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On Dirichlet Series Attached to Holomorphic Cusp Forms on $SO(2, q)$

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§ 0. Introduction

In [1] and [2], A. N. Andrianov has studied the relation of the L-function associated to a Siegel modular form of genus two and its Fourier coefficients, and using this relation he has proved the meromorphic continuation and the functional equation of the L-function. Let F be a Siegel cusp form of genus two of weight k. It has the Fourier expansion:

$$F(Z)=\sum_{T={}^tT>0} a(T)e[\mathrm{Tr}(TZ)], \tag{0.1}$$

where Z is in the Siegel upper half plane of degree two and T runs through all semi-integral symmetric positive definite matrices. We assume that F is a simultaneous eigen function of all the Hecke operators $T_k(m)$:

$$T_k(m)F=\lambda_F(m)F \quad (m=1, 2, \cdots). \tag{0.2}$$

Andrianov proved that, in some right half plane, the Dirichlet series

$$\sum_{m=1}^{\infty}\left\{\sum_{T_i\in H(d)} a(mT_i)\chi(T_i)\right\}m^{-s} \tag{0.3}$$

has the Euler product expansion

$$\left\{\sum_{T_i\in H(d)} a(T_i)\chi(T_i)\right\} L_K(s-k+2, \chi)^{-1}L_F(s). \tag{0.4}$$

Here d is the discriminant of an imaginary quadratic field $K=Q(\sqrt{d})$, $H(d)$ denotes the set of equivalence classes under $SL_2(Z)$ of semi-integral symmetric primitive positive definite matrices with determinant $-d/4$. It forms an abelian group and is identified with the ideal class group of K; χ is a character of $H(d)$, which is regarded as an ideal class character of K, and $L_K(s, \chi)$ denotes the L-function with character χ. $L_F(s)$ is defined by

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$$L_F(s)=\zeta(2s-2k+4)\sum_{m=1}^{\infty}\lambda_F(m)m^{-s}.$$

The main purpose of this paper is to give a generalization of the theorem above by Andrianov in $SO(2, q)$ case (Theorem 1). Note that $Sp(2, \boldsymbol{R})$ is isogenous to $SO(2, 3)$. Let q ($\geqslant 3$) be an integer and

$$\tilde{Q}=\begin{pmatrix} & & 1 \\ & Q & \\ 1 & & \end{pmatrix}$$

be a non-degenerate rational symmetric matrix with 2 positive and q negative eigenvalues. Let G [resp. G^*] be the special orthogonal group of $\tilde{Q}$ [resp. Q]. For each prime p, put

$$K_p=G_p \cap SL_{q+2}(\boldsymbol{Z}_p) \quad \text{and} \quad K_f=\prod_p K_p.$$

In Section 1 we define the space $\mathfrak{S}_k(K_f)$, which consist of holomorphic cusp forms on G_A of weight k with respect to K_f. Each element F in $\mathfrak{S}_k(K_f)$ has the Fourier expansion (cf. (1.11)):

$$F(g_f\,;\mathscr{Z})=\sum_{\substack{\eta\in\hat{L}(g_f)\\ \sqrt{-1}\eta\in\mathscr{D}}} a(g_f\,;\eta)e[Q(\eta,\mathscr{Z})],$$

where $g_f \in G_{A,f}$, $\hat{L}(g_f)$ is a lattice in Q^q, and $\mathscr{D}$ is a complex domain defined in (1.2). We assume that F is a simultaneous eigen function of the Hecke algebra $\mathscr{H}_p$ determined by the pair (G_p, K_p) for almost all p. We fix a $g_f \in G^*_{A,f}$ and a $\xi \in \hat{L}(g_f)$ such that $\sqrt{-1}\xi \in \mathscr{D}$. We define a subgroup $H(\xi)$ of G^* by

$$H(\xi)_Q=\{g \in G^*_Q \mid g\xi=\xi\}.$$

Then $H(\xi)_\infty$, the group of $\boldsymbol{R}$-rational points, is isomorphic to $SO(q-1)$. For each prime p, put

$$M(g_f\,;\xi)_p=H(\xi)_p \cap g_f K_f g_f^{-1} \quad \text{and} \quad M(g_f\,;\xi)_f=\prod M(g_f\,;\xi)_p.$$

Denote by $\mathscr{V}(g_f\,;\xi)$ the space of functions on $H(\xi)_A$, which are left $H(\xi)_Q$ invariant and right $H(\xi)_\infty M(g_f\,;\xi)_f$ invariant. Let $\{u_1, \cdots, u_h\}$ be a complete system of representatives of $H(\xi)_Q\backslash H(\xi)_A/H(\xi)_\infty M(g_f\,;\xi)_f$, such that $u_{i,\infty}=1$ ($i=1, \cdots, h$). Take an f in $\mathscr{V}(g_f\,;\xi)$ and assume that f is a simultaneous eigen function of the Hecke algebra $\mathscr{H}'_p$ determined by the pair $(H(\xi)_p, M(g_f\,;\xi)_p)$ for almost all p. Then the Dirichlet series

$$(0.5)\qquad \sum_{\substack{m=1\\(m,p)=1\\ \forall p\in\mathscr{P}}}^{\infty} \mu(\xi)^{-1}\sum_{i=1}^{h} a(u_i g_f; m\xi)\frac{\overline{f(u_i)}}{e(\xi)_i} m^{-(s+k-q/2)}$$

has the Euler product expansion

$$(0.6)\qquad \left\{\mu(\xi)^{-1}\sum_{i=1}^{h} a(u_i g_f; \xi)\frac{\overline{f(u_i)}}{e(\xi)_i}\right\} L_{\mathscr{P}}(F; s)L_{\mathscr{P}}(\bar{f}; s+1/2)^{-1}$$
$$\times\begin{cases}1 & \text{if } q \text{ is odd,}\\ \zeta_{\mathscr{P}}(2s)^{-1} & \text{if } q \text{ is even.}\end{cases}$$

Here $e(\xi)_i=\#\{H(\xi)_Q\cap M(u_i g_f; \xi)_f\}$ $(1\leqslant i\leqslant h)$, $\mu(\xi)=\sum_{i=1}^{h} e(\xi)_i^{-1}$, $\mathscr{P}$ is a sufficient large finite set of primes, $L_{\mathscr{P}}(F; s)$ [resp. $L_{\mathscr{P}}(\bar{f}: s)$] is the L-function of F [resp. $\bar{f}$], which is defined in 4–1, and $\zeta_{\mathscr{P}}(s)$ denotes the Riemann zeta function neglecting p-factors for p belonging to $\mathscr{P}$.

In Section 2 we recall some basic facts on the Hecke algebras following [5], and prepare two lemmata (Lemma 2 and Lemma 4). The proof of Theorem 1 is reduced to local argument and is similar to [6], in which the case $q=3$ is treated in detail. After we calculate local factors in Section 3, our main result will be stated and proved in Section 4. In Section 5 we study some related problems. The case which has interest for us is that f satisfies the condition $\sum a(u_i g_f; \xi)\overline{(f(u_i)}/e(\xi)_i)\neq 0$. It seems that in general a constant function on $H(\xi)_A$ does not have this property. Indeed, Proposition 3 asserts that if $f=1$, $p\notin\mathscr{P}$ and $n_p\geqslant 2$ (n_p is the Q_p-rank of G_p^*), then some relations, not depending to F, hold between the eigenvalues of the Hecke algebra $\mathscr{H}_p$. Finally, in a quite special situation, we give an integral representation of the Dirichlet series (0.5) of Rankin-Selberg type (Theorem 2).

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Notations. We denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{C}$, respectively, the ring of integers, the rational number field, the real number field, and the complex number field. For an associative ring R with identity element, $R^\times$ denotes the group of all invertible elements. For any set S, $M_{m,n}(S)$ denotes the set of $m\times n$ matrices with entries in S. Put $M_{n,n}(S)=M_n(S)$. If R is a ring with unit element, $M_n(R)$ forms a ring and we denote by 1_n the unity of $M_n(R)$. Put $GL_n(R)=M_n(R)^\times$. If R is commutative, we denote by $SL_n(R)$ the special linear group of degree n. If $Q\in M_n(R)$ is a symmetric matrix, for X, $Y\in M_{n,1}(R)$ we put $Q(X, Y)={}^tXQY$ and $Q[X]=Q(X, X)$.

For each place v of $\boldsymbol{Q}$, we denote by $\boldsymbol{Q}_v$ the v-completion of $\boldsymbol{Q}$, and by $|x|_v$ the module of x for an $x \in Q_v^\times$. Q_A [resp. $Q_A^\times$] means the adele ring of $\boldsymbol{Q}$ [resp. the idele group of Q] and for $x=(x_v) \in \boldsymbol{Q}_A^\times$ put $|x|_A=\prod_v |x|_v$. For an algebraic group G defined over $\boldsymbol{Q}$ and a field K containing $\boldsymbol{Q}$, we denote by G_K the group of K-rational points of G. We abbreviate $G_{\boldsymbol{Q}_v}$ to G_v. We denote by G_A, G_∞, and $G_{A,f}$, the adelized group of G, the infinite part of G_A, and the finite part of G_A, respectively. Each prime p is identified with the corresponding finite place. When L is a $\boldsymbol{Z}$ module, we put $L_p=L\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_p$. For $z \in \boldsymbol{C}$, we put $e[z]=\exp(2\pi\sqrt{-1}z)$. The cardinality of a finite set S is denoted by $\sharp\, S$ or $|S|$.

§ 1. Holomorphic cusp forms on $SO(2, q)$

1-1. Let $q\geqslant 3$ and Q be a non-degenerate rational symmetric matrix with 1 positive and $q-1$ negative eigenvalues. Put $L=\boldsymbol{Z}^q$ (column vectors) and $V=L\otimes_{\boldsymbol{Z}}\boldsymbol{Q}=\boldsymbol{Q}^q$. We set

$$\tilde{Q}=\begin{pmatrix} & & 1\\ & Q & \\ 1 & & \end{pmatrix}. \tag{1.1}$$

Then it has 2 positive and q negative eigenvalues. We denote by G^* [resp. G] the special orthogonal group of Q [resp. $\tilde{Q}$] defined over $\boldsymbol{Q}$: accordingly, the set of $\boldsymbol{Q}$-rational points is

$$G_{\boldsymbol{Q}}^*=\{g \in SL_q(\boldsymbol{Q})\,|\,{}^tgQg=Q\},$$
$$[\text{resp. } G_{\boldsymbol{Q}}=\{g \in SL_{q+2}(\boldsymbol{Q})\,|\,{}^tg\tilde{Q}g=\tilde{Q}\}].$$

We regard G^* as a subgroup of G through the embedding $g\mapsto\begin{pmatrix}1 & & \\ & g & \\ & & 1\end{pmatrix}$.

We denote by $\mathscr{D}$ one of the connected components of

$$\{\mathscr{Z} \in V\otimes_{\boldsymbol{Q}}\boldsymbol{C}\,|\,Q[\operatorname{Im}\mathscr{Z}]>0\}, \tag{1.2}$$

where $\operatorname{Im}\mathscr{Z}$ means the imaginary part of $\mathscr{Z}$. This domain is isomorphic to the irreducible bounded symmetric domain of type IV_q. Let G_∞^0 denote the identity component of G_∞. We define an action $g\langle\mathscr{Z}\rangle$ of G_∞^0 on $\mathscr{D}$ and a scalar valued automorphy factor $J(g, \mathscr{Z})$ on $G_\infty^0\times\mathscr{D}$ by

$$g\begin{pmatrix}-\frac{1}{2}Q[\mathscr{Z}]\\ \mathscr{Z}\\ 1\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}Q[g\langle\mathscr{Z}\rangle]\\ g\langle\mathscr{Z}\rangle\\ 1\end{pmatrix}J(g, \mathscr{Z}) \quad (g \in G_\infty^0,\ \mathscr{Z} \in \mathscr{D}). \tag{1.3}$$

In this manner G_∞^0 acts on $\mathscr{D}$ transitively. We fix an element $\mathscr{Z}_0$ in $\mathscr{D}$ such that the real part of $\mathscr{Z}_0$ is 0, and denote by K_∞ the stabilizer subgroup of $\mathscr{Z}_0$ in G_∞^0. Then $\mathscr{D}$ is isomorphic to G_∞^0/K_∞.

