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Hodge Structures Attached to Geometric Automorphic Forms

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Introduction

This short note is an introduction to representations of discrete series of real semisimple Lie groups, and to the Hodge theory of cohomology groups of discrete subgroups. All the materials in this paper are found in the literature except for minor changes of proofs. For these several years, I have been hoping that someone would write an article which contains "everything that number theorists have always wanted to know about discrete series $\cdots$ but were ashamed to ask". So this is written partly for myself.

To discuss everything on discrete series is out of my power, who have little experience in the representation theory. But in Chapter 1, I attempt to explain the basic results for discrete series: their definition, characters, realizations, and the K-type theorem (i.e. Blattner's conjecture). The proofs are not given. I refer to the papers of Harish-Chandra and the textbook of Warner [41] for the proofs of the fundamental facts on unitary representations and characters of discrete series. Because the realization of Narasimhan-Okamoto [31] is most suitable for our purpose, I discuss it in Section 1.3. Also I refer to the realization of Schmid [38] which is applicable to more general Lie groups. The realization of Parthasarathy [35] by spinors is omitted. The proof of Blattner's conjecture is completed by Schmid [40] and Hecht-Schmid [19]. We recall only its statement in Section 1.4.

In Chapter 2, I discuss automorphic cohomology groups, or automorphic harmonic forms, i.e. automorphic forms which generate representations of discrete series. The most important result in this chapter is the vanishing theorem (2.3.1) of Parthasarathy [34], which is the sharpest improvement of the vanishing results in [15], [31], [38]. It is a pure-dimensionality of cohomology groups with coefficients in certain holomorphic vector bundles on arithmetic quotients of bounded symmetric domains. Geometric automorphic forms in the title are elements of the

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possible non-vanishing cohomology groups.

Chapter 3 is devoted to the discussion of cohomologies of cocompact discrete subgroups of Lie groups, and the Hodge decompositions of cohomologies. Hodge theory for cohomology groups of discrete subgroups were discussed by Matsushima-Murakami [28], [29], more than two decades ago. Now we know that their results were not the best possible in respect of the vanishing of cohomology.

Borel-Wallach (and also Langlands, cf. [6], Chap. VII, § 6) improved this point to obtain a decomposition of cohomology groups. Their proof uses the theory of continuous cohomologies, i.e. the theory of cohomology groups with values in (differentiable vectors of) the representation spaces of infinite dimensional representations of Lie groups. Though it has some advantages (for example, the arguments apply not only for Hermitian symmetric cases, but also non-Hermitian cases), it is not so clear which component of their decomposition has what Hodge type.

Fortunately, we have another way to discuss the Hodge structures of cohomology groups of discrete series, i.e. the Hodge theory for cohomology groups with values in variations of Hodge structure, which is due to Zucker (and Deligne, cf. [42], [43]). In order to deduce a result which is equivalent to the decomposition of Borel-Wallach, from the results of Zucker [43], it suffices to apply the vanishing theorem of Chapter 2, as far as the discrete subgroup is cocompact. The main result is Theorem (3.3.2). In Section 3.4, we discuss the case where the discrete subgroups are cocompact in $Sp_{2n}(\boldsymbol{R})$. Because it is natural to expect that analogous result is also valid for non-cocompact discrete subgroups, we formulate in Section 3.5 some problems for Hodge structures attached to Siegel modular forms.

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Chapter 1. Representations of discrete series

In this chapter, we recall some basic facts on the representations of discrete series of connected real semisimple Lie groups: their definition, characters, realizations, and K-type theorem. We refer to the textbook of Warner [41] for the first two sections (especially see I, Chap. 4; II, Chap. 10).

§ 1.1. Definition and basic properties

Throughout this chapter, we consider a connected real semisimple Lie group G with finite center. Then the group G is a locally compact unimodular topological group satisfying the second axiom of countability. Our first task is to define the notion of square integrable representations for such groups.

(1.1.1) **Definition.** An irreducible unitary representation $\pi: G \to \mathrm{Aut}(H)$ on a Hilbert space H is said to be *square integrable*, if there exists a non-zero vector v in H such that the coefficient $g \mapsto (\pi(g)v, v)$ ($g \in G$, $v \in H$) is square integrable on G.

As we shall see in Theorem (1.1.10), the group G has a maximal compact subgroup K, which is a large compact subgroup of G in the sense of [41], I, Chap. 4. In this case the following is known.

(1.1.2) **Lemma.** ([41], I, Lemma 4.5.9.1). *Let (π, H) be an irreducible unitary representation of G on a Hilbert space H. Suppose that there are two non-zero vectors v_0, w_0 in H such that the coefficient $(\pi(g)v_0, w_0)$ is square integrable on G. Then there exists a (unique) positive real number d_π such that*

$$\int_G |(\pi(g)v, w)|^2 dg = d_\pi^{-1} \|v\|^2 \cdot \|w\|^2$$

for any v, w in H.

(1.1.3) **Definition.** The number d_π is called the *formal degree* of π. It depends on a choice of Haar measures of G.

Remark. When G is compact, and the Haar measure of G is normalized, the formal degree is the true degree of π.

The notions of the square integrability and the formal degree are invariant under unitary equivalences.

(1.1.4) **Definition.** Let $\hat{G}$ be the set of all unitary equivalence classes of irreducible representations of G. Then a unitary equivalence class $\pi \in \hat{G}$ is said to be *discrete*, if every member of this class is square integrable. The set $\hat{G}_d$ of all discrete classes in $\hat{G}$ is called the *discrete series* for G.

The irreducible square integrable unitary representations of G admit an alternative characterization by the following.

(1.1.5) **Proposition** ([41], I, Corollary 4.5.9.2). *Let π be an irreducible unitary representation of G on a Hilbert space H. Then the following conditions are equivalent:*

(i) *π is square integrable;*

(ii) *π is unitarily equivalent to a subrepresentation of the left (or right) regular representation of G on $L^2(G)$.*

The Schur Orthogonality Relations are valid for square integrable representations of G.

(1.1.6) **Theorem** (Godement, cf. [41], I, Theorem 4.5.9.3). *Let π and π' be irreducible square integrable unitary representations of G on Hilbert spaces H and H', respectively. Let $v_1, v_2 \in H$, $w_1, w_2 \in H'$.*

(i) *If π and π' are not unitarily equivalent, then*

$$\int_G (\pi(g)v_1, v_2)\overline{(\pi'(g)w_1, w_2)}dg = 0$$

(ii) *If π and π' are unitarily equivalent under a unitary equivalence $\psi: H \to H'$, then*

$$\int_G (\pi(g)v_1, v_2)\overline{(\pi'(g)w_1, w_2)}dg = d_\pi^{-1}(\psi v_1, w_1)\overline{(\psi v_2, w_2)},$$

where $d_\pi(=d_{\pi'})$ is the formal degree of π.

(1.1.7) **Corollary** ([41], I, Corollary 4.5.9.4). *Let (π, H) be an irreducible square integrable unitary representation of G. Put*

$$\phi_{v,w}(g) = (\pi(g)v, w) \quad (g \in G, v, w \in H).$$

Then for convolutions of coefficients, we have

$$\phi_{v_1,v_2} * \phi_{w_1,w_2} = d_\pi^{-1}(v_1, w_2)\phi_{w_1,v_2}.$$

Let us recall the notion of integrable representations.

(1.1.8) **Proposition** (Harish-Chandra, cf. [41], I, Proposition 4.5.9.5). *Let π be an irreducible unitary representation of G on a Hilbert space H. Then the following conditions are equivalent:*

(i) *There exists a non-zero vector v in H such that the function $g \mapsto (\pi(g)v, v)$ is integrable on G;*

(ii) *There exists a dense subspace H_0 of H such that for all v, w in H_0, the function $g \mapsto (\pi(g)v, w)$ is integrable on G.*

(1.1.9) **Definition.** An irreducible unitary representation (π, H) of G is said to be *integrable*, if it satisfies either one of the equivalent conditions of the preceding proposition.

It is known that an integrable unitary representation of G is square integrable. The converse is false.

Remark. Let (π, H) be an irreducible integrable unitary representation of G. Let H_K be the subspace of H consisting of all K-finite vectors in H. Then for any $v, w \in H_K$, the function $g \mapsto (\pi(g)v, w)$ is integrable.

The proofs of the preceding results are based on the following fundamental results of Harish-Chandra, which tells that a maximal compact subgroup K of a connected semisimple Lie group G with finite center is a uniformly large compact subgroup of G. We denote by $\hat{K}$ the set of all equivalence classes of finite dimensional irreducible representations of K. Also $d(\delta)$ stands for the degree of the equivalence class δ in $\hat{K}$.

(1.1.10) **Theorem** (Harish-Chandra, cf. [41], I, Theorem 4.4.2.11). *Let G be a connected semisimple Lie group with finite center, K a maximal compact subgroup of G. Then a given $\delta \in \hat{K}$ occurs no more than $d(\delta)$ times in the restriction to K of any topologically completely irreducible Banach representation of G.*

Remark 1. Let χ_δ be the character of $\delta \in \hat{K}$, and set $\xi_\delta = d(\delta)\chi_\delta$. Put

$$\pi(\overline{\xi_\delta}) = \int_K \overline{\xi_\delta(k)}\pi(k)dk$$

for any given Banach representation (π, E) of G, where dk is the normalized Haar measure on K, then $\pi(\overline{\xi_\delta})$ is a projection of E onto $E(\delta) = \pi(\overline{\xi_\delta})E$. $E(\delta)$ is the *isotypic K-submodule of E of type δ*. The preceding theorem says that $\dim E(\delta) \leqq d(\delta)^2$ for any $\delta \in \hat{K}$. The subspace $E_K = \sum_{\delta \in \hat{K}} E(\delta)$ of K-finite vectors in E is a dense subspace of E.

Remark 2. A Banach representation (π, H) of G is said to be *topologically completely irreducible*, if given any bounded operator S on H and elements $a_1, \cdots, a_n$ in H, there exists, for every $\varepsilon > 0$, a measure μ on G with compact support such that

$$\|(\pi(\mu) - S)a_i\| < \varepsilon \quad \left(\pi(\mu) = \int_G \pi(x)d\mu(x);\ i = 1, \cdots, n\right).$$

Let π be a unitary representation of G on a Hilbert space H. Then the following statements are equivalent (cf. [41], I, Proposition 4.3.1.7):

(i) H is topologically irreducible;

(ii) H is topologically completely irreducible.

§ 1.2. Characters of the discrete series

In this section, we recall some fundamental results on the characters of the discrete series of semisimple Lie groups.

In the first place, we review some results on the characters, which are valid for any unimodular Lie group G with a uniformly large compact subgroup K. Let us start with the definition of characters.

Let $C_c^\infty(G)$ be the algebra of C^∞-functions on G with compact support. Let $C_*^\infty(G)$ be the subalgebra of $C_c^\infty(G)$ which is the linear span of the $\overline{\xi_{\delta_1}}*C_c^\infty(G)*\overline{\xi_{\delta_2}}$ $(\delta_1, \delta_2 \in \hat{K})$. Then $C_*^\infty(G)$ is dense in $C_c^\infty(G)$ relative to the inductive limit topology.

(1.2.1) **Definition.** Let (π, H) be a Banach representation of G such that $\dim H(\delta)$ is finite for any $\delta \in \hat{K}$. For any $f \in C_*^\infty(G)$, the operator $\pi(f)=\int_G f(g)\pi(g)dg$ is of finite rank. Hence the linear functional T_π on $C_*(G)$:

$$f \in C_*^\infty(G) \longmapsto \mathrm{tr}\,(\pi(f)) \in \mathbf{C}$$

is well defined. We call T_π the *character* of π.

(1,2.2) **Proposition** ([41], I, Corollary 4.5.8.3). *Let π^i $(i=1, 2)$ be topologically irreducible unitary representations of a unimodular Lie group G with a uniformly large compact subgroup K. Then π^1 and π^2 are unitarily equivalent, if and only if they have the same character.*

The following theorem assures that a character of a representation can be extended to a distribution on a unimodular Lie group G.

(1.2.3) **Theorem** (Harish-Chandra. [41], I, Theorem 4.5.8.5). *Let π be a continuous representation of G on a Hilbert space H with the property that there exists an integer $m_H \geqq 1$ such that* $\dim (H(\delta)) \leqq m_H d(\delta)^2$ *for any $\delta \in \hat{K}$. Then for any $f \in C_c^\infty(G)$, the operator $\int_G f(g)\pi(g)dg$ is of the trace class and the mapping T_π defined by the rule*

$$f \longmapsto \mathrm{tr}\left(\int_G f(g)\pi(g)dg\right) \quad (f \in C_c^\infty(G))$$

is a distribution on G.

By definition, the assumption of the preceding theorem is satisfied for any irreducible unitary representation of G, if G is a unimodular Lie group with a uniformly compact subgroup K, especially if G is a connected semisimple Lie group with finite center, thanks to Theorem (1.1.10).

When the representation π is in the discrete series, the character T_π is written in terms of the coefficient of π.

(1.2.4) **Theorem** ([41], I, Theorem 4.5.9.7). *Let G be a unimodular Lie group with a uniformly large compact subgroup K, and π an irreducible square integrable representation of G on a Hilbert space H. Let T_π denote the character of π, d_π the formal degree of π. Then for all v, $w \in H$, we have*

$$(v, w)T_\pi(f)=d_\pi \int_G dg\left(\int_G f(hgh^{-1})(\pi(h)v, w)dh\right) \quad \textit{for all } f \in C_c^\infty(G).$$

In order to formulate the results on the characters of representations of G, we have to introduce some terminology on the distributions on G, and to recall some results on differentiable vectors in representation spaces of Banach representations of G.

(1.2.5) **Definition.** A distribution T on G is said to be *central*, if it is invariant under the inner automorphisms of G.

Let $U(\mathfrak{g}_C)$ denote the universal enveloping algebra of $\mathfrak{g}_C$, $\mathfrak{g}_C$ the complexification of the Lie algebra $\mathfrak{g}$ of G. Let $Z(\mathfrak{g}_C)$ denote the center of $U(\mathfrak{g}_C)$, Z the center of G.

(1.2.6) **Definition.** A distribution T on G is said to be $Z(\mathfrak{g}_C)$-finite, if the space spanned by $X\cdot T$ ($X \in Z(\mathfrak{g}_C)$) is finite dimensional. In particular T is said to be an *eigendistribution* of $Z(\mathfrak{g}_C)$, if there exists a homomorphism $\kappa\colon Z(\mathfrak{g}_C)\to \boldsymbol{C}$ such that $X\cdot T=\kappa(X)T$ (all $X \in Z(\mathfrak{g}_C)$). A distribution T on G is said to be *eigendistribution* of Z, if there exists a homomorphism $\zeta\colon Z\to \boldsymbol{C}^\times$ with properties that $T^z=\zeta(z)T$ for all $z \in Z$. Here T^z denotes the right translate of T by $z \in Z$.

(1.2.7) **Definition.** Let G be a Lie group, H a locally convex Hausdorff topological vector space (over $\boldsymbol{C}$), and $g\mapsto\pi(g)$ ($g \in G$) a continuous representation of G on H. Then a vector $v \in H$ is said to be *differentiable* (for π), if the map $g\mapsto\pi(g)v$ is an H-valued C^∞-function on G.

Let H_∞ denote the set of differentiable vectors in H. Then it is easy to see that $\pi(f)v \in H_\infty$ for $v \in H$ and $f \in C_c^\infty(G)$.

Let π be a continuous representation of G on H. Then π lifts to a representation π_∞ of $U(\mathfrak{g}_C)$ on H_∞ in the following manner.

(1.2.8) **Definition.** For $X \in \mathfrak{g}$, $v \in H_\infty$, define

$$\pi_\infty(X)v=\lim_{t\to 0}\frac{\pi(\exp(tX))v-v}{t} \quad (t \in \boldsymbol{R}),$$

where the limit is taken with respect to the topology of H. The obvious extension to $U(\mathfrak{g}_C)$ is also denoted by the same symbol π_∞.

(1.2.9) **Definition.** Let (π, H) be a Banach representation of a connected Lie group G. Then π is said to be *quasi-simple*, if there exists a homomorphism κ_π of $Z(\mathfrak{g}_C)$ into C such that $\pi_\infty(Z)v=\kappa_\pi(Z)v$ for all $Z \in Z(\mathfrak{g}_C)$ and $v \in H_\infty$. κ_π is then called the *infinitesimal character* of π.

(1.2.10) **Proposition** ([41], I, Propositions 4.4.1.4, 4.4.1.5). *Let π be a topologically completely irreducible Banach representation of a connected Lie group. Then π is quasi-simple.*

(1.2.11) **Corollary** (Segal, cf. [41], Corollary 4.4.1.6). *Let π be a topologically irreducible unitary representation of a connected Lie group. Then π is quasi-simple.*

For differentiable vectors the following fact is known.

(1.2.12) **Theorem** ([41], I, Theorem 4.4.3.1). *Let G denote a Lie group countable at infinity, π a continuous representation of G on a complete locally convex topological vector space H, and H_∞ the space of differentiable vectors in H for π. Let K be a compact subgroup of G. Then the space*

$$\sum_{\delta \in \hat{K}} H_\infty \cap H(\delta)$$

is dense in H.

Similarly as differentiable vectors, we can define analytic vectors in representation spaces.

(1.2.13) **Definition.** Under the same notation and conditions, a vector $v \in H$ is said to be *analytic* (for π), if the map $g \mapsto \pi(g)v$ is an H-valued C^ω-function on G.

Then the following is known.

(1.2.14) **Proposition** ([41], I, Lemma 4.5.5.1). *Let G be a connected unimodular Lie group countable at infinity, K a compact connected subgroup of G. Let π be a Banach representation of G on H such that* $\dim(H(\delta))$ *is finite for any $\delta \in \hat{K}$. Then*

$$H_K = \sum_{\delta \in \hat{K}} H(\delta) \subset H_\omega,$$

where H_ω is the space of analytic vectors in H for π.

Especially $H_K \subset H_\infty$ in this case. Moreover, we can readily show that H_K is π_ω-stable and π_∞-stable under $U(\mathfrak{g}_C)$. Hence we can consider the restriction π_K of the representation π_ω (or π_∞) on H_ω to H_K.

(1.2.15) **Definition** (Harish-Chandra). Let π (resp. π') be a Banach representation of G on H (resp. H') such that dim $(H(\delta))$ (resp. dim $(H'(\delta))$) is finite for any $\delta \in \hat{K}$. Then π and π' are said to be *infinitesimally equivalent*, if the representations π_K and π'_K of $U(\mathfrak{g}_C)$ on H_K and H'_K, respectively, are algebraically equivalent.

Let us return to the characters. In the first place, we know that the character of a representation satisfies differential equations.

(1.2.16) **Proposition** ([41], I, Proposition 4.5.8.6). *Let G be a connected unimodular Lie group, K a uniformly large compact connected subgroup of G. Let π be a topologically completely irreducible Hilbert representation of G on H, T_π the character of π. Let ζ_π be the central character of π, κ_π the infinitesimal character of π. Then T_π is a central eigendistribution on G verifying*

$$T_\pi^z = \zeta_\pi(z) T_\pi \quad (z \in Z), \qquad X \cdot T_\pi = \kappa_\pi(X) T_\pi \quad (X \in Z(\mathfrak{g}_C)).$$

The last assertion of the above proposition is proved in the following way: Form the Fourier component $T_{\pi,\delta}$ of T_π for each $\delta \in \hat{K}$ defined by $T_{\pi,\delta}(f) = T_\pi(f * \overline{\xi_\delta})$ $(f \in C_c^\infty(G))$; Then it is represented by an analytic function ψ_δ^π called the *spherical trace function of type δ for π*; These functions are eigenfunctions $(X\psi_\delta^\pi = \kappa_\pi(X)\psi_\delta^\pi,\ X \in Z(\mathfrak{g}_C))$ and T_π is the sum of them in the sense of distributions (cf. [41], II, § 6.1.2).

