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The Restricted Root System of a Semisimple Symmetric Pair

Toshio Oshima and Jiro Sekiguchi

Contents

§ 0. Introduction

Let $\mathfrak{g}$ be a real semisimple Lie algebra and let σ be an involutive linear automorphism of $\mathfrak{g}$. If $\mathfrak{h}=\{X \in \mathfrak{g};\, \sigma X=X\}$ and $\mathfrak{q}=\{X \in \mathfrak{g};\, \sigma X=-X\}$, we obtain a direct sum decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$. The pair $(\mathfrak{g}, \mathfrak{h})$ is called a (semisimple) symmetric pair. A classification of such pairs was accomplished by M. Berger [Be]. Then it is important to study the fine structure of a symmetric pair. Among other things, the restricted root system of a symmetric pair is to be determined. One of the purpose of this paper is this. Needless to say, the results of this paper will play a basic role in the study of Fourier analysis on a semisimple symmetric space. This will be treated in the subsequent papers.

This paper deals with the study on the basic structure of a symmetric pair. The main part of this paper is the contents in Section 1–Section 6 and the results of Section 7, Section 8 are preparations of the subsequent papers.

We explain the contents shortly. In Section 1, after giving the defi-

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nitions of the dual and associated pairs of $(\mathfrak{g}, \mathfrak{h})$, we examine the relation between these pairs. Let θ be a Cartan involution of $\mathfrak{g}$ commuting with σ and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition. If $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$, we can define the set $\Sigma(\mathfrak{a})$ of the roots with respect to $(\mathfrak{g}, \mathfrak{a})$. It is shown in Section 2 that $\Sigma(\mathfrak{a})$ becomes a root system (cf. Theorem (2.11)). This is already proved by Rossmann [Ro]. We call $\Sigma(\mathfrak{a})$ the restricted root system of $(\mathfrak{g}, \mathfrak{h})$. In Section 3, we introduce the notion of a (θ, σ)-system of roots. As in the case of the restricted root system of a real semisimple Lie algebra, we give a sufficient condition that the totality of the restricted roots of the (θ, σ)-system of roots becomes a root system. As a corollary, we obtain an alternative proof of Theorem (2.11). The dimension of $\mathfrak{a}$ is called the split rank of the pair $(\mathfrak{g}, \mathfrak{h})$. Needless to say, symmetric pairs of split rank 1 are basic among general ones. In Section 4, for a given $\lambda \in \Sigma(\mathfrak{a})$, we construct the symmetric pair $(\mathfrak{g}(\lambda), \mathfrak{h}(\lambda))$ of split rank 1 which is contained in $(\mathfrak{g}, \mathfrak{h})$. Section 5 is devoted to the determination of all the symmetric pairs of split rank 1 based on the classification of Berger (cf. Table II). In the study of restricted roots of a symmetric pair, the signatures of them are important (see Def. (2.14)). In Section 6, we determine the restricted root system as well as the signatures of simple roots of a fundamental system of $\Sigma(\mathfrak{a})$ for a general symmetric pair. At this stage, it must be stressed that the signatures of the simple roots depend on the choice of the order on $\Sigma(\mathfrak{a})$. In order to develop Fourier analysis on the corresponding semisimple symmetric space, we need a property of the Weyl groups for various root systems. This is done in Section 7. Especially, Corollary (7.10) will play a fundamental role in the definition of principal series for the semisimple symmetric space (cf. [O]). In Section 8, we shall examine the Levi part of a parabolic subalgebra of $\mathfrak{g}$ which is particular to the analysis on the semisimple symmetric space. In Appendix A, we shall prove a lemma which is used in the proof of Lemma (7.7). Most parts of the discussion in Section 8 is applicable to an arbitrary parabolic subalgebra of $\mathfrak{g}$. By this reason, we give a structure of the Levi part of a general parabolic subalgebra in Appendix B. The results there are rather independent of the text.

§ 1. Semisimple symmetric pairs

In this section, we define a semisimple symmetric pair and a semisimple symmetric space. Our main concern is a semisimple Lie group. Accordingly we frequently omit the word "semisimple" and therefore we call them a symmetric pair and a symmetric space for brevity. The discussions in this section are based on the classification of the symmetric pairs by M. Berger [Be].

(1.1) Let $\mathfrak{g}$ be a real semisimple Lie algebra and let σ be an involution of $\mathfrak{g}$. Then we obtain a direct sum decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$, where

$$\begin{aligned}\mathfrak{h}&=\{X\in\mathfrak{g};\ \sigma(X)=X\}\\ \mathfrak{q}&=\{X\in\mathfrak{g};\ \sigma(X)=-X\}.\end{aligned}\tag{1.1.1}$$

We call $(\mathfrak{g}, \mathfrak{h})$ a (semisimple) symmetric pair in this paper.

Let $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}', \mathfrak{h}')$ be symmetric pairs. In this paper we define that they are isomorphic if there exists a Lie algebra isomorphism ϕ of $\mathfrak{g}$ to $\mathfrak{g}'$ such that $\phi(\mathfrak{h})=\mathfrak{h}'$. We note that this definition differs from the one in [Be].

A symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is irreducible if the representation of $\mathfrak{h}$ on $\mathfrak{q}$ is irreducible. If otherwise, $(\mathfrak{g}, \mathfrak{h})$ is reducible.

(1.2) It follows from [Be] that there exists a Cartan involution θ of $\mathfrak{g}$ such that $\theta\sigma=\sigma\theta$. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition corresponding to θ. Since $\theta\sigma$ is also an involution of $\mathfrak{g}$, we obtain another direct sum decomposition $\mathfrak{g}=\mathfrak{h}^a+\mathfrak{q}^a$ with respect to $\theta\sigma$, where

$$\begin{aligned}\mathfrak{h}^a&=\{X\in\mathfrak{g};\ \theta\sigma(X)=X\}\\ \mathfrak{q}^a&=\{X\in\mathfrak{g};\ \theta\sigma(X)=-X\}.\end{aligned}\tag{1.2.1}$$

We here note the following relations

$$\begin{aligned}\mathfrak{h}^a&=(\mathfrak{k}\cap\mathfrak{h})+(\mathfrak{p}\cap\mathfrak{q})\\ \mathfrak{q}^a&=(\mathfrak{k}\cap\mathfrak{q})+(\mathfrak{p}\cap\mathfrak{h}).\end{aligned}\tag{1.2.2}$$

On the other hand, if $\mathfrak{g}_C$ is a complexification of $\mathfrak{g}$, we extend σ and θ to $\mathfrak{g}_C$ as C-linear involutions. Then we define

$$\mathfrak{g}^d=(\mathfrak{k}\cap\mathfrak{h})+\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q})+\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h})+(\mathfrak{p}\cap\mathfrak{q}).\tag{1.2.3}$$

It is clear that $\mathfrak{g}^d$ is another real form of $\mathfrak{g}_C$. We consider the restrictions of θ and σ to $\mathfrak{g}^d$ and denote them by the same letters. Then σ is a Cartan involution of $\mathfrak{g}^d$. Moreover if we put

$$\begin{aligned}\mathfrak{k}^d&=(\mathfrak{k}\cap\mathfrak{h})+\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h})\\ \mathfrak{p}^d&=\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q})+(\mathfrak{p}\cap\mathfrak{q})\\ \mathfrak{h}^d&=(\mathfrak{k}\cap\mathfrak{h})+\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q})\\ \mathfrak{q}^d&=\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h})+(\mathfrak{p}\cap\mathfrak{q}),\end{aligned}\tag{1.2.4}$$

then $\mathfrak{g}^d=\mathfrak{k}^d+\mathfrak{p}^d$ (resp. $\mathfrak{g}^d=\mathfrak{h}^d+\mathfrak{q}^d$) is a direct sum decomposition of $\mathfrak{g}^d$ corresponding to σ (resp. θ). We note here that $\mathfrak{k}^d$ is a maximal compact subalgebra of $\mathfrak{g}^d$.

Definition (1.3) (cf. [Be]). The pair $(\mathfrak{g}^a, \mathfrak{h}^a)$ (resp. $(\mathfrak{g}^d, \mathfrak{h}^d)$) is called the associated (resp. dual) symmetric pair of $(\mathfrak{g}, \mathfrak{h})$. Here we put $\mathfrak{g}^a=\mathfrak{g}$.

(1.4) We frequently use the notation $(\mathfrak{g}, \mathfrak{h})^a=(\mathfrak{g}^a, \mathfrak{h}^a)$ and $(\mathfrak{g}, \mathfrak{h})^d=(\mathfrak{g}^d, \mathfrak{h}^d)$. Moreover $(\mathfrak{g}, \mathfrak{h})^{ad}=(\mathfrak{g}^{ad}, \mathfrak{h}^{ad})$ means the dual of $(\mathfrak{g}, \mathfrak{h})^a$ and $\mathfrak{k}^{ad}$ does a maximal compact subalgebra of $\mathfrak{g}^{ad}$ for a Cartan involution of $\mathfrak{g}^{ad}$ commuting with the involution for $\mathfrak{h}^{ad}$. Other notation are in proportion to these. Then it is clear that $(\mathfrak{g}, \mathfrak{h})^{aa}$ and $(\mathfrak{g}, \mathfrak{h})^{dd}$ are isomorphic to $(\mathfrak{g}, \mathfrak{h})$.

In this paper we frequently identify any two symmetric pairs contained in the same isomorphic class. Accordingly, for example, the dual and associated pairs of the given one mean the pairs isomorphic to the ones defined definitely in Definition (1.3).

We give some remarks on the relations between the associated and dual symmetric pairs. By an easy computation, we find the following relations

$$\begin{aligned}
&\mathfrak{g}=\mathfrak{g}^a, \quad \mathfrak{g}^d=\mathfrak{g}^{da}, \quad \mathfrak{g}^{ad}=\mathfrak{g}^{ada}=\mathfrak{g}^{dad},\\
&\mathfrak{h}=\mathfrak{h}^{dad}=\mathfrak{h}^{ada}, \quad \mathfrak{h}^d=\mathfrak{h}^{ad}, \quad \mathfrak{h}^a=\mathfrak{h}^{da}\\
&\mathfrak{q}=\sqrt{-1}\mathfrak{q}^{dad}=\sqrt{-1}\mathfrak{q}^{ada}, \quad \mathfrak{q}^d=\sqrt{-1}\mathfrak{q}^{ad}, \quad \mathfrak{q}^a=\sqrt{-1}\mathfrak{q}^{da}\\
&\mathfrak{k}=\mathfrak{k}^a, \quad \mathfrak{k}^d=\mathfrak{k}^{da}, \quad \mathfrak{k}^{ad}=\mathfrak{k}^{ada}=\mathfrak{k}^{dad}\\
&\mathfrak{p}=\mathfrak{p}^a, \quad \mathfrak{p}^d=\mathfrak{p}^{da}, \quad \mathfrak{p}^{ad}=\mathfrak{p}^{dad}=\mathfrak{p}^{ada}.
\end{aligned} \tag{1.4.1}$$

In particular, for any symmetric pair $(\mathfrak{g}, \mathfrak{h})$, we have the relation $(\mathfrak{g}, \mathfrak{h})^{ada}=(\mathfrak{g}, \mathfrak{h})^{dad}$ and the following diagram (1.4.2):

$$\begin{tikzcd}
(\mathfrak{g}, \mathfrak{h}) \arrow[r, leftrightarrow, "\text{associated}"] \arrow[d, leftrightarrow, "\text{dual}"] & (\mathfrak{g}, \mathfrak{h}^a) \arrow[r, leftrightarrow, "\text{dual}"] & (\mathfrak{g}^{ad}, \mathfrak{h}^d) \arrow[d, leftrightarrow, "\text{associated}"] \\
(\mathfrak{g}^d, \mathfrak{h}^d) \arrow[r, leftrightarrow, "\text{associated}"'] & (\mathfrak{g}^d, \mathfrak{h}^a) \arrow[r, leftrightarrow, "\text{dual}"'] & (\mathfrak{g}^{ad}, \mathfrak{h})
\end{tikzcd} \tag{1.4.2}$$

It rarely occurs that if $(\mathfrak{g}, \mathfrak{h})$ is irreducible, all the six pairs in the diagram (1.4.2) are not isomorphic to each other (cf. (1.16)).

(1.5) Next we consider the homogeneous space of a semisimple Lie group connected with a symmetric pair.

Let G_C be a connected complex semisimple Lie group whose Lie algebra is $\mathfrak{g}_C$ introduced in (1.1). Let G be an analytic subgroup of G_C corresponding to $\mathfrak{g}$. If there exists an analytic automorphism $\tilde{\sigma}$ of G such that $\tilde{\sigma}(\exp X)=\exp(\sigma X)$ for any $X\in\mathfrak{g}$, we say that σ is lifted to G and call $\tilde{\sigma}$ the lifting of σ. We give here a simple lemma.

Lemma. *If G_C is simply connected or is the adjoint group of $\mathfrak{g}_C$, the involution σ is lifted to the group G.*

Proof. Since σ is a C-linear involution of $\mathfrak{g}_C$, it suffices to show that σ is lifted to G_C.

First assume that G_C is simply connected. If g is an element of G_C, there exist elements $X_1, \cdots, X_r$ of $\mathfrak{g}_C$ such that $g=(\exp X_1)\cdots(\exp X_r)$. Then we define $\tilde{\sigma}(g)=(\exp \sigma X_1)\cdots(\exp \sigma X_r)$. Since G_C is simply connected, $\tilde{\sigma}(g)$ is uniquely determined by g and does not depend on the choice of $X_1, \cdots, X_r$. It is clear that $\tilde{\sigma}$ is an analytic isomorphism of G_C. The automorphism $\tilde{\sigma}$ of G_C is the required one. If Z is the center of G_C, then $\tilde{\sigma}$ clearly stabilizes Z. This implies that $\tilde{\sigma}$ induces an automorphism of the adjoint group G_C/Z of $\mathfrak{g}_C$. Hence the lemma is proved.

(1.6) In this paper we always assume that the involution σ of $\mathfrak{g}$ is lifted to G. This depends on the choice of G_C and therefore does not hold in general. We give here a counterexample.

Example. Let $\mathfrak{g}'=\mathfrak{sl}(2, \boldsymbol{R})$ and put $\mathfrak{g}=\mathfrak{g}'\oplus\mathfrak{g}'$. We define an involution σ of $\mathfrak{g}$ by $\sigma(X, Y)=(Y, X)$ $(\forall X, Y \in \mathfrak{g}')$. Let $G=SL(2, \boldsymbol{R})\times PSL(2, \boldsymbol{R})$. Then it is clear that $\mathfrak{g}$ is the Lie algebra of G but σ is not lifted to G.

(1.7) For brevity, we denote the lifting of σ by the same letter. We define $G^\sigma=\{g \in G;\ \sigma(g)=g\}$ and denote by $(G^\sigma)_0$ the identity component of G^σ. We now take a closed subgroup H of G such that $(G^\sigma)_0 \subseteqq H \subseteqq G^\sigma$. Then we define a homogeneous space G/H of G.

Definition. A homogeneous space G/H defined in the above way is called a semisimple symmetric homogeneous space of G. Unless otherwise stated, we call this a symmetric space for brevity.

(1.8) As in the case of Lie algebras, we can define associated and dual symmetric spaces of the given G/H. But in this case, associated and dual symmetric spaces are not uniquely determined because the choices of the closed subgroups whose Lie algebras are $\mathfrak{h}^a$ and $\mathfrak{h}^d$ are not unique. But as to an associated symmetric space of G/H, we can define a standard one (cf. [Ma]). We now construct this.

Let θ be a Cartan involution of G commuting with σ and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ corresponding to θ. Then we can define the maximal compact subgroup K of G whose Lie algebra is $\mathfrak{k}$. It is clear that every element of $K\cap H$ stabilizes $\mathfrak{p}\cap\mathfrak{q}$. This implies that $H^a=(K\cap H)\exp(\mathfrak{p}\cap\mathfrak{q})$ is a closed subgroup of G. By definition, H^a is contained in $G^{\theta\sigma}$ and $\mathfrak{h}^a$ is the Lie algebra of H^a. Hence G/H^a is a symmetric space.

Definition. The symmetric space G/H^a is called the associated symmetric space of G/H.

(1.9) **Examples.**

In this paragraph, we give some examples of symmetric pairs and symmetric spaces.

Example (1.9.1). Let $\mathfrak{g}$ be a compact semisimple Lie algebra and let σ be an involution of $\mathfrak{g}$. Then we define a symmetric pair $(\mathfrak{g}, \mathfrak{h})$ for σ. Let G be a connected Lie group whose Lie algebra is $\mathfrak{g}$. Assume that σ is lifted to G. Then we can define a closed subgroup H of G as we did in (1.7). The pair $(\mathfrak{g}, \mathfrak{h})$ is called a compact symmetric pair and G/H is called a compact symmetric space or a Riemannian symmetric space of the compact type.

Example (1.9.2). Let G be a connected linear semisimple Lie group and let K be a maximal compact subgroup of G. If $\mathfrak{g}$ is the Lie algebra of G and if $\mathfrak{k}$ is that of K, then the involution σ corresponding to the pair $(\mathfrak{g}, \mathfrak{k})$ coincides with the Cartan involution for $\mathfrak{k}$. We call $(\mathfrak{g}, \mathfrak{k})$ and G/K a Riemannian symmetric pair and a Riemannian symmetric space of the non-compact type, respectively. As is easily seen, the dual of $(\mathfrak{g}, \mathfrak{k})$ coincides with $(\mathfrak{g}, \mathfrak{k})$.

Example (1.9.3). Let $(\mathfrak{g}, \mathfrak{k})$ be a Riemannian symmetric pair and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a corresponding direct sum decomposition of $\mathfrak{g}$. We take a maximal abelian subspace $\mathfrak{a}_\mathfrak{p}$ of $\mathfrak{p}$ and denote by $\Sigma(\mathfrak{a}_\mathfrak{p})$ the root system of $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$. Then we can define a symmetric pair $(\mathfrak{g}, \mathfrak{k}_\varepsilon)$ for any signature ε of $\Sigma(\mathfrak{a}_\mathfrak{p})$ in the following way (cf. [O-S]).

Let ε be a signature of roots of $\Sigma(\mathfrak{a}_\mathfrak{p})$. This means that ε is a mapping of $\Sigma(\mathfrak{a}_\mathfrak{p})$ to $\{1, -1\}$ with the following conditions:

$$(1.9.3.1) \qquad \begin{cases} \varepsilon(\alpha+\beta)=\varepsilon(\alpha)\varepsilon(\beta) & \text{if } \alpha,\ \beta,\ \alpha+\beta \in \Sigma(\mathfrak{a}_\mathfrak{p}), \\ \varepsilon(-\alpha)=\varepsilon(\alpha) & \text{for any } \alpha \in \Sigma(\mathfrak{a}_\mathfrak{p}). \end{cases}$$

If θ is the Cartan involution of $\mathfrak{g}$ for $(\mathfrak{g}, \mathfrak{k})$, we can define an involution θ_ε of $\mathfrak{g}$ associated with θ and ε as we did in [O-S]. Namely, $\theta_\varepsilon(X)=\varepsilon(\alpha)\theta(X)$ for $X \in \mathfrak{g}(\mathfrak{a}_\mathfrak{p}; \alpha)$ with $\alpha \in \Sigma(\mathfrak{a}_\mathfrak{p})$ and $\theta_\varepsilon(X)=\theta(X)$ for $X \in Z_\mathfrak{g}(\mathfrak{a}_\mathfrak{p})$. Then we obtain a direct sum decomposition $\mathfrak{g}=\mathfrak{k}_\varepsilon+\mathfrak{p}_\varepsilon$ for θ_ε. The pair $(\mathfrak{g}, \mathfrak{k}_\varepsilon)$ is the required one.

If G is a connected Lie group with the Lie algebra $\mathfrak{g}$, we can choose a (not necessarily connected) closed subgroup K_ε of G whose Lie algebra is $\mathfrak{k}_\varepsilon$ in a standard manner. We have developed a deep analysis on the symmetric space G/K_ε in [O-S].

Example (1.9.4). Let G' be a connected linear semisimple Lie group and put $G=G'\times G'$. Then we define an involution σ of G by $\sigma(g, h)=$

(h, g) $(g, h \in G')$. Putting $H = G^\sigma = \{(g, g) \in G; g \in G'\}$, we obtain a symmetric space G/H. Needless to say, in this case G/H is isomorphic to G' by the correspondence $(g, h)H \to gh^{-1}$. This means that the group G' itself is regarded as a symmetric homogeneous space of the product group $G' \times G'$.

