{e_H \mid H \in \mathscr{A}\}$ corresponding to the hyperplanes in $\mathscr{A}$. For $J \subseteq \mathscr{A}$, $J = \{H_1, \cdots, H_p\}$, we write $e_J = e_{H_1} \wedge \cdots \wedge e_{H_p}$ and

$$\partial e_J = \sum_{i=1}^{p} (-1)^i e_{H_1} \wedge \cdots \wedge \bar{e}_{H_i} \wedge \cdots \wedge e_{H_p},$$

where $-$ denotes deletion. We say J is *dependent* if $p > \operatorname{codim}(\bigcap_{i=1}^{p} H_i)$.

Let $\pi\colon E \to R$ map e_H to ω_H, and let $I = \text{kernel}(\pi)$. It is shown in [20] that I is the ideal of E generated by $\{\partial e_J \mid J \subseteq \mathscr{A}$ is dependent$\}$. Therefore the natural map $E/I \to R \to A^*(M)$ induces an isomorphism on cohomology, where E/I has differential zero. The differential graded algebra E/I is determined combinatorially, and provides a model for $H^*(M)$. These results remain true with rational coefficients.

We now use E/I to construct the one-minimal model $\rho\colon S \to A^*(M)$ for M. By definition [10], S is an increasing union of Hirsch extensions of degree one, and the induced map $\rho^*\colon H^p(S) \to H^p(M)$ is an isomorphism for $p = 1$ and a monomorphism for $p = 2$. Rational coefficients are understood throughout. By the preliminary remarks, we may use for the one-minimal model of $A^*(M)$ the one-minimal model $\rho\colon S \to E/I$ for the algebra E/I. Since E/I has differential zero, $H^p(E/I) = (E/I)^{(p)}$.

The d.g.a. (S, d) and the mapping ρ are constructed inductively as follows:

(i) $S_1 = E$, $d_1 \equiv 0$, and $\rho_1\colon S_1 \to E/I$ is the natural projection;

(ii) for each $n \geq 1$, $S_{n+1} = S_n \otimes \Lambda(V_{n+1})$, where $\Lambda(V_{n+1})$ is the free exterior algebra on the vector space

$$V_{n+1} = \ker(\rho_n^*\colon H^2(S_n) \longrightarrow (E/I)^{(2)})$$

in degree one; $\rho_{n+1}\colon S_{n+1} \to E/I$ is the extension of ρ_n satisfying $\rho_{n+1}|_{V_{n+1}} = 0$; and d_{n+1} is the extension of d_n defined by $d_{n+1}|_{V_{n+1}}\colon V_{n+1} \to S_n^{(2)}$, a linear choice of representative cochains.

Now set $S=\bigcup_{n=1}^{\infty} S_n$, with differential d and $\rho\colon S\to E/I$ determined by d_n and ρ_n. Observe that $\rho_n \circ d_{n+1}|_{V_n}$ is trivial, so ρ_{n+1} is a d.g.a. map. Also, the kernel of $\rho_n^*\colon H^2(S_n)\to(E/I)^{(2)}$ becomes exact in S_{n+1}, hence vanishes in $H^2(S_{n+1})$. And the only closed 1-forms of S are those of S_1, which map isomorphically onto $(E/I)^{(1)}$. Therefore $\rho\colon S\to E/I$ induces an isomorphism $\rho^*\colon H^1(S)\to(E/I)^{(1)}$ and a monomorphism $\rho^*\colon H^2(S)\to(E/I)^{(2)}$.

Note that, because E/I is generated by degree one elements, $\rho^*\colon H^p(S)\to(E/I)^{(p)}$ is surjective for all p. Therefore M is a rational $K(\pi, 1)$ if and only if kernel $(\rho^*\colon H^p(S)\to(E/I)^{(p)})=0$ for all p.

Now, by Sullivan's theory [10], the dimension k_n of V_n is equal to the rank $\varphi_n(M)$ of the n^{th} commutator quotient in the lower central series of $\pi_1(M)$. So S may be used to check the first few terms of the LCS formula

$$\prod_{n=1}^{\infty}(1-t^n)^{-\varphi_n(M)}=\frac{1}{P_M(-t)}$$

for some small arrangements.