For each prime p, put

$$K_p = G_p \cap SL_{q+2}(\boldsymbol{Z}_p), \tag{1.4}$$

and we abbreviate $\prod_{p<\infty} K_p$ to K_f.

Let k be a positive integer. We say that a function F on G_A is a holomorphic cusp form of weight k and with respect to K_f if F satisfies the following three conditions:

(1.5)
- (i) $F(\gamma g u) = F(g)$ for $\forall \gamma \in G_Q$, $\forall u \in K_f$,
- (ii) For any $g = g_\infty g_f$ ($g_\infty \in G_\infty^0$, $g_f \in G_{A,f}$), $F(g_\infty g_f) J(g_\infty, \mathscr{Z}_0)^k$ depends only on g_f and $\mathscr{Z} = g_\infty\langle \mathscr{Z}_0\rangle$, and it is holomorphic on $\mathscr{D}$ as a function of $\mathscr{Z}$,
- (iii) F is bounded on G_A.

We denote by $\mathfrak{S}_k(K_f)$ the space of such functions. We introduce a positive definite hermitian inner product (the Petersson inner product), $\langle\ ,\ \rangle$ by

$$\langle F_1, F_2\rangle = \int_{G_Q\backslash G_A} F_1(g)\overline{F_2(g)}\, d\dot{g}, \tag{1.6}$$

where $F_1, F_2 \in \mathfrak{S}_k(K_f)$ and $d\dot{g}$ is a fixed right G_A-invariant measure on $G_Q\backslash G_A$. Equipped with this inner product, $\mathfrak{S}_k(K_f)$ forms a finite dimensional Hilbert space.

For each $F \in \mathfrak{S}_k(K_f)$ and $g_f \in G_{A,f}$, we put

$$F(g_f; \mathscr{Z}) = F(g_\infty g_f) J(g_\infty, \mathscr{Z}_0)^k \quad (\mathscr{Z} \in \mathscr{D}), \tag{1.7}$$

where $g_\infty \in G_\infty^0$ is chosen so that $\mathscr{Z} = g_\infty\langle\mathscr{Z}_0\rangle$. If we put

$$\Gamma(g_f) = G_Q \cap G_\infty^0 \times g_f K_f g_f^{-1}, \tag{1.8}$$

which is a discrete subgroup of G_∞^0, then $F(g_f; \mathscr{Z})$ satisfies

$$F(g_f; \gamma\langle\mathscr{Z}\rangle) = J(\gamma, \mathscr{Z})^k F(g_f; \mathscr{Z}) \quad \text{for any } \gamma \in \Gamma(g_f). \tag{1.9}$$

For each $X \in V$, we define an element γ_X of G by

$$\gamma_X = \begin{pmatrix} 1 & -{}^tXQ & -\frac{1}{2}Q[X] \\ & 1_q & X \\ & & 1 \end{pmatrix}. \tag{1.10}$$

Since the holomorphic function $F(g_f;\mathscr{Z})$ is invariant under $\mathscr{Z}\mapsto\mathscr{Z}+X$, where X is in the lattice $L(g_f)=\{X\in V_Q|\gamma_X\in\Gamma(g_f)\}$, it has the following Fourier expansion.

$$F(g_f;\mathscr{Z})=\sum_{\substack{\eta\in\hat{L}(g_f)\\ \sqrt{-1}\eta\in\mathscr{D}}} a(g_f;\eta)e[Q(\eta,\mathscr{Z})], \tag{1.11}$$

where $\hat{L}(g_f)=\{X\in V_Q|Q(X,Y)\in\boldsymbol{Z}$ for all $Y\in L(g_f)\}$ is the dual lattice of $L(g_f)$, and the right hand side of (1.11) converges absolutely and uniformly on any compact subset of $\mathscr{D}$.

Let us introduce adelic Fourier coefficients of F. Let $\chi=\prod_v\chi_v$ be the character of $\boldsymbol{Q}_A$ such that $\chi|\boldsymbol{Q}=1$ and $\chi_\infty(x)=e[x]$ for all $x\in\boldsymbol{R}$. For each $\xi\in V_Q$, put

$$F_\chi(g;\xi)=\int_{V_Q\backslash V_A} F(\gamma_X g)\chi(-Q(\xi,X))dX \qquad (g\in G_A), \tag{1.12}$$

where dX is the normalized Haar measure of $V_Q\backslash V_A$. We can easily check that for each $g_\infty\in G_\infty^0$ and $g_f\in G_{A,f}$,

$$F_\chi(g_\infty g_f;\xi)=a(g_f;\xi)J(g_\infty,\mathscr{Z}_0)^{-k}e[Q(\xi,g_\infty\langle\mathscr{Z}_0\rangle)], \tag{1.13}$$

where we understand $a(g_f;\xi)=0$ if $\xi\notin\hat{L}(g_f)$ or $\sqrt{-1}\xi\notin\mathscr{D}$. The next properties follow easily from the above definition:

$$\begin{aligned} &F_\chi(\gamma_X gu;\xi)=\chi(Q(\xi,X))F_\chi(g;\xi) && \text{for } {}^\forall X\in V_A,\ {}^\forall u\in K_f,\\ &F_\chi\left(\begin{pmatrix}\alpha & & \\ & \beta & \\ & & \alpha^{-1}\end{pmatrix}g;\xi\right)=F_\chi(g;\beta^{-1}\xi\alpha) && \text{for } {}^\forall\alpha\in\boldsymbol{Q}^\times,\ {}^\forall\beta\in G_Q^*\\ &F(\gamma_X g)=\sum_{\xi\in V_Q}F_\chi(g;\xi)\chi(Q(\xi,X)) && \text{for } {}^\forall X\in V_A. \end{aligned} \tag{1.14}$$

1-2. Fix a ξ in V_Q such that $\sqrt{-1}\xi\in\mathscr{D}$, and put $V^{(1)}=\boldsymbol{Q}\xi$, $V^{(2)}=\{X\in V|Q(\xi,X)=0\}$. We write $Q^{(i)}=Q|V^{(i)}$ $(i=1,2)$. Since $Q[\xi]$ is positive and Q has only one positive eigenvalue, we see that $Q^{(2)}$ is negative definite. Let us define an algebraic subgroup $H(\xi)$ of G^* by

$$H(\xi)_Q=\{g\in G_Q^*|g\xi=\xi\}. \tag{1.15}$$

It is nothing but the special orthogonal group of $Q^{(2)}$. For an element $g_f\in G_{A,f}$ and a prime p, we put

$$M(g_f;\xi)_p=H(\xi)_p\cap g_fK_fg_f^{-1}, \tag{1.16}$$

and we abbreviate $\prod_p M(g_f;\xi)_p$ to $M(g_f;\xi)_f$. We denote by $\mathscr{V}(g_f;\xi)$ the space of $\boldsymbol{C}$-valued functions on $H(\xi)_A$ satisfying

(1.17) $f(\gamma h h_\infty m_f)=f(h)$ for $^\forall\gamma \in H(\xi)_Q$, $^\forall h_\infty \in H(\xi)_\infty$, $^\forall m_f \in M(g_f;\xi)_f$.

In this space, the Petersson inner product is defined by

$$\langle f_1, f_2\rangle_{H(\xi)}=\int_{H(\xi)_Q\backslash H(\xi)_A} f_1(h)\overline{f_2(h)}d\dot{h} \tag{1.18}$$

where $d\dot{h}$ is the right $H(\xi)_A$ invariant measure on $H(\xi)_Q\backslash H(\xi)_A$ with the total volume 1. Since $|H(\xi)_Q\backslash H(\xi)_A/H(\xi)_\infty M(g_f;\xi)_f|$ is finite, $\mathscr{V}(g_f;\xi)$ forms a finite dimensional Hilbert space.

When f is a left $H(\xi)_Q$-invariant function on $H(\xi)_A$, we put

$$\varphi^f_{F,\xi}(g)=\int_{H(\xi)_Q\backslash H(\xi)_A} F_\chi(ug;\xi)\overline{f(u)}du \quad (g \in G_A). \tag{1.19}$$

Lemma 1. *Let F be a non-zero element of $\mathfrak{S}_k(K_f)$. Then there exist $g_f \in G^*_{A,f}$ and $\xi \in V_Q$ such that $F_\chi(g_f;\xi)\neq 0$. Furthermore there exists an f in $\mathscr{V}(g_f;\xi)$ such that $\varphi^f_{F,\xi}(g_f)\neq 0$.*

Proof. First we note that

$$G_A=G_Q G^*_{A,f} G^0_\infty K_f. \tag{1.20}$$

Indeed, for any prime p, G_p is generated by G^*_p,

$$\begin{pmatrix} a & & \\ & 1_q & \\ & & a^{-1}\end{pmatrix} (a \in Q_p^\times),\ \gamma_X \quad\text{and}\quad \gamma'_X=\begin{pmatrix} 1 & & \\ X & 1_q & \\ -\frac{1}{2}Q[X] & -{}^tXQ & 1\end{pmatrix} (X \in V_p).$$

Hence, (1.20) is an easy consequence of the approximation theorem of valuations. From this, we can take a $g_f \in G^*_{A,f}$ and $g_\infty \in G^0_\infty$ such that $F(g_\infty g_f)\neq 0$. Take a ξ in V_Q such that $F_\chi(g_\infty g_f;\xi)\neq 0$. Then from the property (1.13), we have $F_\chi(g_f;\xi)\neq 0$. Now we define a function f_1 on $H(\xi)_A$ by

$$f_1(u)=F_\chi(ug_f;\xi). \tag{1.21}$$

From (1.13) and (1.14), f_1 belongs to $\mathscr{V}(g_f;\xi)$. Therefore there exists an f in $\mathscr{V}(g_f;\xi)$ such that $\langle f_1, f\rangle_{H(\xi)}\neq 0$ and the function $\varphi^f_{F,\xi}$ has the required property. Q.E.D.

§ 2. Hecke algebra

2-1. In this subsection we recall the definitions and some properties of Hecke albebras following Satake [5]. Let p be a prime number, L a lattice in $\boldsymbol{Q}_p^N$ (column vectors), and S a non-degenerate symmetric matrix of degree N with coefficients in $\boldsymbol{Q}_p$. We say that L is $\boldsymbol{Z}_p$-integral with respect to S if $S[x]/2 \in \boldsymbol{Z}_p$ for all $x \in L$. We denote by $SO(S)$ the special orthogonal group and put

$$(2.1) \qquad SO(S; L)=\{g \in SO(S) \mid gL=L\}.$$

Denote by $\mathscr{L}(S; L)$ the Hecke algebra of the pair $(SO(S), SO(S; L))$; namely, $\mathscr{L}(S; L)$ is the set of bi-$SO(S; L)$-invariant functions on $SO(S)$ with compact support, and it forms a $\boldsymbol{C}$-algebra by the convolution product

$$(2.2) \qquad (\phi_1 * \phi_2)(g)=\int_{SO(S)} \phi_1(gh^{-1})\phi_2(h)dh,$$

where dh is the Haar measure of $SO(S)$ normalized by the condition that the volume of $SO(S; L)$ is 1. If L is a maximal $\boldsymbol{Z}_p$-integral lattice with respect to S, then $SO(S; L)$ is a maximal compact subgroup of $SO(S)$, and the Hecke algebra $\mathscr{L}(S; L)$ is commutative (cf. Satake [5]).