Let $\mathfrak{g}'$ be the Lie algebra of G' and $\mathfrak{h}$ that of H. Then $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}'$ is the Lie algebra of G and $(\mathfrak{g}, \mathfrak{h})$ is the symmetric pair corresponding to G/H. In this case $\mathfrak{q} = \{(X, -X); X \in \mathfrak{g}'\}$. Clearly $\mathfrak{h}$ and $\mathfrak{q}$ are identified with $\mathfrak{g}'$.

Example (1.9.5). Let G and K be as in Example (1.9.2). Let G_C be a complexification of G and let K_C be a connected closed complex subgroup of G_C such that $G \cap K_C$ coincides with K. Then we can define a symmetric space G_C/K_C. This is a dual to the one defined in Example (1.9.4).

(1.10) In the rest of this section, we closely discuss on the irreducible symmetric pairs. First we give a simple lemma.

Lemma (1.10.1). *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a symmetric pair and let* $\mathfrak{k}$ *be a maximal compact subalgebra of* $\mathfrak{g}$. *Assume that* $\mathfrak{k}_C$ *and* $\mathfrak{h}_C$ *are isomorphic. Then* $(\mathfrak{g}, \mathfrak{h})$ *is self-dual. Moreover, in this case,* $(\mathfrak{g}^{ad}, \mathfrak{h}^{ad})$ *is self-associated.*

Proof. We may take $\mathfrak{k}$ so that the involutions for the pairs $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}, \mathfrak{k})$ are commutative (cf. (1.2)). Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q} = \mathfrak{k} + \mathfrak{p}$ be the direct sum decompositions of $\mathfrak{g}$. Then by definition, $\mathfrak{k}^d = (\mathfrak{k} \cap \mathfrak{h}) + \sqrt{-1}(\mathfrak{p} \cap \mathfrak{h})$ is a maximal compact subalgebra of $\mathfrak{g}^d$. Since $\mathfrak{k}^d$ is a compact real form of $\mathfrak{h}_C$, it follows from the assumption that $\mathfrak{k} \simeq \mathfrak{k}^d$. Hence due to [He 2, Ch. X, Th. 6.2], we conclude that $\mathfrak{g}^d \simeq \mathfrak{g}$.

Next we show that $\mathfrak{h}^d \simeq \mathfrak{h}$. Since $\mathfrak{h}^d = (\mathfrak{k} \cap \mathfrak{h}) + \sqrt{-1}(\mathfrak{k} \cap \mathfrak{q})$, due to the assumption we find that the maximal compact subalgebras of $\mathfrak{h}$ and $\mathfrak{h}^d$ are isomorphic. This combined with [He 2, Ch. X, Th. 6.2] implies that $\mathfrak{h}^d \simeq \mathfrak{h}$.

We have thus proved that $(\mathfrak{g}, \mathfrak{h})^d$ is isomorphic to $(\mathfrak{g}, \mathfrak{h})$.

Last we show that $(\mathfrak{g}, \mathfrak{h})^{ad}$ is self-associated. The above discussion implies that $(\mathfrak{g}, \mathfrak{h})^{da} \simeq (\mathfrak{g}, \mathfrak{h})^a$ and $(\mathfrak{g}, \mathfrak{h})^{dad} \simeq (\mathfrak{g}, \mathfrak{h})^{ad}$. Now we remember that $(\mathfrak{g}, \mathfrak{h})^{dad} \simeq (\mathfrak{g}, \mathfrak{h})^{ada}$ (cf. (1.4)). Hence $(\mathfrak{g}, \mathfrak{h})^{ad} \simeq (\mathfrak{g}, \mathfrak{h})^{ada}$. This means that $(\mathfrak{g}, \mathfrak{h})^{ad}$ is self-associated. q.e.d.

(1.11) In [O-S], we defined a symmetric pair $(\mathfrak{g}, \mathfrak{k}_\varepsilon)$ by using the notation there (cf. Example (1.9.3)). Since $\mathfrak{k}_C \simeq (\mathfrak{k}_\varepsilon)_C$, Lemma (1.10.1) implies that $(\mathfrak{g}, \mathfrak{k}_\varepsilon)$ is always self-dual. We consider $(\mathfrak{g}, \mathfrak{k}_\varepsilon)^a$ and $(\mathfrak{g}, \mathfrak{k}_\varepsilon)^{ad}$. Comparing the classification in [Be] with that in [O-S], we observe that a good many symmetric pairs are obtained in this manner. We now assume that

$\mathfrak{g}$ is real simple. Then the properties of the three pairs $(\mathfrak{g}, \mathfrak{k}_\varepsilon)$, $(\mathfrak{g}, \mathfrak{k}_\varepsilon)^a$, $(\mathfrak{g}, \mathfrak{k}_\varepsilon)^{ad}$ become different according as the complexification of $\mathfrak{g}$ is simple or not. Taking this into account, we are going to determine the dual and associated pairs of a given one. For this purpose, it is preferable to decompose the symmetric pairs into some classes. First we treat the pairs of the forms $(\mathfrak{g}, \mathfrak{k}_\varepsilon)$ in (1.12). Next we do those of the forms $(\mathfrak{g}_C, \mathfrak{g})$ in (1.13). Most pairs of the forms $(\mathfrak{g}_C, \mathfrak{g})$ are reduced to the previous ones. But in this case, the dual and associated pairs are very explicitly decided. By this reason, we distinguish those from the previous ones. Last we treat the pairs in (1.14)–(1.16) which are not obtained from the previous two cases.

(1.12) *Type* $(\mathfrak{k}_\varepsilon)$: $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g}$ is real semisimple and $\mathfrak{h}$ is isomorphic to a subalgebra $\mathfrak{k}_\varepsilon$ of $\mathfrak{g}$ defined as in Example (1.9.3).

We consider an irreducible symmetric pair $(\mathfrak{g}, \mathfrak{h})$ or Type $(\mathfrak{k}_\varepsilon)$. We first note that such a pair is completely classified (cf. [O-S, Appendix]). As is noted in (1.11), $(\mathfrak{g}, \mathfrak{h})$ is self-dual in this case. Hence due to (1.4.2), we obtain the following diagram:

$$\begin{tikzcd} (\mathfrak{g}, \mathfrak{h}) \arrow[loop below, "\text{dual}"] \arrow[r, leftrightarrow, "\text{associated}"] & (\mathfrak{g}, \mathfrak{h}^a) \arrow[r, leftrightarrow, "\text{dual}"] & (\mathfrak{g}^{ad}, \mathfrak{h}) \arrow[loop below, "\text{associated}"] \end{tikzcd}$$

Therefore if $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{h}^a$ and $\mathfrak{g}^{ad}$ are given, we can easily determine the three pairs $(\mathfrak{g}, \mathfrak{h})$, $(\mathfrak{g}^a, \mathfrak{h}^a)$, $(\mathfrak{g}^{ad}, \mathfrak{h}^{ad})$. Here we use the relations $(\mathfrak{g}^{ada}, \mathfrak{h}^{ada}) \simeq (\mathfrak{g}^{ad}, \mathfrak{h}^{ad})$ and $\mathfrak{h}^{ada} = \mathfrak{h}$. If $(\mathfrak{g}, \mathfrak{h})$ is a Riemannian symmetric pair of the non-compact type, it follows from the definition that $(\mathfrak{g}, \mathfrak{h})^a = (\mathfrak{g}, \mathfrak{g})$ and $(\mathfrak{g}, \mathfrak{h})^{ad}$ coincides with the compact dual of $(\mathfrak{g}, \mathfrak{h})$. As to these pairs, the reader is refered to [He 1]. On the other hand, we will treat in (1.13) such a pair $(\mathfrak{g}, \mathfrak{h})$ that $\mathfrak{g}$ itself is complex simple. Thus let $(\mathfrak{g}, \mathfrak{h})$ be an irreducible symmetric pair of Type $(\mathfrak{k}_\varepsilon)$, where $\mathfrak{g}_C$ is simple. For such a pair $(\mathfrak{g}, \mathfrak{h})$, we shall give in Table I the complete informations on the Lie algebras $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{h}^a$ and $\mathfrak{g}^{ad}$.

Remark. In Table I, $\mathfrak{t}$ denotes $\sqrt{-1}\boldsymbol{R}$, the Lie algebra of one dimensional compact torus. Other notation follow [He 1] and [O-S, Appendix].

(1.13) *Type* (C, R): $(\mathfrak{g}_C, \mathfrak{g})$, where $\mathfrak{g}$ is a real semisimple Lie algebra and $\mathfrak{g}_C$ is the complexification of $\mathfrak{g}$.

First we give a simple lemma.

Lemma (1.13.1).

(1) $(\mathfrak{g}_C, \mathfrak{g})$ *is self-dual.*

Table I. $(\mathfrak{g}, \mathfrak{h})$: Type $(\mathfrak{k}_\varepsilon)$ and $\mathfrak{g}_C$ is simple

Symbol	$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{h}^a$	$\mathfrak{g}^{ad}$
$A^1_{l,j}(2j \leqq l+1)$	$\mathfrak{sl}(l+1, \boldsymbol{R})$	$\mathfrak{so}(l+1-j, j)$	$\mathfrak{sl}(l+1-j, \boldsymbol{R}) + \mathfrak{sl}(j, \boldsymbol{R}) + \boldsymbol{R}$	$\mathfrak{su}(l+1-j, j)$
$A^4_{l,j}(2j \leqq l+1)$	$\mathfrak{su}^*(2l+2)$	$\mathfrak{sp}(l+1-j, j)$	$\mathfrak{su}^*(2l+2-2j) + \mathfrak{su}^*(2j) + \boldsymbol{R}$	$\mathfrak{su}(2l+2-2j, 2j)$
$BC^{2m,2,1}_{l,j}(j \leqq l)$	$\mathfrak{su}(l+m, l)$	$\mathfrak{su}(l+m-j, j) + \mathfrak{su}(l-j, j) + \mathfrak{t}$	$\mathfrak{su}(l+m-j, l-j) + \mathfrak{su}(j, j) + \mathfrak{t}$	$\mathfrak{su}(2l+m-2j, 2j)$
$C^{2,1}_{l,j}(2j \leqq l)$	$\mathfrak{su}(l, l)$	$\mathfrak{su}(l-j, j) + \mathfrak{su}(l-j, j) + \mathfrak{t}$	$\mathfrak{su}(l-j, l-j) + \mathfrak{su}(j, j) + \mathfrak{t}$	$\mathfrak{su}(2l-2j, 2j)$
$C^{2,1}_{l,A}$	$\mathfrak{su}(l, l)$	$\mathfrak{sl}(l, \boldsymbol{C}) + \boldsymbol{R}$	$\mathfrak{h}$	$\mathfrak{g}$
$B^{m,1}_{l,j}$	$\mathfrak{so}(l+m, l)$	$\mathfrak{so}(l+m-j, j) + \mathfrak{so}(l-j, j)$	$\mathfrak{so}(l+m-j, l-j) + \mathfrak{so}(j, j)$	$\mathfrak{so}(2l+m-2j, 2j)$
$D^1_{l,j}(2j \leqq l)$	$\mathfrak{so}(l, l)$	$\mathfrak{so}(l-j, j) + \mathfrak{so}(l-j, j)$	$\mathfrak{so}(l-j, l-j) + \mathfrak{so}(j, j)$	$\mathfrak{so}(2l-2j, 2j)$
$D^1_{l,A}$	$\mathfrak{so}(l, l)$	$\mathfrak{su}(l, \boldsymbol{C})$	$\mathfrak{sl}(l, \boldsymbol{R}) + \boldsymbol{R}$	$\mathfrak{so}^*(2l)$
$C^{4,1}_{l,j}(2j \leqq l)$	$\mathfrak{so}^*(4l)$	$\mathfrak{su}(2l-2j, 2j) + \mathfrak{t}$	$\mathfrak{so}^*(4l-4j) + \mathfrak{so}^*(4j)$	$\mathfrak{so}(4l-4j, 4j)$
$C^{4,1}_{l,A}$	$\mathfrak{so}^*(4l)$	$\mathfrak{su}^*(2l) + \boldsymbol{R}$	$\mathfrak{h}$	$\mathfrak{g}$
$BC^{4,4,1}_{l,j}$	$\mathfrak{so}^*(4l+2)$	$\mathfrak{su}(2l+1-2j, 2j) + \mathfrak{t}$	$\mathfrak{so}^*(4l+2-4j) + \mathfrak{so}^*(4j)$	$\mathfrak{so}(4l+2-4j, 4j)$
$C^{1,1}_{l,j}(2j \leqq l)$	$\mathfrak{sp}(l, \boldsymbol{R})$	$\mathfrak{su}(l-j, j) + \mathfrak{t}$	$\mathfrak{sp}(l-j, \boldsymbol{R}) + \mathfrak{sp}(j, \boldsymbol{R})$	$\mathfrak{sp}(l-j, j)$
$C^{1,1}_{l,A}$	$\mathfrak{sp}(l, \boldsymbol{R})$	$\mathfrak{sl}(l, \boldsymbol{R}) + \boldsymbol{R}$	$\mathfrak{h}$	$\mathfrak{g}$
$BC^{4m,4,3}_{l,j}$	$\mathfrak{sp}(l+m, l)$	$\mathfrak{sp}(l+m-j, j) + \mathfrak{sp}(l-j, j)$	$\mathfrak{sp}(l+m-j, l-j) + \mathfrak{sp}(j, j)$	$\mathfrak{sp}(2l+m-2j, 2j)$
$C^{4,3}_{l,j}(2j \leqq l)$	$\mathfrak{sp}(l, l)$	$\mathfrak{sp}(l-j, j) + \mathfrak{sp}(l-j, j)$	$\mathfrak{sp}(l-j, l-j) + \mathfrak{sp}(j, j)$	$\mathfrak{sp}(2l-2j, 2j)$
$C^{4,3}_{l,A}$	$\mathfrak{sp}(l, l)$	$\mathfrak{sp}(l, \boldsymbol{C})$	$\mathfrak{su}^*(2l) + \boldsymbol{R}$	$\mathfrak{sp}(2l, \boldsymbol{R})$
$E^1_{6,A}$	$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(4, \boldsymbol{R})$	$\mathfrak{sl}(6, \boldsymbol{R}) + \mathfrak{sl}(2, \boldsymbol{R})$	$\mathfrak{e}_{6(2)}$
$E^1_{6,D}$	$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(2, 2)$	$\mathfrak{so}(5, 5) + \boldsymbol{R}$	$\mathfrak{e}_{6(-14)}$
$F^{2,1}_{4,B}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(4, 2) + \mathfrak{su}(2)$	$\mathfrak{so}(6, 4) + \mathfrak{t}$	$\mathfrak{e}_{6(-14)}$
$F^{2,1}_{4,C}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(3, 3) + \mathfrak{sl}(2, \boldsymbol{R})$	$\mathfrak{h}$	$\mathfrak{g}$
$BC^{8,6,1}_{2,A}$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}^*(10) + \mathfrak{t}$	$\mathfrak{su}(5, 1) + \mathfrak{sl}(2, \boldsymbol{R})$	$\mathfrak{e}_{6(2)}$

(Continued from Table I)

Symbol	$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{h}^a$	$\mathfrak{g}^{ad}$
$BC_{2,B}^{8,6,1}$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(8, 2)+\mathfrak{t}$	$\mathfrak{h}$	$\mathfrak{g}$
$A_{2,A}^{8}$	$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(9, 1)+\boldsymbol{R}$	$\mathfrak{e}_{6(-14)}$
$E_{7,A}^{1}$	$\mathfrak{e}_{7(7)}$	$\mathfrak{sl}(8, \boldsymbol{R})$	$\mathfrak{h}$	$\mathfrak{g}$
$E_{7,D}^{1}$	$\mathfrak{e}_{7(7)}$	$\mathfrak{su}(4, 4)$	$\mathfrak{so}(6, 6)+\mathfrak{sl}(2, \boldsymbol{R})$	$\mathfrak{e}_{7(-5)}$
$E_{7,E}^{1}$	$\mathfrak{e}_{7(7)}$	$\mathfrak{su}^*(8)$	$\mathfrak{e}_{6(6)}+\boldsymbol{R}$	$\mathfrak{e}_{7(-25)}$
$F_{4,B}^{4,1}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}(8, 4)+\mathfrak{su}(2)$	$\mathfrak{h}$	$\mathfrak{g}$
$F_{4,C}^{4,1}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}^*(12)+\mathfrak{sl}(2, \boldsymbol{R})$	$\mathfrak{h}$	$\mathfrak{g}$
$C_{3,A}^{8,1}$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)}+\boldsymbol{R}$	$\mathfrak{h}$	$\mathfrak{g}$
$C_{3,B}^{8,1}$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-14)}+\mathfrak{t}$	$\mathfrak{so}(10, 2)+\mathfrak{sl}(2, \boldsymbol{R})$	$\mathfrak{e}_{7(-5)}$
$E_{8,D}^{1}$	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(8, 8)$	$\mathfrak{h}$	$\mathfrak{g}$
$E_{8,E}^{1}$	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}^*(16)$	$\mathfrak{e}_{7(7)}+\mathfrak{sl}(2, \boldsymbol{R})$	$\mathfrak{e}_{8(-24)}$
$F_{4,B}^{8,1}$	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_{7(-5)}+\mathfrak{su}(2)$	$\mathfrak{so}(12, 4)$	$\mathfrak{e}_{8(8)}$
$F_{4,C}^{8,1}$	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_{7(-25)}+\mathfrak{sl}(2, \boldsymbol{R})$	$\mathfrak{h}$	$\mathfrak{g}$
$F_{4,B}^{1}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(2, 1)+\mathfrak{su}(2)$	$\mathfrak{so}(5, 4)$	$\mathfrak{f}_{4(-20)}$
$F_{4,C}^{1}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(3, \boldsymbol{R})+\mathfrak{sl}(2, \boldsymbol{R})$	$\mathfrak{h}$	$\mathfrak{g}$
$BC_{1,A}^{8,7}$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(8, 1)$	$\mathfrak{h}$	$\mathfrak{g}$
G_2^1	$\mathfrak{g}_{2(2)}$	$\mathfrak{sl}(2, \boldsymbol{R})+\mathfrak{sl}(2, \boldsymbol{R})$	$\mathfrak{h}$	$\mathfrak{g}$

(2) *Let* $\mathfrak{k}$ *be a maximal compact subalgebra of* $\mathfrak{g}$ *and let* $\mathfrak{k}_C$ *be its complexification. Then* $(\mathfrak{g}_C, \mathfrak{k}_C)$ *is associated to* $(\mathfrak{g}_C, \mathfrak{g})$.

(3) $(\mathfrak{g}\oplus\mathfrak{g}, \mathfrak{g})$ *is dual to* $(\mathfrak{g}_C, \mathfrak{k}_C)$. *Here we identify* $\mathfrak{g}$ *with the diagonal subalgebra* $\{(X, X); X \in \mathfrak{g}\}$ *of* $\mathfrak{g}\oplus\mathfrak{g}$.

(4) $(\mathfrak{g}\oplus\mathfrak{g}, \mathfrak{g})$ *is self associated.*

We give a real simple Lie algebra $\mathfrak{g}$. Then Lemma (1.10.1) and Lemma (1.13.1) imply the following diagram.

$$\begin{array}{ccc} (\mathfrak{g}_C, \mathfrak{g}) \xleftrightarrow{\text{associated}} (\mathfrak{g}_C, \mathfrak{k}_C) & \xleftrightarrow{\text{dual}} & (\mathfrak{g}\oplus\mathfrak{g}, \mathfrak{g}) \\ \circlearrowleft & & \circlearrowleft \\ \text{dual} & & \text{associated} \end{array}$$

As is noted in [O-S, Appendix], the pair $(\mathfrak{g}_C, \mathfrak{g})$ is reduced to the one of Type $(\mathfrak{k}_\varepsilon)$ if and only if $\mathfrak{g}$ has a compact Cartan subalgebra.

There are a few symmetric pairs which are not obtained in the above procedure. We now describe these pairs.

(1.14) We consider the following symmetric pairs.

Type	$\mathfrak{g}$	$\mathfrak{h}$
$A^{(p,q)}_{(i,j)(p-i,\,q-j)}$	$\mathfrak{su}(p, q)$	$\mathfrak{su}(i, j)+\mathfrak{su}(p-i, q-j)+\sqrt{-1}\boldsymbol{R}$
$BD^{(p,q)}_{(i,j)(p-i,\,q-j)}$	$\mathfrak{so}(p, q)$	$\mathfrak{so}(i, j)+\mathfrak{so}(p-i, q-j)$
$C^{(p,q)}_{(i,j)(p-i,\,q-j)}$	$\mathfrak{sp}(p, q)$	$\mathfrak{sp}(i, j)+\mathfrak{sp}(p-i, q-j)$

$(i\leqq q\leqq p,\ i, j\geqq 1,\ 2(i+j)\leqq p+q)$

If X denotes one of A, BD or C, we find that the following relation holds:

$$X^{(i+j,\,p+q-i-j)}_{(i,\,p-i)(j,\,q-j)} \xleftrightarrow{\text{dual}} X^{(p,q)}_{(i,j)(p-i,\,q-j)} \xleftrightarrow{\text{associated}} X^{(p,q)}_{(i,\,q-j)(p-i,\,j)}$$

If the pair of type $X^{(p,q)}_{(i,j)(p-i,\,q-j)}$ is self-dual, then $p=i+j$ or $q=i+j$ and in this case the pair is reduced to the one of Type $(\mathfrak{k}_\varepsilon)$.