From the LCS formula one obtains equations for $\varphi_n(M)$ in terms of the betti numbers b_p of M. The first three are

$$\varphi_1(M)=b_1$$
$$\varphi_2(M)=\binom{b_1}{2}-b_2$$
$$\varphi_3(M)=b_3+\frac{1}{3}b_1(b_1^2-3b_2-1).$$

Clearly the first is always satisfied. We can now see that the second is also satisfied in every case. For the kernel V_2 of $\rho_1^*\colon H^2(E)=E^{(2)}\to(E/I)^{(2)}$ is precisely $I^{(2)}$, with dimension

$$k_2=\dim(E^{(2)}-\dim((E/I)^{(2)}))=\binom{b_1}{2}-b_2.$$

To compute $\varphi_3(M)$, one must examine $\rho_2^*\colon H^2(S_2)\to(E/I)^{(2)}$. Given our knowledge of V_2, it is not hard to show [8] that V_3 is isomorphic to the kernel of the map $D\colon E^{(1)}\otimes I^{(2)}\to E^{(3)}$ defined by $D(\sum e_i\otimes r_j)=\sum e_i r_j$. This gives an algorithm to compute $\varphi_3(M)$ which is manageable for arrangements of seven hyperplanes or less.

We now move on to formal arrangements. Given an arrangement $\mathscr{A}$, there is an associated formal arrangement $\mathscr{A}_F$ whose hyperplanes correspond to those of $\mathscr{A}$ in such a way that the intersection lattices agree through codimension two. If the lattices are isomorphic, then $\mathscr{A}$ itself is formal. The arrangement $\mathscr{A}_F$ satisfies as much general position as possible given the restrictions in codimensions one and two.

Let M_F be the complement of the formal arrangement $\mathscr{A}_F$. Consider the combinatorial model E_F/I_F of $H^*(M_F)$. The natural correspondence of $\mathscr{A}_F$ with $\mathscr{A}$ gives a projection $E_F \to E/I \cong H^*(M)$. The relation ideal I_F for $H^*(M_F)$ is contained in the kernel of this map. For one can show that a dependent subset J of $\mathscr{A}_F$ is necessarily dependent when considered as a subset of $\mathscr{A}$. Thus we obtain a surjective map $H^*(M_F) \to H^*(M)$.

A detailed analysis of the lattices $L_F = L(\mathscr{A}_F)$ and $L = L(\mathscr{A})$ yield the following facts:

(i) there is a surjective, order-preserving, rank-reducing map $L_F \to L$, and

(ii) $\operatorname{rank}(L_F) \geq \operatorname{rank}(L)$, with equality only if L is isomorphic to L_F, i.e., $\mathscr{A}$ is formal.

It follows that the map $H^*(M_F) \to H^*(M)$ is an isomorphism if and only if $\mathscr{A}$ is formal.

Observe that the construction of the one-minimal model $\rho\colon S \to E/I$ depended only on the structure of E/I in degrees one and two. Since $\mathscr{A}_F$ agrees with $\mathscr{A}$ in codimensions one and two, the one-minimal model of M_F is identical to that of M.

Now, let $\pi = \pi_1(M)$. As remarked as Section 2, π is isomorphic to $\pi_1(M_F)$, since M and M_F have identical planar sections. Therefore we have a map $M_F \to K(\pi, 1)$ which induces an isomorphism on cohomology in dimension one and a monomorphism in dimension two. ($K(\pi, 1)$ is obtained from a deformation retract of M_F by attaching cells of dimension ≥ 3.)

We can now display the commutative diagram:

$$(*) \qquad \begin{tikzcd} H^*(S) \arrow[r, "\beta"] \arrow[d, "\gamma"'] \arrow[drr, "\tau" near end] & & H^*(M) \\ H^*(M_F) \arrow[urr, "\sigma" near end] & & H^*(\pi) \arrow[ll, "\delta"] \arrow[u, "\alpha"'] \end{tikzcd}$$

The only arrow as yet undefined is the map $\tau\colon H^*(S) \to H^*(\pi)$. This mapping exists for the following reason: suppose $S_\pi \to A^*(K(\pi, 1))$ is the one-minimal model for $K(\pi, 1)$. Then the composite $H^p(S_\pi) \to H^p(\pi) \to H^p(M_F) \to H^p(M)$ is an isomorphism for $p=1$ and a monomorphism for $p=2$. Thus S_π is also a one-minimal model for M. By uniqueness of one-minimal models [10, Theorem 12.3] there is an isomorphism $S \to S_\pi$ such that

$$\begin{tikzcd} H^*(S) \arrow[r, "\cong"] \arrow[d] & H^*(S_\pi) \arrow[d] \arrow[dl] \\ H^*(M) & H^*(\pi) \arrow[l] \end{tikzcd}$$

commutes.