Let G be a connected semisimple Lie group with finite center, and K a maximal compact subgroup of G. For each $\delta \in \hat{K}$, the Fourier component $T_{\pi,\delta}$ of type δ for a square integrable representation π of G is a spherical trace function, which is real analytic and square integrable on G, as well as being K-finite and $Z(\mathfrak{g}_C)$-finite. Then these functions are known to be rapidly decreasing on G (cf. [41], II, Theorem 9.3.1.5). On the other hand, applying the Weak Selberg Principle for invariant integrals, it is known that the existence of a rapidly decreasing, non-zero $Z(\mathfrak{g}_C)$-finite function on G implies that rank(G)=rank(K). Thus $\hat{G}_d \neq \phi$ implies rank(G)=rank(K) for such a group G (cf. [41], II, § 8.5.1, Theorem 8.5.1.7, Corollary 8.5.1.8). The invariant integral of a smooth function $f(x)$ on G with compact support, relative to a Cartan subgroup J, is the function Φ_f on the subset J' of regular elements in J defined by (cf. § 8.5 of [41], II)

$$\Phi_f(j) = \varepsilon_R(j) \Delta_J(j) \int_{G/J_0} f(xjx^{-1}) d_{G/J_0}(\dot{x}) \quad (j \in J').$$

Here J_0 is the center of J, $x \mapsto \dot{x}$ denotes the canonical projection of G onto G/J_0, d_{G/J_0} is the G-invariant measure on G/J_0, and Δ_J is given by

$$\Delta_J(j)=\xi_\rho(j)\prod_{\alpha\in\Phi+}(1-\xi_\alpha(j^{-1}))$$

for characters ξ_ρ, ξ_α of the complexification J_C of J corresponding to the roots ρ, α. We omit the definition of $\varepsilon_R(j)$, which is equal to $+1$ or -1 for each j (cf. § 8.1.1 of [41], II). The Weak Selberg Principle claims that these invariant integrals vanish, if rank $(G)>$ rank (K) (cf. [41], II, Theorem 8.5.1.7, or [17], § 32, p. 76, Lemma 64).

Conversely, assume that rank $(G)=$ rank (K). Moreover assume that G is acceptable, i.e. one half $\rho=\frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha$ of the sum of positive roots in the root system Φ of G with respect to a Cartan subgroup J of G is the differential of a holomorphic character of the complexification J_C of J to $C^\times=C-\{0\}$. The condition of the acceptability is independent of the choice of an ordering for the root system and the choice of a Cartan subgroup J of G.

For such a group G, Harish-Chandra [16] constructed the central eigendistributions which are found to be the characters of the discrete series of G, in an a priori way completely independent of the theory of infinite dimensional representations. Let us formulate this.

Fix a maximal torus H in K. Then H is a compact Cartan subgroup of G. The unitary character group $\hat{H}$ of H is identified with a lattice L_H in the dual space $\sqrt{-1}\,\mathfrak{h}$ ($\mathfrak{h}$ the Lie algebra of H). Thus for a given $\lambda \in L_H$, we can assign a unitary character ξ_λ of H by

$$\xi_\lambda(\exp X)=e^{\lambda(X)} \quad (X\in\mathfrak{h}),$$

and every element of $\hat{H}$ is obtained in this fashion. A unitary character is said to be *regular* or *singular*, according as λ is regular or singular, i.e. according as $\lambda^w\neq\lambda$ for any non-trivial element w of the Weyl group W or not. Let L_H' be the set of regular elements in L_H.

Let $S(\mathfrak{h}_C)$ be the symmetric tensor algebra of the complexification $\mathfrak{h}_C$ of $\mathfrak{h}$, and $I(\mathfrak{h}_C)$ the invariants of $S(\mathfrak{h}_C)$ under the Weyl group W. Let $\gamma\colon Z(\mathfrak{g}_C)\simeq I(\mathfrak{h}_C)$ be the canonical isomorphism. Any linear function λ on $\mathfrak{h}_C$ defines a homomorphism $I(\mathfrak{h}_C)\to C$ by $p\mapsto p(\lambda)$ ($p\in I(\mathfrak{h}_C)$). The composition of this with the isomorphism γ defines a character $Z(\mathfrak{g}_C)\to C$ of $Z(\mathfrak{g}_C)$, which we denote by γ_λ. And any homomorphism $Z(\mathfrak{g}_C)\to C$ arises in this manner from a suitable $\lambda\in\mathrm{Hom}(\mathfrak{h}_C, C)$. Two linear forms λ_1 and λ_2 of $\mathfrak{h}_C$ determine the same character of $Z(\mathfrak{g}_C)$, if and only if $\lambda_1^w=\lambda_2$ for some $w\in W$.

Let ℓ be the rank of G, and t an inderminate. For any $g\in G$, we denote by $D_\ell(g)$ the coefficient of t^ℓ in $\det(t+1-\mathrm{Ad}(g))$. We denote by Δ_H the function on H defined by $\Delta_H(h)=\xi_\lambda(h)\prod_{\alpha\in\Phi_+}(1-\xi_\alpha(h^{-1}))$. Then Δ_H is non-zero on the subset $H'=H\cap G'$ of regular elements in H. Let $N(H)$ be the normalizer of H in G, and $W(G,H)=N(H)/H$. Then $W(G,H)$ is identified with the subgroup of $W=W(\mathfrak{g}_C,\mathfrak{h}_C)$ generated by reflexions with respect to compact roots. For each $w\in W(G,H)$, we define $\varepsilon(w)$ by $\varepsilon(w)=\det(w)$, regarding w as an element of $\mathrm{End}_R(\mathfrak{h})$.

(1.2.17) **Theorem** ([16], Theorem 3 of § 19, [41], II, Theorem 10.1.1.1). *Fix an element $\lambda\in L'_H$. Then there exists exactly one central eigendistribution Θ_λ on G such that*:

(i) $Z\Theta_\lambda=\gamma_\lambda(Z)\Theta_\lambda$ *(all* $Z\in Z(\mathfrak{g}_C)$*)*;

(ii) $\sup_{g\in G'}|D_\ell(g)|^{1/2}|\Theta_\lambda(g)|<\infty$;

(iii) $\Theta_\lambda=\Delta_H^{-1}\sum_{w\in W(G,H)}\varepsilon(w)\xi_{w\lambda}$ *on* H' $(=H\cap G')$.

Note that Θ_λ is represented by a locally summable function F_{Θ_λ} which is analytic on G' the set of regular elements in G (cf. [41], II, Theorem 8.3.3.1). Therefore the condition (iii) of the above theorem makes sense.

The condition (ii) of the theorem implies that the distribution Θ_λ is tempered, i.e. it extends to a continuous linear functional on the space of rapidly decreasing smooth functions on G ([17], § 19, Theorem 7, or [41], II, Theorem 8.3.8.2).

Choose a class $\delta\in\hat{K}$ such that the Fourier component $\Theta_{\lambda,\delta}$ of Θ_λ does not vanish identically. Then $\Theta_{\lambda,\delta}$ lies in $L^2(G)$. Let L be the left regular representation of G on $L^2(G)$. Then the smallest closed L-stable subspace containing $\Theta_{\lambda,\delta}$ is found to be a finite sum of irreducible unitary representations. Hence $\hat{G}_d\neq\phi$. Thus we have the following criterion of Harish-Chandra.

(1.2.18) **Theorem** (Harish-Chandra, [17], § 39, Theorem 13, or [41], II, Theorem 10.2.1.2). *Let G be a connected semisimple (acceptable) Lie group with finite center, K a maximal compact subgroup of G. Then $\hat{G}_d\neq\phi$, if and only if* $\mathrm{rank}(G)=\mathrm{rank}(K)$.

Remark 1. The above theorem is valid, if G is not acceptable. Because such G has a finite covering by an acceptable group.

Remark 2. Oshima, Matsuki [32], together with the construction of Flensted-Jensen [13], generalizes the above criterion of Harish-Chandra for $L^2(G/H')$ to symmetric spaces G/H'.

The tempered central eigendistributions $\{\Theta_\lambda\mid\lambda\in L'_H\}$ not only show the existence of the discrete series, but also enumerate all the characters of the discrete series. The following theorem of Harish-Chandra is fundamental for the discrete series of semisimple Lie groups.

(1.2.19) **Theorem** (Harish-Chandra, [17], § 41, Theorem 16, or [41], II, Theorem 10.2.4.1). *Let G be a connected semisimple acceptable Lie group with finite center, K a maximal compact subgroup of G. Assume that* $\operatorname{rank}(G)=\operatorname{rank}(K)$. *Then*

(i) *The discrete series $\hat{G}_d$ of G is not empty;*

(ii) *For each $\lambda \in L'_H$, there corresponds a unique element $\Pi_\lambda \in \hat{G}_d$ whose character T_{Π_λ} is given by*

$$T_{\Pi_\lambda}=(-1)^{m_G}\varepsilon(\lambda)\Theta_\lambda,$$

where $\varepsilon(\lambda)=\operatorname{sign}\left(\prod_{\alpha\in\Phi_+}\langle\lambda, \alpha\rangle\right)$ *and* $m_G=\frac{1}{2}\dim(G/K)$;

(iii) *The mapping $\lambda\mapsto\Pi_\lambda$ of L'_H into $\hat{G}_d$ is surjective and the formal degree d_{Π_λ} of the class Π_λ is given by*

$$d_{\Pi_\lambda}=\frac{[W(G, H)]}{(2\pi)^r}\Big|\prod_{\alpha\in\Phi_+}\langle\lambda, \alpha\rangle\Big| \quad (r=\tfrac{1}{2}\dim(G/H));$$

(iv) *Finally, $\Pi_{\lambda_1}=\Pi_{\lambda_2}$ $(\lambda_1, \lambda_2 \in L'_H)$, if and only if λ_1 and λ_2 are conjugate under $W(G, H)$. Here the inner product $\langle\ ,\ \rangle$ of $\mathfrak{h}_C^*=\operatorname{Hom}_C(\mathfrak{h}, C)$ is given by $\langle\lambda, \mu\rangle=\lambda(H_\mu)$ $(\lambda, \mu \in \mathfrak{h}_C^*)$ for $H_\mu \in \mathfrak{h}_C$ defined by the condition:*

$$\mu(H)=B(H, H_\mu) \qquad \text{for all } H \in \mathfrak{h}_C$$

with respect to the Killing form B.

Remark 1. In order let (ii) and (iii) of the above theorem make sense, we have to normalize the Haar measure of G. We do not discuss this here, but refer to Section 8 of [41], II.

Remark 2. There are $|W/W(G, H)|$ distinct classes in $\hat{G}_d$ with the same infinitesimal character, by (iv) of the preceding theorem.

The crucial points in the proof of the identification and enumeration of the characters of the discrete series, are the (Schur) Orthogonality Relations for characters and the following expansion theorem which is deduced from the Weak Selberg Principle.

(1.2.20) **Theorem** ([16], Corollary 2 of Lemma 64, [41], II, Theorem 10.1.2.4). *Any smooth rapidly decreasing $Z(\mathfrak{g}_C)$-finite function f on G has an expansion*

$$f(x)=M_G^{-1}\sum_{\lambda\in L_H}\prod_{\alpha\in\Phi_+}\langle\lambda, \alpha\rangle\Theta_\lambda(R(x)f) \quad (x \in G),$$

where $R(x)f$ is the right translation of f, and M_G is a constant depending on the normalization of the Haar measure of G.

Remark. If the Haar measure of G is normalized so that Theorem (1.2.19) is valid. Then $M_G=(-1)^N(2\pi)^r$ with $N=\frac{1}{2}\dim(G/K)$ and $r=\frac{1}{2}\dim(G/H)$.

§ 1.3. Realizations of the discrete series

A number of realizations of the representations of the discrete series are known, such that their representation spaces are certain cohomology groups on a homogeneous space with coefficients in vector bundles corresponding to the representations of a maximal compact subgroup or a compact Cartan subgroup. These realizations are considered as analogues of the Borel-Weil-Bott theorem for compact Lie groups.

Some realizations of the discrete series which are applicable for any semisimple group with condition $\text{rank}(G)=\text{rank}(K)$, are discussed in [21], [35], [38]. But for our purpose, it suffices to discuss the case where the quotient space $X=G/K$ has a G-invariant Hermitian structure. Thus we review here mainly the realization of Narasimhan-Okamoto [31].

Also we impose another restriction on the group G. Though the constructions below are applicable to any connected semisimple Lie group G with finite center such that G/K is Hermitian, we assume moreover that G has a finite dimensional faithful representation, in order to simplify the description of the irreducible representations of a maximal compact group K. Therefore G is the identity component of the group of the real points of a real linear algebraic group. Thus, for instance, the metaplectic groups are excluded from here. But it is easy to see that similar results are valid for such groups *mutatis mutandis*, because these groups are covering groups of the groups considered here. Note that our group G has a natural complexification by assumption.

The notation of the preceding sections is still in force in this section.

(1.3.1) *Definition of holomorphic vector bundles.* The representation spaces are realized as certain square integrable $\bar{\partial}$-cohomology spaces with coefficients in holomorphic vector bundles on the complex manifold X, associated to the representations of the fixed maximal compact subgroup K. Let us recall the definition of these bundles.

Let $\sigma\colon K\to GL(W)$ be a finite dimensional (unitary) representation of K. Then it extends to a holomorphic representation $\sigma\colon K_C\to GL(W)$ of the complexification K_C of K. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively, and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ the Cartan decomposition. Then the tangent space $T_{X,o}$ of X at the point $o=eK$ is identified with $\mathfrak{p}$ in the natural way. Since X is Hermitian symmetric, there exists an element ι of K such that $\text{Ad}(\iota)|_{\mathfrak{p}}$ defines the given complex structure on $T_{X,o}$ via the above identification $T_{X,o} \cong \mathfrak{p}$. Let $\mathfrak{p}_C = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ be the eigenspace decomposition of the complexification $\mathfrak{p}_C$ of $\mathfrak{p}$ with respect to $\mathrm{Ad}(\iota)$ such that

$$\mathrm{Ad}(\iota)|_{\mathfrak{p}_+} = \sqrt{-1} \quad \text{and} \quad \mathrm{Ad}(\iota)|_{\mathfrak{p}_-} = -\sqrt{-1}.$$

Then both $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are abelian subalgebras of $\mathfrak{g}_C$. Put $P_+ = \exp(\mathfrak{p}_+)$ and $P_- = \exp(\mathfrak{p}_-)$. Then in the complexification G_C of G, we have

$$K = (K_C P_+) \cap G \quad \text{and} \quad K = (K_C P_-) \cap G.$$

Therefore, the natural mapping $G/K \to G_C/K_C P_-$ induced from the inclusion $G \hookrightarrow G_C$ is injective. This mapping identifies $X = G/K$ with an open domain in the compact complex manifold $V = G_C/K_C P_-$.

Now let us define a holomorphic vector bundle F_σ on V associated to σ, as the quotient space of the product $G_C \times W$ by the equivalence relation:

$$(gu, w) \sim (g, \sigma(u)w) \quad (g \in G_C, w \in W, u \in K_C P_-),$$

with the structure mapping $F_\sigma \to V$ induced from the first projection $G_C \times W \to G_C$. Note that F_σ has the natural action of G_C compatible with the action of G_C on V via the left multiplication. We denote by the same symbol F_σ the pullback j^*F_σ of F_σ to X with respect to the injection j: $X \hookrightarrow V$. Then F_σ is a holomorphic vector bundle over X with an action of G compatible with that of G on X. If we denote by $\mathscr{F}_\sigma$, the locally free analytic sheaf of local holomorphic sections of F_σ, we may say that the sheaf $\mathscr{F}_\sigma$ has a G-linearization. Moreover, if the representation (σ, W) of K is unitary, the metric on W induces a G-invariant Hermitian structure on the bundle F_σ over X, by the obvious identification:

$$F_\sigma = j^*F_\sigma = (G \times W)/K$$

over X.

Let the representation (σ, W) be an irreducible representation of K with highest weight λ, λ being a dominant weight with respect to an ordering for the root system (K, H). Then we denote F_σ by F_λ.

(1.3.2) *Square integrable $\bar{\partial}$-cohomology spaces.* Before proceding further, let us confirm some notations and definitions for the root system of G. Fix once for all a compact Cartan subgroup H of G contained in K. Let $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{g}$ corresponding to H, and $\mathfrak{h}_C^* = \mathrm{Hom}_C(\mathfrak{h}_C, C)$ the dual linear space of the complexification $\mathfrak{h}_C$ of $\mathfrak{h}$. Let Φ be the root system of $\mathfrak{g}_C$ with respect to $\mathfrak{h}_C$. Then the root system of $\mathfrak{k}_C$ with respect to $\mathfrak{h}_C$ is canonically identified with a subset Φ_K of Φ. The elements of Φ_K are called *compact roots.* The elements of $\Phi_n = \Phi - \Phi_K$

are called *noncompact roots.* In other words, a root $\alpha \in \Phi$ is called compact or noncompact, according as the corresponding root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{k}_C$ or $\mathfrak{p}_C$. Note that the Weyl group $W(G, H)$ of the pair (G, H) is the subgroup of $W=W(\mathfrak{g}_C, \mathfrak{h}_C)$ generated by the reflexions with respect to compact roots. We denote $W(G, H)$ by W_K. Note that W_K holds the set Φ_K.

Let B be the Killing form of $\mathfrak{g}$. For any linear form $\lambda \in \mathfrak{h}_C^*$ on $\mathfrak{h}_C$, we define $H_\lambda \in \mathfrak{h}_C$ by $B(H_\lambda, H)=\lambda(H)$ for all $H \in \mathfrak{h}_C$. Then for any λ, $\mu \in \mathfrak{h}_C^*$, we set $\langle\lambda, \mu\rangle=\lambda(H_\mu)$.

Let us denote by L the lattice of integral linear forms on $\mathfrak{h}_C$:

$$L=\left\{\lambda \in \mathfrak{h}_C^*=\mathrm{Hom}_C(\mathfrak{h}_C, \boldsymbol{C}) \,\middle|\, \frac{2\langle\lambda, \alpha\rangle}{\langle\alpha, \alpha\rangle} \in \boldsymbol{Z} \text{ for any } \alpha \in \Phi\right\}.$$

The elements of L are called *weights*, and L the *weight group* of G.

Let us choose an ordering in Φ such that the positive roots Φ_+ satisfies

$$\mathfrak{p}_+=\sum_{\alpha \in \Phi_{n+}} \mathfrak{g}_\alpha, \qquad \mathfrak{p}_-=\sum_{\alpha \in \Phi_{n-}} \mathfrak{g}_\alpha,$$

with $\Phi_{n+}=\Phi_n \cap \Phi_+$, $\Phi_{n-}=\Phi_n \cap (-\Phi_+)$. Then we set $\rho=\frac{1}{2}\sum_{\alpha \in \Phi_+}\alpha$, as usual, and denote by Φ_{K+} the intersection $\Phi_K \cap \Phi_+$.

Let $X(H_C)$ be the module of the holomorphic characters $H_C \to \boldsymbol{C}^\times$ of the complexification H_C of H. Then $X(H_C)$ is naturally identified with a sublattice of L. The group G_C is simply connected, if and only if $X(H_C)=L$.

We define the set of *dominant weights* of $\mathfrak{g}$ by

$$D=\{\lambda \in L \mid \langle\lambda, \alpha\rangle \geqq 0 \text{ for all } \alpha \in \Phi_+\},$$

and the subsets L' and L'_0 by

$$L'=\{\lambda \in L \mid \langle\lambda+\rho, \alpha\rangle \neq 0 \text{ for all } \alpha \in \Phi\},$$

and

$$L'_0=\{\lambda \in L' \mid \langle\lambda+\rho, \alpha\rangle > 0 \text{ for all } \alpha \in \Phi_{K+}\},$$

respectively.