(1.15) Next we find that if $l\geqq 2$ and if i is an odd number such that $1\leqq i\leqq l$, the following relation holds:

$$\begin{array}{l} (\mathfrak{so}^*(4l),\ \mathfrak{su}(2l-i, i)+\sqrt{-1}\boldsymbol{R}) \circlearrowleft \text{dual} \\ \quad\Big\updownarrow \text{associated} \\ (\mathfrak{so}^*(4l),\ \mathfrak{so}^*(4l-2i)+\mathfrak{so}^*(2i)) \\ \quad\Big\updownarrow \text{dual} \\ (\mathfrak{so}(4l-2i, 2i),\ \mathfrak{su}(2l-i, i)+\sqrt{-1}\boldsymbol{R}) \circlearrowleft \text{associated} \end{array}$$

We note here that the case when i is even, the pair $(\mathfrak{so}^*(4l), \mathfrak{su}(2l-i, i)+\sqrt{-1}\boldsymbol{R})$ is reduced to the one of Type $(\mathfrak{k}_\varepsilon)$ (cf. [O-S, Appendix]).

Remark ([O-S], p. 79]). Let $(\mathfrak{g}, \mathfrak{h})$ be an irreducible symmetric pair. Let $\mathfrak{k}$ be a maximal compact subalgebra of $\mathfrak{g}$. Assume that $\mathfrak{g}_C$ is simple and $\mathfrak{h}_C \simeq \mathfrak{k}_C$. Then $(\mathfrak{g}, \mathfrak{h})$ is not of Type $(\mathfrak{k}_\varepsilon)$ if and only if $(\mathfrak{g}, \mathfrak{h})$ is isomorphic to $(\mathfrak{so}^*(4l), \mathfrak{su}(2l-i, i)+\sqrt{-1}\boldsymbol{R})$ for an odd integer i.

(1.16) Last we consider the following Lie algebras.

$\mathfrak{g}$	$\mathfrak{g}^d$	$\mathfrak{g}^{ad}$	$\mathfrak{h}$	$\mathfrak{h}^a$	$\mathfrak{h}^d$
$\mathfrak{sl}(2l, \boldsymbol{R})$	$\mathfrak{su}^*(2l)$	$\mathfrak{su}(l, l)$	$\mathfrak{sp}(l, \boldsymbol{R})$	$\mathfrak{sl}(l, \boldsymbol{C})+\sqrt{-1}\boldsymbol{R}$	$\mathfrak{so}^*(2l)$
$\mathfrak{e}_{6(6)}$	$\mathfrak{e}_{6(-26)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{su}^*(6)+\mathfrak{su}(2)$	$\mathfrak{sp}(3, 1)$
$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{6(2)}+\sqrt{-1}\boldsymbol{R}$	$\mathfrak{so}^*(12)+\mathfrak{su}(2)$	$\mathfrak{su}(6, 2)$

Then we find that the following relation holds. We stress that any two of the six pairs below are not isomorphic to each other.

$$\begin{tikzcd}
(\mathfrak{g}, \mathfrak{h}) \arrow[r, leftrightarrow, "\text{associated}"] \arrow[d, leftrightarrow, "\text{dual}"] & (\mathfrak{g}, \mathfrak{h}^a) \arrow[r, leftrightarrow, "\text{dual}"] & (\mathfrak{g}^{ad}, \mathfrak{h}^d) \arrow[d, leftrightarrow, "\text{associated}"] \\
(\mathfrak{g}^d, \mathfrak{h}^d) \arrow[r, leftrightarrow, "\text{associated}"'] & (\mathfrak{g}^d, \mathfrak{h}^a) \arrow[r, leftrightarrow, "\text{dual}"'] & (\mathfrak{g}^{ad}, \mathfrak{h})
\end{tikzcd}$$

§ 2. The restricted root system of a symmetric pair

(2.1) Retain the notation in Section 1. Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. In this section, we define the root subspaces of $\mathfrak{g}$ with respect to $\mathfrak{a}$ and examine their elementary properties. In particular we shall show in Theorem (2.11) that the totality of the roots with respect to $(\mathfrak{g}, \mathfrak{a})$ becomes a root system. We call this the restricted root system of the symmetric pair $(\mathfrak{g}, \mathfrak{h})$.

Lemma (2.2).

(i) *If $\mathfrak{a}_\mathfrak{p}$ is a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}$, then $\mathfrak{a}_\mathfrak{p}$ is σ-stable.*

(ii) *If $\mathfrak{j}$ is a maximal abelian subspace of $\mathfrak{q}$ containing $\mathfrak{a}$, then $\mathfrak{j}$ is θ-stable and consists of only semisimple elements of $\mathfrak{g}$.*

Proof. (i) Let X be an element of $\mathfrak{a}_\mathfrak{p}$. Then we put $X=X_1+X_2$ with $X_1 \in \mathfrak{h}$, $X_2 \in \mathfrak{q}$. Since $\mathfrak{h}$ and $\mathfrak{q}$ are θ-stable, we find that $X_1 \in \mathfrak{h} \cap \mathfrak{p}$ and $X_2 \in \mathfrak{p} \cap \mathfrak{q}$. It follows from the assumption that $[X_1, Y]+[X_2, Y]=[X, Y]=0$ for any $Y \in \mathfrak{a}$. But $[X_1, Y] \in \mathfrak{q}$ and $[X_2, Y] \in \mathfrak{h}$. Thus we have $[X_i, Y]=0$ $(i=1, 2)$. In particular $X_2 \in \mathfrak{p} \cap \mathfrak{q}$ and commutes with $\mathfrak{a}$. Therefore the assumption implies that $X_2 \in \mathfrak{a}$. Then $X_1=X-X_2 \in \mathfrak{a}_\mathfrak{p}$. Hence both X_1 and X_2 are contained in $\mathfrak{a}_\mathfrak{p}$. Then $\sigma(X)=\sigma(X_1)+\sigma(X_2)=X_1-X_2 \in \mathfrak{a}_\mathfrak{p}$.

(ii) We can prove that $\mathfrak{j}$ is θ-stable by an argument similar to the one in the proof of (i). Hence it suffices to show that each element of $\mathfrak{j}$

is semisimple. First we note that $\mathfrak{j}=\mathfrak{k}\cap\mathfrak{j}+\mathfrak{p}\cap\mathfrak{j}$ is a direct sum decomposition. Let X be an element of $\mathfrak{j}$. Then there are $X_1\in\mathfrak{k}\cap\mathfrak{j}$ and $X_2\in\mathfrak{p}\cap\mathfrak{j}$ such that $X=X_1+X_2$. By definition, we find that both X_1 and X_2 are semisimple and $[X_1, X_2]=0$. This implies that X is semisimple. q.e.d.

Remark (2.3). It is widely known that any maximal abelian subspace of $\mathfrak{p}$ consists of only semisimple elements of $\mathfrak{g}$. But the claim for $\mathfrak{q}$ similar to this one does not hold in general. Namely there exists a symmetric pair $(\mathfrak{g}, \mathfrak{h})$ and a maximal abelian subspace $\mathfrak{b}$ of $\mathfrak{q}$ such that $\mathfrak{b}$ contains an element which is not semisimple. We give here a simple example.

Let $\mathfrak{g}=\mathfrak{sl}(2, \boldsymbol{R})$ and define an involution σ of $\mathfrak{g}$ by

$$\sigma(X)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}X\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad (\forall X\in\mathfrak{g}).$$

Then $\mathfrak{q}=\left\{\begin{pmatrix}0 & x\\ y & 0\end{pmatrix};\ x, y\in\boldsymbol{R}\right\}$. If we take $\mathfrak{b}=\left\{\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix};\ x\in\boldsymbol{R}\right\}$, then $\mathfrak{b}$ is a maximal abelian subspace of $\mathfrak{q}$ but consists of only nilpotent elements of $\mathfrak{g}$.

Lemma (2.4). (i) *Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$. If $\mathfrak{a}_{\mathfrak{p}}$ (resp. $\mathfrak{j}$) is a maximal abelian subspace of $\mathfrak{p}$ (resp. $\mathfrak{q}$) containing $\mathfrak{a}$, then* $[\mathfrak{a}_{\mathfrak{p}}, \mathfrak{j}]=0$.

(ii) *Let $\bar{\mathfrak{j}}$ and $\bar{\mathfrak{j}}'$ be Cartan subalgebras of $\mathfrak{g}$ such that each of $\bar{\mathfrak{j}}$ and $\bar{\mathfrak{j}}'$ contains maximal abelian subspaces of $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{p}\cap\mathfrak{q}$. Then they are conjugate under the action of $K\cap(G^{\sigma})_0$.*

Proof. First prove (i). It follows from Lemma (2.2) that $\mathfrak{a}_{\mathfrak{p}}$ is σ-stable and $\mathfrak{j}$ is θ-stable. Hence to prove the lemma, it suffices to show that $[\mathfrak{a}_{\mathfrak{p}}\cap\mathfrak{h}, \mathfrak{j}\cap\mathfrak{k}]=0$. Let $X\in\mathfrak{a}_{\mathfrak{p}}\cap\mathfrak{h}$ and $Y\in\mathfrak{j}\cap\mathfrak{k}$. Then it is clear that $[X, Y]$ is contained in $\mathfrak{p}\cap\mathfrak{q}$ and commutes with $\mathfrak{a}$. Since $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}\cap\mathfrak{q}$, we find that $[X, Y]\in\mathfrak{a}$. But for any $Z\in\mathfrak{a}$, we have $\langle[X, Y], Z\rangle=\langle X, [Y, Z]\rangle=0$. Hence $[X, Y]=0$.

Next prove (ii). For brevity, we put $\mathfrak{a}_{\mathfrak{p}}=\bar{\mathfrak{j}}\cap\mathfrak{p}$, $\mathfrak{j}=\bar{\mathfrak{j}}\cap\mathfrak{q}$, $\mathfrak{a}=\bar{\mathfrak{j}}\cap\mathfrak{p}\cap\mathfrak{q}$, $\mathfrak{a}'_{\mathfrak{p}}=\bar{\mathfrak{j}}'\cap\mathfrak{p}$, $\mathfrak{j}'=\bar{\mathfrak{j}}'\cap\mathfrak{q}$, $\mathfrak{a}'=\bar{\mathfrak{j}}'\cap\mathfrak{p}\cap\mathfrak{q}$ and $L=K\cap(G^{\sigma})_0$. Since $\mathfrak{h}^{a}=\mathfrak{h}\cap\mathfrak{k}+\mathfrak{p}\cap\mathfrak{q}$ is a Cartan decomposition and since $\mathfrak{a}$ and $\mathfrak{a}'$ are maximal abelian subspaces of $\mathfrak{p}\cap\mathfrak{q}$, we find that $\mathfrak{a}$ and $\mathfrak{a}'$ are L-conjugate. Hence we may assume that $\mathfrak{a}=\mathfrak{a}'$. Let $\mathfrak{z}_{\mathfrak{h}}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{h}$. Then it follows from the definition that $\mathfrak{a}_{\mathfrak{p}}\cap\mathfrak{h}$ and $\mathfrak{a}'_{\mathfrak{p}}\cap\mathfrak{h}$ are maximal abelian subspaces of $\mathfrak{z}_{\mathfrak{h}}\cap\mathfrak{p}$. Hence we also find that $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{a}'_{\mathfrak{p}}$ are L-conjugate. Then we may assume that $\mathfrak{a}_{\mathfrak{p}}=\mathfrak{a}'_{\mathfrak{p}}$. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}_{\mathfrak{p}}$ in $\mathfrak{k}$ and let $\mathfrak{m}=\mathfrak{m}_{+}+\mathfrak{m}_{-}$ be the direct sum decomposition for the involution σ. Then it follows that $\mathfrak{j}\cap\mathfrak{k}$ and $\mathfrak{j}'\cap\mathfrak{k}$ are maximal abelian subspaces of $\mathfrak{m}_{-}$. Since $(\mathfrak{m}, \mathfrak{m}_{+})$ is a symmetric pair of the compact type, we find that $\mathfrak{j}$ and $\mathfrak{j}'$ are L-

conjugate. Hence we may also assume that $\mathfrak{j}=\mathfrak{j}'$. Last we consider the centralizer $\mathfrak{z}$ of $\mathfrak{a}_\mathfrak{p}+\mathfrak{j}$ in $\mathfrak{g}$. Since the semisimple part of $\mathfrak{z}$ is compact and since both $\tilde{\mathfrak{j}}$ and $\tilde{\mathfrak{j}}'$ are Cartan subalgebras of $\mathfrak{z}$, we easily conclude that $\tilde{\mathfrak{j}}$ and $\tilde{\mathfrak{j}}'$ are L-conjugate. q.e.d.

(2.5) We take $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{j}$ which satisfy the conditions in Lemma (2.4). Let $\tilde{\mathfrak{j}}$ be a Cartan subalgebra of $\mathfrak{g}$ containing both $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{j}$. We have shown in Lemma (2.4) that such a Cartan subalgebra exists uniquely up to a conjugation of $K\cap(G^\sigma)_0$. We fix $\tilde{\mathfrak{j}}$, $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{j}$ from now on. It follows from the definition that $\mathfrak{j}$ contains a maximal abelian subspace of $\mathfrak{m}\cap\mathfrak{q}$, where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{k}$.

Using $\mathfrak{j}$ and $\mathfrak{a}$, we define the rank and the split rank of the pair $(\mathfrak{g}, \mathfrak{h})$.

Definition (2.5.1). We call $l'=\dim\mathfrak{j}$ and $l=\dim\mathfrak{a}$ the rank and the split rank of the symmetric pair $(\mathfrak{g}, \mathfrak{h})$, respectively.

If $\mathfrak{c}$ is a real reductive Lie algebra, we denote by $r(\mathfrak{c})$ the real rank of $\mathfrak{c}$. Then for a given symmetric pair $(\mathfrak{g}, \mathfrak{h})$, the rank and the split rank of $(\mathfrak{g}, \mathfrak{h})$ are $r(\mathfrak{g}^d)$ and $r(\mathfrak{h}^a)$, respectively. Noting this, we can easily determine the rank and the split rank for each pair appeared in the diagram (1.4.2). The results are summarized in the following table.

Table (2.5.2)

	rank	split rank
$(\mathfrak{g}, \mathfrak{h})$	$r(\mathfrak{g}^d)$	$r(\mathfrak{h}^a)$
$(\mathfrak{g}, \mathfrak{h})^a$	$r(\mathfrak{g}^{ad})$	$r(\mathfrak{h})$
$(\mathfrak{g}, \mathfrak{h})^{ad}$	$r(\mathfrak{g})$	$r(\mathfrak{h})$
$(\mathfrak{g}, \mathfrak{h})^{ada}$	$r(\mathfrak{g}^d)$	$r(\mathfrak{h}^d)$
$(\mathfrak{g}, \mathfrak{h})^{da}$	$r(\mathfrak{g}^{ad})$	$r(\mathfrak{h}^d)$
$(\mathfrak{g}, \mathfrak{h})^d$	$r(\mathfrak{g})$	$r(\mathfrak{h}^a)$

Remark (2.5.3). If the pair $(\mathfrak{g}, \mathfrak{h})$ is the one defined as in Example (1.9.4). Use the notation there. Then l' and l coincide with the rank and the real rank of the Lie algebra $\mathfrak{g}'$, respectively.

(2.6) If $\tilde{\mathfrak{a}}$ is a θ-stable linear subspace of $\tilde{\mathfrak{j}}$, we denote by $\tilde{\mathfrak{a}}^*$ the dual of $\tilde{\mathfrak{a}}$ and by $\tilde{\mathfrak{a}}_C$ (resp. $\tilde{\mathfrak{a}}_C^*$) a complexification of $\tilde{\mathfrak{a}}$ (resp. $\tilde{\mathfrak{a}}^*$). For any element λ of $\tilde{\mathfrak{a}}_C^*$, we put

$$\mathfrak{g}_C(\tilde{\mathfrak{a}};\lambda)=\{X\in\mathfrak{g}_C;\ [Y,X]=\lambda(Y)X\ (\forall Y\in\tilde{\mathfrak{a}})\}$$
$$\mathfrak{g}(\tilde{\mathfrak{a}};\lambda)=\mathfrak{g}_C(\tilde{\mathfrak{a}};\lambda)\cap\mathfrak{g}.$$

An element $\lambda \neq 0$ of $\tilde{\mathfrak{a}}_C^*$ is called a root of $(\mathfrak{g}, \tilde{\mathfrak{a}})$ if and only if $\mathfrak{g}_C(\tilde{\mathfrak{a}}; \lambda) \neq 0$ and we denote by $\Sigma(\tilde{\mathfrak{a}})$ the totality of the roots of $(\mathfrak{g}, \tilde{\mathfrak{a}})$. By the Killing form $\langle\ ,\ \rangle$ on $\mathfrak{g}_C$, we always identify $\tilde{\mathfrak{a}}_C^*$ with $\tilde{\mathfrak{a}}_C$. In particular we may regard $\tilde{\mathfrak{a}}_C^*$ as a linear subspace of $\tilde{\mathfrak{j}}_C^*$. It is clear from the definition that $\Sigma(\mathfrak{a}_\mathfrak{p})$ and $\Sigma(\mathfrak{j})$ are the root systems of the Riemannian symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{g}^a, \mathfrak{k}^a)$, respectively.

In the following, we give some basic lemmas on the root subspaces of $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$.

Lemma (2.7). *Let λ be an element of $\Sigma(\mathfrak{a}_\mathfrak{p})$ and assume that $\lambda | \mathfrak{a} = 0$. Then $\mathfrak{g}(\mathfrak{a}_\mathfrak{p}; \lambda)$ is contained in $\mathfrak{h}$.*

Proof. Let X be an element of $\mathfrak{g}(\mathfrak{a}_\mathfrak{p}; \lambda)$. Since $\sigma(\lambda) = \lambda$, it follows that $X - \sigma(X)$ is also contained in $\mathfrak{g}(\mathfrak{a}_\mathfrak{p}; \lambda)$. On the other hand, $Z = (X - \sigma(X)) - \theta(X - \sigma(X))$ is contained in $\mathfrak{p} \cap \mathfrak{q}$ and the assumption implies that $[Z, \mathfrak{a}] = 0$. Therefore we find from the definition of $\mathfrak{a}$ that $Z \in \mathfrak{a}$. But for any $Y \in \mathfrak{a}$, we have that $\langle Z, Y\rangle = 4\langle X, Y\rangle = 0$. Then $Z = 0$. This implies that $X - \sigma X \in \mathfrak{k} \cap \mathfrak{g}(\mathfrak{a}_\mathfrak{p}; \lambda) = 0$, that is, $X - \sigma X = 0$. Hence we conclude that X is contained in $\mathfrak{h}$ and the lemma is proved.

Lemma (2.8). *Let α be a root of $\Sigma(\tilde{\mathfrak{j}})$. If $\alpha | \mathfrak{a} = 0$, then $\alpha | \mathfrak{a}_\mathfrak{p} = 0$ or $\alpha | \mathfrak{j} = 0$.*

Proof. Let $\alpha \in \Sigma(\tilde{\mathfrak{j}})$ such that $\alpha | \mathfrak{a} = 0$. We assume that $\alpha | \mathfrak{a}_\mathfrak{p} \neq 0$ and $\alpha | \mathfrak{j} \neq 0$ and lead a contradiction. For any $X \in \mathfrak{j}_C$ and $Y \in \mathfrak{g}_C(\tilde{\mathfrak{j}}; \alpha)$, we have $[X, Y] = \alpha(X)Y$. On the other hand, it follows from Lemma (2.7) that $\mathfrak{g}_C(\tilde{\mathfrak{j}}; \alpha)$ is contained in $\mathfrak{h}_C$. Hence we see that $[X, Y] \in \mathfrak{q}_C$. Then $[X, Y] = -\sigma([X, Y]) = -\alpha(X)Y$. This implies that $\alpha(X) = 0$ for any $X \in \mathfrak{j}_C$. We have thus a contradiction. q.e.d.