We make the following observations concerning the diagram $(*)$:

(i) α, β, γ, δ, and σ are surjective,

(ii) β is injective if and only if M is a rational $K(\pi, 1)$,

(iii) σ is injective if and only if $\mathscr{A}$ is formal.

The following result is immediate.

Theorem (4.1). *If M is a rational $K(\pi, 1)$, then $\mathscr{A}$ is formal.*

Proof. If β is an isomorphism, then γ is injective. Then γ is also an isomorphism. Then σ is an isomorphism so $\mathscr{A}$ is formal. □

Only slightly more involved is

Theorem (4.2). *If M is a $K(\pi, 1)$, then $\mathscr{A}$ is formal.*

Proof. If M is a $K(\pi, 1)$, there is a map η: $M_F \to M$ which induces the natural isomorphism $\pi_1(M_F) \to \pi_1(M)$. Then the composite

$$H^*(M_F) \xrightarrow{\sigma} H^*(M) \xrightarrow{\eta^*} H^*(M_F)$$

is an algebra map which is the identity on $H^1(M_F)$. Since $H^1(M_F)$ generates $H^*(M_F)$, $\eta^* \circ \sigma$ is the identity. Then σ is injective, so $\mathscr{A}$ is formal. □

The map η^* coincides with $\delta \circ \alpha^{-1}$ in the diagram $(*)$.

In order to formulate several conjectures we consider the homomorphism τ: $H^*(S) \to H^*(\pi)$ in more detail. In [16], Kohno observes that there are injections $S \xrightarrow{\varphi} \mathscr{M}_\pi \xrightarrow{\psi} \mathscr{M}$, where $\mathscr{M}_\pi$ (resp. $\mathscr{M}$) is the minimal model for $K(\pi, 1)$ (resp. M), and φ induces the map τ. He also observes that $\tau = \varphi^*$ is an isomorphism if and only if $\lim_n H^p(\pi/\pi_n) \cong H^p(\pi)$. Thus we make the following definition:

Definition (4.3). An arrangement $\mathscr{A}$ with $\pi = \pi_1(M)$ is called *quasi-nilpotent* if and only if φ^*: $H^*(S) \to H^*(\pi)$ is an isomorphism.

While we are considering $\pi_1(M)$ and its lower central series we make one additional definition. Let G be a finitely presented group, and let $P_G(t)$ be the Poincaré series of a $K(G, 1)$.

Definition (4.4). The lower central series of G is cohomologically determined if and only if

$$\prod_{j=1}^{\infty}(1-t^{j})^{-\varphi_j}=\frac{1}{P_G(-t)}.$$

We will abbreviate this property as LCS(G). A natural question is: what groups have this property? We have shown in [9] that products of free groups and fundamental groups of fiber-type arrangements have property LCS(G).

We conclude with a number of possibly optimistic conjectures concerning the above properties.

Conjectures (4.5). (i) *If $\mathcal{A}$ is a rational $K(\pi, 1)$ arrangement then $\mathcal{A}$ is a $K(\pi, 1)$ arrangement.*

(ii) *$\mathcal{A}$ is a rational $K(\pi, 1)$ arrangement if and only if $\mathcal{A}$ is* LCS.[*]

(iii) *$\mathcal{A}$ is quasinilpotent if and only if $\pi_1(M)$ is* LCS(π).

(iv) *$\mathcal{A}$ is a rational $K(\pi, 1)$ arrangement if and only if $\mathcal{A}$ is a quasinilpotent $K(\pi, 1)$ arrangement.*

As support for these conjectures, aside from naive optimism, we offer the facts that

(a) They are consistent with our examples.

(b) Kohno [16] proves (i) in the case that $\mathcal{A}$ is an arrangement in $\boldsymbol{C}^3$ and $\pi_2(M)\otimes\boldsymbol{Q}$ is finite dimensional over $\boldsymbol{Q}$. This is not as much evidence as it might seem, however, since $\pi_2(M)$, if not trivial, seems to generally have positive rank over $\boldsymbol{Z}(\pi_1(M))$.

(c) They seem to fit rather well into the diagram ($*$).

References

[1] V. I. Arnold, The cohomology of the colored braid group, Mat. Zametki, **5** (1969), 138–140.

[2] N. Bourbaki, Groupes et algèbres de Lie, Chapters 4–6, Hermann, Paris, 1968.