Let S_0 be an anisotropic symmetric matrix of size n_0 over $\boldsymbol{Q}_p$, and assume that $\boldsymbol{Z}_p^{n_0}$ is a maximal $\boldsymbol{Z}_p$-integral lattice with respect to S_0. From the well known property of quadratic forms over local fields, we have $0 \leqslant n_0 \leqslant 4$. For a non-negative integer n, we put

$$(2.3) \qquad S_n=\begin{pmatrix} & & J_n \\ & S_0 & \\ J_n & & \end{pmatrix}, \quad L_n=\boldsymbol{Z}_p^{2n+n_0}, \text{ and } V_n=\boldsymbol{Q}_p^{2n+n_0},$$

where $J_n=\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ (size n). Then L_n is a maximal $\boldsymbol{Z}_p$-integral lattice with respect to S_n. Put $G_n=SO(S_n)$, $K_n=SO(S_n; L_n)$ and $\mathscr{L}_n=\mathscr{L}(S_n; L_n)$. Note that if L is a maximal $\boldsymbol{Z}_p$-integral lattice with respect to S, then $SO(S; L)$ is isomorphic to K_n for a suitable choice of S_0 and n. For an n-tuple of integers $\boldsymbol{r}=(r_1, \cdots, r_n)$, we set

$$(2.4) \qquad \pi^{\boldsymbol{r}}=\operatorname{diag}(p^{r_1}, \cdots, p^{r_n}, \underbrace{1, \cdots, 1}_{n_0}, p^{-r_n}, \cdots, p^{-r_1}) \in G_n.$$

Put

$$(2.5) \quad N_n=\left\{ g=\begin{pmatrix} X & * & * \\ 0 & 1_{n_0} & * \\ 0 & 0 & J_n{}^tX^{-1}J_n \end{pmatrix} \in G_n \,\middle|\, X=\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in GL_n(\boldsymbol{Q}_p) \right\}.$$

Then the following Iwasawa and Cartan decomposition hold.

$$G_n = \bigcup_{r \in Z^n} N_n \pi^r K_n = \bigcup_{r \in Z^n} \pi^r N_n K_n, \tag{2.6}$$

$$G_n = \coprod_{r \in \Lambda} K_n \pi^r K_n \text{ (disjoint)}, \tag{2.7}$$

where

$$\Lambda = \begin{cases} \{r=(r_1, \cdots, r_n) \in Z^n \mid r_1 \geqslant \cdots \geqslant r_n \geqslant 0\} \text{ if } n_0 \neq 0, \\ \{r=(r_1, \cdots, r_n) \in Z^n \mid r_1 \geqslant \cdots \geqslant r_{n-1} \geqslant |r_n|\} \text{ if } n_0 = 0. \end{cases}$$

We often identify the Hecke algebra $\mathscr{L}_n$ with the set of finite C-linear combinations of double K_n cosets. In [5], I. Satake gives an explicit isomorphism between $\mathscr{L}_n$ and an affine algebra. We recall it here. Let $X_1, \cdots, X_n$ be algebraically independent variables over C and $C[X_1^{\pm}, \cdots, X_n^{\pm}]$ be an affine algebra generated by $X_1, X_1^{-1}, \cdots, X_n, X_n^{-1}$. Let $\mathfrak{S}_n$ denote the group of all permutations of the variables $X_1, \cdots, X_n$ and $w^{(i)}$ $(1 \leqslant i \leqslant n)$ denotes the transformation; $X_i \mapsto X_i^{-1}$, $X_j \mapsto X_j$ $(i \neq j)$. For each $g \in G_n$, the double coset $K_n g K_n$ can be decomposed into right K_n cosets in the form

$$K_n g K_n = \coprod_{i \in I} n_i \pi^{r_i} K_n, \tag{2.8}$$

where $r_i = (r_{i,1}, \cdots, r_{i,n}) \in Z^n$, $n_i \in N_n$ and I is a finite index set. The set $\{r_i \mid i \in I\}$ is uniquely determined by $K_n g K_n$. Put

$$\Phi_n(K_n g K_n) = \sum_{i \in I} \prod_{j=1}^{n} (p^{1-n_0/2-j} X_j)^{r_{i,n+1-j}}, \tag{2.9}$$

and extend it to a C-linear mapping from $\mathscr{L}_n$ to $C[X_1^{\pm}, \cdots, X_n^{\pm}]$. Then it gives an algebra isomorphism

$$\Phi_n : \mathscr{L}_n \xrightarrow{\cong} C[X_1^{\pm}, \cdots, X_n^{\pm}]^{W_n}, \tag{2.10}$$

where W_n denotes the group of automorphisms of the algebra $C[X_1^{\pm}, \cdots, X_n^{\pm}]$ generated by $\mathfrak{S}_n$ and $w^{(i)}$ $(1 \leqslant i \leqslant n)$ [resp. $\mathfrak{S}_n$ and $w^{(i)} w^{(j)}$ $(1 \leqslant i, j \leqslant n)$] if $n_0 \geqslant 1$ [resp. $n_0 = 0$], and $C[X_1^{\pm}, \cdots, X_n^{\pm}]^{W_n}$ denotes the subalgebra of all W_n invariants.

Now we set

$$T_n(1) = \{g \in G_n \mid pg \in M_{2n+n_0}(Z_p)\}. \tag{2.11}$$

For each r $(0 \leqslant r \leqslant n)$, we put

$$\tilde{c}_n^{(r)} = \{g \in T_n(1) \mid \operatorname{rank}_{Z_p/pZ_p}(pg) = r\}, \tag{2.12}$$

where $\mathrm{rank}_{Z_p/pZ_p}(pg)$ means the rank of pg in $M_{2n+n_0}(Z_p/pZ_p)$. Then we have

$$(2.13)\qquad T_n(1)=\coprod_{0\leqslant r\leqslant n}\tilde{c}_n^{(r)}\quad\text{(disjoint)},$$

and from the Cartan decomposition (2.7),

$$(2.14)\qquad \tilde{c}_n^{(r)}=\begin{cases}K_n c_n^{(r)}K_n & \text{if } n_0\neq 0 \text{ or } r\neq n,\\ K_n c_n^{(r)}K_n \coprod K_n c_n^{(r)\prime}K_n & \text{if } n_0=0 \text{ and } r=n,\end{cases}$$

where $c_n^{(r)}=\pi^{(1,\cdots,1,0,\cdots,0)}$ (in the upper suffix, 1 appears r times) and $c_n^{(n)\prime}=\pi^{(1,\cdots,1,-1)}$.

2-2. In this subsection we decompose $\tilde{c}_{n+1}^{(r)}$ into right K_{n+1} cosets inductively. For r $(1\leqslant r\leqslant n)$, $R_n^{(r)}$ [resp. $R_n^{(n)\prime}$] denotes a complete set of representatives of $K_n/(c_n^{(r)}K_n c_n^{(r)-1}\cap K_n)$ [resp. $K_n/(c_n^{(n)\prime}K_n c_n^{(n)\prime-1}\cap K_n)$]

Lemma 2. *When $n_0\geqslant 1$ or $0\leqslant r\leqslant n-1$,*

$$(2.15)\qquad \begin{aligned}\tilde{c}_{n+1}^{(r)}=&\coprod_{\substack{\varepsilon\in R_n^{(r-1)}\\ X_1}}\begin{pmatrix}p & & \\ & \varepsilon c_n^{(r-1)} & \\ & & p^{-1}\end{pmatrix}\gamma_{X_1}K_{n+1}\coprod_{\substack{\varepsilon\in R_n^{(r-2)}\\ X_2}}\begin{pmatrix}1 & & \\ & \varepsilon c_n^{(r-2)} & \\ & & 1\end{pmatrix}\gamma_{X_2}K_{n+1}\\ &\coprod_{\substack{\varepsilon\in R_n^{(r-1)}\\ X_3}}\begin{pmatrix}1 & & \\ & \varepsilon c_n^{(r-1)} & \\ & & 1\end{pmatrix}\gamma_{X_3}K_{n+1}\coprod_{\substack{\varepsilon\in R_n^{(r)}\\ X_4}}\begin{pmatrix}1 & & \\ & \varepsilon c_n^{(r)} & \\ & & 1\end{pmatrix}\gamma_{X_4}K_{n+1}\\ &\coprod_{\varepsilon\in R_n^{(r-1)}}\begin{pmatrix}p^{-1} & & \\ & \varepsilon c_n^{(r-1)} & \\ & & p\end{pmatrix}K_{n+1}\quad\textit{(disjoint)}.\end{aligned}$$

Here, for $X\in V_n$, we have put

$$(2.16)\qquad \gamma_X=\begin{pmatrix}1 & -{}^tXS_n & -\frac{1}{2}S_n[X]\\ & 1_{2n+n_0} & X\\ & & 1\end{pmatrix},$$

and $X_1,\cdots,X_4$ runs through the following set, respectively.

$$\left\{X_1=\begin{bmatrix}x_1\\ x_2\\ z\\ y_2\\ 0\end{bmatrix}\in V_n/L_n\;\middle|\;\begin{matrix}x_1\in p^{-2}Z_p^{r-1},\ x_2,\ y_2\in p^{-1}Z_p^{n-r+1}\\ z\in p^{-1}L_0,\end{matrix}\right\},$$

$$\left\{X_2=\begin{bmatrix}x_1\\ x_2\\ z\\ y_2\\ 0\end{bmatrix}\in V_n/L_n\;\middle|\;\begin{matrix}x_1\in p^{-1}Z_p^{r-2},\ x_2,\ y_2\in p^{-1}Z_p^{n-r+2},\ \frac{1}{2}S_n[X]\in p^{-1}Z_p\\ z\in p^{-1}L_0,\ \textit{and } x_2 \textit{ or } y_2\notin Z_p^{n-r+2}\end{matrix}\right\}$$

$$\left\{X_3=\begin{pmatrix}x_1\\0\\z\\0\end{pmatrix}\in V_n/L_n\left|\begin{matrix}x_1\in p^{-1}\boldsymbol{Z}_p^{r-1},\ z\in p^{-1}L_0-L_0\\ \frac{1}{2}S_0[z]\in p^{-1}\boldsymbol{Z}_p\end{matrix}\right.\right\},$$

$$\left\{X_4=\begin{pmatrix}x\\0\end{pmatrix}\in V_n/L_n\,|\,x\in p^{-1}\boldsymbol{Z}_p^r\right\}.$$

We understand that $R_n^{(r)}=\phi$ *if* $r<0$ *or* $r>n$. *When* $n_0=0$ *and* $r=n,\ n+1$, *the identity* (2.15) *holds with an addition of the following* K_{n+1} *cosets to the right hand side*:

$$\coprod_{\substack{\varepsilon\in R_n^{(n)\prime}\\ X_4'}}\begin{pmatrix}1 & \varepsilon c_n^{(n)\prime} & \\ & & 1\end{pmatrix}\gamma_{X_4'}K_{n+1}\qquad \text{if } r=n,$$

$$\coprod_{\substack{\varepsilon\in R_n^{(n)\prime}\\ X_1'}}\begin{pmatrix}p & \varepsilon c_n^{(n)\prime} & \\ & & p^{-1}\end{pmatrix}\gamma_{X_1'}K_{n+1}$$

$$\coprod_{\varepsilon\in R_n^{(n)\prime}}\begin{pmatrix}p^{-1} & \varepsilon c_n^{(n)\prime} & \\ & & p\end{pmatrix}K_{n+1}\qquad \text{if } r=n+1,$$

where X_4', X_1' *runs through the following set, respectively;*

$$\left\{X_4'=\begin{pmatrix}x_1\\0\\y_2\\0\end{pmatrix}\in V_n/L_n\,|\,x_1\in p^{-1}\boldsymbol{Z}_p^{n-1},\ y_2\in p^{-1}\boldsymbol{Z}_p\right\},$$

$$\left\{X_1'=\begin{pmatrix}x_1\\0\\y_2\\0\end{pmatrix}\in V_n/L_n\,|\,x_1\in p^{-2}\boldsymbol{Z}_p^{n-1},\ y_2\in p^{-2}\boldsymbol{Z}_p\right\}.$$

Proof. We assume that $n_0\geq 1$. From the definition of $T_{n+1}(1)$ and the Iwasawa decomposition (2.6), we have

$$T_{n+1}(1)=\coprod_{\substack{-1\leq a\leq 1\\ 0\leq i\leq n}}\ \coprod_{\varepsilon\in R_n^{(i)}}\ \coprod_{X}\begin{pmatrix}p^a & \varepsilon c_n^{(i)} & \\ & & p^{-a}\end{pmatrix}\gamma_X K_{n+1},$$

where X runs through

$$\left\{X\in V_n/L_n\left|\begin{matrix}c_n^{(i)}X\in p^{-1}L_n,\ p^aS_nX\in p^{-1}L_n\\ \frac{1}{2}p^aS_n[X]\in p^{-1}\boldsymbol{Z}_p\end{matrix}\right.\right\}.$$

For each $b=\begin{pmatrix}p^a & \varepsilon c_n^{(i)} & \\ & & p^{-a}\end{pmatrix}\gamma_X$, $\mathrm{rank}_{\boldsymbol{Z}_p/p\boldsymbol{Z}_p}(pb)$ is calculated easily (note that if $S_0[z] \in 2pZ_p$ then $S_0 z \in L_0$), and our assertion follows. The case $n_0=0$ can be treated similarly. Q.E.D.