Lemma (2.9). *Let $\lambda, \mu \in \Sigma(\mathfrak{a}_\mathfrak{p})$ and assume that $\langle\lambda, \mu\rangle < 0$. Then for any $X \in \mathfrak{g}(\mathfrak{a}_\mathfrak{p}; \lambda)$ $(X \neq 0)$ and $Y \in \mathfrak{g}(\mathfrak{a}_\mathfrak{p}; \mu)$ $(Y \neq 0)$, we have $[X, Y] \neq 0$.*

Proof. We put $H = -[X, \theta X]$. Then it is clear that $H \in \mathfrak{a}_\mathfrak{p}$. Hence, multiplying X by a non-zero constant if necessary, we may assume from the first time that

$$(2.9.1) \qquad [H, X] = 2X, \quad [H, \theta X] = -2\theta X, \quad [X, \theta X] = -H.$$

Then it is easy to see that

$$(2.9.2) \qquad [H, Y] = 2\frac{\langle\lambda, \mu\rangle}{\langle\lambda, \lambda\rangle} Y.$$

Assuming that $[X, Y] = 0$, we lead a contradiction. Let $\mathfrak{l}$ be a subalgebra of $\mathfrak{g}$ spanned by H, X and θX. Then it follows from (2.9.1) that

$\mathfrak{l}$ is isomorphic to $\mathfrak{sl}(2, \boldsymbol{R})$. We consider the representation of $\mathfrak{l}$ on the vector space $\mathfrak{g}$ induced from the adjoint representation of $\mathfrak{g}$. Then due to the assumption, we find that Y is a highest weight vector of the representation of $\mathfrak{l}$. This implies that $[H, Y]=cY$ with a non-negative constant c. But owing to (2.9.2), we see that $2(\langle\lambda, \mu\rangle/\langle\lambda, \lambda\rangle)=c\geqq 0$. Hence $\langle\lambda, \mu\rangle\geqq 0$. This contradicts the assumption. This means that X does not commute with Y. q.e.d.

Remark. Hisayoshi Matumoto pointed out that the results of Lemma 11 (iii), (iv) in [Ma, p. 344] are true but their proofs given there are incorrect. Lemma (2.9) is equivalent to Lemma 11 (iii) in [Ma]. The proof of Lemma 11 (iv) is given by an argument similar to that in Lemma (2.9).

Lemma (2.10). *Let λ be an element of $\Sigma(\mathfrak{a}_{\mathfrak{p}})$ and assume that $\langle\sigma(\lambda), \lambda\rangle<0$. Then $\sigma(\lambda)=-\lambda$.*

Proof. Assuming that $\sigma(\lambda)\neq-\lambda$, we lead a contradiction. Needless to say, if $\sigma(\lambda)=\lambda$, then $\langle\sigma(\lambda), \lambda\rangle=\langle\lambda, \lambda\rangle>0$ and therefore we have a contradiction. Hence we may assume that $\sigma(\lambda)\neq\pm\lambda$. Let $X(\neq 0)$ be an element of $\mathfrak{g}(\mathfrak{a}_{\mathfrak{p}}; \lambda)$ and put $Z=[X, \sigma(X)]$. Then Lemma (2.9) implies that $Z\neq 0$. On the other hand, Z is obviously contained in $\mathfrak{g}(\mathfrak{a}_{\mathfrak{p}}; \lambda+\sigma(\lambda))\cap\mathfrak{q}$. However, since the assumption in Lemma (2.7) holds for $\lambda+\sigma(\lambda)$, Z must be contained in $\mathfrak{h}$. This is a contradiction. q.e.d.

Theorem (2.11). *The set $\Sigma(\mathfrak{a})$ becomes a root system of rank* $\dim\mathfrak{a}$.

Remark. This theorem is already obtained by Rossmann [Ro, Th. 5]. But for the sake of completeness, we give here a proof of it. We will treat the related topics to Theorem (2.11) in Section 3.

Proof. Let λ be a root of $\Sigma(\mathfrak{a}_{\mathfrak{p}})$ such that $\lambda|\mathfrak{a}\neq 0$. Then due to Lemma (2.10), we find that λ satisfies one of the following conditions:

(i) $\sigma\lambda=-\lambda$,

(ii) $\langle\lambda, \sigma\lambda\rangle=0$,

(iii) $\langle\lambda, \sigma\lambda\rangle>0$.

Then we can prove the theorem by an argument similar to that in [W, pp. 21–22]. There $\Sigma(\mathfrak{a}_{\mathfrak{p}})$ is assumed to be reduced. But this condition is not used there. q.e.d.

Definition (2.12). We call $\Sigma(\mathfrak{a})$ and its elements the restricted root system of the symmetric pair $(\mathfrak{g}, \mathfrak{h})$ and the restricted roots, respectively.

(2.13) Let λ be an element of $\Sigma(\mathfrak{a})$. Since $\theta\sigma$ leaves $\mathfrak{g}(\mathfrak{a}; \lambda)$ invariant, we obtain a direct sum decomposition

$$(2.13.1) \qquad \mathfrak{g}(\mathfrak{a};\lambda)=\mathfrak{g}^{+}(\mathfrak{a};\lambda)+\mathfrak{g}^{-}(\mathfrak{a};\lambda)$$

by putting $\mathfrak{g}^{\pm}(\mathfrak{a};\lambda)=\{X\in\mathfrak{g}(\mathfrak{a};\lambda);\ \theta\sigma(X)=\pm X\}$. Moreover we put

$$(2.13.2) \qquad \begin{aligned} m^{\pm}(\lambda)&=\dim_{R}\mathfrak{g}^{\pm}(\mathfrak{a};\lambda)\\ m(\lambda)&=m^{+}(\lambda)+m^{-}(\lambda). \end{aligned}$$

Definition (2.14). For any $\lambda\in\Sigma(\mathfrak{a})$, we call $m(\lambda)$ and $(m^{+}(\lambda),\ m^{-}(\lambda))$ the multiplicity and the signature of λ, respectively.

(2.15) Let $(\mathfrak{g},\mathfrak{h})$ be a symmetric pair and $(\mathfrak{g}^{d},\mathfrak{h}^{d})$ its dual. We take a maximal abelian subspace $\mathfrak{a}$ (resp. $\mathfrak{a}^{d}$) of $\mathfrak{p}\cap\mathfrak{q}$ (resp. $\mathfrak{p}^{d}\cap\mathfrak{q}^{d}$). By definition, $\mathfrak{p}\cap\mathfrak{q}=\mathfrak{p}^{d}\cap\mathfrak{q}^{d}$. Hence we may assume that $\mathfrak{a}=\mathfrak{a}^{d}$. Let $\Sigma(\mathfrak{a})$ (resp. $\Sigma(\mathfrak{a}^{d})$) be the restricted root system of $(\mathfrak{g},\mathfrak{h})$ (resp. $(\mathfrak{g}^{d},\mathfrak{h}^{d})$). Then we have the following lemma.

Lemma (2.15.1). *The root systems $\Sigma(\mathfrak{a})$ and $\Sigma(\mathfrak{a}^{d})$ coincide. Moreover for any $\lambda\in\Sigma(\mathfrak{a})$, the signature of λ coincides with that of λ regarded as an element of $\Sigma(\mathfrak{a}^{d})$.*

Proof. Let λ be an element of $\Sigma(\mathfrak{a})$. Then by definition, $\mathfrak{g}^{+}(\mathfrak{a};\lambda)=\mathfrak{h}^{a}\cap\mathfrak{g}(\mathfrak{a};\lambda)=\mathfrak{h}^{da}\cap\mathfrak{g}^{d}(\mathfrak{a};\lambda)$ and $\mathfrak{g}^{-}(\mathfrak{a};\lambda)=\mathfrak{q}^{a}\cap\mathfrak{g}(\mathfrak{a};\lambda)=\sqrt{-1}\,\mathfrak{q}^{da}\cap\mathfrak{g}^{d}(\mathfrak{a};\lambda)$. These imply that $\lambda\in\Sigma(\mathfrak{a}^{d})$ and the signatures of λ regarded as a root of $\Sigma(\mathfrak{a})$ and that of $\Sigma(\mathfrak{a}^{d})$ coincide. The converse is also true.

q.e.d.

(2.16) We give here some remarks on the multiplicities and the signatures of restricted roots.

(1) For any $\lambda\in\Sigma(\mathfrak{a})$, we put $R(\mathfrak{a}_{\mathfrak{p}};\lambda)=\{\alpha\in\Sigma(\mathfrak{a}_{\mathfrak{p}});\ \alpha|\mathfrak{a}=\lambda\}$. Then it follows from the definition that

$$m(\lambda)=\sum_{\alpha\in R(\mathfrak{a}_{\mathfrak{p}};\lambda)}\dim_{R}\mathfrak{g}(\mathfrak{a}_{\mathfrak{p}};\lambda).$$

(2) We introduce a quadratic form on the root space $\mathfrak{g}(\mathfrak{a};\lambda)$ for any restricted root $\lambda\in\Sigma(\mathfrak{a})$ by

$$Q_{\lambda}^{\sigma}(X)=-\langle X,\sigma X\rangle \qquad \text{for any } X\in\mathfrak{g}(\mathfrak{a};\lambda).$$

Then the signature of the quadratic form $Q_{\lambda}^{\sigma}(X)$ on $\mathfrak{g}(\mathfrak{a};\lambda)$ coincides with $(m^{+}(\lambda),\ m^{-}(\lambda))$.

(3) We have already introduced the signature of roots in [O-S] (cf. Example (1.9.3) in § 1). The signature in Definition (2.14) is regarded as a generalization of that in [O-S]. We now explain this. Let $(\mathfrak{g},\mathfrak{k})$ be a Riemannian symmetric pair and let $\Sigma(\mathfrak{a}_{\mathfrak{p}})$ be the root system of $(\mathfrak{g},\mathfrak{k})$. We

take a signature ε of $\Sigma(\mathfrak{a}_{\mathfrak{p}})$ as we did in Example (1.9.3) of Section 1. Let θ_ε be the corresponding involution of $\mathfrak{g}$. Then we define a symmetric pair $(\mathfrak{g}, \mathfrak{k}_\varepsilon)$. In this case, it follows from the definition that $\mathfrak{a}$ coincides with $\mathfrak{a}_{\mathfrak{p}}$. By an easy computation, we find that

$$Q_\lambda^{\theta_\varepsilon}(X)=\varepsilon(\lambda)Q_\lambda^\theta(X) \qquad \text{for any } \lambda \in \Sigma(\mathfrak{a}_{\mathfrak{p}}) \quad \text{and} \quad X \in \mathfrak{g}(\mathfrak{a}_{\mathfrak{p}}; \lambda).$$

Here θ denotes the Cartan involution for $\mathfrak{k}$. This means that for a root $\lambda \in \Sigma(\mathfrak{a}_{\mathfrak{p}})$, $\varepsilon(\lambda)=1$ (resp. $\varepsilon(\lambda)=-1$) if and only if the quadratic form $Q_\lambda^{\theta_\varepsilon}(X)$ on $\mathfrak{g}(\mathfrak{a}_{\mathfrak{p}}; \lambda)$ is positive definite (resp. negative definite). In this sense, the signature defined above is a generalization of that in [O-S].

(2.17) The following lemma will be useful in the determination of $m^+(\lambda)$ and $m^-(\lambda)$.

Lemma (2.17.1). *Let λ be a root of $\Sigma(\mathfrak{a})$. If $2\lambda \in \Sigma(\mathfrak{a})$ and $m^-(2\lambda)>0$, then $m^+(\lambda)=m^-(\lambda)$.*

Proof. It follows from the assumption that there exists an element $Z\neq 0$ of $\mathfrak{g}(\mathfrak{a}; 2\lambda)$ such that $\sigma\theta Z=-Z$. Using Z, we define a linear endomorphism ϕ of $\mathfrak{g}(\mathfrak{a}; \lambda)$ by $\phi(X)=[\theta X, Z]$ for any $X \in \mathfrak{g}(\mathfrak{a}; \lambda)$. It is clear that $\phi(\mathfrak{g}^\pm(\mathfrak{a}; \lambda))\subseteq \mathfrak{g}^\mp(\mathfrak{a}; \lambda)$. Hence to prove the lemma, it suffices to show that ϕ is injective. Assume that $X \in \mathfrak{g}(\mathfrak{a}; \lambda)$ and $\phi(X)=0$. Then $[Z, [\theta Z, X]]=0$. Since $[Z, X]=0$, it follows that $[[Z, \theta Z], X]=0$. On the other hand, $[Z, \theta Z] \in \mathfrak{p}\cap\mathfrak{q}$ and $[Z, \theta Z]$ commutes with $\mathfrak{a}$. These imply that $[Z, \theta Z] \in \mathfrak{a}$. Then $0=[[Z, \theta Z], X]=\lambda([Z, \theta Z])X$. Since $\lambda([Z, \theta Z])\neq 0$, we conclude that $X=0$. q.e.d.

§ 3. The (θ, σ)-system of roots

Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair. We use the notation defined in the previous sections without notice. Let $\Sigma(\tilde{\mathfrak{j}})$ be the root system with respect to $(\mathfrak{g}, \tilde{\mathfrak{j}})$. Then θ and σ induce involutions on $\Sigma(\tilde{\mathfrak{j}})$. We denote them by the same letters. Needless to say, θ and σ commute with each other. They may satisfy additional conditions. Hence it is natural to ask the problem to give conditions on θ and σ such that they are actually induced from a symmetric pair. In this section, we treat this problem.

(3.1) Let V be a finite dimensional real vector space. Let Σ be a root system in V (cf. [W, p. 8]). There exists a positive definite non-degenerate symmetric bilinear form $\langle\cdot, \cdot\rangle$ on V which is invariant under the Weyl group of Σ. We fix this form. Let V^* be the dual of V. Then V^* is identified with V by the inner product $\langle\cdot, \cdot\rangle$. In the sequel, we frequently identify V and V^*.

Let θ and σ be linear involutive isometry of V with $\theta\neq 1$, $\sigma\neq 1$. In the sequel, we always assume that $\theta\Sigma=\sigma\Sigma=\Sigma$. That is, the pair (Σ, θ) (resp. (Σ, σ)) is a θ-system of roots (resp. σ-system of roots) (cf. [W, p. 21]). In addition, if $\theta\sigma=\sigma\theta$, we call the triple (Σ, θ, σ) (or simply Σ) a (θ, σ)-system of roots.

We put $V(\theta)=\{v \in V; \theta v=-v\}$, $V(\sigma)=\{v \in V; \sigma v=-v\}$ and $V(\theta, \sigma)=V(\theta)\cap V(\sigma)$. For any $v \in V$, we define $r_\theta(v)=\frac{1}{2}(v-\theta v)$, $r_\sigma(v)=\frac{1}{2}(v-\sigma v)$ and $r(v)=\frac{1}{4}(v-\theta v-\sigma v+\theta\sigma v)$. Then it is clear that $r_\theta(v) \in V(\theta)$, $r_\sigma(v) \in V(\sigma)$, $r(v) \in V(\theta, \sigma)$ for any $v \in V$. Using these notation, we define

$$\Sigma(\theta)=\{r_\theta(\alpha);\ \alpha \in \Sigma,\ r_\theta(\alpha)\neq 0\}$$
$$\Sigma(\sigma)=\{r_\sigma(\alpha);\ \alpha \in \Sigma,\ r_\sigma(\alpha)\neq 0\}$$
$$\Sigma(\theta, \sigma)=\{r(\alpha);\ \alpha \in \Sigma,\ r(\alpha)\neq 0\}.$$

(3.2) Let Σ be a root system in V and let σ be an involution of V such that $\sigma\Sigma=\Sigma$. We consider the following conditions for Σ and σ:

(N_σ) $\quad \alpha+\sigma\alpha \notin \Sigma$ for any $\alpha \in \Sigma$.

(GN_σ) $\quad$ If α is a root of Σ such that $\langle\alpha, \sigma\alpha\rangle<0$, then $\sigma\alpha=-\alpha$.

Lemma (3.2.1). *Let Σ be a σ-system of roots in V. If the condition (N_σ) holds, so does the condition (GN_σ).*

Proof. Let $\alpha \in \Sigma$. We assume that $\langle\alpha, \sigma\alpha\rangle<0$ and $\sigma\alpha\neq-\alpha$. Then it follows from [W, Prop. 1.1.2.1] that $\alpha+\sigma\alpha$ is a root of Σ. This contradicts the condition (N_σ). q.e.d.

As was already shown in Lemma (2.10), involutions σ satisfying the condition (GN_σ) naturally appear in the course of the examination of symmetric pairs. The classification of such involutions will be treated in another paper.

Lemma (3.3). *Let Σ be a σ-system of roots in V. (We do not assume that Σ is reduced). If the condition (GN_σ) holds, then $\Sigma(\sigma)$ is a root system.*

Replacing Lemma (2.10) with the condition (GN_σ), we can prove Lemma (3.3) by an argument similar to that in Theorem (2.11).

Theorem (3.4). *Let (Σ, θ, σ) be a (θ, σ)-system of roots in V. We assume that the conditions (N_θ) and (N_σ) hold. Moreover we assume that*

(C) Let $\alpha \in \Sigma$. If $r(\alpha)=0$, then $r_\theta(\alpha)=0$ or $r_\sigma(\alpha)=0$.

Then $\Sigma(\theta, \sigma)$ is a root system in $V(\theta, \sigma)$.

Proof. It follows from (N_θ) and [W, Prop. 1.1.3.1] that $\Sigma(\theta)$ is a root system in $V(\theta)$. Since $\theta\sigma=\sigma\theta$, we see that σ induces an involution on $V(\theta)$ which we denote by the same letter. Then $\Sigma(\theta)$ is a σ-system of roots in $V(\theta)$. If the condition (GN_σ) holds for $\Sigma(\theta)$ and σ, it follows from Lemma (3.2.1) that $\Sigma(\theta, \sigma)$ is a root system in $V(\theta, \sigma)$.

Hence it suffices to show the following.

(3.4.1) If $\alpha\in\Sigma$ satisfies the conditions (a) $r_\theta(\alpha)\neq 0$ and (b) $\langle r_\theta(\alpha), \sigma(r_\theta(\alpha))\rangle<0$, then $\sigma(r_\theta(\alpha))=-r_\theta(\alpha)$.

We are going to prove this statement in the cases (i) $\langle\alpha, \theta\alpha\rangle<0$, (ii) $\langle\alpha, \theta\alpha\rangle>0$ and (iii) $\langle\alpha, \theta\alpha\rangle=0$, separately.

(i) *The case where* $\langle\alpha, \theta\alpha\rangle<0$.

In this case, it follows from the condition (N_θ) that $\theta\alpha=-\alpha$. Then (b) is equivalent to the condition $\langle\alpha, \sigma\alpha\rangle<0$. Hence the condition (N_σ) implies the claim.

(ii) *The case where* $\langle\alpha, \theta\alpha\rangle>0$.

It follows from (a) and [W, Prop. 1.1.2.1] that $\beta=\alpha-\theta\alpha$ is also a root of Σ. Since $\beta=2r_\theta(\alpha)$, the conditions (b) and (N_θ) imply the claim.

(iii) *The case where* $\langle\alpha, \theta\alpha\rangle=0$.

We note that (b) is equivalent to the condition (b′) $\langle\alpha, \sigma\alpha\rangle<\langle\alpha, \theta\sigma\alpha\rangle$. If $\langle\alpha, \sigma\alpha\rangle<0$, then (N_σ) implies that $\sigma\alpha=-\alpha$ and therefore we have nothing to prove. Hence we may assume that $\langle\alpha, \sigma\alpha\rangle\geqq 0$. On the other hand, if $\theta\sigma\alpha=\alpha$, it is clear that $\sigma(r_\theta(\alpha))=-r_\theta(\alpha)$. Hence we may also assume that $\theta\sigma\alpha\neq\alpha$. Then $\langle\alpha, \theta\sigma\alpha\rangle<\langle\alpha, \alpha\rangle$. Since α, $\sigma\alpha$ and $\theta\sigma\alpha$ are of the same length, the conditions $0\leqq\langle\alpha, \sigma\alpha\rangle<\langle\alpha, \theta\sigma\alpha\rangle$ implies that $\langle\sigma\alpha, \alpha\rangle=0$. Here we used the properties of roots explained in [W, p. 10]. Since $\langle\alpha, \theta\sigma\alpha\rangle>0$, it follows from [W, Prop. 1.1.2.1] that $\beta=\alpha-\theta\sigma\alpha$ is a root of Σ. It is clear that $r(\beta)=\beta-\theta\beta-\sigma\beta+\theta\sigma\beta=0$. Then the condition (C) implies that $\theta\beta=\beta$ or $\sigma\beta=\beta$. If $\theta\beta=\beta$, then $\alpha-\theta\alpha+\sigma\alpha-\theta\sigma\alpha=0$. On the other hand, if $\sigma\beta=\beta$, then $\alpha+\theta\alpha-\sigma\alpha-\theta\sigma\alpha=0$. In both cases, by taking inner product, we find that $\langle\alpha, \alpha\rangle-\langle\alpha, \theta\sigma\alpha\rangle=0$. This contradicts the assumption $\theta\sigma\alpha\neq\alpha$.