For any $\lambda \in L'_0 \cap X(H_C)$ we denote by σ_λ the irreducible unitary representation of K with highest weight λ on the representation space W_λ. Then we can attach a holomorphic vector bundle F_λ on X as in the previous section.

Let Ω_X^i denote the bundle (or the sheaf) of holomorphic i-forms on X. Then Ω_X^i has the canonical Hermitian (and Kähler) metric induced from that of X, and the canonical G-action (or G-linearization, resp.) for each i. Denote by $\bar{\Omega}_X^i$ the bundle of antiholomorphic i-forms on X. Let $C_c^\infty(X, F_\lambda \otimes \bar{\Omega}_X^i)$ be the space of C^∞-sections of $F_\lambda \otimes \bar{\Omega}_X^i$ with compact support. Then we can consider the $\bar{\partial}$-complex (or Dolbeault complex) with coefficients in F_λ:

$$0 \longrightarrow C_c^\infty(X, F_\lambda) \xrightarrow{\bar{\partial}} C_c^\infty(X, F_\lambda \otimes \bar{\Omega}_X^1) \xrightarrow{\bar{\partial}} C_c^\infty(X, F_\lambda \otimes \bar{\Omega}_X^2) \longrightarrow \cdots$$

Let $L_2(X, F_\lambda \otimes \bar{\Omega}_X^i)$ be the completion of $C_c^\infty(X, F_\lambda \otimes \bar{\Omega}_X^i)$ with respect to the metric:

$$(s_1, s_2) = \int_X \langle s_1(x), s_2(x) \rangle_{F_\lambda \otimes \bar{\Omega}_X^i} d\omega(x) \quad (s_j \in C_c^\infty(X, F_\lambda \otimes \bar{\Omega}_X^i), j=1, 2).$$

Here the inner metric $\langle \ , \ \rangle_{F_\lambda \otimes \bar{\Omega}_X^i}$ on $F_\lambda \otimes \bar{\Omega}_X^i$ is the tensor product of the metrics F_λ and $\bar{\Omega}_X^i$, and $d\omega$ is the G-invariant measure on X unique up to constant multiple. Let

$$\delta \colon L_2(X, F_\lambda \otimes \bar{\Omega}_X^i) \longrightarrow L_2(X, F_\lambda \otimes \bar{\Omega}_X^{i-1})$$

be the formal adjoint of $\bar{\partial}$. Consider again the formal adjoint

$$\bar{\partial} \colon L_2(X, F_\lambda \otimes \bar{\Omega}_X^i) \longrightarrow L_2(X, F_\lambda \otimes \bar{\Omega}_X^{i+1})$$

of δ. Then we have a complex:

$$0 \longrightarrow L_2(X, F_\lambda) \xrightarrow{\bar{\partial}} L_2(X, F_\lambda \otimes \bar{\Omega}_X^1) \xrightarrow{\bar{\partial}} L_2(X, F_\lambda \otimes \bar{\Omega}_X^2) \longrightarrow \cdots$$

of square integrable differential forms of type $(0, *)$ with coefficients in F_λ.

Since these complexes are G-equivariant, the cohomology group

$$H_2^{0,i}(X, F_\lambda) = \{s \in L_2(X, F_\lambda \otimes \bar{\Omega}_X^i) \mid \bar{\partial} s = 0\} / \{\bar{\partial} L_2(X, F_\lambda \otimes \bar{\Omega}_X^{i-1})\}^{cl},$$

is also a G-module for each i, where $\{\bar{\partial} L_2(X, F_\lambda \otimes \bar{\Omega}_X^{i-1})\}^{cl}$ is the closure of $\bar{\partial} L_2(X, F_\lambda \otimes \bar{\Omega}_X^{i-1})$ in $L_2(X, F_\lambda \otimes \bar{\Omega}_X^i)$. Put

$$\square = \delta \circ \bar{\partial} + \bar{\partial} \circ \delta.$$

Then it is known that for any $s \in L_2(X, F_\lambda \otimes \bar{\Omega}_X^i)$ the conditions:

(i) $\square s = 0$; and (ii) $\bar{\partial} s = 0$, $\delta s = 0$

are equivalent. The space of harmonic $(0, i)$-forms with values in F_λ is given by

$$\mathcal{H}_2^{0,i}(X, F_\lambda) = \{s \in L_2(X, F_\lambda \otimes \bar{\Omega}_X^i) \mid \bar{\partial} s = 0, \delta s = 0\}.$$

Then we have a natural isomorphism

$$H_2^{0,i}(X, F_\lambda)\cong\mathscr{H}_2^{0,i}(X, F_\lambda) \qquad \text{for each } i$$

(cf. [33], § 1), and $\mathscr{H}_2^{0,i}(X, F_\lambda)$ is a unitary G-module with respect to the induced metric from $L_2(X, F_\lambda\otimes\bar{\Omega}_X^i)$.

Define a subset W^1 of the Weyl group $W=W(\mathfrak{g}_C, \mathfrak{h}_C)$ by

$$W^1=\{\sigma \in W|\, \sigma(\Phi_+)\supset\Phi_{K+}\}.$$

Then the map $W_K\times W^1\to W$ given by $(s, \sigma)\mapsto s\sigma$ is a bijection. Moreover, for D and L_0', we have a bijection

$$D\times W^1\longrightarrow L_0'$$

given by $(\lambda, \sigma)\mapsto\lambda^{(\sigma)}$, where $\lambda^{(\sigma)}=\sigma(\lambda+\rho)-\rho$.

Then the following result is known by [31] and [34] for the unitary representation of G on $\mathscr{H}_2^{0,i}(X, F_\lambda)$.

(1.3.3) **Theorem.** *Let $\lambda \in L_0'\cap X(H_C)$, and let $\lambda_D \in D$ and $\sigma \in W^1$ be the unique elements such that $\lambda=\lambda_D^{(\sigma)}$. Let q_λ be the number of $\alpha \in \Phi_{n+}$ such that $\langle\lambda+\rho, \alpha\rangle>0$. Then*

$$\text{(1.3.3.1)} \qquad \mathscr{H}_2^{0,q}(X, F_\lambda)=\{0\}, \quad \text{if } q\neq q_\lambda,$$

and the space $\mathscr{H}_2^{0,q_\lambda}(X, F_\lambda)$ is the representation space of an irreducible unitary representation of the discrete series of G with character $\Theta_{\lambda+\rho}$.

Remark 1. [31] assumes that G_C is simply connected. Hence $X(H_C)=L$ in this case. Note also that the complex structure on $X=G/K$ of [31] and [35] are the conjugation of that of ours. Therefore our cohomology groups appear different from those of [31] and [35]. See the final remark (Remark 3 of § 9) of [31] for this point.

Remark 2. The vanishing part (i.e. § 7, Theorem 2) of the main result of [31] was weaker than (1.3.3.1) of the above theorem. Parthasarathy proved a sharper vanishing theorem (Theorem 3, [35]) under the assumption:

$$\langle\sigma\lambda_D, \alpha\rangle\neq 0 \qquad \text{for any } \alpha \in \Phi_{n+}.$$

This assumption is now known to be unnecessary by the following reason. Since the Blattner conjecture is true (by Schmid et al.), one of H^{even} or H^{odd} vanishes as shown in Schmid [39]. The alternating sum formula of traces (cf. Theorem 1, [31]) implies that there is unique q such that

$H_2^{0,q}(X, F_\lambda)\neq\{0\}$. And finally we can check that $q=q_\lambda$ by K-type theorem.

Once the vanishing theorem is improved, Theorem 1 of [31] implies the second statement of Theorem (1.3.3) (see the proof of Theorem 3, § 8, [31]).

(1.3.4) *Other realization.* Let us review other realizations. Let us recall the realization of Schmid ([15], [37], [38]), which is applicable even if G/K has no invariant complex structure.

In the first place, the quotient G/H of G by a compact Cartan subgroup H is made into a homogeneous complex manifold with a G-invariant Hermitian metric, and every character λ of H (in L_0') determines a holomorphic line bundle $L_\lambda\to D=G/H$. Then similarly as in the preceding section, we can consider the $\bar\partial$-complex:

$$\cdots\longrightarrow A_0^i(L_\lambda)\xrightarrow{\bar\partial}A_0^{i+1}(L_\lambda)\longrightarrow\cdots$$

of the spaces $A_0^i(L_\lambda)$ of compactly supported C^∞ L_λ-valued forms of type $(0, i)$ on D. The Hermitian metric on D and the (essentially) unique G-invariant metric of L_λ define a G-invariant inner product on $A_0^i(L_\lambda)$. With respect to it, $\bar\partial$ has a formal adjoint $\bar\partial^*$. Let $L^i(L_\lambda)$ be the completion of $A_0^i(L_\lambda)$ i.e. the space of square-integrable L_λ-valued forms of type $(0, i)$. Then the Laplace operator $\Box=\bar\partial\circ\bar\partial^*+\bar\partial^*\circ\bar\partial$ can be extended to an unbounded self-adjoint operator on the Hilbert space $L^i(L_\lambda)$. Its kernel is a closed subspace, which coincides with the space $\mathcal{H}^i(L_\lambda)$ of square-integrable C^∞ L_λ-valued $(0, i)$-forms ω such that $\bar\partial\omega=0$, $\bar\partial^*\omega=0$.

Let $\Phi_+(\lambda)$ be the positive roots with respect to λ, i.e.

$$\Phi_+(\lambda)=\{\alpha\in\Phi\,|\,\langle\lambda, \alpha\rangle>0\}.$$

Then we define an integer k_λ by

$$k_\lambda=|-\Phi_+(\lambda)\cap\Phi_{K+}|+|\Phi_+(\lambda)\cap\Phi_{n+}|,$$

where Φ_+ is the set of positive roots, by which the complex structure on G/H is determined.

Griffiths-Schmid [15] shows the vanishing of $\mathcal{H}^i(L_\lambda)$ for $i\neq k_\lambda$, Schmid [38] proves that the representation of the discrete series with character $\Theta_{\lambda+\rho}$ is realized on the space $\mathcal{H}^{k_\lambda}(L_\lambda)$ for a sufficiently non-singular λ.

The relation of this realization with that of the previous section is given by the "Leray spectral sequence" for the holomorphic mapping $D=G/H\to X=G/K$ with fiber K/H, in view of the Borel-Weil-Bott theorem for the pair (K, H).

§ 1.4. The K-type theorem for the discrete series

We consider a real connected linear semisimple Lie group with a maximal compact subgroup K such that K has the same rank as G. But we do not assume that G/K has a G-invariant Hermitian structure. Since G is linear, the center of G is finite.

Let us denote by $\hat{K}$ the set of equivalence classes of finite dimensional irreducible representations of K. Then for each $\sigma \in \hat{K}$, we can associate the character χ_σ.

Let π be an irreducible unitary representation of G. Then it is known that the restriction of π to K breaks up discretely with finite multiplicities. And the multiplicity of any $\sigma \in \hat{K}$ in the decomposition of π is bounded by the degree of σ (cf. Theorem (1.1.10)). Set

$$m(\sigma; \pi) = \text{the multiplicity of } \sigma \text{ of } \hat{K} \text{ in } \pi.$$

For each $\sigma \in \hat{K}$, $\chi_\sigma(k)dk$ defines a distribution on K, where dk is the normalized Haar measure on K. Then the boundedness of the multiplicities $m(\sigma; \pi)$ implies that

$$\tau_\pi = \sum_{\sigma \in \hat{K}} m(\sigma; \pi)$$

converges to define a distribution on K. Schmid calls τ_π the *K-character* of π in [40].

When π is a representation of the discrete series of G, Blattner's conjecture, which is proved completely by Schmid [40] and Hecht-Schmid [19], tells an explicit formula for the multiplicity $m(\sigma; \pi)$ of any $\sigma \in \hat{K}$. Let us recall this result in this section. We assume that G_C is simply connected for simplicity.

(1.4.1) Let λ be an element of L'. Then we denote by (π_λ, H) the representation of the discrete series of G with character $\Theta_{\lambda+\rho}$ such that the restriction of it to the compact Cartan subgroup H is given by

$$\Theta_{\lambda+\rho|H} = (-1)^N \Big(\prod_{\substack{\alpha \in \Phi \\ \langle \lambda+\rho, \alpha \rangle > 0}} (e^{\alpha/2} - e^{-\alpha/2}) \Big)^{-1} \sum_{w \in W_K} \varepsilon(w) e^{w(\lambda+\rho)},$$

where $N = \frac{1}{2} \dim (G/K)$.

Let us define a system of positive roots $\Phi_+(\lambda)$ with respect to λ by

$$\Phi_+(\lambda) = \{\alpha \in \Phi \mid \langle \lambda+\rho, \alpha \rangle > 0\},$$

and let us enumerate the set $\Phi_n \cap \Phi_+(\lambda)$ as $\Phi_n \cap \Phi_+(\lambda) = \{\beta_1, \beta_2, \cdots, \beta_q\}$. In order to formulate the multiplicity formula of Blattner, we have to define a function Q on $\sqrt{-1}\mathfrak{h}^*$ in the first place, where $\mathfrak{h}^*$ is the dual space $\mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{h}, \boldsymbol{R})$ of $\mathfrak{h}$. For each $\mu \in \sqrt{-1}\mathfrak{h}^*$, $Q(\mu)$ is the number of distinct ways in which μ can be expressed as a sum

$$\mu = n_1\beta_1 + n_2\beta_2 + \cdots + n_q\beta_q,$$

with nonnegative integral coefficients n_i. The function $Q(\mu)$ is well defined, because $\Phi_+(\lambda)$ spans a cone in $\sqrt{-1}\mathfrak{h}^*$ lying in a half space. We put

$$\rho_c = \frac{1}{2} \sum_{\alpha \in \Phi_K \cap \Phi_+(\lambda)} \alpha.$$

(1.4.2) **Theorem.** *Assume that $\lambda \in L'$. Let $\mu \in \sqrt{-1}\mathfrak{h}^*$ be a weight for K which is dominant with respect to the system of positive root $\Phi_K \cap \Phi_+(\lambda)$ in Φ_K, and let π_λ be a representation of the discrete series with character $\Theta_{\lambda+\rho}$. Then in $\pi_{\lambda|K}$, the irreducible K-module of highest weight μ occurs with multiplicity*

$$\sum_{w \in W} \varepsilon(w) Q(w(\mu + \rho_c) - (\lambda + \rho) - \tfrac{1}{2}(\beta_1 + \cdots + \beta_q)).$$

By a simple argument (cf. [7] e.g.), it is known that an equivalent variant of the preceding theorem is stated as follows in terms of the K-character τ_π.

(1.4.2)′ **Theorem.** *Assume that $\lambda \in L'$, and let π_λ be a discrete series representation with character $\Theta_{\lambda+\rho}$. Then*

$$\tau_\pi = \sum_{0 \leqq n_1, \cdots, n_q < \infty} \Big(\prod_{\alpha \in \Phi_K \cap \Phi_+(\lambda)} (e^{\alpha/2} - e^{-\alpha/2})\Big)^{-1} \times \sum_{w \in W} \varepsilon(w) \exp\left[w(\lambda + \rho + \sum_{i=1}^{q} (n_i + \tfrac{1}{2})\beta_i \right]$$

is the K-character of π_λ.

From this theorem, we can deduce that for each $\lambda \in L_0'$ there exists a unique weight

$$\mu_0 = \lambda + \rho - \rho_c + \frac{1}{2} \sum_{i=1}^{q} \beta_i$$

of K, characterized by the following properties:

(i) $m(\sigma_{\mu_0}; \pi_\lambda) = 1$;

(ii) $m(\sigma_\mu; \pi_\lambda) = 0$ for any μ of the form $\mu = \mu_0 - A$, where A stands for a non-empty sum of roots in $\{\beta_1, \cdots, \beta_q\}$.

Conversely, μ_0 characterizes the representation π_λ. The following theorem is due to Schmid (cf. [39], Theorem 1.3).

(1.4.3) **Theorem.** *Up to infinitesimal equivalences, π_λ is the unique representation of G whose restriction to K contains the irreducible K-module of highest weight μ_0, and does not contain any irreducible K-module with a highest weight of the form $\mu_0 - A$, where A stands for a non-empty sum of roots in $\{\beta_1, \cdots, \beta_q\} = \Phi_+(\lambda) \cap \Phi_n$.*

Remark 1. The condition of the preceding theorem is relaxed for "most" of the discrete series, i.e. for the representation π_λ of the discrete series such that the weight λ is far enough from the walls $\langle \lambda + \rho, \alpha \rangle = 0$ of the Weyl chamber. The result reads as follows.

There exists a constant c depending only on G, such that if λ satisfies $|\langle \lambda + \rho, \alpha \rangle| \geqq c$ for any $\alpha \in \Phi$, then up to infinitesimal equivalences, π_λ is the unique representation of G whose restriction to K contains the irreducible K-module of highest weight μ_0, and does not contain any irreducible K-module with a highest weight of the form $\mu_0 - \beta_i$ $(1 \leqq i \leqq q)$ (cf. [21], [37], [39]).

Remark 2. If G/K is Hermitian and $\lambda \in L_0'$, we have

$$\Phi_K \cap \Phi_+(\lambda) = \Phi_{K+} \quad \text{and} \quad \mu_0 = \lambda + \sum_{\alpha \in \Phi_{n+} \cap \Phi_+(\lambda)} \alpha.$$

Chapter 2. Automorphic forms of discrete series

In this chapter, we discuss some automorphic forms on bounded symmetric domains which generate the representations of the discrete series of semisimple Lie groups. In the first section, we recall the definition of automorphic forms by Harish-Chandra. In the second section, we define "geometric" automorphic forms as sections of cohomology spaces over arithmetic quotients of bounded symmetric domains with coefficients in certain analytic sheaves. And in later sections, we identify them with intertwining operators from the discrete series representations to L^2 spaces of quotients of Lie groups by discrete subgroups.

§ 2.1. Harish-Chandra's definition

Harish-Chandra [18] gave the definition of the most general notion of automorphic forms on real reductive groups, and proved a number of fundamental results about them (see also Borel-Jacquet [5]). The purpose of this section is to recall some of them. In general we should consider the real (or complex) points of reductive algebraic groups defined over algebraic number fields. But thanks to Weil's theory of the restriction of scalars, it suffices to consider algebraic groups over $\boldsymbol{Q}$.

Let $\mathscr{G}$ be a connected reductive algebraic group over $\boldsymbol{Q}$, $\mathscr{Z}$ the greatest $\boldsymbol{Q}$-split subtorus of the center of $\mathscr{G}$. Fix a maximal compact subgroup K of the real points $G=\mathscr{G}(\boldsymbol{R})$ of $\mathscr{G}$, and denote by Z the real points $\mathscr{Z}(\boldsymbol{R})$ of $\mathscr{Z}$. In order to define slowly increasing functions on G, we first define a norm $\| \quad \|$ on G.

(2.1.1) **Definition.** Let $\tau: G \to GL(E)$ be a finite dimensional complex representation with finite kernel and closed image in End (E), and let * denote the adjoint with respect to a Hilbert space structure on E invariant under K. Then a norm $\| \quad \|$ on G is defined by

$$\|g\| = (\mathrm{tr}\, \tau(g)^* \tau(g))^{1/2}.$$

It is easy to check that if τ' is another representation with the same conditions, then there are a constant $C>0$ and a positive integer n such that

$$\|g\|_\tau \leqq C \|g\|_{\tau'}^n \qquad \text{for all } g \in G.$$

Also the norm has the following properties:

(i) $\|gg'\| \leqq \|g\| \cdot \|g'\|$ $(g, g' \in G)$;

(ii) there are positive constants c and N such that $\|g^{-1}\| \leqq c\|g\|^N$ for any $g \in G$.