[3] E. Brieskorn, Sur les groupes de tresses (d'après V.I. Arnold), Séminaire Bourbaki 24e année, 1971–72, Springer Lecture Notes No. **317**, Springer Verlag, Berlin.

[4] D. Chéniot, Le Théorèm de Van Kampen sur le groupe fondamental du complémentaire d'une courbe algébrique projective plane, in Séminaire F. Norguet, 1970–73: Fonctions de Plusiers Variables Complexes, Lecture Notes in Math., no. **409**, Springer-Verlag, Berlin-Heidelberg-New York, 1974, pp. 394–417.

[5] P. Deligne, Les immeubles des groupes de tresses généralisés, Invent. Math., **17** (1972), 273–302.

[6] E. Fadell and L. Neuwirth, Configuration spaces, Math. Scand., **10** (1962), 111–118.

[7] M. Falk, Geometry and topology of hyperplane arrangements, Thesis, Uni-

[*] Added in proof: This is resolved; see 2.8.

versity of Wisconsin, 1983.
[8] ——, The minimal model of the complement of an arrangement of hyperplanes, preprint.
[9] M. Falk and R. Randell, The lower central series of a fiber-type arrangement, Invent. Math., **82** (1985), 77–88.
[10] P. Griffiths and J. Morgan, Rational homotopy theory and differential forms, Birkhäuser, Boston, Basil, Stuttgart, 1981.
[11] B. Grünbaum, Arrangements of hyperplanes, Proc. Second Louisiana Conf. on Combinatories and Graph Theory, Baton Rouge, 1971.
[12] H. Hamm and Lê Dũng Tráng, Un Théorème de Zariski du type de Lefschetz, Ann. Sci. École Norm. Sup., **6** (1973), 317–366.
[13] A. Hattori, Topology of C^n minus a finite number of affine hyperplanes in a general position, J. Fac. Sci. Univ. Tokyo, **22** (1975), 205–219.
[14] F. Hirzebruch, Arrangements of lines and algebraic surfaces, in Arithmetic and Geometry: Papers dedicated to I.R. Shafarevich, Vol. II, Geometry, pp. 113–140.
[15] M. Jambu and H. Terao, Free arrangements of hyperplanes and supersolvable lattices. Adv. in Math., **52** (1984), 248–258.
[16] T. Kohno, On the minimal algebra and $K(\pi, 1)$ property of affine algebraic varieties, personal communication.
[17] ——, Serie de Poincaré-Koszul associée aux groupes de tresse pure, Invent. Math., **82** (1985), 57–75.
[18] W. Magnus, A. Karrass and D. Solitar, Combinatorial group theory, Wiley, 1966.
[19] P. Orlik and L. Solomon, Arrangements defined by unitary reflection groups, Math. Ann., **261** (1982), 339–357.
[20] ——, Combinatorics and topology of complements of hyperplanes, Invent. Math., **56** (1980), 167–189.
[21] ——, Coxeter arrangements, in Symp. Pure Math., **40**, Vol. 2, (1983), 269–291.
[22] ——, Unitary reflection groups and cohomology, Invent, Math., **59** (1980), 77–94.
[23] R. Randell, The fundamental group of the complement of a union of complex hyperplanes, Invent. Math., **69** (1982), 103–108.
[24] G. Shephard and J. Todd, Finite unitary reflection groups, Canad. J. Math., **6** (1954), 274–304.
[25] A. Sommese, On the density of ratios of Chern numbers of algebraic surfaces, Math. Ann., **268** (1984), 207–221.
[26] H. Terao, Arrangements of hyperplanes and their freeness, I, J. Fac. Sci. Univ. Tokyo IA, **27** (1980), 293–320.
[27] ——, Free arrangement of hyperplanes and unitary reflection groups, Proc. Japan Acad. Ser. A, **56** (1980), 389–392.
[28] ——, Generalized exponents of a free arrangement of hyperplanes and Shephard-Todd-Brieskorn formula. Invent. Math., **63** (1981), 159–179.
[29] ——, The exponents of free hypersurface, in Proc. Sym. Pure Math. **40**, Vol. 2 (1983), 561–566.
[30] ——, Modular elements of lattices and topological fibrations, preprint.

Department of Mathematics
University of Iowa
Iowa City, Iowa 52242 USA

Current address of M. Falk:
Department of Mathematics
Northern Arizona University
Flagstaff, AZ 86011 USA