Hence the theorem is completely proved.

Remark (3.5). Let (Σ, θ, σ) be a (θ, σ)-system of roots. We assume that the conditions (N_θ) and (N_σ) hold. Under the assumption, the condition (C) is not necessary to the condition that $\Sigma(\theta, \sigma)$ is a root system. We give here an example of a (θ, σ)-root system (Σ, θ, σ) that (N_θ) and (N_σ) hold, that $\Sigma(\theta, \sigma)$ is a root system but the condition (C) does not hold.

Let $\Sigma=\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}\cup\{\pm\beta_1, \pm\beta_2, \pm(\beta_1+\beta_2)\}$ be a root system of type $A_2\times A_2$. Let θ be an involution on Σ defined by $\theta\alpha_1=-\beta_2$, $\theta\beta_1=-\alpha_2$ and let σ be that defined by $\sigma\alpha_i=-\beta_i$ $(i=1, 2)$. In this case, it is clear that (N_θ) and (N_σ) hold and that $\Sigma(\theta, \sigma)$ is a root system of type A_1 but the condition (C) does not hold. Actually, we see that $\theta\sigma(\alpha_1+\alpha_2) = -(\alpha_1+\alpha_2)$ but $\theta(\alpha_1+\alpha_2)\neq\alpha_1+\alpha_2$, $\sigma(\alpha_1+\alpha_2)\neq\alpha_1+\alpha_2$.

Definition (3.6). Let (Σ, θ, σ) be a (θ, σ)-system of roots. If the conditions (N_θ), (N_σ) and (C) hold, we call it a normal (θ, σ)-system of roots.

(3.7) We give here a remark on the role of Theorem (3.4) in the study of the restricted root system of a symmetric pair. Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair and let $\tilde{\mathfrak{j}}$ be a Cartan subalgebra of $\mathfrak{g}$ satisfying the conditions as we introduced in (2.1). We put $\tilde{\mathfrak{j}}_0=\sqrt{-1}(\mathfrak{k}\cap\tilde{\mathfrak{j}})+\mathfrak{a}_{\mathfrak{p}}$. Then every root of $\Sigma(\tilde{\mathfrak{j}})$ takes real values on $\tilde{\mathfrak{j}}_0$. We now identify V with $\tilde{\mathfrak{j}}_0$ and Σ with $\Sigma(\tilde{\mathfrak{j}})$ and write the restrictions of the complex linear extensions of the involutions θ and σ on $\mathfrak{g}$ to V by the same letters. Then it follows that Σ is a normal (θ, σ)-system of roots in V. In fact, the conditions (N_θ) and (N_σ) are easily checked (cf. [W, Lemma 1.1.3.6]) and the condition (C) is a direct consequence of Lemma (2.8). It is now clear that Theorem (2.11) is a special case of Theorem (3.4).

(3.8) From now on, we introduce an order on the root system Σ which meets our purpose. First recall *a* σ-order on Σ. An order on Σ is called a σ-order if $\sigma\alpha$ is *negative* for any *positive* root α of Σ satisfying $r_\sigma(\alpha)\neq 0$ (cf. [W, p. 23]). Similarly we can define a θ-order on Σ. In general, a θ-order on Σ is not a σ-order. But under the condition (C), we can define an order on Σ which is both a σ-order and a θ-order. To define a standard one, we take elements $Y_- \in V(\theta, \sigma)$, $Y_+ \in V(\theta)\cap V(\sigma)^\perp$, $Z_- \in V(\sigma)\cap V(\theta)^\perp$ and $Z_+ \in V(\theta)^\perp\cap V(\sigma)^\perp$ such that for any root $\alpha \in \Sigma$, we have

$$(3.8.1)\qquad \begin{cases} \alpha(Y_+)=0\Rightarrow\alpha\,|\,V(\theta)\cap V(\sigma)^\perp=0, \\ \alpha(Y_-)=0\Rightarrow\alpha\,|\,V(\theta, \sigma)=0, \\ \alpha(Z_+)=0\Rightarrow\alpha\,|\,V(\theta)^\perp\cap V(\sigma)^\perp=0, \\ \alpha(Z_-)=0\Rightarrow\alpha\,|\,V(\theta)^\perp\cap V(\sigma)=0. \end{cases}$$

Here $V(\theta)^\perp$ and $V(\sigma)^\perp$ denote the orthogonal complements of $V(\theta)$ and $V(\sigma)$ in V, respectively. Then we define an order on Σ such that a root α of Σ is positive if and only if one of the following conditions holds:

$$(3.8.2)\qquad \begin{array}{ll} \text{(i)} & \alpha(Y_-)>0, \\ \text{(ii)} & \alpha(Y_-)=0 \text{ and } \alpha(Y_+)>0, \\ \text{(iii)} & \alpha(Y_-)=0 \text{ and } \alpha(Z_-)>0, \\ \text{(iv)} & \alpha(Y_-)=\alpha(Y_+)=\alpha(Z_-)=0 \text{ and } \alpha(Z_+)>0. \end{array}$$

Due to the condition (C), we find that if $\alpha(Y_-)=0$, then $\alpha(Y_+)=0$ or $\alpha(Z_-)=0$. Hence we can safely define an order on Σ in view of the condition (3.8.2). We call such an order a (θ, σ)-order on Σ. It is clear from the definition that a (θ, σ)-order is both a θ-order and a σ-order. For a given (θ, σ)-order on Σ, let Ψ be its fundamental system of positive roots. Then Ψ is called a (θ, σ)-fundamental system of Σ^+.

The following lemma is a direct consequence of the definition of the (θ, σ)-order on Σ.

Lemma (3.8.3). *Let $\tilde{V}$ be one of the subspaces $V(\theta, \sigma)$, $V(\theta)\cap V(\sigma)^{\perp}$, $V(\theta)^{\perp}\cap V(\sigma)$ of V. Let α and β be roots of Σ such that $\alpha|\tilde{V}=\beta|\tilde{V}\neq 0$. Then $\alpha>0$ if and only if $\beta>0$.*

Let $\tilde{V}$ be the one as in Lemma (3.8.3). For the sake of convenience, we denote by $\Sigma(\tilde{V})$ the root system on $\tilde{V}$ induced from Σ. That is, for example if $\tilde{V}=V(\theta, \sigma)$, then $\Sigma(\tilde{V})=\Sigma(\theta, \sigma)$. We can safely define a compatible order on the root system $\Sigma(\tilde{V})$ such that a root λ of $\Sigma(\tilde{V})$ is positive if and only if there is a positive root α of Σ satisfying $\alpha|\tilde{V}=\lambda$. We denote by Σ^+ and $\Sigma(\tilde{V})^+$ the totality of the positive roots in Σ and $\Sigma(\tilde{V})$, respectively. Then we find the following.

(3.8.4) $$\Sigma(\tilde{V})^+=\{\alpha|\tilde{V};\alpha\in\Sigma^+\}-\{0\}.$$

(3.9) Let (Σ, θ, σ) be a normal (θ, σ)-system of roots in V. Let α be a root of Σ such that $r(\alpha)\neq 0$. Then α satisfies some conditions. We now examine these conditions in detail. Let α be as above. We examine in Lemma (3.10) the case where $\theta(r_\sigma(\alpha))=-r_\sigma(\alpha)$ and that where $\sigma(r_\theta(\alpha))=-r_\theta(\alpha)$ and also examine in Lemma (3.11) the case where $\theta(r_\sigma(\alpha))\neq -r_\sigma(\alpha)$, $\sigma(r_\theta(\alpha))\neq -r_\theta(\alpha)$.

Lemma (3.10). *Let α be a root of Σ such that $r(\alpha)\neq 0$.*

(i) *Assume that $\alpha+\theta\alpha-\sigma\alpha-\theta\sigma\alpha=0$. Then one of the conditions (1)–(6) given below holds.*

(ii) *Assume that $\alpha-\theta\alpha+\sigma\alpha-\theta\sigma\alpha=0$. Then one of the conditions (1′)–(6′) given below holds.*

(1) $\theta\alpha=\sigma\alpha=-\alpha$.	(1′)=(1).
(2) $\theta\alpha=-\alpha$, $\langle\alpha, \sigma\alpha\rangle=0$.	(2′) $\sigma\alpha=-\alpha$, $\langle\alpha, \theta\alpha\rangle=0$.
(3) $\theta\alpha=-\alpha$, $\langle\alpha, \sigma\alpha\rangle>0$.	(3′) $\sigma\alpha=-\alpha$, $\langle\alpha, \theta\alpha\rangle>0$.
(4) $\theta\alpha=\sigma\alpha$, $\langle\alpha, \theta\alpha\rangle=0$.	(4′)=(4).
(5) $\theta\alpha=\sigma\alpha$, $\langle\alpha, \theta\alpha\rangle>0$.	(5′)=(5).
(6) $\langle\alpha, \theta\alpha\rangle=0$, $\langle\alpha, \sigma\alpha\rangle>0$.	(6′) $\langle\alpha, \sigma\alpha\rangle=0$, $\langle\alpha, \theta\alpha\rangle>0$.

Proof. (i) First consider the case where $\langle\alpha, \theta\alpha\rangle<0$. Then (N_θ) implies that $\theta\alpha=-\alpha$. In this case, we derive (1), (2), (3) from the cases

$\langle\alpha, \sigma\alpha\rangle<0$, $\langle\alpha, \sigma\alpha\rangle=0$, $\langle\alpha, \sigma\alpha\rangle>0$, respectively.

Next assume that $\langle\alpha, \theta\alpha\rangle=0$. If $\langle\alpha, \sigma\alpha\rangle<0$, then $\sigma\alpha=-\alpha$. This combined with the condition $\alpha+\theta\alpha-\sigma\alpha-\theta\sigma\alpha=0$ implies that $\theta\alpha=-\alpha$. This contradicts the assumption. Hence, in this case, we have $\langle\alpha, \sigma\alpha\rangle\geqq 0$. If $\langle\alpha, \sigma\alpha\rangle=0$, by taking the inner product, we find that $\langle\alpha, \alpha\rangle-\langle\alpha, \theta\sigma\alpha\rangle=0$. Since α and $\theta\sigma\alpha$ are of the same length, it follows that $\theta\sigma\alpha=\alpha$. Then (4) follows. On the other hand, if $\langle\alpha, \sigma\alpha\rangle>0$, then (6) follows.

Last consider the case when $\langle\alpha, \theta\alpha\rangle>0$. It is clear from the assumption that $\sigma\alpha\neq-\alpha$. Since α, $\theta\alpha$, $\sigma\alpha$ and $\theta\sigma\alpha$ are of the same length, we see that $2\langle\alpha, \theta\alpha\rangle=\langle\alpha, \alpha\rangle$ and the equation $\langle\alpha, \alpha\rangle+\langle\alpha, \theta\alpha\rangle-\langle\alpha, \sigma\alpha\rangle-\langle\alpha, \theta\sigma\alpha\rangle=0$ which follows from the assumption implies that $\langle\alpha, \sigma\alpha\rangle>0$ and $\theta\sigma\alpha=\alpha$. Then (5) follows.

Exchanging the roles of θ and σ, we can prove (ii) similarly. q.e.d.

Lemma (3.11). *Let α be a root of Σ such that $r(\alpha)\neq 0$ and that $\sigma(r_\theta(\alpha))\neq-r_\theta(\alpha)$, $\theta(r_\sigma(\alpha))\neq-r_\sigma(\alpha)$. Then α satisfies one of the following conditions.*

(7) $\langle\alpha, \theta\alpha\rangle=0$, $\langle\alpha, \sigma\alpha\rangle>0$, $\langle\alpha, \theta\sigma\alpha\rangle=0$.

(7′) $\langle\alpha, \theta\alpha\rangle>0$, $\langle\alpha, \sigma\alpha\rangle=0$, $\langle\alpha, \theta\sigma\alpha\rangle=0$.

(8) $\langle\alpha, \theta\alpha\rangle=\langle\alpha, \sigma\alpha\rangle=0$, $\langle\alpha, \theta\sigma\alpha\rangle<0$.

(9) $\langle\alpha, \theta\alpha\rangle=\langle\alpha, \sigma\alpha\rangle=\langle\alpha, \theta\sigma\alpha\rangle=0$.

Proof. If $\langle\alpha, \theta\alpha\rangle<0$, then the condition (N_θ) implies that $\theta\alpha=-\alpha$. This contradicts the assumption. Hence $\langle\alpha, \theta\alpha\rangle\geqq 0$. By the same reason, we find that $\langle\alpha, \sigma\alpha\rangle\geqq 0$.

We now show that $\langle\alpha, \theta\sigma\alpha\rangle\leqq 0$. If otherwise, we have $\langle\alpha, \theta\sigma\alpha\rangle>0$. The assumption implies that $\theta\sigma\alpha\neq\alpha$. Hence it follows from [W, Prop. 1.1.2.1] that $\beta=\alpha-\theta\sigma\alpha\in\Sigma$. Then $r(\beta)=0$ and the condition (C) implies that $\theta\beta=\beta$ or $\sigma\beta=\beta$. We may assume $\theta\beta=\beta$. Then $\sigma(r_\theta(\alpha))=-r_\theta(\alpha)$. This is a contradiction. Therefore $\langle\alpha, \theta\sigma\alpha\rangle\leqq 0$. We have thus shown that

$$\langle\alpha, \theta\alpha\rangle\geqq 0, \quad \langle\alpha, \sigma\alpha\rangle\geqq 0, \quad \langle\alpha, \theta\sigma\alpha\rangle\leqq 0. \tag{3.11.1}$$

Now assume that $\langle\alpha, \theta\alpha\rangle>0$. Then $\beta=\alpha-\theta\alpha\in\Sigma$. Since $r(\alpha)\neq 0$, $\sigma\beta\neq\beta$. On the other hand, we have that $\langle\beta, \sigma\beta\rangle=2\langle\alpha, \sigma\alpha\rangle-2\langle\alpha, \theta\sigma\alpha\rangle$. If $\langle\alpha, \sigma\alpha\rangle=\langle\alpha, \theta\sigma\alpha\rangle=0$ does not hold, then $\langle\beta, \sigma\beta\rangle>0$. This implies that $\gamma=\beta-\sigma\beta\in\Sigma$. Then $r(\gamma)=4r(\alpha)$. This contradicts the condition that Σ is a root system. We have thus proved that if $\langle\alpha, \theta\alpha\rangle>0$, then $\langle\alpha, \sigma\alpha\rangle=\langle\alpha, \theta\sigma\alpha\rangle=0$. This combined with (3.11.1) implies the lemma.

(3.12) Let (Σ, θ, σ) be a normal (θ, σ)-system of roots and let Ψ be a (θ, σ)-fundamental system for Σ. For brevity, $S(\Sigma, \Psi, \theta)$ denotes the Satake diagram for the triple (Σ, Ψ, θ). Similarly, $S(\Sigma, \Psi, \sigma)$ denotes that for (Σ, Ψ, σ). Put $\Psi(\theta)=\{r_\theta(\alpha); \alpha \in \Psi\}$. This is a fundamental system for the root system $\Sigma(\theta)$. Then we can define a diagram $S(\Sigma(\theta), \Psi(\theta), \sigma)$ for the triple $(\Sigma(\theta), \Psi(\theta), \sigma)$ as the Satake diagram is done for the above case. For the sake of convenience, we call this the Satake diagram for $(\Sigma(\theta), \Psi(\theta), \sigma)$.

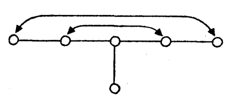
The purpose of this paragraph is to explain a method to determine the Satake diagram $S(\Sigma(\theta), \Psi(\theta), \sigma)$ from the information on $S(\Sigma, \Psi, \sigma)$. In fact, it is easy to check the following facts concerning $S(\Sigma(\theta), \Psi(\theta), \sigma)$ from the normality assumption for (Σ, θ, σ).

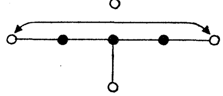
(3.12.1) The node corresponding to $\lambda \in \Psi(\theta)$ is black if and only if there exists an $\alpha \in \Psi$ such that $r_\theta(\alpha)=\lambda$ and that the node of $S(\Sigma, \Psi, \sigma)$ corresponding to α is black.

(3.12.2) Let $\lambda, \mu \in \Psi(\theta)$ and assume that $\lambda \neq \mu$. Then the nodes corresponding to λ and μ are connected by an arrow if and only if there exist $\alpha, \beta \in \Psi$ such that $r_\theta(\alpha)=\lambda$, $r_\theta(\beta)=\mu$ and that the nodes of $S(\Sigma, \Psi, \sigma)$ corresponding to α and β are connected by an arrow.

We give here an example for the Satake diagram of $(\Sigma(\theta), \sigma)$ in the case where (Σ, θ, σ) comes from a symmetric pair.

Consider the symmetric pair $(\mathfrak{g}, \mathfrak{h})=(\mathfrak{e}_{6(-14)}, \mathfrak{su}(5, 1)+\mathfrak{sl}(2, \boldsymbol{R}))$. Retain the notation of the previous sections. Let Σ be the root system of $\mathfrak{g}_C$. Then the type of Σ is E_6. Take a (θ, σ)-fundamental system Ψ for Σ. Then the Satake diagrams $S(\Sigma, \Psi, \theta)$ and $S(\Sigma, \Psi, \sigma)$ are given as follows:

$S(\Sigma, \Psi, \theta)$: 

$S(\Sigma, \Psi, \sigma)$:  .

It follows from (3.12.1) and (3.12.2) that the Satake diagram $S(\Sigma(\theta), \Psi(\theta), \sigma)$ is given as follows:

$S(\Sigma(\theta), \Psi(\theta), \sigma)$:  .

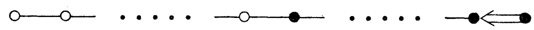
It is easy to check that the involution σ on $\Sigma(\theta)$ satisfies the condition (GN_σ) but does not (N_σ).

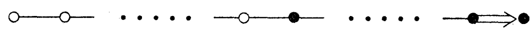
We are going to give a complete list for the Satake diagrams which are obtained by the procedure explained above but are not the ones for

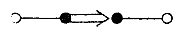
σ-normal systems of roots. We may restrict our attention to the irreducible root systems which come from symmetric pairs. Then the result is given as follows:

(I) B_l (l: even) 

(I)′ BC_l (l: even) 

(II) C_l ($l>1$) 

(II)′ BC_l ($l>1$) 

(III) F_4 

In fact, the symmetric pairs given below are examples with the Satake diagrams given above.

(I) $(\mathfrak{so}(2p, 4n-2p), \mathfrak{su}(p, 2n-p)+\sqrt{-1}\boldsymbol{R})$ $p<n$, $l=2p$.

(I)′ $(\mathfrak{su}(2p, 2n-2p), \mathfrak{sp}(p, n-p))$ $2p<n$, $l=2p$.

(II) $(\mathfrak{so}^*(4n), \mathfrak{so}^*(2p)+\mathfrak{so}^*(4n-2p))$ $p<n$, $l=n$.

(II)′ $(\mathfrak{so}^*(4n+2), \mathfrak{so}^*(2p)+\mathfrak{so}^*(4n-2p+2))$ $p<n$, $l=n$.

(III) $(\mathfrak{e}_{6(-14)}, \mathfrak{su}(5, 1)+\mathfrak{sl}(2, \boldsymbol{R}))$.

§ 4. A reduction to the case of split rank 1

(4.1) Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair. Retain the notation in the previous sections without notice.

First collect here some notation which will be used in this and the subsequent sections.

$$\Sigma(\tilde{\mathfrak{j}})_\theta=\{\alpha\in\Sigma(\tilde{\mathfrak{j}});\ \alpha\,|\,\mathfrak{a}_\mathfrak{p}=0\}$$
$$\Sigma(\tilde{\mathfrak{j}})_\sigma=\{\alpha\in\Sigma(\tilde{\mathfrak{j}});\ \alpha\,|\,\mathfrak{j}=0\}$$
$$\Sigma(\tilde{\mathfrak{j}})_{\theta,\sigma}=\{\alpha\in\Sigma(\tilde{\mathfrak{j}});\ \alpha\,|\,\mathfrak{a}=0\}$$
$$\Sigma(\mathfrak{a}_\mathfrak{p})_\sigma=\{\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p});\ \alpha\,|\,\mathfrak{a}=0\}$$
$$\Sigma(\mathfrak{j})_\theta=\{\alpha\in\Sigma(\mathfrak{j});\ \alpha\,|\,\mathfrak{a}=0\}$$

It is clear from the definition that each of these sets is a root system. As to $\Sigma(\tilde{\mathfrak{j}})_{\theta,\sigma}$, we have the following lemma which is a direct consequence of the condition (C).