Put

$$L_0'=\left\{z \in V_0 \,\middle|\, \frac{1}{2} S_0[z] \in p^{-1}Z_p\right\}. \tag{2.17}$$

Then L_0'/L_0 is a vector space over Z_p/pZ_p. We denote by $\partial=\partial(S_0)$ its dimension ($0 \leqslant \partial \leqslant n_0$). From Lemma 2 and the definitions of the Satake isomorphism Φ_n ((2.9)), we have

Lemma 3.

$$\begin{aligned}\Phi_{n+1}(\tilde{c}_{n+1}^{(r)})=&p^{n+n_0/2}(X_{n+1}+X_{n+1}^{-1})\Phi_n(\tilde{c}_n^{(r-1)})\\&+p^{r-1}(p^{\partial}-1)\Phi_n(\tilde{c}_n^{(r-1)})+p^r\Phi_n(\tilde{c}_n^{(r)})\\&+p^{r-2}(p^{n-r+2}-1)(p^{n-r+1+n_0}+p^{\partial})\Phi_n(\tilde{c}_n^{(r-2)}).\end{aligned}$$

Especially, since $\#\{\tilde{c}_n^{(r)}/K_n\}$ is given by the value $\Phi_n(\tilde{c}_n^{(r)})$ for $X_i=p^{n_0/2+i-1}$ ($1 \leqslant i \leqslant n$), we can prove

$$\#\{\tilde{c}_n^{(r)}/K_n\}=\prod_{j=1}^{r}\frac{p^{j-1}(p^{n-j+1}-1)(p^{n-j+n_0}+p^{\partial})}{p^j-1} \tag{2.18}$$

by using this lemma and induction on n. This formula will be used in Section 5.

2-3. Let T be an indeterminate. Since each coefficient of

$$\prod_{j=1}^{n}(1-X_jT)(1-X_j^{-1}T) \tag{2.19}$$

is invariant under W_n, there uniquely exists a polynomial

$$P_n(T)=P_{S_n}(T)=\sum_{k=0}^{2n}(-1)^k\alpha_n(k)T^k \quad (\alpha_n(k) \in \mathscr{L}_n) \tag{2.20}$$

such that

$$\sum_{k=0}^{2n}(-1)^k\Phi_n(\alpha_n(k))T^k=\prod_{j=1}^{n}(1-X_jT)(1-X_j^{-1}T).$$

From the reciprocity of (2.19), we have

$$\alpha_n(2n-k)=\alpha_n(k) \quad (0 \leqslant k \leqslant 2n).$$

In this subsection we determine $\alpha_n(k)$ inductively.

Lemma 4.

(i) *$\alpha_n(k)$ is written in the form*

$$\alpha_n(k)=\sum_{0\leqslant r\leqslant n} a_{n,k}(r)\tilde{c}_n^{(r)} \quad \textit{with} \quad a_{n,k}(r)\in C.$$

(ii) $$a_{n+1,k}(r)=p^{-(n+n_0/2)}a_{n,k-1}(r-1) \qquad \textit{if } r\geqslant 1,$$

$$=a_{n,k}(0)+a_{n,k-2}(0)-\frac{p^\partial-1}{p^{n+n_0/2}}a_{n,k-1}(0)$$

$$-\frac{(p^n-1)(p^{n-1+n_0}+p^\partial)}{p^{n+n_0/2}}a_{n,k-1}(1) \qquad \textit{if } r=0.$$

(iii) *When $0\leqslant k\leqslant 2n+2$ and $1\leqslant r\leqslant n$, the following relations hold.*

$$a_{n,k}(r)+a_{n,k-2}(r)=\frac{p^r(p^\partial-1)}{p^{n+n_0/2}}a_{n,k-1}(r)+\frac{p^r}{p^{n+n_0/2}}a_{n,k-1}(r-1)$$

$$+\frac{p^r(p^{n-r}-1)(p^{n-r-1+n_0}+p^\partial)}{p^{n+n_0/2}}a_{n,k-1}(r+1).$$

Here we understand that $a_{n,k'}(r')=0$ unless $0\leqslant k'\leqslant 2n$ or unless $0\leqslant r'\leqslant n$.

Proof. If $n=0$, (i) is trivial. We shall prove our assertions by induction on n. Assume that (i) holds for n. Since

$$\Phi_{n+1}(P_{n+1}(T))=\{1-(X_{n+1}+X_{n+1}^{-1})T+T^2\}\sum_{k=0}^{2n}(-1)^k\Phi_n(\alpha_n(k))T^k,$$

we have

$$\Phi_{n+1}(\alpha_{n+1}(k))=\Phi_n(\alpha_n(k))+(X_{n+1}+X_{n+1}^{-1})\Phi_n(\alpha_n(k-1))+\Phi_n(\alpha_n(k-2))$$
$$(0\leqslant k\leqslant 2n+2).$$

From Lemma 3,

$$\Phi_{n+1}(\alpha_{n+1}(k))-\frac{1}{p^{n+n_0/2}}\sum_{1\leqslant r\leqslant n+1}a_{n,k-1}(r-1)\Phi_{n+1}(\tilde{c}_{n+1}^{(r)})$$

$$-\Big\{a_{n,k}(0)+a_{n,k-2}(0)-\frac{p^\partial-1}{p^{n+n_0/2}}a_{n,k-1}(0)$$

$$-\frac{(p^n-1)(p^{n-1+n_0}+p^\partial)}{p^{n+n_0/2}}a_{n,k+1}(1)\Big\} \tag{2.21}$$

$$=\sum_{1\leqslant r\leqslant n}\Phi_n(\tilde{c}_n^{(r)})\Big\{a_{n,k}(r)+a_{n,k-2}(r)-\frac{p^r(p^\partial-1)}{p^{n+n_0/2}}a_{n,k-1}(r)$$

$$-\frac{p^r}{p^{n+n_0/2}}a_{n,k-1}(r-1)-\frac{p^r(p^{n-r}-1)(p^{n-r-1+n_0}+p^\partial)}{p^{n+n_0/2}}a_{n,k-1}(r+1)\Big\}.$$

Since the left hand side of (2.21) belongs to $C[X_1^{\pm}, \cdots, X_{n+1}^{\pm}]^{\mathfrak{S}_{n+1}}$ and the right hand side belongs to $C[X_1^{\pm}, \cdots, X_n^{\pm}]^{\mathfrak{S}_n}$, it must be a constant. From the fact that 1 and $\tilde{c}_n^{(r)}$ $(1 \leqslant r \leqslant n)$ are linearly independent over C, the coefficient of $\Phi_n(\tilde{c}_n^{(r)})$ must be 0 $(1 \leqslant r \leqslant n)$, so (iii) is proved. As Φ_{n+1} is an isomorphism, (i) holds for $n+1$ and (ii) is also proved. Q.E.D.

Let us define the local L-factor. As in Section 1, let L be a maximal Z_p-integral lattice in Q_p^N with respect to S. Denote by n the Witt index of S and put $n_0 = N - 2n$. If we take a suitable S_0, then S is represented in the form (2.3) and $\mathscr{L}(S; L)$ is isomorphic to $\mathscr{L}_n$. We put $\partial = \partial(S; L) = \partial(S_0)$ ($\partial(S_0)$ is defined after (2.17)). Through this isomorphism, we define a polynomial $P_S(T)$ in $\mathscr{L}(S; L)[T]$ (see (2.20)). When σ is a homomorphism from $\mathscr{L}(S; L)$ to C, we obtain a polynomial $P_S(T; \sigma)$ in $C[T]$ replacing each coefficient of $P_S[T]$ by its σ-image. For $s \in C$, we put

$$(2.22)\quad L_p(S; \sigma; s) = \begin{cases} P_S(p^{-s}; \sigma)^{-1} & \text{if } n_0 = 0 \text{ or } 1, \\ P_S(p^{-s}; \sigma)^{-1}(1 - p^{-s+1-n_0/2})^{-1}(1 + p^{-s+1+\partial-n_0/2})^{-1} & \\ & \text{if } n_0 = 2 \text{ or } 3, \\ P_S(p^{-s}; \sigma)^{-1}(1 - p^{-s})^{-1}(1 - p^{-s-1})^{-1}(1 + p^{-s+1})^{-1} & \\ \qquad \times (1 + p^{-s+2})^{-1} & \text{if } n_0 = 4. \end{cases}$$

Thus $L_p(S; \sigma; s)^{-1}$ is of degree N [resp. $N-1$] as a polynomial in p^{-s} if N is even [resp. odd].

§ 3. Euler factor

3-1. In this section we use the same notations as in Section 2. We denote by $\hat{L}_n$ the dual lattice of L_n with respect to S_n; so $\hat{L}_n = S_n^{-1} L_n$. Let ξ be a primitive element of $\hat{L}_n$ and fix it throughout this section. We denote by $N(\xi)$ an element of Z_p such that $N(\xi)\xi$ is a primitive element of L_n. We assume that $N(\xi) S_n[\xi]$ is a unit of Z_p. Note that $N(\xi) \in 2Z_p$. Put

$$(3.1)\qquad H(\xi) = \{g \in G_n \mid g\xi = \xi\},$$

and

$$(3.2)\qquad \begin{aligned} W_{n+1,\xi}^{\chi} = \{\varphi \colon & H(\xi) \cap K_n \backslash G_{n+1} / K_{n+1} \longrightarrow C \mid \\ & \varphi(\gamma_X g) = \chi(S_n(\xi, X)) \varphi(g) \text{ for all } X \in V_n\}, \end{aligned}$$

where $\chi = \chi_p$ is a character of Q_p whose conductor is Z_p. The Hecke algebra $\mathscr{L}_{n+1}$ acts on $W_{n+1,\xi}^{\chi}$ by the right convolution;

$$(3.3) \qquad (\varphi * \phi)(g) = \int_{G_p} \varphi(gh^{-1})\phi(h)dh \quad (\varphi \in W^{\chi}_{n+1,\xi},\ \phi \in \mathscr{L}_{n+1}).$$

Furthermore, when we denote by $\mathscr{L}'$ the Hecke algebra determined by the pair $(H(\xi), H(\xi) \cap K_n)$, $\mathscr{L}'$ acts on $W^{\chi}_{n+1,\xi}$ by the left convolution;

$$(3.4) \qquad (\phi * \varphi)(g) = \int_{H(\xi)} \phi(h)\varphi(h^{-1}g)dh \quad (\varphi \in W^{\chi}_{n+1,\xi},\ \phi \in \mathscr{L}').$$

In Proposition 1 and Proposition 2, we shall calculate the formal power series

$$(3.5) \qquad F_{\varphi}(T) = \sum_{l=0}^{\infty} \varphi\left(\begin{pmatrix} p^l & & \\ & 1 & \\ & & p^{-l} \end{pmatrix}\right) T^l,$$

when φ is a left $\mathscr{L}'$ and right $\mathscr{L}_{n+1}$ eigen function.

For $b \in G_n$, we denote by m_b the minimal integer such that $p^{m_b} b^{-1}\xi \in \hat{L}_n$. We can easily check that

$$(3.6) \qquad m_b \geqslant 0.$$

Lemma 5.

$$\bigcup_{\varphi \in W^{\chi}_{n+1,\xi}} \operatorname{supp} \varphi \subset \bigcup_{l \geqslant 0} \bigcup_{\substack{b \in G_n \\ X \in V_n}} \gamma_X \begin{pmatrix} p^{m_b + l} & & \\ & b & \\ & & p^{-(m_b + l)} \end{pmatrix} K_{n+1},$$

where supp φ *means the support of* φ.

Proof. Take any element g in G_{n+1} such that $\varphi(g) \neq 0$. From the Iwasawa decomposition (2.6) and the definition of $W^{\chi}_{n+1,\xi}$, we may assume that

$$(3.7) \qquad g = \begin{pmatrix} p^a & & \\ & b & \\ & & p^{-a} \end{pmatrix},$$

where $a \in \boldsymbol{Z}$ and $b \in G_n$. Since for $X \in L_n$, $\varphi(g\gamma_X) = \varphi(g)$, $p^{a - m_b} S_n(p^{m_b} b^{-1}\xi, X)$ must be an integer. From the choice of m_b, we have $a \geqslant m_b$ and our assertion is verified. Q.E.D.

