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On the Stark-Shintani Conjecture and Certain Relative Class Numbers

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§ 1. Introduction

1.1. In [4], [5], H. M. Stark introduced certain ray class invariants of real quadratic fields with the use of special values at $s=0$ of the derivatives of some zeta functions, and presented a remarkable conjecture on the arithmetic of the ray class invariants (his treatment covers the cases of totally real fields). T. Shintani established the conjecture independently and solved it in a special but non-trivial significant case (see [3]). The solved case of the conjecture owing to Shintani might be of some importance in connection with certain $\boldsymbol{Z}_p$-extensions of ray class fields over real quadratic fields (see J. Nakagawa [1], [2]). In this note we obtain a certain relative class number formula of the ray class fields under the assumption that the Stark-Shintani conjecture is valid. Such a class number formula will have some application in the study of $\boldsymbol{Z}_p$-extensions of the ray class fields (cf. [1], [2]).

1.2. We summarize our results. Let F be a real quadratic field embedded in the real number field $\boldsymbol{R}$. Let $E(F)$ (resp. $E^+(F)$) be the group of units (resp. totally positive units) of F. For each $\alpha \in F$, α' denotes the conjugate of α in F. For an integral ideal $\mathfrak{f}$ of F, let $H_F(\mathfrak{f})$ denote the group of narrow ray classes modulo $\mathfrak{f}$ of F. Take a totally positive integer ν of F such that $\nu+1 \in \mathfrak{f}$, and denote by $\nu(\mathfrak{f})$ the ray class of $H_F(\mathfrak{f})$ represented by the principal ideal (ν). For each class $c \in H_F(\mathfrak{f})$, let $\zeta_F(s, c)$ be the partial zeta function defined by $\zeta_F(s, c)=\sum N(\mathfrak{a})^{-s}$, where $\mathfrak{a}$ is taken over all integral ideals of F belonging to the class c. It is known that $\zeta_F(s, c)$ is holomorphic in the whole complex plane except for a simple pole at $s=1$. Let $\zeta'_F(s, c)$ denote the derivative of $\zeta_F(s, c)$. Set, for each $c \in H_F(\mathfrak{f})$,

$$X_{\mathfrak{f}}(c)=\exp\,(\zeta'_F(0, c)-\zeta'_F(0, c\nu(\mathfrak{f}))).$$

The invariant $X_{\mathfrak{f}}(c)$ is intensively studied by Stark [4] and Shintani [3].

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From $X_{\mathfrak{f}}(c)$, another invariant $Y_{\mathfrak{f}}(c)$ ($c \in H_F(\mathfrak{f})$) is introduced in [3, § 2]: Let $\mathfrak{P}(\mathfrak{f})$ be the set of prime divisors of $\mathfrak{f}$. For each subset S of $\mathfrak{P}(\mathfrak{f})$, denote by $\mathfrak{f}(S)$ the intersection of all the divisors of $\mathfrak{f}$ which are prime to any $\mathfrak{p}$ of $\mathfrak{P}(\mathfrak{f})-S$. If we write $\mathfrak{f}=\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{f})}\mathfrak{p}^{e(\mathfrak{p})}$ ($e(\mathfrak{p})>0$), then, $\mathfrak{f}(S)=\prod_{\mathfrak{p}\in S}\mathfrak{p}^{e(\mathfrak{p})}$. Put $n(S)=|H_F(\mathfrak{f})|/|H_F(\mathfrak{f}(S))|$ (for any finite set A, $|A|$ denotes the cardinality of A). Set, for each $c \in H_F(\mathfrak{f})$,

$$Y_{\mathfrak{f}}(c)=\prod_S X_{\mathfrak{f}(S)}(\tilde{c}\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{f})-S}\mathfrak{p}^{-1})^{1/n(S)}, \tag{1.1}$$

where S runs over all subsets of $\mathfrak{P}(\mathfrak{f})$, and for each S, $\tilde{c}$ denotes the image of c under the natural homomorphism of $H_F(\mathfrak{f})$ onto $H_F(\mathfrak{f}(S))$.

We impose the following conditions (1.2), (1.3) on $\mathfrak{f}$:

(1.2) for any $u \in E^+(F)$, $u+1 \notin \mathfrak{f}$,

(1.3) no unit u of F satisfies $u>0$, $u'<0$ and $u-1 \in \mathfrak{f}$.