Lemma (4.1.1). *$\Sigma(\tilde{\mathfrak{j}})_{\theta,\sigma}$ is the disjoint union of* $(\Sigma(\tilde{\mathfrak{j}})-\Sigma(\tilde{\mathfrak{j}})_\theta)\cap\Sigma(\tilde{\mathfrak{j}})_\sigma$, $(\Sigma(\tilde{\mathfrak{j}})-\Sigma(\tilde{\mathfrak{j}})_\sigma)\cap\Sigma(\tilde{\mathfrak{j}})_\theta$ *and* $\Sigma(\tilde{\mathfrak{j}})_\theta\cap\Sigma(\tilde{\mathfrak{j}})_\sigma$.

Let $\lambda \in \Sigma(\mathfrak{a})$ and let $\tilde{\mathfrak{a}}$ be one of $\tilde{\mathfrak{j}}$, $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{j}$. Then we define the following sets.

$$R(\tilde{\mathfrak{a}}; \lambda) = \{\alpha \in \Sigma(\tilde{\mathfrak{a}}); \alpha|\mathfrak{a} = \lambda\}$$
$$\tilde{R}(\tilde{\mathfrak{a}}; \lambda) = \text{the union of } R(\tilde{\mathfrak{a}}; m\lambda) \text{ such that } m \in \boldsymbol{R} \text{ and } m\lambda \in \Sigma(\mathfrak{a}).$$

For any $\alpha \in \Sigma(\tilde{\mathfrak{j}})$, we define $r_\theta(\alpha)$, $r_\sigma(\alpha)$ and $r(\alpha)$ as we did in (3.1).

A subset M of $\Sigma(\tilde{\mathfrak{j}})$ is said to be connected if M is not decomposed into two mutually orthogonal parts. Similarly we define the notion of θ-connected, σ-connected and (θ, σ)-connected subsets of $\Sigma(\tilde{\mathfrak{j}})$. A subset M of $\Sigma(\tilde{\mathfrak{j}})$ is said to be θ-connected (resp. σ-connected, (θ, σ)-connected) if and only if $\theta M = M$ (resp. $\sigma M = M$, $\theta M = \sigma M = M$) and M is not decomposed into mutually orthogonal θ-invariant (resp. σ-invariant, (θ, σ)-invariant) subsets. Since σ acts on $\Sigma(\mathfrak{a}_\mathfrak{p})$ and θ acts on $\Sigma(\tilde{\mathfrak{j}})$, we can similarly define connected and σ-connected subsets of $\Sigma(\mathfrak{a}_\mathfrak{p})$ and connected and θ-connected subsets of $\Sigma(\mathfrak{j})$.

Lemma (4.2). *Let λ be a root of $\Sigma(\mathfrak{a})$. Then $\tilde{R}(\tilde{\mathfrak{j}}; \lambda)$ is (θ, σ)-connected.*

Proof. It is clear that $\theta(\tilde{R}(\tilde{\mathfrak{j}}; \lambda)) = \sigma(\tilde{R}(\tilde{\mathfrak{j}}; \lambda)) = \tilde{R}(\tilde{\mathfrak{j}}; \lambda)$.

We first consider $\tilde{R}(\mathfrak{a}_\mathfrak{p}; \lambda)$ instead of $\tilde{R}(\tilde{\mathfrak{j}}; \lambda)$. For any $\mu, \nu \in \tilde{R}(\mathfrak{a}_\mathfrak{p}; \lambda)$, we show that $\langle \mu, \nu \rangle \neq 0$ or $\langle \mu, \sigma\nu \rangle \neq 0$. In fact, it follows from the definition that $\langle \mu - \sigma\mu, \nu - \sigma\nu \rangle = mn\langle \lambda, \lambda \rangle \neq 0$. (Here m, n are the integers defined by $\mu - \sigma\mu = m\lambda$, $\nu - \sigma\nu = n\lambda$.) This implies that $\langle \mu, \nu \rangle \neq \langle \mu, \sigma\nu \rangle$ and therefore $\langle \mu, \nu \rangle \neq 0$ or $\langle \mu, \sigma\nu \rangle \neq 0$. Now we take an element $\mu \in \tilde{R}(\mathfrak{a}_\mathfrak{p}; \lambda)$ and fix it once for all. Putting $N = \{\nu \in \tilde{R}(\mathfrak{a}_\mathfrak{p}; \lambda); \langle \mu, \nu \rangle \neq 0\}$, we find that $\tilde{R}(\mathfrak{a}_\mathfrak{p}; \lambda) = N \cup \sigma N$. We take an element $\alpha \in \tilde{R}(\tilde{\mathfrak{j}}; \lambda)$ such that $\alpha|\mathfrak{a}_\mathfrak{p} = \mu$. If β is an element of $\tilde{R}(\tilde{\mathfrak{j}}; \lambda)$, then $\beta|\mathfrak{a}_\mathfrak{p}$ is contained in N or in σN. If $\beta|\mathfrak{a}_\mathfrak{p} \in N$, it follows that $\langle \alpha - \theta\alpha, \beta - \theta\beta \rangle \neq 0$. This implies that $\langle \alpha, \beta \rangle \neq 0$ or $\langle \alpha, \theta\beta \rangle \neq 0$. On the other hand, if $\beta|\mathfrak{a}_\mathfrak{p} \in \sigma N$, by an argument similar to the above, we find that $\langle \alpha, \sigma\beta \rangle \neq 0$ or $\langle \alpha, \theta\sigma\beta \rangle \neq 0$. Putting $M = \{\beta \in \tilde{R}(\tilde{\mathfrak{j}}; \lambda); \langle \alpha, \beta \rangle \neq 0\}$, we eventually conclude that

$$\tilde{R}(\tilde{\mathfrak{j}}; \lambda) = M \cup \theta M \cup \sigma M \cup \theta\sigma M. \qquad \text{q.e.d.}$$

(4.3) It is clear that $\Sigma' = \tilde{R}(\tilde{\mathfrak{j}}; \lambda) \cup \Sigma(\tilde{\mathfrak{j}})_{\theta,\sigma}$ is a closed subsystem of $\Sigma(\tilde{\mathfrak{j}})$. Namely, Σ' is a root system and satisfies that 1) if $\alpha \in \Sigma'$, then $-\alpha \in \Sigma'$ and that 2) if $\alpha, \beta \in \Sigma'$ and $\alpha + \beta \in \Sigma$, then $\alpha + \beta \in \Sigma'$ (cf. [Ar, p. 7]). Let $\Sigma(\tilde{\mathfrak{j}}; \lambda)$ be the (θ, σ)-connected component of Σ' containing $\tilde{R}(\tilde{\mathfrak{j}}; \lambda)$. Then there exists an irreducible closed subsystem Σ of $\Sigma(\tilde{\mathfrak{j}})$ such that $\Sigma(\tilde{\mathfrak{j}}; \lambda) = \Sigma \cup \theta\Sigma \cup \sigma\Sigma \cup \theta\sigma\Sigma$.

(4.4) Similar to (4.3), we see that $\tilde{R}(\mathfrak{a}_\mathfrak{p}; \lambda) \cup \Sigma(\mathfrak{a}_\mathfrak{p})_\sigma$ is a closed subsystem of $\Sigma(\mathfrak{a}_\mathfrak{p})$. Denote by $\Sigma(\mathfrak{a}_\mathfrak{p}; \lambda)$ the σ-connected component of $\tilde{R}(\mathfrak{a}_\mathfrak{p}; \lambda) \cup \Sigma(\mathfrak{a}_\mathfrak{p})_\sigma$ containing $\tilde{R}(\mathfrak{a}_\mathfrak{p}; \lambda)$ (for the definition of the σ-component, see [Ar]).

We note here the following well-known statement (cf. [Ar]).

(4.4.1) If R is a closed subsystem of $\Sigma(\tilde{\mathfrak{j}})$ such that $\theta R = R$, then $r_\theta(R - \Sigma(\tilde{\mathfrak{j}})_\theta)$ is a closed subsystem of $\Sigma(\mathfrak{a}_\mathfrak{p})$. Furthermore if R is θ-connected, then $r_\theta(R - \Sigma(\tilde{\mathfrak{j}})_\theta)$ is connected.

Lemma (4.5). $r_\theta(\Sigma(\tilde{\mathfrak{j}}; \lambda) - \Sigma(\tilde{\mathfrak{j}})_\theta) = \Sigma(\mathfrak{a}_\mathfrak{p}; \lambda)$.

Proof. For any $\alpha \in \Sigma(\tilde{\mathfrak{j}}; \lambda) - \Sigma(\tilde{\mathfrak{j}})_\theta$, we see that

$$r_\theta(\alpha) \in \tilde{R}(\mathfrak{a}_\mathfrak{p}; \lambda) \cup \Sigma(\mathfrak{a}_\mathfrak{p})_\sigma.$$

Hence it follows that

$$r_\theta(\Sigma(\tilde{\mathfrak{j}}; \lambda) - \Sigma(\tilde{\mathfrak{j}})_\theta) \subseteqq \tilde{R}(\mathfrak{a}_\mathfrak{p}; \lambda) \cup \Sigma(\mathfrak{a}_\mathfrak{p})_\sigma.$$

Since (4.4.1) implies that $r_\theta(\Sigma(\tilde{\mathfrak{j}}; \lambda) - \Sigma(\tilde{\mathfrak{j}})_\theta)$ is a closed subsystem of $\Sigma(\mathfrak{a}_\mathfrak{p})$, it is clear that

$$\Sigma(\mathfrak{a}_\mathfrak{p}; \lambda) \subseteqq r_\theta(\Sigma(\tilde{\mathfrak{j}}; \lambda) - \Sigma(\tilde{\mathfrak{j}})_\theta).$$

Hence if we show that $r_\theta(\Sigma(\tilde{\mathfrak{j}}; \lambda) - \Sigma(\tilde{\mathfrak{j}})_\theta)$ is σ-connected, the lemma follows.

We are going to prove that $r_\theta(\Sigma(\tilde{\mathfrak{j}}; \lambda) - \Sigma(\tilde{\mathfrak{j}})_\theta)$ is σ-connected. We use the notation in (4.3) without notice. Then $\Sigma(\tilde{\mathfrak{j}}; \lambda) = \Sigma \cup \theta\Sigma \cup \sigma\Sigma \cup \theta\sigma\Sigma$ and Σ is an irreducible closed subsystem of $\Sigma(\tilde{\mathfrak{j}})$. First assume that $\theta\Sigma = \Sigma$. It follows from (4.4.1) that $\sigma(\Sigma - \Sigma(\tilde{\mathfrak{j}})_\theta) = \sigma\Sigma - \Sigma(\tilde{\mathfrak{j}})_\theta$. Then we find that $r_\theta(\sigma\Sigma - \Sigma(\tilde{\mathfrak{j}})_\theta) = \sigma(r_\theta(\Sigma - \Sigma(\tilde{\mathfrak{j}})_\theta))$. Hence $r_\theta(\Sigma(\tilde{\mathfrak{j}}; \lambda) - \Sigma(\tilde{\mathfrak{j}})_\theta)$ is σ-connected. Next assume that $\theta\Sigma \cap \Sigma = \emptyset$. Since $\theta(\Sigma - \Sigma(\tilde{\mathfrak{j}})_\theta) = \theta\Sigma - \Sigma(\tilde{\mathfrak{j}})_\theta$, it also follows from (4.4.1) that $r_\theta(\Sigma - \Sigma(\tilde{\mathfrak{j}})_\theta)$ is an irreducible closed subsystem of $\Sigma(\mathfrak{a}_\mathfrak{p})$. Then by an argument similar to that in the previous case, we conclude that $r_\theta(\Sigma(\tilde{\mathfrak{j}}; \lambda) - \Sigma(\tilde{\mathfrak{j}})_\theta)$ is σ-connected. q.e.d.

(4.6) Denote by $\Sigma(\mathfrak{j}; \lambda)$ the σ-connected component of $\tilde{R}(\mathfrak{j}; \lambda) \cup \Sigma(\mathfrak{j})_\theta$ containing $\tilde{R}(\mathfrak{j}; \lambda)$ (cf. (4.4)). Then the following lemma is shown by an argument similar to that in the proof of Lemma (4.5).

Lemma. $r_\sigma(\Sigma(\tilde{\mathfrak{j}}; \lambda) - \Sigma(\tilde{\mathfrak{j}})_\sigma) = \Sigma(\mathfrak{j}; \lambda)$.

Theorem (4.7). *For any $\lambda \in \Sigma(\mathfrak{a})$, we denote by $\mathfrak{g}(\lambda)$ the subalgebra of $\mathfrak{g}$ generated by $\{\mathfrak{g}(\mathfrak{a}_\mathfrak{p}; \mu); \mu \in \tilde{R}(\mathfrak{a}_\mathfrak{p}; \lambda)\}$. Then we have the following.*

(i) $\mathfrak{g}(\lambda)$ *is a semisimple Lie algebra of the non-compact type and*

$\theta(\mathfrak{g}(\lambda))=\sigma(\mathfrak{g}(\lambda))=\mathfrak{g}(\lambda)$.

(ii) $\theta(\lambda)=\theta|\mathfrak{g}(\lambda)$ *is a Cartan involution of* $\mathfrak{g}(\lambda)$. *Let* $\mathfrak{g}(\lambda)=\mathfrak{k}(\lambda)+\mathfrak{p}(\lambda)$ *be the corresponding Cartan decomposition.*

(iii) $\sigma(\lambda)=\sigma|\mathfrak{g}(\lambda)$ *is an involution of* $\mathfrak{g}(\lambda)$. *And* $\sigma(\lambda)$ *is non-trivial on each simple factors of* $\mathfrak{g}(\lambda)$. *Let* $\mathfrak{g}(\lambda)=\mathfrak{h}(\lambda)+\mathfrak{q}(\lambda)$ *be the corresponding direct sum decomposition.*

(iv) $\mathfrak{a}(\lambda)=\mathfrak{a}\cap\mathfrak{g}(\lambda)$ *is a maximal abelian subspace of* $\mathfrak{p}(\lambda)\cap\mathfrak{q}(\lambda)$. *And* $\dim_R\mathfrak{a}(\lambda)=1$.

(v) $\mathfrak{a}_{\mathfrak{p}}(\lambda)=\mathfrak{a}_{\mathfrak{p}}\cap\mathfrak{g}(\lambda)$ *is a maximal abelian subspace of* $\mathfrak{p}(\lambda)$ *containing* $\mathfrak{a}(\lambda)$.

(vi) $\mathfrak{j}(\lambda)=\mathfrak{j}\cap\mathfrak{g}(\lambda)$ *is a maximal abelian subspace of* $\mathfrak{q}(\lambda)$ *containing* $\mathfrak{a}(\lambda)$.

(vii) $\tilde{\mathfrak{j}}(\lambda)=\tilde{\mathfrak{j}}\cap\mathfrak{g}(\lambda)$ *is a Cartan subalgebra of* $\mathfrak{g}(\lambda)$ *containing both* $\mathfrak{a}_{\mathfrak{p}}(\lambda)$ *and* $\mathfrak{j}(\lambda)$.

Proof. Let $\Psi(\mathfrak{a}_{\mathfrak{p}})$ be the fundamental system of $\Sigma(\mathfrak{a}_{\mathfrak{p}})$. (Needless to say, we may assume that the orders on $\Sigma(\tilde{\mathfrak{j}})$, $\Sigma(\mathfrak{a}_{\mathfrak{p}})$, $\Sigma(\mathfrak{j})$ and $\Sigma(\mathfrak{a})$ are so taken that they are compatible.) Then we see that $\Psi(\mathfrak{a}_{\mathfrak{p}})\cap\Sigma(\mathfrak{a}_{\mathfrak{p}};\lambda)$ is a fundamental system of $\Sigma(\mathfrak{a}_{\mathfrak{p}};\lambda)$. Hence it follows from [W, Lemma 1.2.3.14] that $\mathfrak{g}(\lambda)$ is semisimple of the non-compact type and $\theta(\lambda)$ is a Cartan involution of $\mathfrak{g}(\lambda)$. Moreover [W, Lemma 1.2.3.15] implies that $\mathfrak{a}_{\mathfrak{p}}(\lambda)=\mathfrak{a}_{\mathfrak{p}}\cap\mathfrak{g}(\lambda)$ is a maximal abelian subspace of $\mathfrak{p}(\lambda)$. Since σ leaves $\Sigma(\mathfrak{a}_{\mathfrak{p}};\lambda)$ invariant, we find that $\sigma(\mathfrak{g}(\lambda))=\mathfrak{g}(\lambda)$. These show (i) and (ii).

By definition, $\mathfrak{g}(\mathfrak{a};\lambda)$ is contained in $\mathfrak{g}(\lambda)$. Since $\theta\sigma$ leaves $\mathfrak{g}(\mathfrak{a};\lambda)$ invariant, we can take an element $X\in\mathfrak{g}(\mathfrak{a};\lambda)$ $(X\neq 0)$ such that $\theta\sigma X=X$ or $\theta\sigma X=-X$. Then it is clear that $Y=[X,\theta X]$ $(\neq 0)$ is contained in $\mathfrak{a}\cap\mathfrak{p}(\lambda)\cap\mathfrak{q}(\lambda)=\mathfrak{a}(\lambda)$. In particular this implies that $\sigma(\lambda)$ is not trivial on $\mathfrak{g}(\lambda)$. By multiplying Y by a non-zero constant if necessary, we may assume that $\lambda(Y)=1$. We now show that $\mathfrak{a}(\lambda)=\boldsymbol{R}Y$. Let $Z\in\mathfrak{a}(\lambda)$ such that $\lambda(Z)=0$. Fix a $\mu\in\Sigma(\mathfrak{a}_{\mathfrak{p}};\lambda)$. If $\sigma\mu=\mu$, then $\mu(Z)=\frac{1}{2}(\mu+\sigma\mu)(Z)=0$. On the other hand, if $\sigma\mu\neq\mu$, we have that $\mu-\sigma\mu=2\lambda$. Then $\mu(Z)=\frac{1}{2}(\mu-\sigma\mu)(Z)=\lambda(Z)=0$. Accordingly, we find that $\mu(Z)=0$ for any $\mu\in\Sigma(\mathfrak{a}_{\mathfrak{p}};\lambda)$. This implies that $Z=0$ and therefore $\mathfrak{a}(\lambda)=\boldsymbol{R}Y$. We have thus shown (iv). Now (v) is clear.

Let $\mathfrak{g}(\lambda)_C$ be the complexification of $\mathfrak{g}(\lambda)$ in $\mathfrak{g}_C$. For any $\alpha\in\Sigma(\tilde{\mathfrak{j}};\lambda)$, let X_α $(\neq 0)$ be an element of $\mathfrak{g}_C(\tilde{\mathfrak{j}};\alpha)$. It follows from Lemma (4.5) that $\mathfrak{g}(\lambda)_C$ is generated by $\{X_\alpha;\alpha\in\Sigma(\tilde{\mathfrak{j}};\lambda)\}$. Then it is clear that the subspace $\tilde{\mathfrak{j}}'(\lambda)_C$ of $\mathfrak{g}(\lambda)_C$ spanned by $\{[X_\alpha,X_{-\alpha}];\alpha\in\Sigma(\tilde{\mathfrak{j}};\lambda)\}$ is a Cartan subalgebra of $\mathfrak{g}(\lambda)_C$. Since $\tilde{\mathfrak{j}}'(\lambda)_C$ is contained in $\tilde{\mathfrak{j}}_C$, we find that $\tilde{\mathfrak{j}}'(\lambda)_C\subseteq\tilde{\mathfrak{j}}(\lambda)_C$. Since $\tilde{\mathfrak{j}}(\lambda)$ is abelian, this implies that $\tilde{\mathfrak{j}}'(\lambda)_C=\mathfrak{j}(\lambda)_C$. Hence $\tilde{\mathfrak{j}}(\lambda)$ is a Cartan subalgebra of $\mathfrak{g}(\lambda)$. This proves (vii).