Let us describe the action of some elements of $\mathscr{L}_{n+1}$ on $W^{\chi}_{n+1,\xi}$. For $l \in \boldsymbol{Z}$ and r $(0 \leqslant r \leqslant n)$, put

$$(3.8)\qquad \varphi(r,l)=\begin{cases}\displaystyle\sum_{\varepsilon\in R_n^{(r)}}\varphi\left(\begin{pmatrix}p^l & & \\ & \varepsilon c_n^{(r)} & \\ & & p^{-l}\end{pmatrix}\right) & \text{if } n_0\neq 0 \text{ or } r\neq n,\\ \displaystyle\sum_{\varepsilon\in R_n^{(n)}}\varphi\left(\begin{pmatrix}p^l & & \\ & \varepsilon c_n^{(n)} & \\ & & p^{-l}\end{pmatrix}\right)+\sum_{\varepsilon\in R_n^{(n)\prime}}\varphi\left(\begin{pmatrix}p^l & & \\ & \varepsilon c_n^{(n)\prime} & \\ & & p^{-l}\end{pmatrix}\right) & \text{if } n_0=0 \text{ and } r=n.\end{cases}$$

Note that if l is negative $\varphi(r, l)=0$.

Lemma 6. *For $l\geqslant 0$ and r $(0\leqslant r\leqslant n+1)$, the following identity holds.*

$$(\varphi*\tilde{c}_{n+1}^{(r)})\left(\begin{pmatrix}p^l & & \\ & 1 & \\ & & p^{-l}\end{pmatrix}\right)=p^{2n+n_0}\varphi(r-1,l+1)+p^r\varphi(r,l)+\varphi(r-1,l-1)$$

$$+\begin{cases}p^{r-2}(p^{n-r+2}-1)(p^{n-r+1+n_0}+p^{\partial})\varphi(r-2,l)+p^{r-1}(p^{\partial}-1)\varphi(r-1,l) & \text{if } l\geqslant 1,\\ \varphi'(r-2,0)-p^{r-2}\varphi''(r-2,0)+p^{r-1}\varphi''(r-1,0)-p^{r-1}\varphi(r-1,0) & \text{if } l=0.\end{cases}$$

Here we have put

$$(3.9)\qquad \varphi'(r,0)=\sum_{\substack{\varepsilon\in R_n^{(n)}\\ X\in p^{-1}L_n/L_n\\ c_n^{(r)}X\in p^{-1}L_n\\ \frac{1}{2}S_n[X]\in p^{-1}Z_p}}\chi(S_n(\xi,\varepsilon c_n^{(r)}X))\varphi\left(\begin{pmatrix}1 & & \\ & \varepsilon c_n^{(r)} & \\ & & 1\end{pmatrix}\right),\quad (0\leqslant r\leqslant n-1)$$

$$(3.10)\qquad \varphi''(r,0)=\sum_{\substack{\varepsilon\in R_n^{(r)}\\ z\in p^{-1}L_0/L_0\\ \frac{1}{2}S_0[z]\in p^{-1}Z_p\\ X=\begin{pmatrix}0\\ z\\ 0\end{pmatrix}}}\chi(S_n(\xi,\varepsilon c_n^{(r)}X))\varphi\begin{pmatrix}1 & & \\ & \varepsilon c_n^{(r)} & \\ & & 1\end{pmatrix}\quad \text{if } n_0\neq 0 \text{ or } r\neq n,$$

and $\varphi''(n, 0)=\varphi(n, 0)$ *if $n_0=0$ and $r=n$.*

This lemma is a direct consequence of Lemma 2. In the right hand side of (3.9) and (3.10), every element ε of $R_n^{(r)}$ which contributes the sum, must satisfy $c_n^{(r)-1}\varepsilon^{-1}\xi\in\hat{L}_n$ (cf. Lemma 5).

3-2. In this subsection we assume that φ is an eigen function of $\mathscr{L}_{n+1}$. We denote by σ_φ the homomorphism of $\mathscr{L}_{n+1}$ to $\boldsymbol{C}$ determined by φ:

$$(3.11)\qquad \varphi*\phi=\sigma_\varphi(\phi)\varphi\qquad(\phi\in\mathscr{L}_{n+1}).$$

We put

$$Q_\varphi(T)=P_{S_{n+1}}(p^{-(n+n_0/2)}T;\sigma_\varphi)=\sum_{k=0}^{2n+2}(-1)^k\sigma_\varphi(\alpha_{n+1}(k))\left(\frac{T}{p^{n+n_0/2}}\right)^k, \tag{3.12}$$

and

$$P_\varphi(T)=F_\varphi(T)\times Q_\varphi(T), \tag{3.13}$$

where $F_\varphi(T)$ is the formal power series defined in (3.5).

Proposition 1. *Notation being as above, we have*

$$P_\varphi(T)=\sum_{k=0}^{2n+1}(-1)^k\left(\frac{T}{p^{n+n_0/2}}\right)^k\Big(\sum_{0\leqslant r\leqslant n}B_{\varphi,k}(r)\Big),$$

where

$$\begin{aligned}B_{\varphi,k}(r)=&\Big\{a_{n,k}(r)-\frac{p^r(p^{n-r}-1)(p^{n-r-1+n_0}+p^\partial)}{p^{n+n_0/2}}a_{n,k-1}(r+1)\\&-p^{r+\partial-(n+n_0/2)}a_{n,k-1}(r)\Big\}\varphi(r,0)\\&+p^{-(n+n_0/2)}a_{n,k-1}(r+1)\varphi'(r,0)\\&+p^{r-(n+n_0/2)}\{a_{n,k-1}(r)-a_{n,k-1}(r+1)\}\varphi''(r,0).\end{aligned}$$

Proof. Put

$$P_\varphi(T)=\sum_{l=0}^{\infty}(-1)^lB'_\varphi(l)T^l.$$

From Lemma 6, we have

$$\begin{aligned}B'_\varphi(l)=&(-1)^l\sum_{k=0}^{2n+2}(-1)^kp^{-k(n+n_0/2)}\\&\times\sum_{0\leqslant r\leqslant n+1}a_{n+1,k}(r)\sigma_\varphi(\tilde{c}_{n+1}^{(r)})\varphi\left(\begin{pmatrix}p^{l-k}&&\\&1&\\&&p^{k-l}\end{pmatrix}\right)\\=&\sum_{k=0}^{2n+2}(-1)^{k+l}p^{-k(n+n_0/2)}\sum a_{n+1,k}(r)\\&\times\{\delta(l\geqslant k)p^{2n+n_0}\varphi(r-1,l-k+1)+\delta(l\geqslant k)p^r\varphi(r,l-k)\\&+\delta(l\geqslant k)\varphi(r-1,l-k-1)+\delta(l>k)p^{r-1}(p^\partial-1)\varphi(r-1,l-k)\\&+\delta(l>k)p^{r-2}(p^{n-r+2}-1)(p^{n-r+1+n_0}+p^\partial)\varphi(r-2,l-k)\\&+\delta(l=k)\varphi'(r-2,0)-\delta(l=k)p^{r-2}\varphi''(r-2,0)\\&+\delta(l=k)p^{r-1}\varphi''(r-1,0)-\delta(l=k)p^{r-1}\varphi(r-1,0)\},\end{aligned} \tag{3.14}$$

where the symbol $\delta((*))$ means 1 or 0 according as the condition $(*)$ is satisfied or not. We write the right hand side of (3.14) as

$$\sum_{\substack{0\leq m\leq l\\0\leq r\leq n}} u_{l,m}(r)\varphi(r,m)+\sum_{0\leq r\leq n-1} u'_l(r)\varphi'(r,0)+\sum_{0\leq r\leq n} u''_l(r)\varphi''(r,0)$$

From (iii) of Lemma 4, we have $u_{l,m}(r)=0$ if $m\geq 1$, and

$$\begin{aligned}u_{l,0}(r)&=p^{-(n+n_0/2)(l-1)}\{a_{n+1,l+1}(r+1)-p^{r-(n+n_0/2)}(p^{n-r}-1)\\&\quad\times(p^{n-r-1+n_0}+p^{\partial})a_{n+1,l}(r+2)-p^{r+\partial-(n+n_0/2)}a_{n+1,l}(r+1)\}\\&=p^{-(n+n_0/2)l}\{a_{n,l}(r)-p^{r-(n+n_0/2)}(p^{n-r}-1)(p^{n-r-1+n_0}+p^{\partial})\\&\quad\times a_{n,l-1}(r+1)-p^{r+\partial-(n+n_0/2)}a_{n,l-1}(r)\}.\end{aligned}$$

Here we used the inductive property (ii) in Lemma 4. The values of $u'_l(r)$ and $u''_l(r)$ are easily seen, and our assertion is verified. Q.E.D.

3-3. Changing a $\boldsymbol{Z}_p$ basis of L_n, we may assume that

$$(3.15)\qquad S_n=\begin{pmatrix} & & J_{n'}\\ & \tilde{S}'_0 & \\ J_{n'} & & \end{pmatrix},\quad \tilde{S}'_0=\begin{pmatrix} N(\xi)s & \\ & S'_0\end{pmatrix},\quad \xi=\begin{pmatrix}0\\N(\xi)^{-1}\\0\\0\end{pmatrix},$$

where s is a unit of $\boldsymbol{Z}_p$, S'_0 is an anisotropic symmetric matrix of size n'_0 ($n'_0=n_0+1$ or n_0-1) and n' is the $\boldsymbol{Q}_p$-rank of $H(\xi)$. We fix such a realization, and put

$$S'_{n'}=\begin{pmatrix} & & J_{n'}\\ & S'_0 & \\ J_{n'} & & \end{pmatrix}.$$

We define $G'_{n'}$, $K'_{n'}$, $T'_{n'}(1)$, $\tilde{e}^{(n)}_{n'}$, $e^{(r)}_{n'}$ or $P'_{n'}(T)$ in the same way as G_n, K_n, $T_n(1)$, $\tilde{c}^{(r)}_n$, $c^{(r)}_n$ or $P_n(T)$, respectively.

Lemma 7.

$$\{g\in T_n(1)\,|\,g^{-1}\xi\in\hat{L}_n\}=T'_{n'}(1)K_n.$$

Proof. Take any element g in $T_n(1)$ such that $g^{-1}\xi\in\hat{L}_n$. We shall prove that $g\in T'_{n'}(1)K_n$. From the Iwasawa decomposition (2.6) we may assume that

$$g\in\begin{pmatrix} a & & \\ & b & \\ & & J_{n'}{}^{t}a^{-1}J_{n'}\end{pmatrix}\begin{pmatrix}1_{n'} & X_Z & Y_Z+Y\\ & 1_{n_0'+1} & Z\\ & & 1_{n'}\end{pmatrix},$$

where $a \in GL_{n'}(\boldsymbol{Q}_p)$, $b \in SO(\tilde{S}'_0)$, $Z \in M_{n_0'+1,n'}(\boldsymbol{Q}_p)$, $X_Z = -J_{n'}{}^tZ\tilde{S}'_0$, $Y_Z = -\frac{1}{2}J_{n'}{}^tZ\tilde{S}'_0Z$ and $Y \in M_{n'}(\boldsymbol{Q}_p)$ satisfying $J_{n'}Y + {}^tYJ_{n'} = 0$. Let us show that if $b^{-1}\xi \in \hat{L}_n$, then b is in $M_{n_0'+1}(\boldsymbol{Z}_p)$. Put $b = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where $\alpha \in \boldsymbol{Q}_p$, $\beta \in M_{1,n_0'}(\boldsymbol{Q}_p)$, $\gamma \in M_{n_0',1}(\boldsymbol{Q}_p)$ and $\delta \in M_{n_0'}(\boldsymbol{Q}_p)$. We know

$$(3.16) \qquad \begin{pmatrix} N(\xi)s & \\ & S'_0 \end{pmatrix} = \begin{pmatrix} N(\xi)s\alpha^2 + S'_0[\gamma] & N(\xi)s\alpha\beta + {}^t\gamma S'_0\delta \\ N(\xi)s\,{}^t\beta\alpha + {}^t\delta S'_0\gamma & N(\xi)s\,{}^t\beta\beta + {}^t\delta S'_0\delta \end{pmatrix}.$$

Since $b^{-1}\begin{pmatrix} N(\xi)^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha/N(\xi) \\ sS_0'^{-1}\,{}^t\beta \end{pmatrix} \in \tilde{S}_0'^{-1}\boldsymbol{Z}_p^{n_0'+1}$, we obtain $\alpha \in \boldsymbol{Z}_p$ and $\beta \in M_{1,n_0'}(\boldsymbol{Z})$. Comparing (1, 1) block of (3.16) we have $S'_0[\gamma] \in 2\boldsymbol{Z}_p$ (here we have used the fact that $N(\xi) \in 2\boldsymbol{Z}_p$). Since S'_0 is anisotropic, we have $\gamma \in M_{n_0',1}(\boldsymbol{Z}_p)$. Similarly by comparing (2, 2) block of (3.16), we know that $\delta \in M_{n_0'}(\boldsymbol{Z}_p)$, and b is in $M_{n_0'+1}(\boldsymbol{Z}_p)$. Thus we may assume that $b=1$. Put $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, where $z_1 \in M_{1,n'}(\boldsymbol{Q}_p)$ and $z_2 \in M_{n_0',n'}(\boldsymbol{Q}_p)$. Since $g^{-1}\xi \in \hat{L}_n$, we know that $z_1 \in M_{1,n'}(\boldsymbol{Z}_p)$, and we may assume that $z_1 = 0$. Then g belongs to $G'_{n'} \cap T_n(1)$, the above statement has been checked. Q.E.D.