(2.1.2) **Definition.** A function $f(g)$ on G is said to be *slowly increasing* if there exist a norm $\| \quad \|$ on G, a constant C and a positive integer n such that

$$|f(g)| \leqq C\|g\|^n, \qquad \text{for all } g \in G.$$

Obviously this condition does not depend on the choice of the norm (but n should be replaced).

Now let us define automorphic forms. Let Γ be a discrete subgroup of G, $\sigma: K \to GL(V)$ be a finite dimensional (irreducible) unitary representation of K, and $\lambda: Z(\mathfrak{g}_{\boldsymbol{C}}) \to \boldsymbol{C}$ be an algebra homomorphism. Let χ be a quasi-character of $Z/\Gamma \cap Z$ (i.e. a continuous homomorphism $\chi: Z/\Gamma \cap Z \to \boldsymbol{C}^\times$).

(2.1.3) **Definition.** A V-valued C^∞-function $f(g)$ on G is called an *automorphic form* for (Γ, K) of type (σ, λ, χ), if it satisfies the following conditions:

(i) $g \in G \mapsto \|f(g)\|$ is a slowly increasing function on G, where $\|f(g)\|$ is the norm of the vector $f(g)$ of V with respect to the given Hilbert space structure on V.

(ii) $f(\gamma g)=f(g)$ $(\gamma \in \Gamma,\ g \in G)$, and $f(zg)=\chi(z)f(g)$ $(z \in Z,\ g \in G)$.

(iii) $f(gk)=\sigma(k^{-1})f(g)$ $(k \in K,\ g \in G)$.

(iv) $f*X=\lambda(X)f$ for any $X \in Z(\mathfrak{g}_C)$, where $f*X$ is the right action of X on f.

(cf. [18], § 2, Definition).

Usually the group Γ is an arithmetic subgroup of $\mathscr{G}(\boldsymbol{Q})$. We assume this from now on. Let $\mathscr{A}(\Gamma, K; \sigma, \lambda)_\chi$ denote the space of automorphic forms of type (σ, λ, χ).

A variant of the preceding definition is the following (cf. [5]). Let dk be the normalized Haar measure on K. For an irreducible finite dimensional representation σ of K with character χ_σ, we associate a measure $d_\sigma \chi_\sigma dk$ on K, where d_σ is the dimension of σ. A finite sum of such measures defines an idempotent of the convolution algebra $\mathscr{H}(G, K)$ of distributions on G with supports in K.

(2.1.4) **Definition.** A complex valued C^∞-function $f(g)$ on G is an *automorphic form* for (Γ, K) of type $(\xi, \mathscr{J}, \chi)$, if it satisfies the conditions:

(i) f is slowly increasing on G.

(ii) $f(\gamma g)=f(g)$ $(\gamma \in \Gamma,\ g \in G)$, and $f(zg)=\chi(z)f(g)$ $(z \in Z,\ g \in G)$.

(iii) There is an idempotent ξ in $\mathscr{H}(G, K)$ defined by certain representations of K, such that $f*\xi=f$.

(iv) There exists an ideal $\mathscr{J}$ of finite codimension in $Z(\mathfrak{g}_C)$ which annihilates f: $f*X=0$ $(X \in \mathscr{J})$.

We denote by $\mathscr{A}(\Gamma, K; \xi, \mathscr{J})_\chi$ the space of automorphic forms of type $(\xi, \mathscr{J}, \chi)$ for (Γ, K).

It is easy to see that (2.1.3) and (2.1.4) are substantially equivalent.

One of the main results in [18] is the following finiteness theorem:

(2.1.5) **Theorem.** *The space $\mathscr{A}(\Gamma, K; \sigma, \lambda)_\chi$, or $\mathscr{A}(\Gamma, K; \xi, \mathscr{J})_\chi$ is finite dimensional (over $\boldsymbol{C}$).*

Actually [18] discusses only semisimple groups. But the methods work for reductive groups.

(2.1.6) **Definition.** An automorphic form f is called a *cusp form*, if

$$\int_{(\Gamma \cap \mathscr{N}(\boldsymbol{R}))\backslash \mathscr{N}(\boldsymbol{R})} f(ng)dn=0$$

for all $g \in G$, where $\mathscr{N}$ is the unipotnet radical of any parabolic $\boldsymbol{Q}$-subgroup of $\mathscr{G}$. In fact it suffices to require this condition for any proper maximal parabolic $\boldsymbol{Q}$-subgroup of $\mathscr{G}$ (cf. Lemma 2, § 2 of [18]). We denote by $\mathscr{A}_0(\Gamma, K; \sigma, \lambda)_\chi$ (resp. $\mathscr{A}_0(\Gamma, K; \xi, \mathscr{J})_\chi$) the space of cusp forms in $\mathscr{A}(\Gamma, K; \sigma, \lambda)_\chi$ (resp. $\mathscr{A}(\Gamma, K; \xi, \mathscr{J})_\chi$).

(2.1.7) **Remark.** A more intrinsic version of the growth condition (i) in the definition of automorphic forms is given as follows. Let A be the identity component of the real points $\mathscr{S}(\boldsymbol{R})$ of a maximal $\boldsymbol{Q}$-split torus $\mathscr{S}$ of $\mathscr{G}$ and Φ_Q the system of roots of $\mathscr{G}$ with respect to $\mathscr{S}$. Choose an ordering on Φ_Q and let Δ be the set of simple roots with respect to this ordering. Given a positive real number t, put

$$A_t = \{a \in A \mid |\alpha(a)| > t, \text{ for all } \alpha \in \Delta\}.$$

Let f be a function satisfying (2.1.4) (ii), (iii), and (iv). Then the growth condition (i) is equivalent to:

(i)$'$ For any given compact subset R of G, and any $t > 0$, there exist a constant C and a positive integer m such that

$$|f(xa)| < C|\alpha(a)|^m, \qquad \text{for all } a \in A_t,\ \alpha \in \Delta,\ x \in R.$$

The equivalence is shown in [18], Lemma 6 of Section 3 by using Siegel sets.

§ 2.2. Representations generated by automorphic forms

Let f be an automorphic form which belongs to $\mathscr{A}(\Gamma, K; \xi, \mathscr{J})_\chi$. Then the subspace of the right translations $\{f(xg) \mid g \in G\}$ of the smooth function $f(x)$ on $\Gamma\backslash G$ in $C^\infty(\Gamma\backslash G)$ defines a subrepresentation of G in the right regular representation of G on $C^\infty(\Gamma\backslash G)$. We want to recall some basic facts about such representations.

In order to formulate some statements, let us introduce more terminology for representations of Lie groups.

Let E be a locally convex Hausdorff topological vector space, and π a continuous representation of G on E. Then for each differentiable vector v of E (i.e. $v \in E_\infty$), we denote by $\tilde{v}$ the map $\tilde{v}: g \in G \mapsto \pi(g)v \in E$, which belongs to $C^\infty(G; E)$ (cf. (1.2.7)).

(2.2.1) **Definition.** Let π be a continuous representation of G on E. Then π is said to be *differentiable*, if $E = E_\infty$ and the map $v \mapsto \tilde{v}$ is a topological isomorphism of E into $C^\infty(G; E)$.

If $\pi: G \to \mathrm{Aut}(E)$ is a differentiable representation, we can associate a representation π_∞ of $\mathfrak{g}$ (or of $U(\mathfrak{g})$) on E (cf. Definition (1.2.8)). We denote π_∞ also by $\pi|_{\mathfrak{g}}$.

(2.2.2) **Definition.** Let $\pi: K \to \mathrm{Aut}(E)$ be a continuous representation of K. Then a vector $v \in E$ is *K-finite*, if it is contained in a finite dimensional subspace stable under K. The representation π is called *locally finite*, if every vector in E is K-finite.

Let $\pi: G \to \mathrm{Aut}(E)$ be a continuous representation, and let E_K the subspace of K-finite vectors in E. Then it is a semisimple K-module. The subspace $E_\infty \cap E_K$ is stable under $\mathfrak{g}$. It is known that if an isotypic subspace $E(\delta)$ ($\delta \in \hat{K}$) of K in E is finite dimensional, it is contained in E_∞.

(2.2.3) **Definition.** We say that a representation π is *admissible*, if the isotypic subspaces $E(\delta)$ ($\delta \in K$) in E are all finite dimensional. In this case $E_K \subset E_\infty$.

(2.2.4) **Definition.** A $(\mathfrak{g}, K)$-module E is a real or complex vector space E which is a bimodule of $\mathfrak{g}$ and K, and also a locally finite and semisimple K-module with the compatibility conditions:

(i) $\pi(k)(\pi(X))v = \pi(\mathrm{Ad}(k)X)\pi(k)v$ ($k \in K$, $X \in U(\mathfrak{g}_C)$, $v \in E$);

(ii) If F is a K-stable finite dimensional subspace of E, the representation of K on F is differentiable, and has $\pi|_{\mathfrak{k}}$ as its differential.

A $(\mathfrak{g}, K)$-module is *admissible*, if all the isotypic subspaces $E(\delta)$ for K are finite dimensional ($\delta \in \hat{K}$). By Theorem (1.1.10), any $(\mathfrak{g}, K)$-module E_K of a topologically irreducible unitary representation (π, E) is admissible.

Let $\mathscr{H}(G, K)$ be the convolution algebra of left and right K-finite distributions on G with supports in K. Since any element of $\mathscr{H}(G, K)$ is a transversal derivative of an extension of a distribution on K with respect to the embedding $K \hookrightarrow G$, it is not difficult to see that $\mathscr{H}(G, K)$ is isomorphic to $U(\mathfrak{g}_C) \otimes_{U(\mathfrak{k}_C)} A_K$ as vector spaces, where A_K is the algebra of finite measures on K.

Let f be an automorphic form. Then, as a function of $C^\infty(\Gamma \backslash G)$, it is a K-finite and $Z(\mathfrak{g}_C)$-finite function via the right action of G on $C^\infty(\Gamma \backslash G)$.

(2.2.5) **Proposition** (cf. [5], § 2). *The subspace* $f * \mathscr{H}(G, K)$ *of* $C^\infty(\Gamma \backslash G)$ *is an admissible* $(\mathfrak{g}, K)$*-module* (*or* $\mathscr{H}(G, K)$*-module*) *consisting of automorphic forms. Moreover if* f *is a cusp form, then* $f * \mathscr{H}(G, K)$ *consists of cusp forms.*

Remark. If f satisfies $f * X = \gamma(X) f$ for any $X \in Z(\mathfrak{g}_C)$ with respect to a character $\gamma: Z(\mathfrak{g}_C) \to C$ of $Z(\mathfrak{g}_C)$, then the representation on $f * \mathscr{H}(G, K)$ has the infinitesimal character γ.

§ 2.3. Geometric automorphic forms

Let $\lambda \in L_0'$, and let F_λ be the G-bundle or the G-linearized locally free analytic sheaf on X associated to λ. If the sheaf F_λ descends to an analytic sheaf on $\Gamma\backslash X$ with respect to the quotient mapping $X \to \Gamma\backslash X$, we denote the induced sheaf on $\Gamma\backslash X$ by the same symbol F_λ. Let Z_0 be the kernel of the canonical homomorphism $G \to G^{\mathrm{ad}}$ of G to its adjoint group G^{ad}. Let σ_λ be the irreducible representation of K with highest weight λ. Then, if $\sigma_\lambda(Z_0 \cap \Gamma)$ is not trivial, F_λ on $\Gamma\backslash X$ never exists. If $\sigma_\lambda(Z_0 \cap \Gamma) = \{1\}$ and the quotient group $\Gamma/(Z_0 \cap \Gamma)$ is torsion-free, Γ has no fixed point on X. Hence F_λ on $\Gamma\backslash X$ exists in this case.

Assume that $\sigma_\lambda(Z_0 \cap \Gamma) = \{1\}$ and that Γ has fixed points on X. Then we can consider a dense open subset X_0 of X invariant under Γ, not containing any fixed point of Γ. Consider the descent $F_{\lambda 0}$ of $F_\lambda|_{X_0}$ with respect to $X_0 \to \Gamma\backslash X_0$, and put $F_\lambda = j_*(F_{\lambda 0})$ for $j\colon \Gamma\backslash X_0 \hookrightarrow \Gamma\backslash X$ in this case.

In any case, the sheaf F_λ is coherent analytic on $\Gamma\backslash X$, and locally free except on a closed analytic subset of $\Gamma\backslash X$.

The following vanishing theorem of the cohomology groups $H^i(\Gamma\backslash X, F_\lambda)$ is due to Parthasarathy [34], [35] and Hotta-Parthasarathy [20], [21].

(2.3.1) **Theorem.** *Assume that $\Gamma\backslash X$ is compact. If $\lambda \in L_0'$, λ_D and σ are unique elements in D and W^1, respectively, such that $\lambda = \lambda_D^{(\sigma)} = \sigma(\lambda+\rho)-\rho$. Assume moreover that λ_D satisfies the condition $\langle \tau(\lambda_D), \alpha\rangle \neq 0$ for any $\tau \in W^1$ and any $\alpha \in \Phi_n$. Then*

$$H^i(\Gamma\backslash X, F_\lambda) = \{0\} \qquad \text{for } i \neq q_\lambda.$$

Remark. Under the same assumption as the above theorem, the space $H^i(\Gamma\backslash X, F_\lambda)$ is naturally identified with the space of Γ-invariant harmonic forms on X of type $(0, i)$ with values in F_λ.

(2.3.2) **Definition.** We call an element of $H^{q_\lambda}(\Gamma\backslash X, F_\lambda)$ a *geometric automorphic form* of type λ on X with respect to λ.

In the rest of this chapter, we see that the space $H^{q_\lambda}(\Gamma\backslash X, F_\lambda)$ is canonically identified with the intertwing space of G-modules from a representation of the discrete series of G to the space $L^2(\Gamma\backslash G)$, and geometric automorphic forms generate the representations of the discrete series. Also we see the relation between geometric automorphic forms and the automorphic forms of Harish-Chandra.

§ 2.4. Serre duality

Under the same notation as in the previous section, we assume that

Γ has no fixed point on X, hence $\Gamma\backslash X$ is smooth. Let

$$\tau_+ = \mathrm{Ad}|_{\mathfrak{p}_+} : K \longrightarrow GL(\mathfrak{p}_+)$$

be the restriction of the adjoint representation of K on $\mathfrak{g}_C$ to $\mathfrak{p}_+$. Then the bundle F_{τ_+} associated to this representation is isomorphic to the holomorphic tangent bundle of X or of $\Gamma\backslash X$. Passing to the duals, we have isomorphisms of bundles with G-actions on X:

$$F_{\tau_-} \cong \Omega_X^1, \quad \text{and} \quad F_{\tau_-^p} \cong \Omega_X^p,$$

where τ_- and τ_-^p are the adjoint representation

$$\tau_- = \mathrm{Ad}|_{\mathfrak{p}_-} : K \longrightarrow GL(\mathfrak{p}_-),$$

and its p-th wedge product

$$\tau_-^p = \overset{p}{\wedge} \mathrm{Ad}|_{\mathfrak{p}_-} : K \longrightarrow GL(\overset{p}{\wedge} \mathfrak{p}_-),$$

respectively. Especially, for $p=N$ the representation τ_-^N is the irreducible representation of K with highest weight $-2\rho_n$. Hence

$$F_{-2\rho_n} \cong \Omega_X^N \qquad \text{as } G\text{-bundles on } X.$$

Now we recall that in the Weyl group W there is a unique element w_- such that $w_-(\Phi_+) = \Phi_-$, and also that there is a unique element w_{K-} in $W_K = W(G, H)$ such that $w_{K-}(\Phi_{K+}) = \Phi_{K-}$. Put $\sigma_{\mathrm{op}} = w_{K-} w_- \in W^1$, and for $\lambda \in L_0'$ set

$$\lambda_{\mathrm{op}} = \lambda^{(\sigma_{\mathrm{op}})} = \sigma_{\mathrm{op}}(\lambda + \rho) - \rho = \sigma_{\mathrm{op}}(\lambda) + \sigma_{\mathrm{op}}(\rho) - \rho = \rho_{\mathrm{op}}(\rho) - 2\rho_n.$$

Note that $\sigma_{\mathrm{op}}(\rho_c) = \rho_c$ and $\rho_{\mathrm{op}}(\rho_n) = -\rho_n$ here. Then $\lambda_{\mathrm{op}} \in L_0'$ and $q_{\lambda_{\mathrm{op}}} = N - q_\lambda$. Moreover we can check that the representation of K with highest weight λ_{op} is the contragradient representation of the representation with highest weight λ.

Thus we have an isomorphism of G-bundle on X:

$$F_{\lambda_{\mathrm{op}}} = F_{\sigma_{\mathrm{op}}(\lambda)} \otimes F_{-2\rho_n} \cong F_\lambda^\vee \otimes \Omega_X^N.$$

Hence, if Γ has no fixed point on X, there is a perfect pairing

$$F_\lambda \times F_{\lambda_{\mathrm{op}}} \longrightarrow \Omega_S^N$$

of analytic locally free sheaves on $S = \Gamma\backslash X$.

Therefore in this case the Serre duality reads as follows.

(2.4.1) **Theorem.** *Assume that $\Gamma\backslash X$ is compact. There is an isomorphism*:

$$H^{q_\lambda}(\Gamma\backslash X, F_\lambda) \cong \mathrm{Hom}_C(H^{N-q_\lambda}(\Gamma\backslash X, F_{\lambda_{\mathrm{op}}}), C),$$

or equivalently there is a perfect pairing:

$$H^{q_\lambda}(\Gamma\backslash X, F_\lambda) \times H^{N-q_\lambda}(\Gamma\backslash X, F_{\lambda_{\mathrm{op}}}) \longrightarrow C.$$

When Γ has no fixed point on X, $\Gamma\backslash X$ is smooth. In this case the above theorem is the Serre duality. When Γ has fixed points on X, we can find an index-finite normal subgroup Γ' in Γ such that Γ' has no fixed point on X. Then $H^{q_\lambda}(\Gamma\backslash X, F_\lambda) = H^{q_\lambda}(\Gamma'\backslash X, F_\lambda)^{\Gamma\backslash\Gamma'}$ implies the above result.

Let us denote by $\mathscr{H}^{N-q_\lambda}(X, F_{\lambda_{\mathrm{op}}})$ the space of harmonic $(N-q_\lambda)$-forms of type $(0, N\text{-}q_\lambda)$ on X with values in $F_{\lambda_{\mathrm{op}}}$. Then $H^{N-q_\lambda}(\Gamma\backslash X, F_{\lambda_{\mathrm{op}}})$ is identified with the space

$$\mathscr{H}^{N-q_\lambda}(X, F_{\lambda_{\mathrm{op}}})^\Gamma$$

of Γ-invariant elements in $\mathscr{H}^{N-q_\lambda}(X, F_{\lambda_{\mathrm{op}}})$.

(2.4.2) **Corollary.** *Suppose that $\lambda \in L_0'$ and $\Gamma\backslash X$ is compact. Then*

$$\begin{aligned}\dim_C H^{q_\lambda}(\Gamma\backslash X, F_\lambda) &= \dim_C H^{N-q_\lambda}(\Gamma\backslash X, F_{\lambda_{\mathrm{op}}}) = \dim_C \mathscr{H}^{N-q_\lambda}(X, F_{\lambda_{\mathrm{op}}})^\Gamma \\ &= \dim_C \mathscr{H}^{q_\lambda}(X, F_\lambda)^\Gamma.\end{aligned}$$

§ 2.5. Frobenius reciprocity

This subsection is an extract from Section 4 of Chapter 8 of the textbook of Borel-Wallach [6], which reformulates the result of the book [14] of Gelfand, Graev, and Piatetski-Shapiro, in a more general form.

Let G be a connected semisimple Lie group, and a discrete subgroup of G such that $\Gamma\backslash G$ is compact.