We have proved that $\Sigma(\tilde{\mathfrak{j}};\lambda)$ is the root system of $(\mathfrak{g}(\lambda)_C,\tilde{\mathfrak{j}}(\lambda)_C)$. It follows from the arguments in (4.3) that $\Sigma(\bar{\mathfrak{j}}; \lambda)$ is (θ, σ)-irreducible. We use the notation there. First assume that $\sigma\Sigma=\Sigma$. If σ is trivial on Σ, it follows that σ is trivial on $r_\theta(\Sigma-\Sigma(\bar{\mathfrak{j}})_\theta)$. This implies that $\sigma\,|\,\mathfrak{g}(\lambda)$ is trivial. But we have already remarked that there exists $Y\in\mathfrak{p}(\lambda)$ $(Y\neq 0)$ such that $\sigma Y=-Y$. This is a contradiction. Hence σ is not trivial on Σ. In this case it is clear that $\mathfrak{g}(\lambda)_C$ is simple. Hence we conclude that $\sigma(\lambda)$ is non-trivial on the simple Lie algebra $\mathfrak{g}(\lambda)$. Next assume that $\sigma\Sigma\cap\Sigma=\emptyset$. Let $\mathfrak{g}'(\lambda)$ be the subalgebra of $\mathfrak{g}(\lambda)$ generated by $\{\mathfrak{g}(\mathfrak{a}_\mathfrak{p}; \mu); \mu\in r_\theta(\Sigma-\Sigma(\bar{\mathfrak{j}})_\theta)\}$. Then it follows that $\mathfrak{g}(\lambda)=\mathfrak{g}'(\lambda)+\sigma\mathfrak{g}'(\lambda)$ is a direct sum decomposition and clearly σ is not trivial on each simple factor of $\mathfrak{g}(\lambda)$. Hence (iii) is proved.

Finally we show (vi). Let $(\mathfrak{g}^d, \mathfrak{h}^d)$ be the dual of $(\mathfrak{g}, \mathfrak{h})$ defined in (1.2). Let $\mathfrak{g}^d(\lambda)$ be the subalgebra of $\mathfrak{g}^d$ generated by

$$\{\mathfrak{g}^d(\mathfrak{a}^d_\mathfrak{p}; \mu); \mu\in\Sigma(\mathfrak{a}^d_\mathfrak{p}; \lambda)\}.$$

Here we put $\mathfrak{a}^d_\mathfrak{p}=\mathfrak{a}+\sqrt{-1}(\mathfrak{k}\cap\mathfrak{j})$. By definition, $(\mathfrak{a}^d_\mathfrak{p})_C=\mathfrak{j}_C$ and therefore $\Sigma(\mathfrak{a}^d_\mathfrak{p}; \lambda)=\Sigma(\mathfrak{j}; \lambda)$. Then by an argument similar to the above one, we find that $\mathfrak{a}^d_\mathfrak{p}\cap\mathfrak{g}^d(\lambda)$ is a maximal abelian subspace of $\mathfrak{p}^d\cap\mathfrak{g}^d(\lambda)$. Since $\mathfrak{j}(\lambda)_C=(\mathfrak{a}^d_\mathfrak{p}\cap\mathfrak{g}^d(\lambda))_C$, we conclude that $\mathfrak{j}(\lambda)$ is a maximal abelian subspace of $\mathfrak{q}(\lambda)$. Thus (vi) is proved. q.e.d.

(4.8) In the above discussions, we have shown the following statements.

(4.8.1) Use the notation in Theorem (4.7). Assume that $\mathfrak{g}$ is of the non-compact type. Then $\mathfrak{g}$ is generated by $\{\mathfrak{g}(\mathfrak{a}; \lambda); \lambda\in\Sigma(\mathfrak{a})\}$. In particular, if $(\mathfrak{g}, \mathfrak{h})$ is of split rank 1, then $\mathfrak{g}$ is generated by $\mathfrak{g}(\lambda)$ and $\mathfrak{g}(-\lambda)$, where λ is the simple root of $\Sigma(\mathfrak{a})$.

(4.8.2) Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair. Assume that $(\mathfrak{g}, \mathfrak{h})$ is irreducible. Then the restricted root system $\Sigma(\mathfrak{a})$ is irreducible.

§ 5. The irreducible symmetric pairs of split rank 1

Needless to say, the irreducible symmetric pairs of split rank 1 are basic among general symmetric pairs. By this reason, it is preferable to study them. This will be done in this section.

(5.1) The irreducible symmetric pairs of split rank 1 are enumerated in Table II. (We follow the notation in [He 1].) We can prove this claim by deciding the split ranks of all the irreducible symmetric pairs. As was noted in (2.4), for a given symmetric pair $(\mathfrak{g}, \mathfrak{h})$, its split rank is identified with the real rank of $\mathfrak{h}^a$.

Let $(\mathfrak{g}, \mathfrak{h})$ be an irreducible symmetric pair of split rank 1 and let $(\mathfrak{g}^d, \mathfrak{h}^d)$ be its dual. In Table II, we always take so that $r(\mathfrak{g}^d)\leqq r(\mathfrak{g})$. Then

Table II. Irreducible symmetric pairs of split rank 1

I_1:	$(\mathfrak{so}(p+1, q+1), \mathfrak{so}(p+1, q))$
I_1^d:	$(\mathfrak{so}(p+q+1, 1), \mathfrak{so}(p+1)+\mathfrak{so}(q, 1))$
I_2:	$(\mathfrak{su}(p+1, q+1), \mathfrak{su}(p+1, q)+\sqrt{-1}\boldsymbol{R})$
I_2^d:	$(\mathfrak{su}(p+q+1, 1), \mathfrak{su}(p+1)+\mathfrak{su}(q, 1)+\sqrt{-1}\boldsymbol{R})$
I_3:	$(\mathfrak{sp}(p+1, q+1), \mathfrak{sp}(p+1, q)+\mathfrak{sp}(1))$
I_3^d:	$(\mathfrak{sp}(p+q+1, 1), \mathfrak{sp}(p+1)+\mathfrak{sp}(q, 1))$
I_4^1:	$(\mathfrak{f}_{4(-20)}, \mathfrak{so}(9))$
I_4^2:	$(\mathfrak{f}_{4(-20)}, \mathfrak{so}(8, 1))$
II_1:	$(\mathfrak{sl}(m+2, \boldsymbol{R}), \mathfrak{sl}(m+1, \boldsymbol{R})+\boldsymbol{R})$
II_1^d:	$(\mathfrak{su}(m+1, 1), \mathfrak{so}(m+1, 1))$
II_2:	$(\mathfrak{sp}(m+2, \boldsymbol{R}), \mathfrak{sp}(m+1, \boldsymbol{R})+\mathfrak{sp}(1, \boldsymbol{R}))$
II_2^d:	$(\mathfrak{sp}(m+1, 1), \mathfrak{su}(m+1, 1)+\sqrt{-1}\boldsymbol{R})$
II_3:	$(\mathfrak{f}_{4(4)}, \mathfrak{so}(5, 4))$
II_3^d:	$(\mathfrak{f}_{4(-20)}, \mathfrak{sp}(2, 1)+\mathfrak{su}(2))$
III_1:	$(\mathfrak{so}(m+2, \boldsymbol{C}), \mathfrak{so}(m+1, \boldsymbol{C}))$
III_1^d:	$(\mathfrak{so}(m+1, 1)+\mathfrak{so}(m+1, 1), \mathfrak{so}(m+1, 1))$
III_2:	$(\mathfrak{sl}(m+2, \boldsymbol{C}), \mathfrak{sl}(m+1, \boldsymbol{C})+\boldsymbol{C})$
III_2^d:	$(\mathfrak{su}(m+1, 1)+\mathfrak{su}(m+1, 1), \mathfrak{su}(m+1, 1))$
III_3:	$(\mathfrak{sp}(m+2, \boldsymbol{C}), \mathfrak{sp}(m+1, \boldsymbol{C})+\mathfrak{sp}(1, \boldsymbol{C}))$
III_3^d:	$(\mathfrak{sp}(m+1, 1)+\mathfrak{sp}(m+1, 1), \mathfrak{sp}(m+1, 1))$
III_4:	$(\mathfrak{f}_4, \mathfrak{so}(9, \boldsymbol{C}))$
III_4^d:	$(\mathfrak{f}_{4(-20)}+\mathfrak{f}_{4(-20)}, \mathfrak{f}_{4(-20)})$
IV_1:	$(\mathfrak{so}^*(2(m+2)), \mathfrak{so}^*(2(m+1))+\mathfrak{so}^*(2))$
IV_1^d:	$(\mathfrak{so}(2(m+1), 2), \mathfrak{su}(m+1, 1)+\sqrt{-1}\boldsymbol{R})$
IV_2:	$(\mathfrak{su}^*(2(m+2)), \mathfrak{su}^*(2(m+1))+\mathfrak{su}^*(2)+\boldsymbol{R})$
IV_2^d:	$(\mathfrak{su}(2(m+1), 2), \mathfrak{sp}(m+1, 1))$
IV_3:	$(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9, 1)+\boldsymbol{R})$
IV_3^d:	$(\mathfrak{e}_{6(-14)}, \mathfrak{f}_{4(-20)})$
V_1:	$(\mathfrak{sl}(3, \boldsymbol{C}), \mathfrak{sl}(3, \boldsymbol{R}))$
V_2:	$(\mathfrak{su}(3, 3), \mathfrak{sp}(3, \boldsymbol{R}))$
V_2^d:	$(\mathfrak{su}^*(6), \mathfrak{sl}(3, \boldsymbol{C})+\sqrt{-1}\boldsymbol{R})$
V_3:	$(\mathfrak{e}_{6(2)}, \mathfrak{f}_{4(4)})$
V_3^d:	$(\mathfrak{e}_{6(-26)}, \mathfrak{su}^*(6)+\mathfrak{su}(2))$

we observe that $r(\mathfrak{g}^d)\leqq 2$. If $r(\mathfrak{g}^d)=1$, then $(\mathfrak{g}, \mathfrak{h})$ is contained in one of the classes I and II. On the other hand, if $r(\mathfrak{g}^d)=2$, then $(\mathfrak{g}, \mathfrak{h})$ is contained in one of the classes III, IV and V. If X denotes one of I–V, X^d denotes the class of the dual pairs to those in X.

Let $(\mathfrak{g}, \mathfrak{h})$ be an irreducible symmetric pair of split rank 1. We use the notation in the previous sections. By the assumption, the restricted root system $\Sigma(\mathfrak{a})$ coincides with $\{\pm\lambda\}$ or $\{\pm\lambda, \pm 2\lambda\}$. Here λ denotes a unique positive simple root of $\Sigma(\mathfrak{a})$. We always fix it in this section.

(5.2) The irreducible symmetric pairs of split rank 1 and of Type ($\mathfrak{k}_\varepsilon$) are contained in $\mathrm{I}_1-\mathrm{I}_4$. We give here concrete correspondences. Needless to say, a symmetric pair of Type ($\mathfrak{k}_\varepsilon$) is self-dual.

(5.2.i) $\mathrm{I}_1(q=0)=\mathrm{I}_1^d(q=0)$: $(\mathfrak{so}(p+1, 1), \mathfrak{so}(p+1))$

(5.2.i)′ $\mathrm{I}_1(p=0)=\mathrm{I}_1^d(p=0)$: $(\mathfrak{so}(1, q+1), \mathfrak{so}(1, q))$

(5.2.ii) $\mathrm{I}_2(q=0)=\mathrm{I}_2^d(q=0)$: $(\mathfrak{su}(p+1, 1), \mathfrak{su}(p+1)+\sqrt{-1}\boldsymbol{R})$

(5.2.ii)′ $\mathrm{I}_2(p=0)=\mathrm{I}_2^d(p=0)$: $(\mathfrak{su}(1, q+1), \mathfrak{su}(1, q)+\sqrt{-1}\boldsymbol{R})$

(5.2.iii) $\mathrm{I}_3(q=0)=\mathrm{I}_3^d(q=0)$: $(\mathfrak{sp}(p+1, 1), \mathfrak{sp}(p+1)+\mathfrak{sp}(1))$

(5.2.iii)′ $\mathrm{I}_3(p=0)=\mathrm{I}_3^d(p=0)$: $(\mathfrak{sp}(1, q+1), \mathfrak{sp}(1)+\mathfrak{sp}(1, q))$

(5.2.iv) I_4^1: $(\mathfrak{f}_{4(-20)}, \mathfrak{so}(9))$

(5.2.iv)′ I_4^2: $(\mathfrak{f}_{4(-20)}, \mathfrak{so}(8, 1))$

(5.3) *Special isomorphisms.*

Because of the isomorphisms in [He 2, p. 519], there are some overlaps in Table II in addition to those described in (5.2). We derive the following isomorphisms.

(5.3.i) $\mathrm{I}_1(p=1, q=0)=\mathrm{I}_1^d(p=1, q=0)=\mathrm{I}_2(p=q=0)=\mathrm{I}_2^d(p=q=0)$

(5.3.ii) $\mathrm{I}_1=(p=0, q=1)=\mathrm{I}_1^d(p=0, q=1)=\mathrm{II}_1(m=0)=\mathrm{II}_1^d(m=0)$
$=\mathrm{IV}_1(m=0)=\mathrm{IV}_1^d(m=0)$

(5.3.iii) $\mathrm{I}_1(p=3, q=0)=\mathrm{I}_3(p=q=0)$

(5.3.iv) $\mathrm{I}_1^d(p=3, q=0)=\mathrm{I}_3^d(p=q=0)$

(5.3.v) $\mathrm{I}_1(p=1, q=2)=\mathrm{II}_2(m=0)$

(5.3.vi) $\mathrm{I}_1^d(p=1, q=2)=\mathrm{II}_2^d(m=0)$

(5.3.vii) $\mathrm{IV}_1(m=2)=\mathrm{IV}_1^d(m=2)$

(5.3.viii) $\mathrm{IV}_2(m=0)=\mathrm{I}_1^d(p=3, q=1)$

(5.3.viii)′ $\mathrm{IV}_2^d(m=0)=\mathrm{I}_1(p=3, q=1)$

(5.4) We consider the sets $R(\tilde{\mathfrak{j}}; \lambda)$ and $R(\tilde{\mathfrak{j}}; 2\lambda)$. If α is a root of $\Sigma(\tilde{\mathfrak{j}})$ contained in $R(\tilde{\mathfrak{j}}; \lambda)\cup R(\tilde{\mathfrak{j}}; 2\lambda)$, then we have already shown that α satisfies one of the conditions (1)–(9) and (1′)–(9′) in Lemmas (3.10) and (3.11). Hereafter we frequently use this notation without any comments. By definition, we find that if α satisfies one of the conditions (2)–(5), then $-\sigma\alpha$ is different from α but $-\theta\alpha$ or $\theta\sigma\alpha$ coincides with α. On the other hand, if α satisfies one of the conditions (6)–(9) in Lemmas (3.10) and (3.11), then any two of the quartet $(\alpha, -\theta\alpha, -\sigma\alpha, \theta\sigma\alpha)$ are mutually different. Even if α satisfies one of the conditions (1′)–(9′) instead of (1)–(9), the situation is not changed.

We give in Table III the number of roots satisfying the condition (1), that of pairs satisfying one of (2)–(5), and that of quartet satisfying one of the conditions (6)–(9) for each irreducible symmetric pair of split rank 1.

Table III. Classification of roots

Class	$m(1)$	$m(2)$	$m(3)$	$m(4)$	$m(5)$	$m(6)$	$n(1)$	$n(2)$
I_1 ($p+q$: odd)	1	$\min(p,q)$	0	$\frac{\lvert p-q\rvert-1}{2}$	0	0	0	0
I_1 ($p+q$: even)	0	$\min(p,q)$	0	$\frac{\lvert p-q\rvert}{2}$	0	0	0	0
I_2	0	0	0	0	$\lvert p-q\rvert$	$\min(p,q)$	1	0
I_3	0	0	0	0	$2\lvert p-q\rvert$	$2\min(p,q)$	1	1
I_4	0	0	0	0	4	0	1	3
II_1	0	0	m	0	0	0	1	0
II_2	0	0	$2m$	0	0	0	1	1
II_3	0	0	4	0	0	0	1	3

Class	$m(2)$	$m(6)$	$m(7)$	$m(8)$	$m(9)$	$n(1)$	$n(2)$	$n(4)$	$n(9)$
III_1^d (m: odd)	1	0	0	0	$\frac{m-1}{2}$	0	0	0	0
III_1^d (m: even)	0	0	0	0	$\frac{m}{2}$	0	0	0	0
III_2^d	0	0	m	0	0	0	1	0	0
III_3^d	0	0	$2m$	0	0	0	1	0	1
III_4^d	0	0	4	0	0	0	1	0	3
IV_1^d (m: odd)	0	1	0	0	$m-1$	1	0	0	0
IV_1^d (m: even)	0	0	0	0	m	1	0	0	0
IV_2^d	0	0	$2m$	0	0	0	1	1	0
IV_3^d	0	0	4	0	0	0	1	3	0
V_1	0	0	0	1	0	0	0	1	0
V_2	0	0	0	2	0	0	1	1	0
V_3	0	0	0	4	0	0	3	1	0

There we pay attention mainly to the symmetric pairs contained in the classes I, II, III^d, IV^d, V. We give here comments on the notation used there. The number $m(i)$ $(1\leqq i\leqq 9)$ means that of the roots, the pairs or the quartets of roots contained in $R(\tilde{\mathfrak{j}};\lambda)$ and satisfying the condition (i). Similarly $n(i)$ means that of the roots, the pairs or the quartets contained in $R(\tilde{\mathfrak{j}}:2\lambda)$ and satisfying the condition (i). Hence the multiplicity $m(\lambda)$ of λ and that of $m(2\lambda)$ are obtained from the following formulas:

$$m(\lambda)=m(1)+2\sum_{i=2}^{5}m(i)+4\sum_{i=6}^{9}m(i)$$

$$m(2\lambda)=n(1)+2n(2)+2n(4)+4n(9).$$

As to the dual of the given symmetric pair, these informations are obtained by replacing $m(i)$, $n(j)$ with $m(i')$, $n(j')$, respectively. Here $m(i')$ and $n(j')$ are the numbers defined similarly as $m(i)$ and $n(j)$.

(5.5) We give here some observations which are obtained from Table III.

(5.5.0) Let $(\mathfrak{g},\mathfrak{h})$ be a symmetric pair of split rank 1 satisfying the condition: $\theta\alpha=\sigma\alpha$ for any $\alpha\in R(\tilde{\mathfrak{j}};\lambda)$. Then $(\mathfrak{g},\mathfrak{h})$ is one of the pairs in I_i $(p=0$ or $q=0)$ $(i=1,2,3)$ and I_4^j $(j=1,2)$. In these cases, $\theta=\sigma$ and the pairs are of Type $(\mathfrak{k}_\varepsilon)$.

(5.5.1) The pairs in I_i and I_i^d $(i=1,2,3,4)$ are characterized by the condition: There exists a root α of $\Sigma(\tilde{\mathfrak{j}})$ contained in $R(\tilde{\mathfrak{j}};\lambda)$ such that $\theta\alpha=\sigma\alpha$.

(5.5.2) The pairs in I_1 $(|p-q|\leqq 1)$, II_i $(i=1,2,3)$ are characterized by the condition: $\theta\alpha=-\alpha$ for any $\alpha\in\Sigma(\tilde{\mathfrak{j}})$. This is clear from the reason that $\mathfrak{g}$ is a normal real form in this case.

(5.5.3) The pairs in III_i $(i=1,2,3)$ are characterized by the condition: $\langle\alpha,\theta\alpha\rangle=\langle\alpha,\theta\sigma\alpha\rangle=0$ for any $\alpha\in\Sigma(\tilde{\mathfrak{j}})$. This follows from the reason that $\mathfrak{g}$ is a complex semisimple Lie algebra.

(5.5.4) The pairs in I_1^d $(p=q\geqq 2)$, IV_i $(i=1,2,3)$ are characterized by the condition: $\langle\alpha,\theta\alpha\rangle=\langle\alpha,\theta\sigma\alpha\rangle$ for any $\alpha\in R(\tilde{\mathfrak{j}};\lambda)$ and $\Sigma(\tilde{\mathfrak{j}})$ is connected.

(5.5.5) The pairs in I_1, I_1^d $(p+q$: even) and V_i, V_i^d $(i=1,2,3)$ are characterized by the condition: $\langle\alpha,\theta\alpha\rangle=\langle\alpha,\sigma\alpha\rangle=0$ for any $\alpha\in R(\tilde{\mathfrak{j}};\lambda)$.

Lemma (5.6). *Let α be a root of $\Sigma(\tilde{\mathfrak{j}})$ such that $\mu=\alpha\,|\,\mathfrak{a}\neq 0$.*

(i) *Assume that $\theta\alpha\neq\sigma\alpha$ and $\sigma\alpha=-\alpha$. Then the subspace*

$$\mathfrak{g}\cap(\mathfrak{g}_C(\tilde{\mathfrak{j}};\alpha)+\mathfrak{g}_C(\tilde{\mathfrak{j}};-\theta\alpha))$$

of $\mathfrak{g}(\mathfrak{a};\mu)$ is two dimensional and is spanned by such vectors X and Y that

$\theta\sigma X = X$ and $\theta\sigma Y = -Y$.