We denote by $\mathscr{L}'_{n'}$ the Hecke algebra determined by the pair $(G'_{n'}, K'_{n'})$. Hereafter we suppose that $\varphi \in W^{\chi}_{n+1,\xi}$ is a simultaneous eigen function of $\mathscr{L}'_{n'}$, and denote by σ'_φ the homomorphism of $\mathscr{L}'_{n'}$ to $\boldsymbol{C}$ determined by φ:

$$(3.17) \qquad \phi * \varphi = \sigma'_\varphi(\phi)\varphi \quad (\phi \in \mathscr{L}'_{n'})$$

Lemma 8.

(i) $\varphi(r, 0) = \sigma'_\varphi(\tilde{e}^{(r)}_{n'})\varphi(1)$,

(ii) $\varphi'(r, 0) = p^{n'}C'\sigma'_\varphi(\tilde{e}^{(r)}_{n'})\varphi(1)$,

(iii) $\varphi''(r, 0) = C''\sigma'_\varphi(\tilde{e}^{(r)}_{n'})\varphi(1)$,

where $C' = \sum_{\substack{z \in p^{-1}\boldsymbol{Z}_p^{n_0'+1}/\boldsymbol{Z}_p^{n_0'+1} \\ \frac{1}{2}\tilde{S}_0'[z] \in p^{-1}\boldsymbol{Z}_p}} \chi\left(\tilde{S}'_0\left(\begin{pmatrix} N(\xi)^{-1} \\ 0 \end{pmatrix}, z\right)\right)$, *and*

$$C'' = \sum_{\substack{z \in p^{-1}L_0/L_0 \\ \frac{1}{2}S_0[z] \in p^{-1}\boldsymbol{Z}_p}} \chi\left(S_n\left(\xi, \begin{pmatrix} 0 \\ z \\ 0 \end{pmatrix}\right)\right).$$

Proof. Lemma 7 assures that we can take a set of representatives of $\tilde{e}^{(r)}_{n'}/K'_{n'}$ as that of $\{g \in \tilde{c}^{(r)}_n \mid g^{-1}\xi \in \hat{L}_n\}/K_n$. Thus we have

$$\varphi(r, 0) = \sum_{g \in \tilde{e}^{(r)}_{n'}/K'_{n'}} \varphi\left(\begin{pmatrix} 1 & & \\ & g & \\ & & 1 \end{pmatrix}\right) = \sigma'_\varphi(\tilde{e}^{(r)}_{n'})\varphi(1).$$

We shall prove (ii). From the definition (3.9), we get

$$\varphi'(r, 0)=\sum_{\substack{g\in e_{n'}^{(r)}/K'_{n'}\\ X\in p^{-1}L_n/L_n\\ gX\in p^{-1}L_n\\ \frac{1}{2}S_n[X]\in p^{-1}\boldsymbol{Z}_p}} \chi(S_n(\xi, X))\varphi\left(\begin{pmatrix}1 & g\\ & 1\end{pmatrix}\right)$$

$$=\Big\{\sum_{\substack{X\in p^{-1}L_n/L_n\\ e_{n'}^{(r)}X\in p^{-1}L_n\\ \frac{1}{2}S_n[X]\in p^{-1}\boldsymbol{Z}_p}} \chi(S_n(\xi, X))\Big\}\varphi(r, 0).$$

It is easy to see that the coefficient of $\varphi(r, 0)$ coincides to $p^{n'}C'$. (iii) is proved quite similarly. Q.E.D.

Let ∂' denote the dimension of the vector space $\{z\in p^{-1}\boldsymbol{Z}_p^{n_0'}\,|\,\frac{1}{2}S_0'[X]\in p^{-1}\boldsymbol{Z}_p\}/\boldsymbol{Z}_p^{n_0'}$ over $\boldsymbol{Z}_p/p\boldsymbol{Z}_p$ (i.e., $\partial'=\partial(S_0')$).

Lemma 9.

(i) $\partial'=\begin{cases}\partial \text{ or } \partial-1 & \text{if } n_0'=n_0-1,\\ \partial & \text{if } n_0'=n_0+1,\end{cases}$

(ii) $C''=\begin{cases}p^{\partial} & \text{if } \partial'=\partial,\\ 0 & \text{if } \partial'=\partial-1,\end{cases}$

(iii) $C'=\begin{cases}p^{\partial} & \text{if } \partial'=\partial \text{ and } n_0'=n_0-1,\\ -p^{n_0} & \text{if } \partial'=\partial \text{ and } n_0'=n_0+1,\\ 0 & \text{if } \partial'=\partial-1.\end{cases}$

This lemma is easily checked by using the complete list of S_0 in [3, Satz 9.7].

Proposition 2. *Let φ be an element of $W_{n+1,\xi}^{\chi}$. Assume that φ is an eigen function of $\mathscr{L}'_{n'}$ and $\mathscr{L}_{n+1}$, and denote by σ'_φ and σ_φ the homomorphisms defined by* (3.17) *and* (3.11), *respectively. Then the following identity holds.*

$$P_\varphi(T)=Q_\varphi(T)F_\varphi(T)=P'_{n'}\left(\frac{T}{p^{n+(n_0+1)/2}};\sigma'_\varphi\right)\varphi(1)$$

$$\times\begin{cases}1 & \text{if } n'=n \text{ and } \partial'=\partial,\\ (1+p^{\partial-(n+n_0)}T) & \text{if } n'=n \text{ and } \partial'=\partial-1,\\ (1-p^{-(n+n_0)}T)(1+p^{\partial-(n+n_0)}T) & \text{if } n'=n-1 \text{ and } \partial'=\partial,\end{cases}$$

where $P'_{n'}(T;\sigma'_\varphi)$ denotes the image of $P'_{n'}(T)$ by σ'_φ.

Proof. Suppose that $n'=n$ and $\partial'=\partial$. We can write $P'_n(T)$ in the form

$$P'_n(T)=\sum_{k=0}^{2n}(-1)^k(\sum_{0\leqslant r\leqslant n} b_{n,k}(r)\tilde{e}_n^{(r)})T^k.$$

By induction on n, we shall prove

(3.18) $\quad B_{\varphi,k}(r)=p^{-k/2}b_{n,k}(r)\sigma'_\varphi(\tilde{e}_n^{(r)})\varphi(1).\quad (0\leqslant k\leqslant 2n,\ 0\leqslant r\leqslant n)$

From the above two lemmata, we know

$$B_{\varphi,k}(r)=\{a_{n,k}(r)-(p^{n-r-1+n_0/2}-p^{-1+n_0/2})a_{n,k-1}(r+1)\}\sigma'_\varphi(e_n^{(r)})\varphi(1).$$

Clearly (3.18) holds for $n=0$; so we assume that (3.18) holds for n. If $r\geqslant 1$, then

(3.19) $\quad a_{n+1,k}(r)-p^{n_0/2}(p^{n-r}-p^{-1})a_{n+1,k-1}(r+1)-p^{-k/2}b_{n+1,k}(r)$
$(0\leqslant k\leqslant 2n+2)$

is equal to

$$p^{-(n+n_0/2)}\{a_{n,k-1}(r-1)-p^{n_0/2}(p^{n-(r-1)-1}-p^{-1})a_{n,k-2}(r) -p^{-(k-1)/2}b_{n,k-1}(r-1)\},$$

from (ii) of Lemma 4, and it vanishes by the induction assumption. Let r be 0. Then (3.19) is equal to

$$\begin{aligned} &a_{n,k}(0)+a_{n,k-2}(0)-p^{-(n+n_0/2)}(p^\partial-1)a_{n,k-1}(0)\\ &\quad -p^{-(n+n_0/2)}(p^n-1)(p^{n-1+n_0}+p^\partial)a_{n,k-1}(1)-(1-p^{-1-n})a_{n,k-2}(0)\\ &\quad -p^{-k/2}\{b_{n,k}(0)+b_{n,k-2}(0)-p^{-(n+(n_0-1)/2)}(p^\partial-1)b_{n,k-1}(0)\\ &\quad -p^{-(n+(n_0-1)/2)}(p^n-1)(p^{n-1+n_0-1}+p^\partial)b_{n,k-1}(1)\}.\end{aligned}$$

Using the induction assumption and the fact

$$\begin{aligned} &p^{1-(n+n_0/2)}(p^{n-1}-1)(p^{n-1+n_0-1}+p^\partial)a_{n,k-2}(2)\\ &\quad =a_{n,k-1}(1)+a_{n,k-3}(1)-p^{1-(n+n_0/2)}(p^\partial-1)a_{n,k-2}(1)\\ &\qquad -p^{1-(n+n_0/2)}a_{n,k-2}(0),\end{aligned}$$

we know that (3.19) is 0. Hence our assertion is proved. The other cases follow similarly. Q.E.D.

§ 4. Main Theorem

4-1. In this section we shall state our main theorem and its proof. We use the same notations as in Section 1. For each prime p, we denote by $\mathscr{H}_p$ the Hecke algebra $\mathscr{L}(\tilde{Q}; Z_p^{q+2})$. Let $\mathscr{H}_p$ act on $\mathfrak{S}_k(K_f)$ by

(4.1) $$(F*\phi)(g)=\int_{G_p} F(gh^{-1})\phi(h)dh \quad (F\in \mathfrak{S}_k(K_f),\ \phi\in \mathscr{H}_p).$$

Note that

(4.2) $$\langle F_1*\phi, F_2\rangle=\langle F_1, F_2*\tilde{\phi}\rangle \quad (F_1, F_2\in \mathfrak{S}_k(K_f),\ \phi\in \mathscr{H}_p),$$

where $\tilde{\phi}(g)=\overline{\phi(g^{-1})}$ ($\overline{}$ denotes the complex conjugation). Thus, if $\mathscr{H}_p$ is commutative, then each element of $\mathscr{H}_p$ acts on $\mathfrak{S}_k(K_f)$ as a normal operator with respect to the Petersson inner product $\langle\ ,\ \rangle$.

We denote by $\mathscr{P}_1$ the set of all primes p such that L_p is not maximal $\boldsymbol{Z}_p$-integral with respect to Q. Since $Q\in GL_q(\boldsymbol{Z}_p)$ for almost all p, $\sharp\mathscr{P}_1$ is finite. We note that if $p\notin\mathscr{P}_1$ then $\mathscr{H}_p$ is commutative. Hereafter we assume that $F\in\mathfrak{S}_k(K_f)$ is a simultaneous eigen function of all $\mathscr{H}_p$ such that $p\notin\mathscr{P}_1$. We denote by $\sigma_{F,p}$ the homomorphism from $\mathscr{H}_p$ to $\boldsymbol{C}$ determined by F:

(4.3) $$F*\phi=\sigma_{F,p}(\phi)F \quad (\phi\in\mathscr{H}_p).$$

For any finite set $\mathscr{P}$ containing $\mathscr{P}_1$, and for $s\in\boldsymbol{C}$, we define the L-function $L_{\mathscr{P}}(F; s)$ by

(4.4) $$L_{\mathscr{P}}(F; s)=\prod_{p\notin\mathscr{P}} L_p(\tilde{Q}; \sigma_{F,p}; s),$$

where $L_p(\tilde{Q}; \sigma_{F,p}; s)$ is defined in (2.22). Since every coefficient of $L_p(\tilde{Q}; \sigma_{F,p}; s)^{-1}$ in p^{-s} is bounded by p^{A_F}, where A_F is a positive constant not depending on p, the product in (4.4) converges absolutely in some right half plane.