(2.5.1) **Definition.** If (π, H) is a unitary representation of G, then H_∞ denotes the space of C^∞-vectors for (π, H) with C^∞-topology. $(H_\infty)^*$ denotes the space of continuous linear functionals on H_∞. If (π_i, H_i) $(i=1, 2)$ are unitary representations of G, then $\mathrm{Hom}_G(H_1, H_2)$ denotes the space of all bounded linear operators $A: H_1 \to H_2$ such that

$$A \circ \pi_1(g) = \pi_2(g) \circ A \qquad \text{for } g \in G.$$

Let Γ be a cocompact discrete subgroup of G. Let π_Γ denote the right regular representation of G on $L^2(\Gamma\backslash G)$ (here we fix a biinvariant measure dg on G which induces a right invariant measure $d(\Gamma g)$ on $\Gamma\backslash G$). We recall that the space of C^∞-vectors of $(\pi_\Gamma, L^2(\Gamma\backslash G))$ is precisely the space $C^\infty(\Gamma\backslash G)$ of C^∞-functions on $\Gamma\backslash G$ with C^∞-topology.

If (π, H) is a unitary representation of G, set

$$(H_\infty)^{*\Gamma}=\{\lambda \in (H_\infty)^* \mid \lambda \circ \pi(\gamma)=\lambda \text{ for } \gamma \in \Gamma\}.$$

(2.5.2) **Theorem** (Gelfand, Graev, Piatetski-Shapiro). *Let* (π, H) *be an irreducible unitary representation of* G. *If*

$$A \in \mathrm{Hom}_G(H, L^2(\Gamma\backslash G)),$$

set $\lambda_A(v)=A(v)(\Gamma\cdot 1)$ *for* $v \in H_\infty$. *The the map*

$$A \longmapsto \lambda_A$$

is a bijection from $\mathrm{Hom}_G(H, L^2(\Gamma\backslash G))$ *to* $(H_\infty)^{*\Gamma}$.

(2.5.3) **Remark.** The preceding theorem is considered as the Frobenius reciprocity. Regard $\boldsymbol{C}$ as the trivial Γ-module. Then $C^\infty(\Gamma\backslash G)$ $=\mathrm{Ind}_\Gamma^G \boldsymbol{C}$ in the smooth category (cf. [6], III. 2.1). Hence

$$\mathrm{Hom}_G(H_\infty, C^\infty(\Gamma\backslash G))=\mathrm{Hom}_\Gamma(H_\infty, \boldsymbol{C})=(H_\infty)^{*\Gamma},$$

where H_∞ is any admissible smooth G-module.

§ 2.6. Frobenius reciprocity for the discrete series

We can identify the space of geometric automorphic forms for Γ with a intertwing space from a square integrable representation to the right regular representation on $L^2(\Gamma\backslash G)$ by the following.

(2.6.1) **Theorem.** *Let* $\lambda \in L'_0 \cap X(H_C)$, *and assume that* $\Gamma\backslash X$ *is compact. Then, under the same assumption on* λ_D *as in Theorem* (2.3.1), *we have*

$$\begin{aligned} H^{q_\lambda}(\Gamma\backslash X, F_\lambda) &\cong \mathrm{Hom}_C(\mathscr{H}^{N-q_\lambda}(X, F_{\lambda_{\mathrm{op}}})^\Gamma, \boldsymbol{C}) \\ &\cong \mathrm{Hom}_\Gamma(\mathscr{H}_2^{0,N-q_\lambda}(X, F_{\lambda_{\mathrm{op}}})_\infty, \boldsymbol{C}) \\ &\cong \mathrm{Hom}_G(\mathscr{H}_2^{0,N-q_\lambda}(X, F_{\lambda_{\mathrm{op}}}), L^2(\Gamma\backslash G)). \end{aligned}$$

The first isomorphism is the Serre duality (Theorem (2.4.1)). The last isomorphism

$$\mathrm{Hom}_\Gamma(\mathscr{H}_2^{0,N-q_\lambda}(X, F_{\lambda_{\mathrm{op}}})_\infty, \boldsymbol{C}) \cong \mathrm{Hom}_G(\mathscr{H}_2^{0,N-q_\lambda}(X, F_{\lambda_{\mathrm{op}}}), L^2(\Gamma\backslash G))$$

is shown in the previous section (Theorem (2.5.2)). It suffices to show that

$$H^{q_\lambda}(\Gamma\backslash X, F_\lambda) \cong \mathrm{Hom}_G(\mathscr{H}_2^{0, N-q_\lambda}(X, F_{\lambda_{\mathrm{op}}}), L^2(\Gamma\backslash G)),$$

which is a paraphrase of what is shown in the proof of Theorem 2 of Schmid [38].

(2.6.2) **Remark.** In the proof of Theorem 2 of [38], Schmid assumed that λ is sufficiently non-singular. We have to explain why the condition on λ is relaxed as in the preceding theorem.

The reason to assume that λ is very non-singular was twofold: One is that the vanishing theorem was not sharp enough (cf. Theorem 7.8 of [15], and Corollary 1 of Lemma 5 of [38]), and the other is that the characterization of the discrete series by the non-vanishing of harmonic forms with values in the unitary representations of G was not verified without the assumption that λ is very non-singular (cf. Lemma 7 and Corollary of Lemma 9 of [38]). However the vanishing theorem of Parthasarathy (Theorem 3 of [34]) improves Theorem 1, Corollary 1 and Corollary 2 of Lemma 5 of [38], so that they are valid under the same condition on λ as in Theorem (1.3.3) or (2.3.1). Thus the first point is settled.

The other point is to check that among all irreducible representations occurring in the direct sum decomposition of $L^2(\Gamma\backslash G)$, only the discrete series representations can contribute to $H^{q_\lambda}(\Gamma\backslash X, F_\lambda)$. The proof of [38] is to show that the non-vanishing of the space $\mathscr{H}^i(\pi)_{-\lambda}$ of harmonic forms of a unitary representation of G implies that π is of the discrete series under the assumption that λ is very non-singular (cf. Lemmas 7–9 of [38]). One should note that $\mathscr{H}^i(\pi)_{-\lambda} \neq \{0\}$ in the notation of [38] implies that the Lie algebra cohomology group $H^*(\mathfrak{g}, \mathfrak{k}, H_{\pi,0} \otimes F) \neq \{0\}$ for some finite dimensional representation F of G, where $H_{\pi,0}$ is the space of K-finite vectors in the representation space H_π of π. Then Theorem 6.4 of II in Borel-Wallach [6] implies that π is of the discrete series. Once this point is established, the rest of the proofs of [38] works even if λ is not very non-singular.

(2.6.3) **Remark.** Gelfand, Graev, and Piatetski-Shapiro [14] finds the isomorphism of Theorem (2.6.1) for elliptic modular forms (Piatetski-Shapiro [36], Section 4 discusses the special case for elliptic modular forms of weight 2, and attributes this kind of isomorphisms to Gelfand and Fomin), and Deligne [9], Scholie 2.1.2 regards the isomorphism of the theorem as the definition of (holomorphic) elliptic modular forms through the evaluation mapping of the next section.

§ 2.7. Evaluation mapping and Harish-Chandra's definition

In this section, we discuss the relation between the notion of geometric automorphic forms and automorphic forms in the sense of Harish-Chandra considered in (2.1).

Choose an element $\lambda \in L_0' \cap X(H_C)$, and put $H = H_2^{0, N-q_\lambda}(X, F_\lambda)$. Let us consider the subspace H_K consisting of K-finite vectors in H. Let W be a K-submodule of the isotypic component $H(\hat{\delta})$ of H for some $\hat{\delta} \in \hat{K}$, and δ the representation of K on W. Then by restriction, we have the canonical homomorphism

$$e_W : H^{q_\lambda}(\Gamma \backslash X, F_\lambda) \cong \mathrm{Hom}_G(H, L^2(\Gamma \backslash G)) \longrightarrow \mathrm{Hom}_K(W, C^\infty(\Gamma \backslash G)_K),$$

which we call the *evaluation mapping* at W. If we denote by W^* the dual space of W, the last space $\mathrm{Hom}_K(W, C^\infty(\Gamma \backslash G)_K)$ is canonically identified with $(W^* \otimes C^\infty(\Gamma \backslash G))^K$. Via this identification, we may consider e_W a homomorphism of $H^{q_\lambda}(\Gamma \backslash X, F_\lambda)$ to $(W^* \otimes C^\infty(\Gamma \backslash G))^K$. Moreover, since $\gamma_{\lambda_{\mathrm{op}}+\rho} : Z(\mathfrak{g}_C) \to C$ is the infinitesimal character of $\pi_{\lambda_{\mathrm{op}}}$, we have

$$\pi_{\lambda_{\mathrm{op}}}(Z)v = \gamma_{\lambda_{\mathrm{op}}+\rho}(Z)v \qquad \text{for any } Z \in Z(\mathfrak{g}_C) \text{ and } v \in H_K \subset H_\infty.$$

Therefore the image of e_W belongs to the subspace

$$(W^* \otimes C^\infty(\Gamma \backslash G; \gamma_{\lambda+\rho}))^K$$

of $(W^* \otimes C^\infty(\Gamma \backslash G))^K$, where $C^\infty(\Gamma \backslash G; \gamma_{\lambda+\rho})$ is the subspace of $C^\infty(\Gamma \backslash G)$ consisting of the functions f such that $f * X = \gamma_{\lambda+\rho}(X) f$ for any $X \in Z(\mathfrak{g}_C)$. Note that $\gamma_{\lambda+\rho} = \gamma_{\lambda_{\mathrm{op}}+\rho}$ here. When $\Gamma \backslash G$ is compact, the space $(W^* \otimes C^\infty(\Gamma \backslash G; \gamma_{\lambda+\rho}))^K$ is no other than the space of automorphic forms $\mathscr{A}(\Gamma, K; \delta^*, \gamma_{\lambda+\rho})$ in Section 2.1. Here δ^* is the contragradient representation of δ on W^*, and we drop the subscript of the character of the center Z of G in the symbol $\mathscr{A}(\Gamma, K; *, *)$, because it is trivial in this case.

Now let us recall the K-type theorem of Section 1.4. Let μ_0 be the lowest K-type as in Theorem (1.4.3), and δ_0 the corresponding irreducible representation of K on W_0. Then by Theorem (1.4.3), the irreducible $U(\mathfrak{g}_C)$-module H_K is generated by $W_0 = H(\delta_0)$. Therefore the canonical homomorphism

$$\mathrm{Hom}_{(\mathfrak{g}, K)}(H_K, C^\infty(\Gamma \backslash G)_K) \longrightarrow \mathrm{Hom}_K(W_0, C^\infty(\Gamma \backslash G; \gamma_{\lambda+\rho}))$$

is injective. Accordingly the evaluation mapping

$$e_{W_0} : H^{q_\lambda}(\Gamma \backslash X, F_\lambda) \longrightarrow \mathscr{A}(\Gamma, K; \delta_0^*, \gamma_{\lambda+\rho})$$

is also injective.

Then Theorem (2.6.1) says that the representation of G generated by the automorphic forms in $e_{W_0}(H^{q_\lambda}(\Gamma\backslash X, F_\lambda))$ or $e_{W_0}(\mathscr{H}^{q_\lambda}(\Gamma\backslash X, F_\lambda)^\Gamma)$ is a representation of the discrete series of G, which is equivalent to the representation on $\mathscr{H}_2^{0,N-q_\lambda}(X, F_{\lambda_{\mathrm{op}}})$.

(2.7.1) **Remark.** Especially, when $q_\lambda=0$, the space $e_{W_0}(H^0(\Gamma\backslash X, F_\lambda))$ is identified with a space of holomorphic automorphic forms on X for Γ with values in W_0^*. The representation generated by a non-zero element of this space is equivalent to the representation of G on $H_2^{0,N}(X, F_{\lambda_{\mathrm{op}}})$. Note that it is contragradient to the representation on $H_2^{0,0}(X, F_\lambda)$, and there exists a conjugate-linear isometry of Hilbert spaces with G-actions

$$\sharp : \mathscr{H}_2^{0,N}(X, F_{\lambda_{\mathrm{op}}}) \longrightarrow \mathscr{H}_2^{0,0}(X, F_\lambda)$$

(cf. Theorem 1.2 of [33]).

Consider another embedding $X \hookrightarrow G_C/K_C P_+$ in the place of the embedding $X \hookrightarrow G_C/K_C P_-$ to define a complex structure on X by pull-back. Then the new complex structure on X is the conjugation of the old one. If we denote by $\bar{F}_\lambda$ the holomorphic bundle corresponding to the representation σ_λ of K with respect to the new complex structure on X, then there exists a conjugate-linear isometry of Hilbert spaces with G actions

$$b : \mathscr{H}_2^{0,0}(X, \bar{F}_\lambda) \longrightarrow \mathscr{H}_2^{0,0}(X, F_\lambda).$$

The composition $\sharp \cdot b^{-1}$ gives an G-isomorphisms of Hilbert spaces

$$\mathscr{H}_2^{0,N}(X, F_{\lambda_{\mathrm{op}}}) \cong \mathscr{H}_2^{0,0}(X, \bar{F}_\lambda).$$

Thus the representation of G on $H_2^{0,N}(X, F_{\lambda_{\mathrm{op}}})$ belongs to the "*antiholomorphic*" discrete series of G.

(2.7.2) **Remark.** By a similar reason as in the preceding remark, we have a conjugate-linear isomorphism

$$\sharp : H^{q_\lambda}(\Gamma\backslash X, F_\lambda) \longrightarrow H^{N-q_\lambda}(\Gamma\backslash X, F_{\lambda_{\mathrm{op}}}).$$

If $\langle\ ,\ \rangle$ is the pairing $H^{q_\lambda}(\Gamma\backslash X, F_\lambda) \times H^{N-q_\lambda}(\Gamma\backslash X, F_{\lambda_{\mathrm{op}}}) \to \boldsymbol{C}$ of the Serre duality (Theorem (2.5.1)), then $(\eta_1, \eta_2) = \langle \eta_1, \sharp\eta_2 \rangle$ for $\eta_i \in H^{q_\lambda}(\Gamma\backslash X, F_\lambda)$ $(i=1, 2)$ coincides with the Petersson metric.

Chapter 3. Cohomology groups of discrete subgroups

In this chapter, we consider a local system over an arithmetic quotient of a bounded symmetric domain, which corresponds to a finite dimensional representation of the isometry group of the domain. And we discuss the Hodge decomposition of the cohomology groups with coefficients in that local system. Each Hodge component of it is identified with a space of geometric automorphic forms considered in the previous chapter.

The de Rham-Hodge spectral sequence for such cohomology groups is due to Deligne (cf. Zucker [42]). And Zucker [43] investigates the Hodge decomposition of these cohomology groups, and shows that it coincides with the Hodge decomposition of cohomology groups of discrete subgroups, which is first investigated by Matsushima-Murakami [28], [29], by means of square integrable harmonic forms.

§ 3.1. Spectral sequences for Hodge decomposition

Let us recall some basic spectral sequences for Hodge structures of the cohomology groups over a (projective) algebraic variety with coefficients in a family of Hodge structures.

Let S be a quasi-projective algebraic variety over $\boldsymbol{C}$. Then we denote by the same symbol S the analytic manifold associated to S. Let V be a local system of real vector spaces over S, i.e. a locally constant sheaf on $S=S(\boldsymbol{C})^{\mathrm{an}}$ whose fibers are finite dimensional real vector spaces. We denote by V_s the fiber of V at a point s of S.

(3.1.1) **Definition** (cf. [10], [42], [43]). A *real variation of Hodge structure of weight n* over S is a local system V of real vector spaces over S with the following data:

(i) (Hodge filtration) The associated holomorphic vector bundle $\mathscr{V}=\mathcal{O}_S\otimes_{\boldsymbol{R}} V$ has a finite holomorphic filtration F:

$$\mathscr{V}=F^0\mathscr{V}\supset F^1\mathscr{V}\supset F^2\mathscr{V}\supset\cdots\supset\{0\},$$

such that at each point s of S, F defines a Hodge structure of weight n on V_s.

(ii) (Transversality) The natural (=Gauss-Manin) connection $\nabla=\partial\otimes 1$ on $\mathscr{V}=\mathcal{O}_S\otimes_{\boldsymbol{R}} V$ is a flat connection, hence defines a complex:

$$\text{(3.1.1.1)}\quad \mathscr{V}\xrightarrow{\nabla}\Omega_S^1(V)=\Omega_S^1\otimes_{\boldsymbol{R}} V\cong\Omega_S^1\otimes_{\mathcal{O}_S}\mathscr{V}\xrightarrow{\nabla}\Omega_S^2(V)\xrightarrow{\nabla}\Omega_S^3(V)\longrightarrow$$

which satisfies the transversality axiom:

(3.1.1.2) $$\nabla(F^p\mathcal{V})\subset\Omega_S^1\otimes_{\mathcal{O}_S}F^{p-1}\mathcal{V}$$

of Griffiths for each p.

Deligne (cf. Zucker [42]) defined a Hodge filtration on the de Rham complex $\{\Omega_S^{\cdot}(V)\}$ with values in V by

(3.1.1.3) $$[F^p\Omega_S^{\cdot}(V)]^r=\Omega_S^r(F^{p-r}\mathcal{V})=\Omega_S^r\otimes_{\mathcal{O}_S}F^{p-r}\mathcal{V}.$$

Here p is the filtered degree, and r is the graded degree of the subcomplex $F^p\Omega_S^{\cdot}(V)$. Then there is the spectral sequence of the hypercohomology groups for the filtered complex $(\Omega_S^{\cdot}(V), F^{\cdot})$:

(3.1.1.4) $$E_1^{p,q}=\mathbf{H}^{p+q}(S, \mathrm{Gr}_F^p\Omega_S^{\cdot}(V))\Longrightarrow\mathbf{H}^{p+q}(S, \Omega_S^{\cdot}(V))\cong H^{p+q}(S, V)$$

The following theorem is due to Deligne ([42], Theorem (5.9)).

(3.1.2) **Theorem.** *If S is projective, the above spectral sequence* (3.1.1.4) *degenerates at E_1 term.*

A polarization of the variation of Hodge structure V is a bilinear form ψ; $V\times V\to\boldsymbol{R}(-n)$ with values in the constant variation of Hodge structure $\boldsymbol{R}(-n)$ such that at each point s of V it defines the polarization of V_s. Here $\boldsymbol{R}(-n)$ is the real Hodge structure of Tate of weight $2n$.

The above notion of variations of Hodge structure is too restrictive and a bit inconvenient to consider local systems which are direct summands of higher direct images of analytic families of abelian varieties over arithmetic quotients of bounded symmetric domains (say, the complex hyperballs). Thus Deligne and Zucker (cf. [43]) introduced the following notion.

(3.1.3) **Definition.** A complex variation of Hodge structure of weight n over S is a local system V of complex vector spaces over S with the following data:

(i) The associated holomorphic vector bundle $\mathcal{V}=\mathcal{O}_S\otimes_{\boldsymbol{C}}V$ has a finite holomorphic decreasing filtration F^* as in (i) of (3.1.1).

(i)′ Let $\bar{\mathcal{O}}_S$ be the sheaf of germs of anti-holomorphic functions on S. Then $\bar{\mathcal{V}}=\bar{\mathcal{O}}_S\otimes_{\boldsymbol{C}}V$ is considered as the sheaf of local anti-holomorphic sections of an anti-holomorphic vector bundle, which we denote by the same symbol $\bar{\mathcal{V}}$. Then $\bar{\mathcal{V}}$ has a finite filtration:

$$\bar{\mathcal{V}}=\bar{F}^0\bar{\mathcal{V}}\supset\bar{F}^1\bar{\mathcal{V}}\supset\cdots\supset\{0\},$$

by anti-holomorphic subbundles $\bar{F}^i\bar{\mathcal{V}}$.