Under the assumption (1.2), the ray class $\nu(\mathfrak{f})$ is of order two in $H_F(\mathfrak{f})$. Denote by $K_F(\mathfrak{f})$ the maximal narrow ray class field over F defined modulo $\mathfrak{f}$ and let σ be the Artin canonical isomorphism from $H_F(\mathfrak{f})$ onto the Galois group $\mathrm{Gal}(K_F(\mathfrak{f})/F)$ of $K_F(\mathfrak{f})$ over F. For any subgroup G of $H_F(\mathfrak{f})$, let $K_F(\mathfrak{f}, G)$ be the subfield of $K_F(\mathfrak{f})$ corresponding to G: $K_F(\mathfrak{f}, G)=\{\theta \in K_F(\mathfrak{f}) \mid \theta^{\sigma(g)}=\theta \text{ for all } g \in G\}$. Set

$$X_{\mathfrak{f}}(c, G)=\prod_{g\in G} X_{\mathfrak{f}}(cg), \quad Y_{\mathfrak{f}}(c, G)=\prod_{g\in G} Y_{\mathfrak{f}}(cg) \quad (c \in H_F(\mathfrak{f})).$$

Take an integer μ of F such that $\mu<0$, $\mu'>0$ and $\mu-1 \in \mathfrak{f}$. Denote by $\mu(\mathfrak{f})$ the ray class of $H_F(\mathfrak{f})$ represented by the principal ideal (μ). Then $\mu(\mathfrak{f})$ is of order at most two. Let the subgroup G of $H_F(\mathfrak{f})$ satisfy the condition:

$$\mu(\mathfrak{f}) \in G \quad \text{and} \quad \nu(\mathfrak{f}) \notin G. \tag{1.4}$$

Set $M=K_F(\mathfrak{f}, G)$ and $M^+=K_F(\mathfrak{f}, \langle G, \nu(\mathfrak{f})\rangle)$, where $\langle G, \nu(\mathfrak{f})\rangle$ denotes the subgroup of $H_F(\mathfrak{f})$ generated by G and $\nu(\mathfrak{f})$. Then exactly one of the infinite primes of F which corresponds to the prescribed embedding of F into $\boldsymbol{R}$ splits in M over F, and M^+ is the maximal totally real subfield of M. Under the assumptions (1.2), (1.3) on $\mathfrak{f}$ and (1.4) on G, the Stark-Shintani conjecture is formulated as follows (we follow [3]):

Conjecture. *There exists a rational positive integer m which satisfies the following conditions:*

(i) *$X_{\mathfrak{f}}(c, G)^m$ is a unit of M for each $c \in H_F(\mathfrak{f})$,*

(ii) *$\{X_{\mathfrak{f}}(c, G)^m\}^{\sigma(c')}=X_{\mathfrak{f}}(cc', G)^m$ for any c, $c' \in G$.*

Shintani proved the conjecture when M^+ is abelian over the rational number field $\boldsymbol{Q}$ ([3, Theorem 2]).

Let $E(M)$ (resp. $E(M^+)$) be the unit group of M (resp. M^+). Denote by $h(M)$ (resp. $h(M^+)$) the class number of M (resp. M^+). Then, $h(M)$ is divided by $h(M^+)$ by class field theory. In view of the definition (1.1) of $Y_{\mathfrak{f}}(c)$ and the conjecture, we may assume that, for some positive integer m, $Y_{\mathfrak{f}}(c, G)$ satisfies

$$(1.5)\qquad \begin{cases} Y_{\mathfrak{f}}(c, G)^m \in E(M) \text{ for any } c \in H_F(\mathfrak{f}), \\ \{Y_{\mathfrak{f}}(c, G)^m\}^{\sigma(c')} = Y_{\mathfrak{f}}(cc', G)^m \text{ for any } c, c' \in H_F(\mathfrak{f}). \end{cases}$$

Under the assumption (1.5), we shall get a certain formula which connects the relative class number of M/M^+ with the invariants $Y_{\mathfrak{f}}(c, G)$.

Theorem. *Let $\mathfrak{f}$ satisfy the conditions* (1.2), (1.3) *and let G be a subgroup of $H_F(\mathfrak{f})$ with the condition* (1.4). *Assume that, for a suitable positive integer m, the invariants $Y_{\mathfrak{f}}(c, G)$ satisfy the relation* (1.5). *Denote by n the degree of the extension of M^+ over F ($n=[M^+:F]$). Then we have*

$$h(M)/h(M^+)=2^{1-2n}m^{-n}[E(M): E_{Y,m}(M)],$$

where $E_{Y,m}(M)$ is the subgroup of $E(M)$ generated by $E(M^+)$ and $Y_{\mathfrak{f}}(c, G)^m$ ($c \in H_F(\mathfrak{f})/\langle G, \nu(\mathfrak{f})\rangle$), and $[E(M): E_{Y,m}(M)]$ denotes the group index of $E(M)$ to $E_{Y,m}(M)$.