We fix $g_f\in G^*_{A,f}$ and $\xi\in V_Q$ such that $F_\chi(g_f; \xi)\neq 0$. Denote by $\mathscr{H}'_p$ the Hecke algebra $\mathscr{L}(Q^{(2)}; L(g_f)\cap V^{(2)})$. It acts on $\mathscr{V}(g_f; \xi)$ by the convolution, and has a property similar to (4.2). Let $N(\xi)$ be the minimal (positive) integer such that $N(\xi)\xi$ is in L. Assume that ξ is a primitive element of $\hat{L}_p$; then $L_p=(V^{(1)}\cap L)_p\oplus(V^{(2)}\cap L)_p$ if and only if $N(\xi)Q[\xi]\in\boldsymbol{Z}_p^\times$. Denote by $\mathscr{P}(g_f; \xi)$ the minimal set such that if p does not belong to $\mathscr{P}(g_f; \xi)$, then

(4.5) $$\begin{cases} \text{i)} & p\notin\mathscr{P}_1, \\ \text{ii)} & \xi \text{ is a primitive element in } \hat{L}_p, \\ \text{iii)} & N(\xi)Q[\xi]\in\boldsymbol{Z}_p^\times, \\ \text{iv)} & \text{the } p\text{-part of } g_f \text{ is in } K_p. \end{cases}$$

Clearly the number of elements of $\mathscr{P}(g_f; \xi)$ is finite, and if $p\notin\mathscr{P}(g_f; \xi)$ then $\mathscr{H}'_p$ is commutative. When an element f of $\mathscr{V}(g_f; \xi)$ is an eigen function for such $\mathscr{H}'_p$, we denote by $\sigma_{f,p}$ the homomorphism from $\mathscr{H}'_p$ to $\boldsymbol{C}$ determined by f:

$$f*\phi=\sigma_{f,p}(\phi)f \quad (\phi \in \mathscr{H}'_p). \tag{4.6}$$

For any finite set $\mathscr{P}$ of primes containing $\mathscr{P}(g_f; \xi)$ and $s \in \boldsymbol{C}$, we define $L_{\mathscr{P}}(f; s)$ by

$$L_{\mathscr{P}}(f; s)=\prod_{p \notin \mathscr{P}} L_p(Q^{(2)}; \sigma_{f,p}; s). \tag{4.7}$$

Now we state our main result.

Theorem 1. *Let F be an element of $\mathfrak{S}_k(K_f)$, and assume that it is a simultaneous eigen function of $\mathscr{H}_p$ ($\forall p \notin \mathscr{P}_1$). Fix a $g_f \in G^*_{A,f}$ and a $\xi \in V_Q$, and put $\mathscr{P}_2=\mathscr{P}(g_f; \xi)$. Take an element f of $\mathscr{V}(g_f; \xi)$, which is a simultaneous eigen function of all $\mathscr{H}'_p$ ($\forall p \notin \mathscr{P}_2$). Take a finite set $\mathscr{P}$ of primes containing $\mathscr{P}_2$. Then the following Euler product expansion holds in some right half plane.*

$$\begin{aligned} &\sum_{\substack{m=1 \\ (m,p)=1 \\ \text{for } \forall p \in \mathscr{P}}}^{\infty} \mu(\xi)^{-1} \sum_{i=1}^{h} a(u_i g_f; m\xi) \frac{\overline{f(u_i)}}{e(\xi)_i} m^{-(s+k-q/2)} \\ &= \left\{\mu(\xi)^{-1} \sum_{i=1}^{h} a(u_i g_f; \xi) \frac{\overline{f(u_i)}}{e(\xi)_i}\right\} L_{\mathscr{P}}(F; s) L_{\mathscr{P}}(\bar{f}; s+\tfrac{1}{2})^{-1} \\ &\times \begin{cases} \displaystyle\prod_{\substack{p \notin \mathscr{P} \\ \partial_p \neq \partial'_p}} (1+p^{-s+\partial_p-1/2}) & \text{if } q \text{ is odd,} \\ \displaystyle\prod_{p \notin \mathscr{P}} (1-p^{-s})(1+p^{-s+\partial_p}) \prod_{\substack{p \notin \mathscr{P} \\ \partial_p \neq \partial'_p}} (1+p^{-s-1+\partial_p}) & \text{if } q \text{ is even.} \end{cases} \end{aligned} \tag{4.8}$$

Here $a(\ ;\ )$ is defined in (1.11), *$\{u_1, \cdots, u_h\}$ is a complete set of representatives of $H(\xi)_Q \backslash H(\xi)_A / H(\xi)_\infty M(g_f; \xi)_f$ such that $u_{i,\infty}=1$ $(1 \leqslant i \leqslant h)$, $e(\xi)_i=\#\{H(\xi)_Q \cap M(u_i g_f; \xi)_f\}$, and $\mu(\xi)=\sum_{i=1}^{h} e(\xi)_i^{-1}$. For $p \notin \mathscr{P}_2$ we put $\partial_p=\partial(\tilde{Q}; Z_p^{q+2})$ and $\partial'_p=\partial(Q^{(2)}; (V^{(2)} \cap L)_p)$.*

Remark. (i) If f is an eigen function of $\mathscr{H}'_p$, then $\bar{f}$ is also an eigen function of $\mathscr{H}'_p$.

(ii) $\partial_p=\partial'_p=0$ for almost all p.

(iii) Note that

$$\varphi^f_{F,\xi}(g_f)=\mu(\xi)^{-1} \sum_{i=1}^{h} a(u_i g_f; \xi) \frac{\overline{f(u_i)}}{e(\xi)_i} e[Q(\xi, \mathscr{Z}_0)].$$

Therefore the identity (4.8) is not trivial for a suitable choice of g_f, ξ and f (Lemma 1).

This theorem is reduced to some local problems. Let p be a prime not belonging to $\mathscr{P}$, and put

$$W(g_f;\xi)_p^\chi=\{\varphi;\ M(g_f;\xi)_p\backslash G_p/K_p\longrightarrow \boldsymbol{C}\mid \varphi(\gamma_X g)=\chi(Q(\xi,X))\varphi(g)\quad \forall X\in V_p\}. \tag{4.9}$$

Note that as a function on G_p, $\varphi_{F,\xi}^f$ belongs to $W(g_f;\xi)_p^\chi$. Since p is not in $\mathscr{P}_2$, this space is nothing but the space $W_{n_p+1,\xi}^\chi$ (n_p is the Witt index of Q over $\boldsymbol{Q}_p$), so we can use the results in Section 3.

4-2. Now we are going to prove Theorem 1. Let $c(t)$ denote the characteristic function of $\boldsymbol{R}^\times\times\prod_{p\in\mathscr{P}}\boldsymbol{Z}_p^\times\times\prod'_{p\notin\mathscr{P}}\boldsymbol{Q}_p^\times$ in $\boldsymbol{Q}_A^\times$, where $\prod'$ means the usual restricted product. In two manners we calculate

$$\int_{Q_A^\times} c(t)\varphi_{F,\xi}^f\left(\begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1}\end{pmatrix} g_f\right)|t|_A^{s-q/2}d^\times t, \tag{4.10}$$

where $d^\times t=\prod d^\times t_v$ is a Haar measure of $Q_A^\times$. First, this integral is equal to

$$\int_{Q^\times\backslash Q_A^\times}\sum_{m\in Q^\times} c(mt)\varphi_{F,m\xi}^f\left(\begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1}\end{pmatrix} g_f\right)|t|_A^{s-q/2}d^\times t$$

$$=\int_0^\infty \sum_{\substack{m\in Q^\times\\ m\in Z_p^\times\\ (\forall p\in\mathscr{P})}}\left\{\mu(\xi)^{-1}\sum_{i=1}^h a(u_ig_f;m\xi)\frac{\overline{f(u_i)}}{e(\xi)_i}t^{s+k-q/2}e[mtQ(\xi,\mathscr{X}_0)]\right\}d^\times t.$$

Since ξ is primitive in $\hat{L}_p$ for $p\notin\mathscr{P}$, $a(u_ig_f;m\xi)=0$ unless $m\in\boldsymbol{Z}_p$. Hence (4.10) is equal to

$$\sum_{\substack{m=1\\ (m,p)=1\\ (\forall p\in\mathscr{P})}}^{\infty}\mu(\xi)^{-1}\sum_{i=1}^h a(u_ig_f;m\xi)\frac{\overline{f(u_i)}}{e(\xi)_i}m^{-(s+k-q/2)}\frac{\Gamma(s+k-q/2)}{A^{s+k-q/2}}, \tag{4.11}$$

where we have put $2\pi\sqrt{-1}Q(\xi,\mathscr{X}_0)=-A$ $(A>0)$. On the other hand, if $p\notin\mathscr{P}_2$, the function

$$\varphi(g)=\varphi_{F,\xi}^f(g'g)\quad (g\in G_p),$$

where g is a fixed element of G_A whose p-part is 1, belongs to $W_{n_p+1,\xi}^\chi$ (n_p is the Witt index of Q over $\boldsymbol{Q}_p$). Note that

$$\begin{aligned}\phi*\varphi&=\sigma_{\bar{f},p}(\bar{\check{\phi}})\varphi\quad (\phi\in\mathscr{H}'_p)\\ \varphi*\phi&=\sigma_{F,p}(\phi)\varphi\quad (\phi\in\mathscr{H}_p).\end{aligned} \tag{4.12}$$

From Proposition 2, we have

$$\int_{Q_p^\times} \varphi^f_{F,\xi}\left(g'\begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1}\end{pmatrix}\right)|t|_p^{s-q/2}d^\times t$$
$$=\varphi^f_{F,\xi}(g')P_\varphi(p^{-(s-q/2)})Q_\varphi(p^{-(s-q/2)})^{-1}$$
$$=\varphi^f_{F,\xi}(g')L_p(\tilde{Q};\sigma_{F,p};s)L_p(Q^{(2)};\sigma_{\bar{f},p};s+\tfrac{1}{2})^{-1}$$
$$\times\begin{cases} 1 & \text{if } q \text{ is odd and } \partial'_p=\partial_p,\\ (1+p^{-s+\partial_p-1/2}) & \text{if } q \text{ is odd and } \partial'_p=\partial_p-1,\\ (1-p^{-s})(1+p^{-s+\partial_p}) & \text{if } q \text{ is even and } \partial'_p=\partial_p,\\ (1-p^{-s})(1+p^{-s+\partial_p})(1+p^{-s-1+\partial_p}) & \text{if } q \text{ is even and } \partial'_p=\partial_p-1.\end{cases}$$

Therefore (4.10) is equal to

$$\frac{\Gamma(s+k-q/2)}{A^{s+k-q/2}}\mu(\xi)^{-1}\sum_{i=1}^{h}a(u_ig_f;\xi)\frac{\overline{f(u_i)}}{e(\xi)_i}L_{\mathscr{P}}(F;s)L_{\mathscr{P}}(\bar{f};s+\tfrac{1}{2})^{-1}$$

$$\text{(4.13)}\qquad \times\begin{cases} \prod\limits_{\substack{\partial'_p\neq\partial_p\\ p\notin\mathscr{P}}}(1+p^{-s+\partial_p-1/2}) & \text{if } q \text{ is odd,}\\ \prod\limits_{p\notin\mathscr{P}}(1-p^{-s})(1+p^{-s+\partial_p})\prod\limits_{\substack{p\notin\mathscr{P}\\ \partial'_p\neq\partial_p}}(1+p^{-s-1+\partial_p}) & \text{if } q \text{ is even.}\end{cases}$$

Comparing (4.11) with (4.13), we obtain our theorem. Q.E.D.

§ 5. Some related problems

5-1. In this subsection, we prove Proposition 3, which asserts that in Theorem 1 if we can take a constant function as f, then F must be a kind of old form. We use the same notations as in Theorem 1. For each prime p, we denote by n_p the $\boldsymbol{Q}_p$-rank of G^* (thus the $\boldsymbol{Q}_p$-rank of G is n_p+1), and we put $n_{0,p}=q-2n_p$. Let $\mathscr{P}_3$ be a finite set of primes including $\mathscr{P}_2=\mathscr{P}(g_f;\xi)$ and satisfies the condition: if $p\notin\mathscr{P}_3$ then $Q\in GL_q(\boldsymbol{Z}_p)$. Therefore if p is not in $\mathscr{P}_3$, then $0\leqslant n_{0,p}\leqslant 2$ and $\partial_p=0$.