(ii) The natural connection $\nabla=\partial\otimes 1$ on $\mathcal{V}$, which is flat, defines a complex (3.1.1.1) satisfying the transversality (3.1.1.2).

(ii)′ Let $\bar{\Omega}_S^i$ be the bundle of anti-holomorphic i-forms on S. Then the natural connection $\bar{\nabla}=\bar{\partial}\otimes 1$ on $\bar{V}$ is a flat connection, hence defines a complex:

$$(3.1.3.1)\qquad \bar{\mathscr{V}}\xrightarrow{\bar{\nabla}}\bar{\Omega}_S^1(V)=\bar{\Omega}_S^i\otimes_C V=\bar{\Omega}_S^1\otimes_{\bar{\mathcal{O}}_S}\bar{\mathscr{V}}\xrightarrow{\bar{\nabla}}\bar{\Omega}_S^2\longrightarrow\cdots$$

which satisfies the transversality condition:

$$(3.1.3.2)\qquad \bar{\nabla}(\bar{F}^p\bar{\mathscr{V}})\subset\bar{\Omega}_S^1\otimes_{\bar{\mathcal{O}}_S}\bar{F}^{p-1}\bar{\mathscr{V}}.$$

The main ingredient in the above definition is that we do *not* assume the Hodge symmetry $\overline{H_s^{p,q}}=H_s^{q,p}$ for the Hodge components of the decomposition $V_s=\bigoplus_{p+q=n}H_s^{p,q}$ at each point $s\in S$. A polarization of the complex Hodge structure is a sesquilinear form ψ: $V\times V\to C$ such that $\psi(Cv, v)>0$ for $v\neq 0$, where C is the C operator of Weil.

We note that the spectral sequence (3.1.1.4) and its conjugate one are also defined for a complex variation of Hodge structure. Recall also that there exists the second spectral sequence of hypercohomologies for the complex $\{\mathrm{Gr}_F^k\Omega_S^\cdot(V)\}$:

$$(3.1.4.1)\qquad {}_{\mathrm{II}}E_2^{p,q}=H^p(S,\mathscr{H}^q(\mathrm{Gr}_F^k\Omega_S^\cdot(V)))\Longrightarrow\mathbf{H}^{p+q}(S,\mathrm{Gr}_F^k\Omega_S^\cdot(V)),$$

and its conjugation

$$(3.1.4.2)\qquad {}_{\mathrm{II}}\bar{E}_2^{p,q}=H^p(S,\mathscr{H}^q(\mathrm{Gr}_{\bar{F}}^k\bar{\Omega}_S^\cdot(V)))\Longrightarrow\mathbf{H}^{p+q}(S,\mathrm{Gr}_{\bar{F}}^k\bar{\Omega}_S^\cdot(V))).$$

In the subsequent sections of this chapter, we discuss the degeneracy of these spectral sequences for variations of Hodge structure over arithmetic quotients of bounded symmetric domains, applying the vanishing theorem (2.3.1).

§ 3.2. Variations of Hodge structure over arithmetic quotients

In this section, we construct local systems and variations of Hodge structure associated to the finite dimensional representations of G. We refer to Matsushima-Murakami [28], [29], and Zucker [43] for them.

(3.2.1) *Local systems.* Assume that the group G is the (topological) identity component of a real semisimple linear algebraic group $\mathscr{G}$, such that the quotient G/K of G by a maximal compact subgroup K of G has a G-invariant Hermitian structure. Let $A=\boldsymbol{R}$, or $A=\boldsymbol{C}$, and let

$$\rho\colon \mathscr{G}\longrightarrow GL(V)$$

be a finite dimensional representation of $\mathscr{G}$ on an A-vector space V defined over A.

Let Γ be a discrete subgroup of G, and Z_0 the kernel of the canonical homomorphism $G \to G^{\mathrm{ad}}$ as in Section 2.3. Assume that $\rho(Z_0 \cap \Gamma)=\{1\}$. Then, if Γ has no fixed point on X, we can define a local system as the quotient space of $X \times V$ by the relation:

$$(x, v) \sim (\gamma x, \rho(\gamma)v) \quad (\gamma \in \Gamma),$$

which we denote by the same symbol V. If the arithmetic subgroup Γ has fixed points on X, then we choose a Γ-invariant open dense subset X_0 of X without fixed points under Γ, and we first define the local system $V_0=\Gamma\backslash(X_0 \times V)$ over $\Gamma\backslash X_0$. Putting $V=j_*(V_0)$ with respect to the canonical immersion $j\colon \Gamma\backslash X_0 \to \Gamma\backslash X$, we have a constructible sheaf V on $S=\Gamma\backslash X$ in this case.

(3.2.2) *Variations of Hodge structure.* Assume that Γ has no fixed point on X for a while, and let us see that the local system V has a natural structure of a variation of Hodge structure over S, when the pair (G, ρ) satisfies some conditions.

Before discussing such conditions on (G, ρ), let us see that the holomorphic bundle $\mathscr{V}$ or the locally free analytic sheaf $\mathcal{O}_S(V)$ consisting of local holomorphic sections of V has a canonical holomorphic filtration and a connection with the transversality condition.

Let us consider the trivial local system $V_X=X \times V$ on X, which has the trivial extension $V_{X^c}=X^c \times V$ to the compact dual $X^c=G_C/K_C P_-$ of X. Here P_- is the subgroup of G_C defined in Section 1.3.1. Then the holomorphic bundle $\mathscr{V}_{X^c}$ associated to V_{X^c} over X^c is a trivial bundle, which is isomorphic to the bundle F_ρ over X^c defined for the representation $(\rho \otimes_A C)|_U$ of $U=K_C P_-$

$$(\rho \otimes_A C)|_U\colon U \longrightarrow GL(V \otimes_A C),$$

which is the restriction of the representation $\rho \otimes_A C$ of G_C to U.

Note that U and P_- are a solvable and a nilpotent Lie group, respectively. Hence there exists a decreasing filtration $\{F^{\cdot}(V \otimes_A C)\}$ on $V \otimes_A C$ stable under U, such that P_- acts trivially on each graded module $F^i(V \otimes_A C)/F^{i+1}(V \otimes_A C)$.

In order to make the situation more precise and to define a connection on $\mathscr{V}$ in the next paragraph, we recall the following lemma (cf. Lemma (5.2) of Part II of [29]).

(3.2.3) **Lemma.** *Suppose that a finite dimensional complex vector space V is an irreducible G_C-module via $\rho\colon G_C \to GL(V)$. Then there exists*

K_C-submodules $S_0, S_1, \cdots, S_m$ of V with the following properties:

(i) *V is the direct sum of S_i $(0 \leqq i \leqq m)$, i.e. $V = \bigoplus_{i=0}^{m} S_i$.*

(ii) *$\rho(g)v - v \in S_{i-1}$ for any $v \in S_i$ and $g \in P_+$, and $\rho(g)v - v \in S_{i+1}$ for any $v \in S_i$ and $g \in P_-$ $(i = 0, 1, \cdots, m)$. Here we set $S_{-1} = \{0\}$ and $S_{m+1} = \{0\}$.*

(iii) *S_0 and S_m are the irreducible K_C-modules given by*

$$S_0 = \{v \in V \mid \rho(g)v = v \text{ for all } g \in P_+\},$$

and

$$S_m = \{v \in V \mid \rho(g)v = v \text{ for all } g \in P_-\}.$$

Moreover, if V is the irreducible G_C-module with highest weight $\lambda \in D \cap X(H_C)$, S_0 is an irreducible K_C-module with highest weight λ.

If $\rho \otimes_A C$ is irreducible, we can apply the above lemma to $V_C = V \otimes_A C$, and define a U-stable filtration by $F^p(V) = \bigoplus_{i \geqq p} S_i$, and a $\bar{U}$-stable filtration by $\bar{F}^q(V) = \bigoplus_{j \geqq m-q} S_j$, where $U = K_C P_-$ and $\bar{U} = K_C P_+$. These filtrations induce a holomorphic filtration on $\mathcal{O}_{X^c}(V) = V \otimes_C \mathcal{O}_{X^c}$ and an anti-holomorphic filtration on $\bar{\mathcal{O}}_{X^c}(V) = V \otimes_{X^c} \bar{\mathcal{O}}_{X^c}$.

Now let us recall the definition of the connection ∇. Since $V_{X^c} = X^c \times V$ is a constant local system, the tensor product of V with the de Rham complex of X^c defines a complex of sheaves:

$$\mathcal{O}_{X^c}(V) \xrightarrow{\nabla = \partial \otimes 1} \Omega^1_{X^c}(V) \xrightarrow{\nabla} \Omega^2_{X^c}(V) \xrightarrow{\nabla} \cdots \tag{3.2.3.1}$$

We can define a similar complex for $\bar{\nabla}$. Since the following arguments also apply for this conjugate complex by changing the complex structure of X via the embedding $X \hookrightarrow G_C/\bar{U}$, we consider only the holomorphic case in the sequels.

The fibers of the *bundles* $\Omega^i_{X^c}(V)$ and *C-linear* mappings at a point $x_0 = [K_C P_-] \in X^c = G_C/K_C P_-$ are given as follows. Let $\{E_\alpha \mid \alpha \in \Phi_n\}$ be a set of elements in $\mathfrak{g}_C$ such that

$$CE_\alpha = \mathfrak{g}_\alpha \quad \text{and} \quad B(E_\alpha, E_{-\alpha}) = 1 \qquad \text{for all } \alpha \in \Phi_n.$$

Here $\mathfrak{g}_\alpha$ is the root space for α, and B the Killing form of $\mathfrak{g}_C$. Then $\{E_\alpha \mid \alpha \in \Phi_{n+}\}$ and $\{E_{-\alpha} \mid \alpha \in \Phi_{n+}\}$ make basis of $\mathfrak{p}_+$ and $\mathfrak{p}_-$, respectively. Let $\overset{i}{\wedge}\mathfrak{p}_-$ be the component of degree i of the exterior algebra $\overset{*}{\wedge}\mathfrak{p}_-$ of $\mathfrak{p}_-$. Then for each i, we denote by

$$e(X)\colon \overset{i}{\wedge}\mathfrak{p}_- \longrightarrow \overset{i+1}{\wedge}\mathfrak{p}_-$$

the exterior product $Y \mapsto X \wedge Y$ $(Y \in \overset{i}{\wedge}\mathfrak{p}_-)$. For any element $w = \sum_j v_j \otimes Y_j$

($v_j \in V$, $Y_j \in \overset{i}{\wedge}\mathfrak{p}_-$) of $V\otimes\overset{i}{\wedge}\mathfrak{p}_-$, we define ∇_0: $V\otimes\overset{i}{\wedge}\mathfrak{p}_- \to V\otimes\overset{i+1}{\wedge}\mathfrak{p}_-$ by

$$\nabla_0(w)=\sum_j\sum_\alpha \rho(E_\alpha)v_j\otimes e(E_{-\alpha})(Y_j).$$

Then the fiber of the complex (3.2.3.1) at x_0 is given by

$$(\sharp)\quad V\xrightarrow{\nabla_0}V\otimes\mathfrak{p}_-\xrightarrow{\nabla_0}V\otimes\overset{2}{\wedge}\mathfrak{p}_-\longrightarrow\cdots$$

of K-modules. Note the isomorphism $F_{\tau p}\cong\Omega^p_{X^c}$ of Section 2.4 here. The Lemma (3.2.3) implies the transversality condition:

$$(3.2.3.2)\qquad \nabla(F^p(\mathcal{O}_{X^c}(V))\otimes_C\Omega^{i-1}_{X^c})\subset F^{p-1}(\mathcal{O}_{X^c}(V))\otimes_C\Omega^i_{X^c},$$

since the induced homomorphism

$$\nabla\colon \mathrm{Gr}^*_F(\mathcal{O}_{X^c}(V))\otimes_C\Omega^{i-1}_{X^c}\longrightarrow\mathrm{Gr}^*_F(\mathcal{O}_{X^c}(V))\otimes_C\Omega^i_{X^c}$$

is $\mathcal{O}_{X^c}$-linear.

In order to compute the spectral sequence (3.1.3.1) in the next section, it is necessary to know the sheaf $\mathscr{H}^q(\mathrm{Gr}^k_F\Omega^\cdot_S(V))$. Zucker [43] computes it as follows.

In the first place, we note that the fiber at x_0 of the complex

$$\cdots\longrightarrow\bigoplus_{k=0}^m\mathrm{Gr}^k_F(\mathcal{O}_{X^c}(V))\otimes_C\Omega^{i-1}_{X^c}\xrightarrow{\nabla}\bigoplus_{k=0}^m\mathrm{Gr}^k_F(\mathcal{O}_{X^c}(V))\otimes_C\Omega^i_{X^c}\xrightarrow{\nabla}\cdots$$

with $\mathcal{O}_{X^c}$*-linear* boundary homomorphisms ∇ is also given by ($\sharp$). Hence the sheaf $\mathscr{H}^q(\mathrm{Gr}^k_F\Omega^\cdot_S(V))$ is isomorphic to the sheaf $F_{\sigma^{q,k}}$ on X^c associated to the representation $\sigma^{q,k}$ of K on the cohomology group of the complex:

$$(\sharp\sharp)\quad S_{k-q+1}\otimes\overset{q-1}{\wedge}\mathfrak{p}_-\xrightarrow{\nabla_0}S_{k-q}\otimes\overset{q}{\wedge}\mathfrak{p}_-\xrightarrow{\nabla_0}S_{k-q-1}\otimes\overset{q+1}{\wedge}\mathfrak{p}_-$$

(cf. Theorem (5.29) of [43]). Here we put $S_i=\{0\}$, if $i\notin\{0,1,\cdots,m\}$.

The last cohomology group is a direct summand of the q-th cohomology group of the complex ($\sharp$). Note that

$$V\otimes\overset{q}{\wedge}\mathfrak{p}_-\cong C^q(\mathfrak{p}_+,V)=\{q\text{-alternating mapping of }\mathfrak{p}_+\text{ to }V\},$$

for any q, and that the complex ($\sharp$) is isomorphic to the standard complex of the Lie algegra $\mathfrak{p}_+$ with values in V (note also that $\mathfrak{p}_+$ is abelian). Then the cohomology groups of ($\sharp$) are identified with $H^*(\mathfrak{p}_+,V)$, and a theorem of Kostant describes them as K-modules.

(3.2.4) **Theorem** (Theorem 5.14 of [22], or [41], I, Chap. 2, p. 175). *Let δ be a dominant weight in $X(H_C)$, and V the irreducible G-module with* *highest weight λ. Then for each q, we have an isomorphism of K-modules*

$$H^q(\mathfrak{p}^+, V) \cong \bigoplus_{w \in W^1(q)} W_{w(\lambda+\rho)-\rho}.$$

Here W_ν is the irreducible K-module with highest weight ν, and

$$W^1(q) = \{w \in W(\mathfrak{g}_C, \mathfrak{h}_C) \mid w(\Phi_+) \supset \Phi_{K+} \text{ and } |w(-\Phi_+) \cap \Phi_{n+}| = q\}.$$

(3.2.5) In order to characterize the sheaf $\mathscr{H}^q(\mathrm{Gr}_F^k(\mathcal{O}_{X^c}(V)))$ as a subsheaf of $\bigoplus_{w \in W^1(q)} F_{w(\lambda+\rho)-\rho}$, it is necessary to characterize the cohomology group of the complex (##) as a submodule of $H^q(\mathfrak{p}_+, V)$. This is done in Zucker [43] by investigating the action of a special subtorus of dimension 1 in the center of K on V. Let us recall this.

In the first place, we define an anisotropic subtorus Z of dimension 1 over $\boldsymbol{R}$ in the center Z_K of the maximal compact subgroup K of G, in the following way. Note that there is an element ι in Z_K such that $\mathrm{Ad}(\iota)|_{\mathfrak{p}}$ induces the complex structure on the tangent space $T_{X,o}$ of $X = G/K$ at $o = [K]$ via the identification $\mathfrak{p} \cong T_{X,0}$. Denote by $\hat{Z}_K$ the group of continuous characters of Z_K. Then $\hat{Z}_K$ is a torsion free (discrete) abelian group of finite rank. We consider a submodule $\hat{Z}_K{}^{\perp\iota}$ of Z_K of corank 1 defined by

$$\hat{Z}_K{}^{\perp\iota} = \{\chi \in \hat{Z}_K \mid \chi(\iota) = 1\},$$

and define a subgroup Z of Z_K by

$$Z = (\hat{Z}_K{}^{\perp\iota})^{\perp} = \{z \in Z_K \mid \chi(z) = 1 \text{ for all } \chi \in \hat{Z}_K{}^{\perp\iota}\}.$$

Then we can see that Z is isomorphic to $T^1 = \{z \in \boldsymbol{C}^\times \mid |z| = 1\}$ as topological groups, and contains ι. Note here that if X is decomposed as a product $X = \prod_{i=1}^m X_i$ of m irreducible symmetric domains X_i, then $Z_K \cong (T^1)^m$.

Let ρ be a representation of G on a finite-dimensional complex vector space V. Then for each character χ of Z,we put

$$(*) \qquad V\langle\chi\rangle = \{v \in V \mid \rho(z)v = \chi(z)v \text{ for all } z \in Z\},$$

and obtain the obvious decomposition:

$$V = \bigoplus_{\chi \in \hat{Z}} V\langle\chi\rangle.$$

Each $V\langle\chi\rangle$ is invariant under K. If (σ, W) is a representation of K, we define $W\langle\chi\rangle$ for $\chi \in \hat{Z}$ similarly as in $(*)$. If W is irreducible, then $W = W\langle\chi\rangle$ for some $\chi \in \hat{Z}$ by Schur's Lemma. If we fix an isomorphism $Z \cong T^1$, then $\hat{Z} \cong \hat{T}^1 \cong \mathrm{Hom}(T^1, T^1) \cong \boldsymbol{Z}$. In this case, if $\chi \in \hat{Z}$ corresponds to $n \in \boldsymbol{Z}$ via the isomorphism $\hat{Z} \cong \boldsymbol{Z}$, we write $W\langle n\rangle$ for $W\langle \chi\rangle$. For the irreducible K-module W with highest weight λ, we denote by χ_λ (resp. n_λ) the element of $\hat{Z}$ (resp. $\boldsymbol{Z}$) such that $W=W$ (resp. $W\langle \chi_\lambda\rangle = W\langle n_\lambda\rangle$).

Let us consider the adjoint action of Z on $\mathfrak{g}_C$. Suppose that X is written as a product

$$X=\prod_{i=1}^{s} X_i$$

of irreducible bounded symmetric domains of non-compact type. Then G and K are written as

$$G=\left(\prod_{i=1}^{s} G_i\right)\times K_0$$

and

$$K=\left(\prod_{i=1}^{s} K_i\right)\times K_0,$$

where the Lie algebra of G_i is simple and G_i/K_i is a non-compact Hermitian symmetric space for each i, and K_0 is compact. The center Z_K of K is written as

$$Z_K=\prod_{i=1}^{s} Z_{K_i}\times Z_{K_0}$$

with $\dim_{\boldsymbol{R}} Z_{K_i}=1$ for each i $(1\leqq i\leqq s)$.

Let $G\to G^{\mathrm{ad}}$ be the natural covering of G on its adjoint group G^{ad}. Let Z_K^{ad} and $Z_{K_i}^{\mathrm{ad}}$ be the subgroups of G^{ad} which are the images of Z_K and Z_{K_i} by $G\to G^{\mathrm{ad}}$, respectively. We denote by μ_i the covering degree of $Z_{K_i}\to Z_{K_i}^{\mathrm{ad}}$ for each i.

From now on, *we assume that μ_i are equal for all i* $(1\leqq i\leqq s)$, and we put $\mu=\mu_i$. Note that this assumption is satisfied, if the Lie algebra of G is simple, or $G=G^{\mathrm{ad}}$.