If M^+ is abelian over $\boldsymbol{Q}$, then the relation (1.5) for a certain positive integer m has been verified by Shintani ([3, Proposition 5]). In this solved case the relative class number formula in Theorem actually holds. Recently, Nakagawa obtained a series of cyclotomic $\boldsymbol{Z}_p$-extensions $\bigcup_{n=0} M_n$ such that, for each M_n, the Stark-Shintani conjecture is valid with the index $m=1$. Moreover, with respect to such $\boldsymbol{Z}_p$-extensions, he obtained a more precise version of our Theorem (see [1], and [2, Theorem 1]).

§ 2. Proof of Theorem

We keep the notation used in the introduction. Let $\zeta_M(s)$, $\zeta_{M^+}(s)$ be the Dedekind zeta functions of M, M^+, respectively. For each character χ of the group $H_F(\mathfrak{f})$, we denote by $\mathfrak{f}_\chi$ the conductor of χ and by $\tilde{\chi}$ the primitive character of the group $H_F(\mathfrak{f}_\chi)$ corresponding to χ in a natural manner. Let $L_F(s, \tilde{\chi})$ be the Hecke L-function associated with $\tilde{\chi}$. It is well-known by class field theory that

$$\zeta_M(s)=\prod_{\chi} L_F(s, \tilde{\chi}),$$

where χ runs over all characters of the group $H_F(\mathfrak{f})$ with the condition $\chi(G)=1$. A similar identity holds for $\zeta_{M^+}(s)$. Therefore we get the expression for $\zeta_M(s)/\zeta_{M^+}(s)$:

$$\zeta_M(s)/\zeta_{M^+}(s)=\prod_{\chi}{}' L_F(s, \tilde{\chi}), \tag{2.1}$$

where the product $\prod'$ means that χ is taken over all characters of the group $H_F(\mathfrak{f})$ with the conditions $\chi(G)=1$, $\chi(\nu(\mathfrak{f}))=-1$. Denote by $R(M)$, $R(M^+)$ the regulators of M, M^+, respectively. It is easily derived from the well-known residue formula of $\zeta_M(s)$ at $s=1$ that

$$\lim_{s\to 0}\frac{\zeta_M(s)}{s^{3n-1}}=-\frac{h(M)R(M)}{2}.$$

Similarly we have

$$\lim_{s\to 0}\frac{\zeta_{M^+}(s)}{s^{2n-1}}=-\frac{h(M^+)R(M^+)}{2}.$$

Therefore we easily get

$$\frac{h(M)}{h(M^+)}=\frac{R(M^+)}{R(M)}\prod_{\chi}{}'\left\{\frac{d}{ds}L_F(s, \tilde{\chi})\right\}_{s=0},$$

where we note that $L_F(0, \tilde{\chi})=0$. It is known by Shintani [3, Proposition 3] that, for each character χ of the group $H_F(\mathfrak{f})$ with $\chi(\nu(\mathfrak{f}))=-1$, the value $\{(d/ds)L_F(s, \tilde{\chi})\}_{s=0}$ is expressed as follows:

$$\left\{\frac{d}{ds}L_F(s, \tilde{\chi})\right\}_{s=0}=\sum_{c\in H_F(\mathfrak{f})/\langle G,\nu(\mathfrak{f})\rangle}\chi(c)\log Y_{\mathfrak{f}}(c, G)$$

(we note that the definition (1.1) of $Y_{\mathfrak{f}}(c)$ coincides with that of Shintani [3, (18)], since $X^{\mathfrak{f}(S)}(\tilde{c})$ ($c\in H_F(\mathfrak{f})$) is trivially one unless $\mathfrak{f}(S)$ satisfies the condition (1.2)). Thus we get

$$\frac{h(M)}{h(M^+)}=\frac{R(M^+)}{m^n R(M)}\prod_{\chi}{}'\{\sum_{c\in H_F(\mathfrak{f})/\langle G,\nu(\mathfrak{f})\rangle}\chi(c)\log Y_{\mathfrak{f}}(c, G)^m\}. \tag{2.2}$$