Proposition 3. *Notations being as above, and we assume that*

$$\text{(5.1)}\qquad \sum_{i=1}^{h}a(u_ig_f;\xi)e(\xi)_i^{-1}\neq 0.$$

Let p be a prime not belonging to $\mathscr{P}_3$ and assume that $n_p\geqslant 2$. Then between $\sigma_{F,p}(\tilde{c}^{(r)}_{n_p+1})$ $(0\leqslant r\leqslant n_p+1)$, the following (n_p-1) relations hold:

$$\text{(5.2)}\qquad \sigma_{F,p}(\tilde{c}^{(r)}_{n_p+1})=A_p^{(r)}\sigma_{F,p}(\tilde{c}^{(2)}_{n_p+1})-B_p^{(r)}\sigma_{F,p}(\tilde{c}^{(1)}_{n_p+1})+C_p^{(r)}\sigma_{F,p}(\tilde{c}^{(0)}_{n_p+1})$$
$$(3\leqslant r\leqslant n_p+1),$$

Here

$$A_p^{(r)}=\frac{p^{r-1}-1}{p^{r-2}(p^{n_p}-1)(p^{n_p+n_{0,p}-1}+1)}\times D_p^{(r)},$$

$$B_p^{(r)}=\frac{p^{r-2}-1}{p^{r-2}(p-1)}\times D_p^{(r)},$$

$$C_p^{(r)}=\frac{(p^{r-2}-1)(p^{r-1}-1)(p^{n_p+1}-1)(p^{n_p+n_{0,p}}+1)}{p^{r-2}(p-1)(p^2-1)(p^r-1)}\times D_p^{(r)},$$

$$D_p^{(r)}=\#(\tilde{c}_{n_p}^{(r-1)}/K_{n_p})$$

$$=\prod_{j=1}^{r-1}\frac{p^{j-1}(p^{n_p-j+1}-1)(p^{n_p-j+n_{0,p}}+1)}{p^j-1}.$$

Proof. We fix p ($p \notin \mathscr{P}_3$), and abbreviate n_p to n. We assume that $n_{0,p}=1$, since the other cases are proved quite similarly. We use the same notations as in Section 3. To prove this assertion, it is sufficient to show that if $\varphi \in W_{n+1,\xi}^{\chi}$ is left $H(\xi)$-invariant and $\varphi(1)\neq 0$, then the relation (5.2) holds. We may assume that

$$\xi=\begin{pmatrix}0\\ \xi_1\\ 0\end{pmatrix},\qquad \xi_1=\begin{pmatrix}1\\ 0\\ c\end{pmatrix}\in \hat{L}_1.$$

First we reformulate Lemma 5 in a more precise form. Put for l, $m\geqslant 0$,

$$h'_m=\begin{pmatrix}p^m & & \\ & 1 & \\ & & p^{-m}\end{pmatrix}\in G_1,\qquad h_m=\begin{pmatrix}1_{n-1} & & \\ & h'_m & \\ & & 1_{n-1}\end{pmatrix}\in G_n,$$

and

$$g_{m,l}=\begin{pmatrix}p^{m+l} & & \\ & h_m & \\ & & p^{-(m+l)}\end{pmatrix}\in G_{n+1}.$$

Then we have

$$\bigcup_{\varphi\in W_{n+1,\xi}^{\chi}}\operatorname{supp}\varphi\subset\bigcup_{\substack{l\geqslant 0\\ m\geqslant 0}}NH(\xi)g_{m,l}K_{n+1},\tag{5.3}$$

where $N=\{\gamma_X\,|\,X\in V_n\}$. Indeed, from Lemma 5 it remains to prove that for each $b\in G_n$,

$$b\in H(\xi)h_mK_n\tag{5.4}$$

where $m=m_b$. We put $b^{-1}\xi=p^{-m}\xi'$, where ξ' is a primitive element of $\hat{L}_n$. Then there exists a k in K_n such that

$$k\xi'=\begin{pmatrix}0\\ \xi_1'\\ 0\end{pmatrix},\qquad \xi_1'=\begin{pmatrix}1\\ 0\\ c'\end{pmatrix}\in \hat{L}_1$$

Since $\frac{1}{2}S_n[\xi]=c=p^{-2m}c'$, we have

$$\xi_1'=h_m'^{-1}p^m\xi_1.$$

If we put $bk^{-1}h_m^{-1}=u$, then u belongs to $H(\xi)$, and (5.4) is proved. It is easily seen that the right hand side of (5.3) is a disjoint union. Now, we prove our proposition by using Lemma 6. Since $\varphi(\in W^{\chi}_{n+1,\xi})$ is left $H(\xi)$ invariant, each term appearing in the right hand side in Lemma 6 is written in terms of $\varphi(g_{m',l'})$. For $1\leqslant r\leqslant n+1$, we have

$$\sigma_{F,p}(\tilde{c}^{(r)}_{n+1})\varphi(1)=p^{2n+1}\varphi(r-1,\,1)+p^r\varphi(r,\,0)+\varphi'(r-2,\,0)-p^{r-2}\varphi(r-2,\,0). \tag{5.5}$$

From Lemma 7, Lemma 8, and Lemma 9, we know that

$$\varphi(j,\,0)=\#(\tilde{e}^{(j)}_{n'}/K'_{n'})\varphi(1),\qquad \varphi'(j,\,0)=\#(\tilde{e}^{(j)}_{n'}/K'_{n'})p^{n'}C'\varphi(1),$$

and

$$\varphi(j,\,1)=\#(\tilde{e}^{(j)}_{n'}/K'_{n'})\varphi(g_{0,1})+\{\#(\tilde{c}^{(j)}_{n}/K_n)-\#(\tilde{e}^{(j)}_{n'}/K'_{n'})\}\varphi(g_{1,0}),$$

where n' is the $\boldsymbol{Q}_p$-rank of $H(\xi)$ and C' is given in Lemma 9. Therefore the right hand side of (5.5) is written as a linear combination of $\varphi(1)$, $\varphi(g_{0,1})$ and $(g_{1,0})$, and their coefficients are easily calculated by (2.18). Cancelling $\varphi(g_{0,1})$ and $\varphi(g_{1,0})$ by using (5.5) for $r=1$ and $r=2$, we obtain our assertion. Q.E.D.

5-2. In a special case we give an integral representation of Rankin-Selberg type of the Dirichlet series in Theorem 1. We put

$$Q=\begin{pmatrix}N & 0\\ 0 & T\end{pmatrix}\quad\text{and}\quad \xi=\begin{pmatrix}N^{-1}\\ 0\end{pmatrix},$$

where N is a positive even integer and T is an even integral symmetric negative definite matrix of degree $q-1$, and assume that for each prime p, $\boldsymbol{Z}_p^q$ is a maximal $\boldsymbol{Z}_p$-integral lattice with respect to Q. Furthermore for the sake of simplicity, we assume that $\mathscr{Z}_0=\sqrt{-1}\xi$ ($\mathscr{Z}_0$ is the origin of $\mathscr{P}$). Note that in this case, $\mathscr{P}_2=\mathscr{P}(1;\xi)=\phi$. We denote by G'' the special orthogonal group of $\begin{pmatrix} & & 1\\ & T & \\ 1 & & \end{pmatrix}$, regarded as a subgroup of G. We define a maximal parabolic subgroup B'' of G'' by

$$B''_Q=\left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in G''_Q\right\}. \tag{5.6}$$

For any prime p,

$$K''_p=G''_p\cap SL_{q+1}(Z_p)$$

is a maximal compact subgroup of G''_p and G''_p has the Iwasawa decomposition $G''_p=B''_pK''_p$. We put

$$K''_\infty=K_\infty\cap G''_\infty,$$

where K_∞ is the stabilizer subgroup of $\mathscr{Z}_0$ in G^0_∞. Then we have $G''_\infty=B''_\infty K''_\infty$. We put $K''=\prod_v K''_v$, For any $g\in G''_A$, we put

$$g=\begin{pmatrix} t(g) & * & * \\ 0 & u(g) & * \\ 0 & 0 & t(g)^{-1}\end{pmatrix}k(g),$$

where $t(g)\in Q^\times_A$, $u(g)\in H(\xi)_A=SO(T)_A$ and $k(g)\in K''$.

Let F [resp. f] be an element of $\mathfrak{S}_k(K_f)$ [resp. $\mathscr{V}(1;\xi)$], and assume that F [resp. f] is a simultaneous eigen function of $\mathscr{H}_p$ [resp. $\mathscr{H}'_p$] for all p.

Theorem 2. *Let the assumptions be as above. Then*

$$\begin{aligned}&\left\{\mu(\xi)^{-1}\sum_{i=1}^{h}a(u_i;\xi)\frac{\overline{f(u_i)}}{e(\xi)_i}\right\}\times\left(\frac{2\pi}{N}\right)^{-(s+k-q/2)}\\ &\times\Gamma(s+k-q/2)\times L_\phi(F;s)\times L_\phi(\bar{f};s+\tfrac{1}{2})^{-1}\\ &\times\begin{cases}\prod_{\partial_p\neq\partial_p'}(1+p^{-s+\partial_p-1/2}) & \text{if } q \text{ is odd,}\\ \prod(1-p^{-s})(1+p^{-s+\partial_p})\prod_{\partial_p\neq\partial_p'}(1+p^{-s-1+\partial_p}) & \text{if } q \text{ is even,}\end{cases}\end{aligned}\tag{5.7}$$

has the following integral representation in some right half plane:

$$\int_{G''_Q\backslash G''_A}F(g)E(g,s-1/2:\bar{f})d\dot{g}. \tag{5.8}$$

Here $E(g,s;\bar{f})=\sum_{\gamma\in B''_Q\backslash G''_Q}|t(\gamma g)|_A^{s+(q-1)/2}\overline{f(u(\gamma g))}$, *and the other notations are the same as in Theorem* 1.

Proof. We start from (4.10). Put

$$\Phi^f_{F,\xi}(s)=\int_{Q^\times_A}\varphi^f_{F,\xi}\left(\begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1}\end{pmatrix}\right)|t|_A^{s-q/2}d^\times t. \tag{5.9}$$

As we have already seen in 4–2, it is enough to prove that $\Phi^f_{F,\xi}(s)$ has the integral representation (5.8). The right hand side of (5.9) is equal to

$$\int_{Q^\times\backslash Q_A^\times} \sum_{\varepsilon\in Q^\times} \varphi^f_{F,\varepsilon\xi}\left(\begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1}\end{pmatrix}\right)|t|_A^{s-q/2}d^\times t.$$

We can easily see that

$$\sum_{\varepsilon\in Q} F_\chi(g;\varepsilon\xi)=\int_{V_Q^{(2)}\backslash V_A^{(2)}} F(\gamma_Y g)dY \quad (g\in G_A).$$

Therefore we have

$$\text{(5.10)}\qquad \begin{aligned}\Phi^f_{F,\xi}(s)=&\int_{Q^\times\backslash Q_A^\times}\int_{H(\xi)_Q\backslash H(\xi)_A}\int_{V_Q^{(2)}\backslash V_A^{(2)}}\\ &F\left(\gamma_Y\begin{pmatrix} t & & \\ & u & \\ & & t^{-1}\end{pmatrix}\right)|t|_A^{s-q/2}\overline{f(u)}dY\,du\,d^\times t.\end{aligned}$$

Since $G''_A=B''_AK''$, taking a suitable right G''_A invariant measure on $B''_Q\backslash G''_A$, the right hand side of (5.10) is equal to

$$\begin{aligned}&\int_{B''_Q\backslash G''_A} F(g)|t(g)|_A^{s-1+q/2}\overline{f(u(g))}d\dot{g}\\ &=\int_{G''_Q\backslash G''_A} F(g)\{\sum_{\gamma\in B''_Q\backslash G''_Q}|t(\gamma g)|_A^{s-1+q/2}\overline{f(u(\gamma g))}\}d\dot{g}.\end{aligned}$$

So our theorem has been proved. Q.E.D.

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