If we consider the adjoint representation $\rho=\mathrm{Ad}$ of G on $V=\mathfrak{p}\otimes_{\boldsymbol{R}}\boldsymbol{C}$, then $\mathfrak{p}_+=V\langle\mu\rangle$, $\mathfrak{k}_C=V\langle 0\rangle$, and $\mathfrak{p}_-=V\langle-\mu\rangle$ for this integer μ, replacing the isomorphism $Z\cong T^1$ if necessary.

Under these notations, we can paraphrase Lemma (3.2.2) in the following manner.

Let V be an irreducible G-module (ovre $\boldsymbol{C}$) with highest weight λ. We denote by the same symbol ρ the representations of G and $\mathfrak{g}$ on V. Then for V the following holds (cf. Zucker [43], (1.7)):

(i) $\rho(\mathfrak{p}_+)V\langle n\rangle\subset V\langle n+\mu\rangle$, $\rho(\mathfrak{k})V\langle n\rangle\subset V\langle n\rangle$, $\rho(\mathfrak{p}_-)V\langle n\rangle\subset V\langle n-\mu\rangle$.

(ii) $\{n \mid V\langle n\rangle\neq\{0\}\}=\{n_\lambda, n_\lambda-\mu, n_\lambda-2\mu, \cdots, n_\lambda-m\mu\}$.

The subspace S_i of V is equal to $V\langle n_\lambda-i\mu\rangle$. Therefore, we have

$$S_{k-q}\otimes \overset{q}{\wedge}\mathfrak{p}_- = (S_{k-q}\otimes \overset{q}{\wedge}\mathfrak{p}_-)\langle n_\lambda - k\mu\rangle.$$

Hence the cohomology group of the complex (##) of (3.2.3) is given by $H^q(\mathfrak{p}_+, V)\langle n_\lambda - k\mu\rangle$, which is isomorphic to

$$\bigoplus_{w\in W^1(q)} W_{w(\lambda+\rho)-\rho}\langle n_\lambda - k\mu\rangle$$

as a K-module. Note that $W_{w(\lambda+\rho)-\rho}\langle n_\lambda - k\mu\rangle$ is equal to $W_{w(\lambda+\rho)-\rho}$ or to $\{0\}$, since $W_{w(\lambda+\rho)-\rho}$ is an irreducible K-module.

Thus we have the following proposition.

(3.2.6) **Proposition** (Corollary (5.32) of [43]). *The holomorphic vector bundle $\mathscr{H}^q(\mathrm{Gr}_F^k\Omega_S^\cdot(V))$ is the bundle associated with the K-module*

$$\bigoplus_{w\in W^1(q)} W_{w(\lambda+\rho)-\rho}\langle n_\lambda - k\mu\rangle.$$

(3.2.7) **Remark.** Deligne [10] tells that the complex structure of G/K is defined by a homomorphism of real algebraic groups

$$h\colon \underline{S}\longrightarrow \mathscr{G},$$

and K is given as the centralizer of h in G (=the identity component of $\mathscr{G}(\boldsymbol{R})$). Our group Z is the image of $\underline{S}(\boldsymbol{R})$.

(3.2.8) **Remark.** Let us assume that the representation (ρ, V) is defined over $\boldsymbol{R}$, i.e. there exists a real vector space $V_{\boldsymbol{R}}$ and a representation $\rho_{\boldsymbol{R}}\colon G\to GL(V_{\boldsymbol{R}})$ such that $\rho=\rho_{\boldsymbol{R}}\otimes_{\boldsymbol{R}}\boldsymbol{C}$ and $V=V_{\boldsymbol{R}}\otimes_{\boldsymbol{R}}\boldsymbol{C}$. Then, in addition to the conditions (i) and (ii) for $V\langle n\rangle$, we have the Hodge symmetry

$$\overline{V\langle n\rangle} = V\langle -n\rangle.$$

Hence $n_\lambda - m\mu = -n_\lambda$, i.e. $m\mu = 2n_\lambda$.

§ 3.3. Degeneracy of the spectral sequences

Let V be a finite dimensional complex vector space which is an irreducible G-module with highest weight $\lambda\in D\cap X(H_{\boldsymbol{C}})$. Consider the pull-back of the complex (3.2.3.1) with respect to the open immersion $X\to X^c$, and assume that its descent with respect to $X\to S=\Gamma\backslash X$ exists for an arithmetic discrete subgroup Γ of G which has no fixed point on X. We denote the induced bundle and sheaf on S by the same symbol as the original one on X^c, if they are obtained in the above way. So we can consider the bundle $\mathscr{H}^q(\mathrm{Gr}_F^k\Omega_S^\cdot(V))$ on S. The induced complex on S from (3.2.3.1) defines a complex variation of Hodge structure. Zucker

[43] shows the degeneracy of the spectral sequence (3.1.4.1) for such a variation of Hodge structure.

(3.3.1) **Proposition** (Proposition (5.19) of [43]). *If $S\backslash X$ is compact, the spectral sequence*

$$_{\mathrm{II}}E_2^{p,q}=H^p(S,\mathscr{H}^q(\mathrm{Gr}_F^k\Omega_S^{\cdot}(V)))\Longrightarrow \mathbf{H}^{p+q}(S,\mathrm{Gr}_F^k\Omega_S^{\cdot}(V))$$

degenerates at E_2.

Note that Theorem (3.1.2) is also valid for complex variations of Hodge structure. Then the following is immediate.

(3.3.2) **Corollary** (Remark (5.20) of [43]). *If $S=\Gamma\backslash X$ is compact,*

$$\dim H^n(S,V)=\sum_k\sum_{p+q=n}\dim H^p(S,\mathscr{H}^q(\mathrm{Gr}_F^k\Omega_S^{\cdot}(V))).$$

In the rest of this section, we make the preceding proposition more precise, applying the vanishing theorem (2.3.1).

From now on, we assume that the Lie algebra of G is simple.

Let us compute the E_2 term of (3.1.4.1). Assume that the highest weight $\lambda\in D\cap X(H_C)$ of V satisfies the condition:

$$\langle\sigma(\lambda),\alpha\rangle\neq 0\qquad\text{for any }\sigma\in W^1\text{ and any }\alpha\in\Phi_{n+}.$$

For any $w\in W^1$, we put $q_w=q_{w(\lambda+\rho)-\rho}$. Then we have $q_w=N-q$ for any $w\in W^1(q)$ by the definition of $W^1(q)$ and q_w. Let us define a complex vector space $H_w(q,k)$ by

$$H_w(q,k)=\begin{cases}H^{q_w}(S,F_{w(\lambda+\rho)-\rho}), & \text{if } W_{w(\lambda+\rho)-\rho}\langle n_\lambda-k\mu\rangle=W_{w(\lambda+\rho)-\rho};\\ \{0\}, & \text{if } W_{w(\lambda+\rho)-\rho}\langle n_\lambda-k\mu\rangle=\{0\}.\end{cases}$$

Then by Theorem (2.3.1), $_{\mathrm{II}}E_2^{p,q}$ of the spectral sequence (3.1.4.1) is given by

$$_{\mathrm{II}}E_2^{p,q}=\begin{cases}\bigoplus_{w\in W^1(q)}H_w(q,k), & \text{if } p=q_w\ (=N-q);\\ \{0\}, & \text{if } p\neq q_w\ (=N-q).\end{cases}$$

Therefore we have $_{\mathrm{II}}E_2^{p,q}=\{0\}$, if $p+q\neq N$. Hence the homomorphism

$$d_2^{p,q}:{}_{\mathrm{II}}E_2^{p,q}\longrightarrow{}_{\mathrm{II}}E_2^{p+2,q-1}$$

is the zero homomorphism for any p, q.

Thus we have another proof of the degeneracy of (3.1.4.1), when λ satisfies the preceding condition. Moreover

$$\mathbf{H}^{p+q}(S, \mathrm{Gr}_F^k \Omega_S^{\cdot}(V)) = \{0\}$$

for any k, if $p+q \neq N$. And if $p+q=N$, we have an isomorphism of vector spaces

$$\mathbf{H}^{p+q}(S, \mathrm{Gr}_F^k \Omega_S^{\cdot}(V)) \cong \bigoplus_{w \in W^1} H_w(q(w), k),$$

where $q(w) = |w(-\Phi_+) \cap \Phi_{n+}|$.

Combining with the degeneracy of (3.1.1.4) (cf. Theorem (3.1.2)), we have the following theorem.

(3.3.2) **Theorem.** *Assume that $S = \Gamma \backslash X$ is compact, and λ satisfies the condition: $\langle \sigma(\lambda), \alpha \rangle > 0$ for any $\sigma \in W^1$ and any $\alpha \in \Phi_{n+}$. Then*

(i) *$H^i(S, V) = \{0\}$, if $i \neq N$.*

(ii) *The Hodge filtration $F^{\cdot}$ in $H^N(S, V)$ of* (3.1.1.4) *gives an isomorphism*

$$F^k H^N(S, V)/F^{k+1} H^N(S, V) \cong \bigoplus_{w \in W^1} H_w(q(w), k)$$

for each k. This induces an isomorphism of vector spaces

$$H^N(S, V) \cong \bigoplus_{w \in W^1} H^{q_w}(S, F_{w(\lambda+\rho)-\rho}).$$

(iii) *If there exists a representation $\rho_{\mathbf{R}}: G \to GL(V_{\mathbf{R}})$ on a real vector space $V_{\mathbf{R}}$ such that $V = V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ and $\rho = \rho_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$, then the space $H^N(S, V_{\mathbf{R}})$ has a real Hodge structure of weight $N+m$ via the filtration $F^{\cdot}$ and its conjugate of* (3.1.1.4), *such that the $(N+m-k, k)$ component of $H^N(S, V_{\mathbf{R}}) \otimes_{\mathbf{R}} \mathbf{C}$ is isomorphic to*

$$\bigoplus_{w \in W^1} H_w(q(w), k).$$

Here $m = 2n_\lambda/\mu$ is the number defined in Remark (3.2.8).

(3.3.3) **Remark.** The result (i) of the preceding theorem was shown by Matsushima-Murakami (cf. Theorem 12.1 of [29]). The last isomorphism of (ii) is considered as a generalization of the Eichler-Shimura isomorphism for modular forms of one variable. Borel-Wallach constructed an isomorphism (cf. [6], Chap. VII, Theorem 6.9):

$$H^N(S, V) \cong \bigoplus_{w \in W^1} \mathrm{Hom}_G(H_2^{0, q_w}(X, F_{w(\lambda+\rho)-\rho}), L^2(\Gamma \backslash G))$$

under the assumption that $\lambda+\rho$ is strongly Φ_{K+}-dominant, by investigating the cohomology groups with values in the irreducible constituents of the G-module $L^2(\Gamma\backslash G)$. Theorem (2.6.1) implies that it is equivalent to the last isomorphism of (ii).

An advantage of the proof given here is that we can tell the Hodge type of $H^{q_w}(S, F_{w(\lambda+\rho)-\rho})$, which does not seem to be clear in the proof of [6].

There is another way to equip a Hodge structure on $H^N(S, V)$ via square-integrable cohomology groups as it is done in Matsushima-Murakami [28], [29]. But even in this case, it is important to check that this Hodge structure coincides with the geometric one defined above. This check was done by Zucker, which is one of the merits of [43].

The edge component of the Hodge decomposition in the above theorem is identified with a space of holomorphic automorphic forms.

Though we have assumed that the Lie algebra of G is simple for simplicity, Theorem (3.3.2) is valid without modification if μ_i are equal for all quasi-simple non-compact components G_i of G. The simplest case is the case where $G_i = SL_2(\boldsymbol{R})$ for all i. This case is already investigated in Matsushima-Shimura [30].

§ 3.4. Example. The case of discrete subgroups of the symplectic groups

Now we apply Theorem (3.3.2) to discrete subgroups Γ of $G = Sp_{2n}(\boldsymbol{R})$. Let us recall the basic facts on the root system of $Sp_{2n}(\boldsymbol{R})$.

Consider the Euclid space $\boldsymbol{R}^n$ of dimensional n with the usual orthonomal basis:

$$\{e_i = (1, \cdots, 0, \overset{i\text{-th}}{1}, 0, \cdots, 0) \mid 1 \leqq i \leqq n\}.$$

Then the root system Φ is $\Phi = \{\pm 2e_i, \pm e_i \pm e_j \mid 1 \leqq i, j \leqq n, i \neq j\}$, the set of positive roots is

$$\Phi_+ = \{2e_i, e_i + e_j \mid 1 \leqq i, j \leqq n\} \cup \{e_i - e_j \mid 1 \leqq i < j \leqq n\},$$

the set of compact roots is

$$\Phi_K = \{e_i - e_j \mid 1 \leqq i, j \leqq n, i \neq j\},$$

and the set of non-compact roots is

$$\Phi_n = \{\pm 2e_i, \pm (e_i + e_j) \mid 1 \leqq i, j \leqq n\}.$$

A half of the sum of positive roots is

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha = \sum_{i=1}^{n} (n+1-i) e_i = (n, n-1, n-2, \cdots, 2, 1).$$

The weight lattice L of $Sp_{2n}(\boldsymbol{R})$ is identified with $\boldsymbol{Z}^n$:

$$L=\{\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \boldsymbol{Z}^n\},$$

and the set D of dominant weights is given by

$$D=\{\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n) \in L \mid \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n\}.$$

Note that any irreducible finite dimensional representation of G is defined over $\boldsymbol{R}$, since G has a split Cartan subgroup over $\boldsymbol{R}$. Let $\rho_{\boldsymbol{R}}: G \to GL(V_{\boldsymbol{R}})$ be the irreducible representation of G on a *real* vector space $V_{\boldsymbol{R}}$ with highest weight λ in D. Let $\rho_0: G \to GL_{2n}(\boldsymbol{R})$ be the standard representation of G on $\boldsymbol{R}^n$ induced from the natural monomorphism $Sp_{2n}(\boldsymbol{R}) \hookrightarrow GL_{2n}(\boldsymbol{R})$. Then any point x of $X=G/K$ defines a complex structure on the real vector space $\boldsymbol{R}^{2n}$, i.e. x defines a Hodge structure of weight 1 of type $\{(1. 0), (0, 1)\}$. The $\boldsymbol{R}$-local system V_0 over X associated to ρ_0 is a *real* variation of Hodge structure. Let $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Then the G-module V is identified with a submodule of the tensor product $T^{n_\lambda}(\boldsymbol{R}^{2n})$ of degree $n_\lambda=\sum_{i=1}^n \lambda_i$ of the G-module $\boldsymbol{R}^{2n}$. Thus we can see that the $\boldsymbol{R}$-local system $V_{\boldsymbol{R}}$ associated to the representation $\rho_{\boldsymbol{R}}$ is a real variation of Hodge structure over X of weight n_λ.

In our case, the dimension is $N=\dim_{\boldsymbol{C}} X=n(n+1)/2$, and $\mathfrak{p}_+=\mathfrak{p}_{\boldsymbol{C}}\langle 2\rangle$, $\mathfrak{p}_-=\mathfrak{p}_{\boldsymbol{C}}\langle -2\rangle$. And the module $W_{w(\lambda+\rho)-\rho}\langle k\rangle$ is given by

$$W_{w(\lambda+\rho)-\rho}\langle k\rangle=\begin{cases} W_{w(\lambda+\rho)-\rho}, & \text{if } w(\lambda+\rho)-\rho=(\mu_1, \cdots, \mu_n) \\ & \text{satisfies } \mu_1+\cdots+\mu_n=k; \\ \{0\}, & \text{otherwise.}\end{cases}$$

Note that X is biholomorphically equivalent to the Siegel upper half space of degree n. Then by Theorem (3.3.2), we have the following.

(3.4.1) **Theorem.** *Assume that X is the Siegel upper half space of degree n, and Γ is a discrete subgroup of $Sp_{2n}(\boldsymbol{R})$ without fixed points on X such that $S=\Gamma\backslash X$ is compact. Assume that $\lambda=(\lambda_1, \cdots, \lambda_n) \in D$ satisfies $\langle\sigma(\lambda), \alpha\rangle \neq 0$ for any $\sigma \in W^1$ and $\alpha \in \Phi_{n+}$. Assume moreover that the local system associated to the representation $(\rho_{\boldsymbol{R}}, V_{\boldsymbol{R}})$ exists on S. Then $H^N(S, V_{\boldsymbol{R}})$ is a real Hodge structure of weight $N+n_\lambda$ such that $(N+n_\lambda-k, k)$-type component of $H^N(S, V_{\boldsymbol{R}})$ is canonically isomorphic to*

$$\bigoplus_{w \in W^1[k]} H^{q_w}(S, F_{w(\lambda+\rho)-\rho}),$$

where $W^1[k]$ is the subset of W^1 defined by

$$W^1[k]=\left\{w \in W^1 \mid w(\lambda+\rho)=(\nu_1, \nu_2, \cdots, \nu_n),\ \sum_{i=1}^n \nu_i=N+n_\lambda-k\right\},$$

and $n_\lambda = \sum_{i=1}^{n} \lambda_i$.

Let us reformulate the above theorem in a slightly different way. Suppose that $\{e_1, \cdots, e_n\}$ is the standard basis of the Euclid space $\boldsymbol{R}^n$. For any element w of the Weyl group W of G_C, we can associate a permutation σ of the set $\{1, \cdots, n, -1, \cdots, -n\}$ by

$$\sigma(i)=j, \text{ if } w(e_i)=e_j, \quad \text{and} \quad \sigma(i)=-j, \text{ if } w(e_i)=-e_j$$

for each i. Then the group W is identified with the set of the permutations of the set $\{1, \cdots, n, -1, \cdots, -n\}$ such that

$$-\sigma(i)=\sigma(-i) \qquad \text{for any } i \in \{1, \cdots, n, -1, \cdots, -n\}.$$

Let ε: $\{1, 2, \cdots, n\} \to \{\pm 1\}$ be a function on the set $\{1, 2, \cdots, n\}$ with values in $\{\pm 1\}$. We define an element σ_ε of W by

$$\sigma_\varepsilon(i)=\varepsilon(i)i, \qquad \sigma_\varepsilon(-i)=-\varepsilon(i)i$$

for any i $(1 \leqq i \leqq n)$, or we set

$$\sigma_\varepsilon = \begin{pmatrix} 1, \cdots\cdots, n, -1, \cdots\cdots, -n \\ \varepsilon(1)1, \cdots, \varepsilon(n)n, -\varepsilon(1)1, \cdots, -\varepsilon(n)n \end{pmatrix}.$$

Then there exists a unique element w_ε of W^1 such that

$$W_K \sigma_\varepsilon \ni w_\varepsilon.$$

We put $\lambda_\varepsilon = w_\varepsilon(\lambda+\rho)-\rho$ for any $\lambda \in D$, and set $q_\varepsilon = q_{\lambda_\varepsilon} = |\Phi_{n+} \cap w_\varepsilon(\Phi_{n+})= =|\Phi_{n+} \cap \sigma_\varepsilon(\Phi_{n+})|$. Then it is easy to see that Theorem (3.4.1) is equivalent to the following.

(3.4.1) **bis Theorem.** *Under the same notation and assumptions as in Theorem* (3.4.1), *the Hodge decomposition of* $H^N(S, V_R)$ *is given by*

$$H^N(S, V_R) \otimes_R C = \bigoplus_{\varepsilon \in E} H^{q_\varepsilon}(S, F_{\lambda_\varepsilon})$$

such that $H^{q_\varepsilon}(S, F_{\lambda_\varepsilon})$ *is the Hodge component of type*

$$\Big(\sum_{\substack{\varepsilon(i)=-1 \\ 1\leqq i \leqq n}} \{\lambda_i+(n+1-i)\}, \sum_{\substack{\varepsilon(i)=+1 \\ 1\leqq i \leqq n}} \{\lambda_i+(n+1-i)\}\Big).$$

Here E is the set of all functions on $\{1, \cdots, n\}$ *with values in* $\{\pm 1\}$.