Let $\{c_1, c_2, \cdots, c_n\}$ be a complete set of representatives of $H_F(\mathfrak{f})/\langle G, \nu(\mathfrak{f})\rangle$. Let τ be any embedding of M into the complex number field C which is an extension of the non-trivial automorphism ι of F. Then all mutually distinct embeddings of M into C are exhausted by

$$\begin{cases}\sigma(c_1), \cdots, \sigma(c_n), \sigma(c_1\nu), \cdots, \sigma(c_n\nu),\\ \sigma(c_1)\tau, \cdots, \sigma(c_n)\tau, \sigma(c_1\nu)\tau, \cdots, \sigma(c_n\nu)\tau,\end{cases}$$

where we write ν instead of $\nu(\mathfrak{f})$. Note that $\sigma(c_i\nu)\tau$ is the complex conjugate of $\sigma(c_i)\tau$ $(1\leqq i\leqq n)$. Let $\{u_1, u_2, \cdots, u_{2n-1}\}$ be a system of fundamental units of M^+. For simplicity we write $Y(c_j)=Y_{\mathfrak{f}}(c_j, G)^m$ $(1\leqq j\leqq n)$. Now we calculate the regulator $R[u_1, \cdots, u_{2n-1}, Y(c_1), \cdots, Y(c_n)]$ of the units $u_1, \cdots, u_{2n-1}, Y(c_1), \cdots, Y(c_n)$ of M. We define matrices Y, W, U and U' by putting

$$Y=(\log|Y(c_j)^{\sigma(c_i)}|)_{1\leqq i,j\leqq n},$$
$$W=(\log|Y(c_j)^{\sigma(c_i)\tau}|)_{1\leqq i\leqq n-1, 1\leqq j\leqq n},$$
$$U=(\log|u_j^{\sigma(c_i)}|)_{1\leqq i\leqq n, 1\leqq j\leqq 2n-1},$$
$$U'=(\log|u_j^{\sigma(c_i)\tau}|)_{1\leqq i\leqq n-1, 1\leqq j\leqq 2n-1}.$$

Immediately we have

$$Y(c_j)^{\sigma(c_i)}=Y(c_ic_j),\quad Y(c_j)^{\sigma(c_i\nu)}=Y(c_ic_j)^{-1},\quad u_j^{\sigma(c_i\nu)}=u_j^{\sigma(c_i)}.$$

It is easy to see from the definition of the regulator that

$$R[u_1, \cdots, u_{2n-1}, Y(c_1), \cdots, Y(c_n)]=\det\begin{bmatrix} Y & U \\ -Y & U \\ 2W & 2U' \end{bmatrix}$$
$$=\det\begin{bmatrix} 2Y & 0 \\ -Y & U \\ 2W & 2U' \end{bmatrix}$$
$$=2^{2n-1}\det(Y)\det\begin{bmatrix} U \\ U' \end{bmatrix}.$$

Thus we get

$$(2.3)\qquad R[u_1, \cdots, u_{2n-1}, Y(c_1), \cdots, Y(c_n)]=\pm 2^{2n-1}R(M^+)\det(Y).$$

We need a lemma which is only a modification of the Dedekind determinant relation.

Lemma 2.1. *Let H be a finite abelian group and K a subgroup of H. Let $h_1, h_2, \cdots, h_r$ $(r=|H/K|)$ be a complete set of representatives of the quotient H/K. Take a character ψ of K. Then, for any function f on H such that*

$$f(kh)=\psi(k)f(h)\quad \text{for all } k\in K,\ h\in H,$$

we have

$$\det(f(h_ih_j^{-1}))=\prod_{\chi}\left\{\sum_{i=1}^{r}\chi(h_i)f(h_i^{-1})\right\}$$

where χ runs over all characters of H such that the restriction of χ onto K coincides with ψ.

We omit the proof, which is quite similar for instance to that of Lemma 5.26 of [6].

Now let $H=H_F(\mathfrak{f})$ and $K=\langle G, \nu(\mathfrak{f})\rangle$. Define a character ψ of K by putting $\psi(G)=1$ and $\psi(\nu(\mathfrak{f}))=-1$. Applying Lemma 2.1 to our situation, we get

$$\det(\log Y(c_ic_j^{-1}))=\prod_{\chi}{}'\left\{\sum_{i=1}^{n}\chi(c_i^{-1})\log Y(c_i)\right\}, \tag{2.4}$$

where $\prod_{\chi}'$ means the same as in (2.1). We note that

$$\det(\log Y(c_ic_j^{-1}))=\pm\det(\log Y(c_ic_j))=\pm\det(Y).$$

Taking the relations (2.2), (2.3), (2.4) into account of, we get

$$\frac{h(M)}{h(M^+)}=\pm\frac{R[u_1, \cdots, u_{2n-1}, Y(c_1), \cdots, Y(c_n)]}{2^{2n-1}m^nR(M)},$$

which implies the assertion of Theorem. Q.E.D.

Remark. It seems better to obtain a relative class number formula in Theorem by using the original invariants $X_{\mathfrak{f}}(c, G)$ instead of $Y_{\mathfrak{f}}(c,G)$.

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