(3.4.2) **Remark.** The Serre duality tells that there is a perfect pairing

$$H^{q_\varepsilon}(S, F_{\lambda_\varepsilon}) \times H^{q_{-\varepsilon}}(S, F_{\lambda_{-\varepsilon}}) \longrightarrow C \quad (q_\varepsilon + q_{-\varepsilon} = N)$$

of complex vector spaces for each $\varepsilon \in E$, and the natural conjugate linear isomorphism

$$\sharp\colon H^{q_\varepsilon}(S, F_{\lambda_\varepsilon}) \cong H^{q_{-\varepsilon}}(S, F_{\lambda_{-\varepsilon}})$$

of $H^{q_\varepsilon}(S, F_{\lambda_\varepsilon})$ to $H^{q_{-\varepsilon}}(S, F_{\lambda_{-\varepsilon}})$ for each E gives the Hodge symmetry.

In the rest of this section, let us observe a relation between the Hodge type of the preceding theorem and the Langlands parameter of the discrete series representations of $Sp_{2n}(\boldsymbol{R})$. We refer to the article of Borel [4] for the definition and basic properties of Langlands parameter. Let $\bar{G}$ be the adjoint group $PSp_{2n}(\boldsymbol{R})$ of G. Then the identity component ${}^L\bar{G}^0$ of the L-group of $\bar{G}$ is isomorphic to the complex spinor group Spin $(2n+1, \boldsymbol{C})$, which is a double cover of the complex orthogonal group $SO(2n+1, \boldsymbol{C})$.

Let $W_{\boldsymbol{C}/\boldsymbol{R}} = \boldsymbol{C}^\times \cup \boldsymbol{C}^\times j$ $(j^2 = -1; jzj^{-1} = \bar{z},\ z \in \boldsymbol{C}^\times)$ be the Weil group of $\boldsymbol{R}$, and let

$$\varphi_\lambda\colon W_{\boldsymbol{C}/\boldsymbol{R}} \longrightarrow {}^L\bar{G}^0 \text{ modulo conjugation}$$

be the Langlands parameter of the representation π_λ of the discrete series of $\bar{G}$ with infinitesimal character $\Theta_{\lambda+\rho}$. Then $\varphi_\lambda(j) = -1$ (cf. Example 10.5 of [4]). We consider the restriction of φ_λ to the identity component $\boldsymbol{C}^\times$ of $W_{\boldsymbol{C}/\boldsymbol{R}}$:

$$\rho_\infty(\pi_\lambda)\colon \boldsymbol{C}^\times \longrightarrow {}^L\bar{G}^0.$$

Then the composition $\sigma_{\mathrm{spin}} \circ \rho_\infty(\pi_\lambda)$ of $\rho_\infty(\pi_\lambda)$ with the spin representation

$$\sigma_{\mathrm{spin}}\colon {}^L\bar{G}^0 \longrightarrow GL(2^n, \boldsymbol{C})$$

is equivalent to the homomorphism α of $\boldsymbol{C}^\times$ to the diagonal matrices of $GL(2^n, \boldsymbol{C})$ given as follows:

Let $\lambda \in L_0$ and $\lambda + \rho = (\mu_1, \mu_2, \cdots, \mu_n)$. For each $\varepsilon \in E$, we put

$$\mu_\varepsilon = \frac{1}{2} \sum_{i=1}^{n} \varepsilon(i)\mu_i \in \boldsymbol{Q}.$$

Then for each $z \in \boldsymbol{C}^\times$, $\alpha(z)$ is the diagonal matrix in $GL(2^n, \boldsymbol{C})$ whose 2^n diagonal elements are given by $\{z^{\mu_\varepsilon}\bar{z}^{-\mu_\varepsilon} \mid \varepsilon \in E\}$.

(3.4.3) **Observation.** The "Hodge type" of

$$\alpha\colon \boldsymbol{C}^\times \cong \underline{S}(\boldsymbol{R}) \longrightarrow GL(2^n, \boldsymbol{C})$$

is given by $\{(\mu_\varepsilon, -\mu_\varepsilon) \mid \varepsilon \in E\}$. Under the assumption of Theorem (3.4.1), the Hodge type of $H^N(S, V_{\boldsymbol{R}})$ is given by

$$\{(\mu_+ + \mu_\varepsilon, \mu_+ - \mu_\varepsilon) \mid \varepsilon \in E\},$$

where $+ \in E$ is the unique function on $\{1, \cdots, n\}$ with the constant value $+1$.

In the next section, we discuss some problems for the Hodge decomposition of cohomology groups of Siegel modular groups, expecting that the result of this section should have an analogy for non-cocompact subgroups. For the elliptic modular case, these problems were already solved. See Deligne [8] and Zucker [42] for this case. It is also useful to see a formulation in terms of Lie algebra cohomologies discussed by Langlands [26] for the elliptic modular case.

§ 3.5. Problems in the Siegel modular case

It is natural to suspect that the analogy of Theorem (3.4.1) is also valid for discrete subgroups which are not cocompact in $Sp_{2n}(\boldsymbol{R})$. Let us formulate this as problems for discrete subgroups of $Sp_{2n}(\boldsymbol{R})$ commensurable with $Sp_{2n}(\boldsymbol{Z})$.

Let us start with an irreducible representation

$$\rho_{\boldsymbol{Q}}: Sp_{2n}(\boldsymbol{Q}) \longrightarrow GL(V_{\boldsymbol{Q}})$$

of $Sp_{2n}(\boldsymbol{Q})$ with highest weight $\lambda \in D$ on a finite-dimensional $\boldsymbol{Q}$-vector space $V_{\boldsymbol{Q}}$. Then for a discrete subgroup $\Gamma \subset Sp_{2n}(\boldsymbol{Q})$ which is commensurable with $Sp_{2n}(\boldsymbol{Z})$ and has no fixed points on X, we can construct a local system $V_{\boldsymbol{Q}}$ over $S = \Gamma \backslash X$ with fibers of $\boldsymbol{Q}$-vector spaces corresponding to $\rho_{\boldsymbol{Q}}$.

Recall that these $\boldsymbol{Q}$-local systems appear naturally by the following geometric construction. If Γ is a congruence subgroup of $Sp_{2n}(\boldsymbol{Z})$, then we can construct a canonical analytic family $A \xrightarrow{f} S$ of principally polarized abelian varieties over S with a level structure corresponding to Γ. For each i $(1 \leqq i \leqq n)$, we have an isomorphism $R^i f_* \boldsymbol{Q} \cong V_{\boldsymbol{Q}}$ of local systems with $\lambda = (\underbrace{1, 1, \cdots, 1}_{i}, 0, \cdots, 0)$. In general any local system $V_{\boldsymbol{Q}}$ over S is a direct summand of the higher direct image $R^m f^{(k)}_* \boldsymbol{Q}$ of some degree m for some k-tuple fiber product $f^{(k)}: A \times_S A \times \cdots \times_S A \rightarrow S$ of f.

Let us consider the image

$$\tilde{H}^N(S, V_{\boldsymbol{Q}}) = \text{Image}\,(H^N_c(S, V_{\boldsymbol{Q}}) \longrightarrow H^N(S, V_{\boldsymbol{Q}}))$$

of the natural homomorphism of the cohomology group with compact supports to the cohomology groups (without any conditions on supports). Then the first problem is the following.

(3.5.1) **Problem.** Show that $\tilde{H}^N(S, V_Q)$ has a rational Hodge structure of weight $N+n_\lambda$, where

$$n_\lambda=\sum_{i=1}^n \lambda_i \qquad \text{for } \lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n) \in D\subset L\cong \boldsymbol{Z}^n.$$

Remark. The above problem is considered as a special case of the analogue for Hodge structures of the ℓ-adic purity theorem, which is the main result of Deligne [12].

The analytic manifold S has a natural structure of a quasi-projective algebraic variety over $\boldsymbol{C}$. Let S^* be a smooth compactification of the algebraic variety S. The bundle F_λ is considered as an algebraic bundle over S.

(3.5.2) **Problem.** Suppose that $\lambda \in D$ satisfies the assumption of Theorem (3.4.1). Set $E_\lambda^*=j_*(F_\lambda)$ for the immersion $j\colon S\hookrightarrow S^*$. Find a sheaf of ideal J_λ on S^* with support in S^*-S for each λ, such that there exists an isomorphism

$$\iota\colon \tilde{H}^N(S, V_Q)\otimes_Q \boldsymbol{C}\cong \bigoplus_{w\in W^1} H^{q_w}(S^*, F^*_{w(\lambda+\rho)-\rho}\otimes J_{w(\lambda+\rho)-\rho})$$

compatible with the Hodge decomposition of $H^N(S, V_Q)$, and for each $w \in W^1$, $H^{q_w}(S^*, F^*_{w(\lambda+\rho)-\rho}\otimes J_{w(\lambda+\rho)-\rho})$ is contained in the

$$\left(\sum_{i=1}^n \nu_i,\ N+n_\lambda-\sum_{i=1}^n \nu_i\right) \quad (w(\lambda+\rho)=(\nu_1, \cdots, \nu_n) \in \boldsymbol{Z}^n)$$

type component of $\tilde{H}^N(S, V_Q)\otimes_Q \boldsymbol{C}$ via ι.

Similarly as for elliptic modular forms or Hilbert modular forms, one might proceed to attach Hodge structures to primitive Siegel modular forms.

Let f be an automorphic cusp form on $G^{\mathrm{ad}}=PSp_{2n}(\boldsymbol{R})$ for Γ, which generates an irreducible automorphic representation (ω, V_ω) in $\mathscr{A}(PSp_{2n}(\boldsymbol{Q})\backslash PSp_{2n}(\boldsymbol{A}))$ such that the infinite component ω_∞ of $\omega=\bigotimes_{v\in P}\omega_v$ is equivalent to $\pi_{\lambda'_{\mathrm{op}}}$ for some $\lambda' \in L'_0\cap X(H_C^{\mathrm{ad}})$. Here $\boldsymbol{A}$ is the adele ring of $\boldsymbol{Q}$, P the set of all non-trivial places of $\boldsymbol{Q}$, and H_C^{ad} the complexification of a compact Cartan subgroup H^{ad} of G^{ad}. For the given weight $\lambda' \in L'_0$, there exist unique $\lambda \in D$ and $w \in W^1$ such that $\lambda'=\lambda^{(w)}=w(\lambda+\rho)-\rho$.

If f (or the representation ω) is "generic" in some sense, it seems natural to expect the following.

(3.5.3) **Problem.** For each $w \in W^1$, find an automorphic form f_w on G^{ad} for Γ which generates an irreducible automorphic representation $\omega_w = \otimes_{v \in P} \omega_{w,v}$ such that

$$\omega_{w,v} \cong \omega_v \quad \text{for all finite place } v \in P,$$

and

$$\omega_{w,\infty} \cong \pi_{\{w(\lambda+\rho)-\rho\}^{\mathrm{op}}} \quad \text{for the infinite place } \infty.$$

Let K_ω be the subfield of $\boldsymbol{C}$, generated over $\boldsymbol{Q}$ by the eigenvalues a_T of f with respect to the Hecke operators T: $T(f) = a_T f$. Assume that K_ω is totally real. Then for each embedding $\sigma: K_\omega \hookrightarrow \boldsymbol{C}$, we expect that there exists a companion f^σ, which is an automorphic form on G^{ad} for Γ such that $T(f^\sigma) = a_T^\sigma f^\sigma$ for all Hecke operators T. We write f_w^σ for $(f^\sigma)_w$.

(3.5.4) **Problem.** Define a rational sub-Hodge structure $H^{n(\omega)}(M_\omega, \boldsymbol{Q})$ of weight $n(\omega) = N + n_\lambda$ in $\tilde{H}^N(S, V_{\boldsymbol{Q}})$ for f (or for ω) with the following properties:

(i) There is an algebra homomorphism

$$\theta_\omega: K_\omega \longrightarrow \mathrm{End}\,(H^{n(\omega)}(M_\omega, \boldsymbol{Q}))$$

of K_ω to the endomorphism algebra of the Hodge structure $H^{n(\omega)}(M_\omega, \boldsymbol{Q})$ induced from the action of the Hecke operators on $\tilde{H}^N(S, V_{\boldsymbol{Q}})$. Moreover,

$$\dim_{K_\omega} H^{n(\omega)}(M_\omega, \boldsymbol{Q}) = 2^n \quad (= |W^1|).$$

(ii) For each embedding $\sigma: K_\omega \hookrightarrow \boldsymbol{C}$, the σ-eigencomponent

$$H^{n(\omega)}(M_\omega, \boldsymbol{Q}) \otimes_{K_\omega, \sigma} \boldsymbol{C} \quad \text{of} \quad H^{n(\omega)}(M_\omega, \boldsymbol{Q}) \otimes_{\boldsymbol{Q}} \boldsymbol{C}$$

with respect to K_ω has a natural identification

$$H^{n(w)}(M_\omega, \boldsymbol{Q}) \otimes_{K_\omega, \sigma} \boldsymbol{C} \cong \bigotimes_{w \in W^1} \boldsymbol{C} f_w^\sigma$$

such that $\boldsymbol{C} f_w^\sigma$ constitutes a part of the $(\sum_{i=1}^n \nu_i, N + n_\lambda - \sum_{i=1}^n \nu_i)$ type component of $H^{n(\omega)}(M_\omega, \boldsymbol{Q}) \otimes_{K_\omega, \sigma} \boldsymbol{C}$. Here $(\nu_1, \cdots, \nu_n) = w(\lambda + \rho)$.

In order to justify (3.5.3), it seems natural to expect the following.

(3.5.5) **Problem.** Fix a dominant weight λ. Show that the traces of the Hecke operator T:

$$\mathrm{tr}\,\{T \mid H^{q_\omega}(S^*, F^*_{w(\lambda+\rho)-\rho} \otimes J_{w(\lambda+\rho)-\rho})\}$$

on $H^{q_\omega}(S^*, F^*_{w(\lambda+\rho)-\rho} \otimes J_{w(\lambda+\rho)-\rho})$ are independent of a choice of $w \in W^1$, possibly, modulo some terms corresponding to "degenerate" automorphic forms.

(3.5.6) **Remark.** We find that the contribution from elliptic elements of Γ *does* depend on $w \in W^1$ in the dimension formula of automorphic forms of Hotta-Parthasarathy [21].

(3.5.7) **Remark.** If the modular forms f_w in Problem (3.5.3) exist, the L-functions for the f_w coincide with that of f. In this sense, we cannot tell f_w from f by their L-functions. This problem seems to be different from the L-indistinguishability of Labesse-Langlands [23].

In the rest of this section, let us add a few words about the gamma factors of L-functions of Siegel modular forms. The L-group of G^{ad} for $\boldsymbol{Q}$ is isomorphic to $\mathrm{Spin}(2n+1, \boldsymbol{C})$. Let $L(s, \omega, \mathrm{spin})$ be the L-function of the automorphic representation ω for the spinor representation spin: $\mathrm{Spin}(2n+1, \boldsymbol{C}) \to GL(2^n, \boldsymbol{C})$. The Euler factors of $L(s, \omega, \mathrm{spin})$ are defined except for a finite number of bad primes for ω (cf. Langlands [24], [25], and Borel [4]). Assume that the Euler factors at bad primes are also defined in some way.

When the variety $S = \Gamma \backslash X$ has a canonical model $S_{\boldsymbol{Q}}$ defined over $\boldsymbol{Q}$, we can consider the ℓ-adic analogy $\tilde{H}^N_{\text{ét}}(S_{\boldsymbol{Q}} \times \bar{\boldsymbol{Q}}, V_{\boldsymbol{Q}_\ell})$ of $\tilde{H}^N(S, V_{\boldsymbol{Q}})$ on which the absolute Galois group $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ acts. Let $H^{n(\omega)}(M_\omega, \boldsymbol{Q}_\ell)$ be the $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$-submodule of $\tilde{H}^N_{\text{ét}}(S_{\boldsymbol{Q}} \times \bar{\boldsymbol{Q}}, V_{\boldsymbol{Q}_\ell})$, corresponding to $H^{n(\omega)}(M_\omega, \boldsymbol{Q})$ by the comparison theorem. For a given embedding $\sigma: K_\omega \to \bar{\boldsymbol{Q}}_\ell$, we put

$$V_{\omega,\ell} \cong H^{n(\omega)}(M_\omega, \boldsymbol{Q}_\ell) \otimes_{K_\omega, \sigma} \bar{\boldsymbol{Q}}_\ell.$$

Then we expect that it defines an ℓ-adic representation

$$\rho_\omega: \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q}) \longrightarrow \mathrm{Aut}_{\bar{\boldsymbol{Q}}_\ell}(V_{\omega,\ell})$$

on $V_{\omega,\ell}$ with $\dim_{\bar{\boldsymbol{Q}}_\ell} V_{\omega,\ell} = |W^1| = 2^n$. For each isomorphism $\iota: \bar{\boldsymbol{Q}}_\ell \to \boldsymbol{C}$, we can define the L-function

$$L(s, \rho_\omega, \iota \circ \sigma)$$

associated to ρ_ω, which is the product of the characteristic polynomials of the Frobenius elements in $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$. The fundamental problem for L-functions of geometric Siegel modular forms is to show the equality

$$L\left(s + \frac{n(\omega)}{2}, \omega, \mathrm{spin}\right) = L(s, \rho_\omega, \iota \circ \sigma)$$

for a given ω, by choosing suitable σ and ι. Let $L_\infty(s, \rho_\omega)$ be the Euler factor at the infinity place ∞. Then the expected functional equation for $\Lambda(s, \rho_\omega, \iota\circ\sigma)=L_\infty(s, \rho_\omega)L(s, \rho_\omega, \iota\circ\sigma)$ is

$$\Lambda(n(\omega)+1-s, \rho_\omega^\vee, \iota\circ\sigma)=\varepsilon(s)\Lambda(s, \rho_\omega, \iota\circ\sigma).$$

The factor $L_\infty(s, \rho_\omega)$ is conjectured by Serre and Deligne (cf. [11] Section 5.3): when $n(\omega)$ is odd, $L_\infty(s, \rho_\omega)$ is a product of 2^{n-1} gamma functions $\Gamma(s-i)$ $(i\in \boldsymbol{Z})$

$$\Gamma(s, \rho_\omega)=\prod_{\substack{\varepsilon\in E\\ \Sigma_\varepsilon^+<\Sigma_\varepsilon^-}} \Gamma(s \quad \Sigma_\varepsilon^+),$$

where

$$\Sigma_\varepsilon^+=\sum_{\substack{\varepsilon\in E\\ \varepsilon(i)=+1}}\{\lambda_i+(n+1-i)\} \quad \text{and} \quad \Sigma_\varepsilon^-=\sum_{\substack{\varepsilon\in E\\ \varepsilon(i)=-1}}\{\lambda_i+(n+1-i)\}.$$

When $n=2$, the gamma factors of the L-functions for holomorphic Siegel modular forms in Andrianov [1], [2] and Arakawa [3] are of the form conjectured above.

§ 3.6. A remark for the case of unitary groups

Let us suppose that G is the adjoint group of the unitary group $U(p, q)$. Set $n=p+q$. Then the dual group of G is $SL(n, \boldsymbol{C})$. On the other hand, the number of Hodge components of the cohomology group of a discrete subgroup is $|W^1|=C_{n,p}=n!/(p!q!)$. Note that in this case the Hodge symmetry is lost. The computation of the Hodge type suggests that if we can attach a Hodge structure H to a primitive geometric automorphic form, it might be a p-th (or q-th) exterior product of another Hodge structure over the field of eigenvalues. Thus, when $p\geqq 2$ and $q\geqq 2$, one might imagine a period relation for H different from the Riemann-Hodge relation, i.e. a Plücker relation between the entries of the period matrix of H.

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