(ii) *Assume that any two of the quartet $(\alpha, -\theta\alpha, -\sigma\alpha, \theta\sigma\alpha)$ are mutually different. Then the subspace*

$$\mathfrak{g} \cap (\mathfrak{g}_C(\tilde{\mathfrak{j}}; \alpha) + \mathfrak{g}_C(\tilde{\mathfrak{j}}; -\theta\alpha) + \mathfrak{g}_C(\tilde{\mathfrak{j}}; -\sigma\alpha) + \mathfrak{g}_C(\tilde{\mathfrak{j}}; \theta\sigma\alpha))$$

of $\mathfrak{g}(\mathfrak{a}; \mu)$ is four dimensional and is spanned by such vectors X_i, Y_i $(i=1, 2)$ that $\theta\sigma X_i = X_i$ and $\theta\sigma Y_i = -Y_i$ $(i=1, 2)$.

Proof. (i) Assume that $\alpha \in \Sigma(\tilde{\mathfrak{j}})$ satisfies $\theta\alpha \neq \sigma\alpha$ and $\sigma\alpha = -\alpha$. Let $Z \in \mathfrak{g}_C(\tilde{\mathfrak{j}}; \alpha)$, $Z \neq 0$. Then its complex conjugate with respect to $\mathfrak{g}$ is contained in $\mathfrak{g}_C(\tilde{\mathfrak{j}}; -\theta\alpha)$. In this case, $\sigma\alpha = -\alpha$ and therefore $\theta\sigma Z \in \mathfrak{g}_C(\tilde{\mathfrak{j}}; -\theta\alpha)$. Hence by multiplying Z by a constant if necessary, we may assume that $\theta\sigma Z$ is the conjugate of Z. Then $X = Z + \theta\sigma Z$ and $Y = \sqrt{-1}(Z - \theta\sigma Z)$ are a required basis of $\mathfrak{g} \cap (\mathfrak{g}_C(\tilde{\mathfrak{j}}; \alpha) + \mathfrak{g}_C(\tilde{\mathfrak{j}}; -\theta\alpha))$.

Next prove (ii). As in the case of (i), there exist $X \in \mathfrak{g}_C(\tilde{\mathfrak{j}}; \alpha)$ and $Y \in \mathfrak{g}_C(\tilde{\mathfrak{j}}; -\theta\alpha)$ such that $X+Y$ and $\sqrt{-1}(X-Y)$ form a basis of $\mathfrak{g} \cap (\mathfrak{g}_C(\tilde{\mathfrak{j}}; \alpha) + \mathfrak{g}_C(\tilde{\mathfrak{j}}; -\theta\alpha))$. Then

$$X_1 = X + Y + \theta\sigma(X+Y), \qquad X_2 = \sqrt{-1}\,(X - Y + \theta\sigma(X-Y)),$$
$$Y_1 = X - Y - \theta\sigma(X-Y), \qquad Y_2 = \sqrt{-1}\,(X - Y - \theta\sigma(X-Y))$$

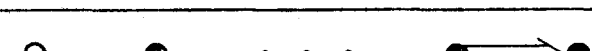
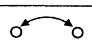
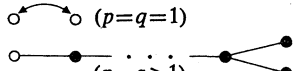
are required ones. q.e.d.

Proposition (5.7). *Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair of split rank 1 and let $\mu \in \Sigma(\mathfrak{a})$. If $\theta\alpha \neq \sigma\alpha$ for any $\alpha \in R(\tilde{\mathfrak{j}}; \mu)$, then $m^+(\mu) = m^-(\mu)$.*

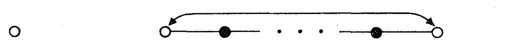
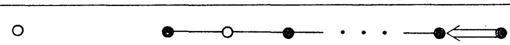
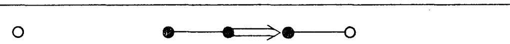
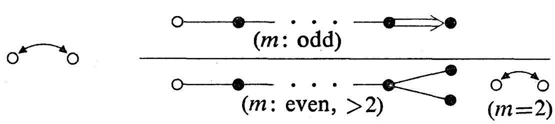
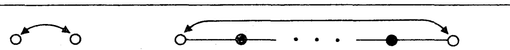
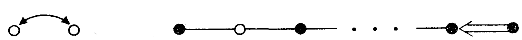
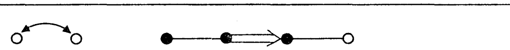
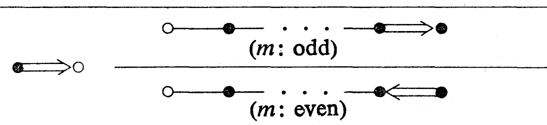
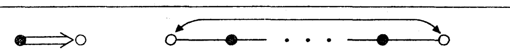
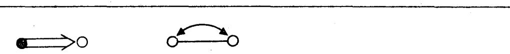
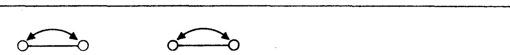
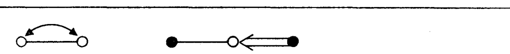
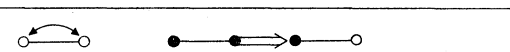
Proof. This is a direct consequence of Lemma (5.6).

(5.8) Last we explain Table IV. Let $(\mathfrak{g}, \mathfrak{h})$ be an irreducible symmetric pair of split rank one. Let $\Psi(\mathfrak{j})$ (resp. $\Psi(\mathfrak{a}_\mathfrak{p})$) be the $\theta\sigma$-fundamental system of $\Sigma(\mathfrak{j})$ (resp. $\Sigma(\mathfrak{a}_\mathfrak{p})$) for the given order. Then we can define a diagram for the pair $(\Sigma(\mathfrak{a}_\mathfrak{p}), \theta\sigma)$ similar to the Satake diagram

Table IV

	$(\Sigma(\mathfrak{j}), \theta\sigma)$	$(\Sigma(\mathfrak{a}_\mathfrak{p}), \theta\sigma)$	$\begin{pmatrix} m^+(\lambda) & m^+(2\lambda) \\ m^-(\lambda) & m^-(2\lambda) \end{pmatrix}$
I_1	○	 $(p \neq q)$  $(p=q=1)$  $(p=q>1)$	$\begin{pmatrix} p & 0 \\ q & 0 \end{pmatrix}$
I_2	○		$\begin{pmatrix} 2p & 1 \\ 2q & 0 \end{pmatrix}$

(Continued from Table IV)

	$(\Sigma(\mathfrak{j}), \theta\sigma)$ — $(\Sigma(\mathfrak{a}_\mathfrak{p}), \theta\sigma)$	$\begin{pmatrix} m^+(\lambda) & m^+(2\lambda) \\ m^-(\lambda) & m^-(2\lambda) \end{pmatrix}$
I_3		$\begin{pmatrix} 4p & 3 \\ 4q & 0 \end{pmatrix}$
I_4^1		$\begin{pmatrix} 8 & 7 \\ 0 & 0 \end{pmatrix}$
I_4^2		$\begin{pmatrix} 0 & 7 \\ 8 & 0 \end{pmatrix}$
II_1		$\begin{pmatrix} m & 0 \\ m & 1 \end{pmatrix}$
II_2		$\begin{pmatrix} 2m & 1 \\ 2m & 2 \end{pmatrix}$
II_3		$\begin{pmatrix} 4 & 3 \\ 4 & 4 \end{pmatrix}$
III_1		$\begin{pmatrix} m & 0 \\ m & 0 \end{pmatrix}$
III_2		$\begin{pmatrix} 2m & 1 \\ 2m & 1 \end{pmatrix}$
III_3		$\begin{pmatrix} 4m & 3 \\ 4m & 3 \end{pmatrix}$
III_4		$\begin{pmatrix} 8 & 7 \\ 8 & 7 \end{pmatrix}$
IV_1		$\begin{pmatrix} 2m & 1 \\ 2m & 0 \end{pmatrix}$
IV_2		$\begin{pmatrix} 4m & 3 \\ 4m & 1 \end{pmatrix}$
IV_3		$\begin{pmatrix} 8 & 7 \\ 8 & 1 \end{pmatrix}$
V_1		$\begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$
V_2		$\begin{pmatrix} 4 & 1 \\ 4 & 3 \end{pmatrix}$
V_3		$\begin{pmatrix} 8 & 3 \\ 8 & 5 \end{pmatrix}$

for a real form of a complex semisimple Lie algebra. We give this one in Table IV. Similarly we can define a diagram for the pair $(\Sigma(\mathfrak{j}), \theta\sigma)$. This is given in the left-hand side of that for $(\Sigma(\mathfrak{a}_\mathfrak{p}), \theta\sigma)$. Accordingly, to obtain these informations on the dual $(\mathfrak{g}^d, \mathfrak{h}^d)$, it is sufficient to look at the one whose θ and σ, $\mathfrak{j}$ and $\mathfrak{a}_\mathfrak{p}$ are changed into σ and θ, $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{j}$, respectively. We also collect the signatures of λ and 2λ in Table IV.

§ 6. Determination of the restricted root system

This section is devoted to a determination of the restricted root system of a general symmetric pair.

(6.1) Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair. As usual, let σ be the involution for $(\mathfrak{g}, \mathfrak{h})$ and let θ be a Cartan involution of $\mathfrak{g}$ commuting with σ. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ and let $\Sigma(\mathfrak{a})$ be the restricted root system of $(\mathfrak{g}, \mathfrak{h})$. For a signature ε of $\Sigma(\mathfrak{a})$, we define an involution σ_ε from σ by

$$\sigma_\varepsilon(X) = \begin{cases} \sigma(X) & \text{for } X \in Z_\mathfrak{g}(\mathfrak{a}) \\ \varepsilon(\lambda)\sigma(X) & \text{for } X \in \mathfrak{g}(\mathfrak{a}; \lambda),\ \lambda \in \Sigma(\mathfrak{a}) \end{cases}$$

where $Z_\mathfrak{g}(\mathfrak{a}) = \{X \in \mathfrak{g}; [X, \mathfrak{a}] = 0\}$ (cf. Example (1.9.3)). Let $\mathfrak{g} = \mathfrak{h}_\varepsilon + \mathfrak{q}_\varepsilon$ be the direct sum decomposition of $\mathfrak{g}$ corresponding to σ_ε. By definition, σ_ε commutes with θ and $\mathfrak{a}$ is also a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}_\varepsilon$. This implies that $\Sigma(\mathfrak{a})$ is also the restricted root system of $(\mathfrak{g}, \mathfrak{h}_\varepsilon)$. However, the signatures of the restricted roots are changed in general. Namely, if $(m^+(\lambda, \varepsilon), m^-(\lambda, \varepsilon))$ denotes the signature of $\lambda \in \Sigma(\mathfrak{a})$ as a root of $(\mathfrak{g}, \mathfrak{h}_\varepsilon)$, then $(m^+(\lambda, \varepsilon), m^-(\lambda, \varepsilon)) = (m^+(\lambda), m^-(\lambda))$ in the case where $\varepsilon(\lambda) = 1$ and $(m^+(\lambda, \varepsilon), m^-(\lambda, \varepsilon)) = (m^-(\lambda), m^+(\lambda))$ in the case where $\varepsilon(\lambda) = -1$. We note here that the complexifications of $\mathfrak{h}$ and $\mathfrak{h}_\varepsilon$ are isomorphic (cf. [O–S, Lemma 1.3]).

For a symmetric pair $(\mathfrak{g}, \mathfrak{h})$, we denote by $F((\mathfrak{g}, \mathfrak{h}))$ the totality of symmetric pairs $(\mathfrak{g}, \mathfrak{h}_\varepsilon)$ for all signatures ε of $\Sigma(\mathfrak{a})$ and call it an ε-family of symmetric pairs (obtained from $(\mathfrak{g}, \mathfrak{h})$).

It is clear from the definition that if $(\mathfrak{g}, \mathfrak{h})$ is irreducible, so is each member of $F((\mathfrak{g}, \mathfrak{h}))$.

(6.2) It is not clear whether for different signatures ε, ε' of $\Sigma(\mathfrak{a})$, the pairs $(\mathfrak{g}, \mathfrak{h}_\varepsilon)$ and $(\mathfrak{g}, \mathfrak{h}_{\varepsilon'})$ are isomorphic or not. If $(\mathfrak{g}, \mathfrak{h})$ is a Riemannian symmetric pair, then $F((\mathfrak{g}, \mathfrak{h}))$ consists of those pairs defined in Example (1.9.3). On the other hand, we find from the classification that if $m^+(\lambda) = m^-(\lambda)$ for any $\lambda \in \Sigma(\mathfrak{a})$, all the pairs of $F((\mathfrak{g}, \mathfrak{h}))$ are isomorphic to each other. For example, this is actually the case when $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{sl}(2l+2, \boldsymbol{R}), \mathfrak{sp}(l+1, \boldsymbol{R}))$ (cf. Table V). In general, $\mathfrak{h}$ is a reductive Lie algebra and let $\mathfrak{h} = \mathfrak{h}_c + \mathfrak{h}_n + \mathfrak{z}(\mathfrak{h})$ be the direct decomposition, where $\mathfrak{h}_c$ (resp. $\mathfrak{h}_n$) is a semisimple Lie algebra of the compact (resp. non-compact) type and

$\mathfrak{z}(\mathfrak{h})$ is the center of $\mathfrak{h}$. Then for the sake of convenience, we call $\mathfrak{h}_n$ the non-compact part of $\mathfrak{h}$.

Lemma (6.3). *Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair and let $\Sigma(\mathfrak{a})$ be its restricted root system as above. Let Ψ be a fundamental system for $\Sigma(\mathfrak{a})$ and let $W(\mathfrak{a})$ be the Weyl group of $\Sigma(\mathfrak{a})$. Assume that $m^+(\lambda)\geq m^-(\lambda)$ for any $\lambda \in \Psi$. Then $m^\pm(w\lambda)=m^\pm(\lambda)$ for any $\lambda \in \Sigma(\mathfrak{a})$ and $w \in W(\mathfrak{a})$. In particular, $m^+(\lambda)\geq m^-(\lambda)$ for any $\lambda \in \Sigma(\mathfrak{a})$ such that $\frac{1}{2}\lambda \notin \Sigma(\mathfrak{a})$.*

Proof. Let $(\mathfrak{g}, \mathfrak{h}^a)$ be the associated pair of $(\mathfrak{g}, \mathfrak{h})$, namely, $\mathfrak{h}^a=\mathfrak{k}\cap\mathfrak{h}+\mathfrak{p}\cap\mathfrak{q}$ (cf. § 1, (1.2.1)). Let $\mathfrak{h}^a_n$ be the non-compact part of $\mathfrak{h}^a$. Then σ and θ stabilize $\mathfrak{h}^a_n$. Let $\mathfrak{h}^a_n=\mathfrak{k}_n+\mathfrak{p}_n$ be the Cartan decomposition for θ. By definition, $\mathfrak{a}$ is also a maximal abelian subspace of $\mathfrak{p}_n$. The assumption implies that $m^+(\lambda)>0$ for any $\lambda \in \Psi$. On the other hand, it follows that for any $\lambda \in \Sigma(\mathfrak{a})$, $m^+(\lambda)$ is the multiplicity of λ as a restricted root of $\mathfrak{h}^a_n$. Then we find that $m^+(w\lambda)=m^+(\lambda)$ for any $\lambda \in \Sigma(\mathfrak{a})$ and $w \in W(\mathfrak{a})$. But it is clear that $m(w\lambda)=m(\lambda)$ for any $\lambda \in \Sigma(\mathfrak{a})$ and $w \in W(\mathfrak{a})$ (cf. Lemma (7.2) (ii)). Hence we also find that $m^-(w\lambda)=m^-(\lambda)$. Since for any $\lambda \in \Sigma(\mathfrak{a})$, with $\frac{1}{2}\lambda \notin \Sigma(\mathfrak{a})$, there exists a $w \in \Sigma(\mathfrak{a})$ such that $w\lambda \in \Psi$, the claim follows. q.e.d.

Definition (6.4). A symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is called basic if $m^+(\lambda)\geq m^-(\lambda)$ for any $\lambda \in \Sigma(\mathfrak{a})$ such that $\frac{1}{2}\lambda \notin \Sigma(\mathfrak{a})$.

It is clear from the definition that any Riemannian symmetric pair is basic.

Proposition (6.5). *Let F be an ε-family of symmetric pairs. Then there exists a basic symmetric pair of F unique up to isomorphisms.*

Proof. It suffices to prove the claim when each symmetric pair of F is irreducible. Hence we may assume that F contains only irreducible symmetric pairs and show the existence and the uniqueness.

(Existence) Let $(\mathfrak{g}, \mathfrak{h}) \in F$ and let $\Sigma(\mathfrak{a})$ be the restricted root system of $(\mathfrak{g}, \mathfrak{h})$. Take a fundamental system $\Psi=\{\lambda_1, \cdots, \lambda_l\}$ of $\Sigma(\mathfrak{a})$. We may assume that $m^+(\lambda_i)\geq m^-(\lambda_i)$ if $i\leq k$ and $m^+(\lambda_i)<m^-(\lambda_i)$ if $i>k$. Let ε be a mapping of Ψ to $\{1, -1\}$ defined by $\varepsilon(\lambda_i)=1$ if $i\leq k$ and $\varepsilon(\lambda_i)=-1$ if $i>k$. By definition, ε is uniquely extended to a signature of $\Sigma(\mathfrak{a})$ (cf. [O–S, Def. 1.1]). Denote it by the same letter. Then it is clear from the definition and Lemma (6.3) that $(\mathfrak{g}, \mathfrak{h}_\varepsilon)$ is basic.

(Uniqueness) Uniqueness of a basic symmetric pair contained in F follows from the classification of irreducible symmetric pairs. q.e.d.

(6.6) Let $(\mathfrak{g}, \mathfrak{h})$ be a basic irreducible symmetric pair and let F be the ε-family obtained from $(\mathfrak{g}, \mathfrak{h})$. Let $\Sigma(\mathfrak{a})$ be as above. Here we recall the following (cf. [O–S, Appendix]).

(6.6.1) Let ε be any non-trivial signature of $\Sigma(\mathfrak{a})$. Then there exist a fundamental system Ψ of positive roots of $\Sigma(\mathfrak{a})$ and a unique $\lambda \in \Psi$ such that $\varepsilon(\lambda)=-1$ and $\varepsilon(\mu)=1$ for $\mu \in \Psi-\{\lambda\}$.

Let $\Psi=\{\lambda_1, \cdots, \lambda_l\}$ be a fundamental system for $\Sigma(\mathfrak{a})$ and fix it once for all. Noting that $(\mathfrak{g}, \mathfrak{h})$ is basic, we may assume that $m^+(\lambda_i)>m^-(\lambda_i)$ $(i\leqq l')$ and $m^+(\lambda_i)=m^-(\lambda_i)$ $(i>l')$. Then we have the following observation.

(6.6.2) Put $\Sigma'(\mathfrak{a})=(\Sigma_{i=1}^{l'} \boldsymbol{R}\lambda_i)\cap\Sigma(\mathfrak{a})$. Then $\Sigma'(\mathfrak{a})$ is an irreducible root system and its fundamental system is $\Psi'(\mathfrak{a})=\{\lambda_1, \cdots, \lambda_{l'}\}$.

For $1\leq i\leq l$, let ε_i be the signature of $\Sigma(\mathfrak{a})$ such that $\varepsilon_i(\lambda_j)=1$ if $j\neq i$ and $\varepsilon_i(\lambda_i)=-1$. Then (6.6.1) implies the following.

(6.6.3) Let $(\mathfrak{g}, \mathfrak{h}')$ be any symmetric pair contained in F. Then there exists an i $(1\leq i\leq l)$ such that $(\mathfrak{g}, \mathfrak{h}')$ is isomorphic to $(\mathfrak{g}, \mathfrak{h}_{\varepsilon_i})$.

In the case where the ε-family F contains a Riemannian symmetric pair, namely, each of F is of Type $(\mathfrak{k}_\varepsilon)$, the mutually non-isomorphic pairs contained in F is determined in [O–S, Appendix]. In the general case, by the choice of simple roots, we have the following observation.

(6.6.4) If $l'<i\leq l$, then $(\mathfrak{g}, \mathfrak{h}_{\varepsilon_i})$ is isomorphic to $(\mathfrak{g}, \mathfrak{h})$. On the other hand, if $i\leq l'$, then $(\mathfrak{g}, \mathfrak{h}_{\varepsilon_i})$ is not basic. Furthermore, it frequently occurs that $(\mathfrak{g}, \mathfrak{h}_{\varepsilon_i})$ and $(\mathfrak{g}, \mathfrak{h}_{\varepsilon_j})$ are isomorphic to each other even if $i\neq j$ and $i, j\leq l'$.

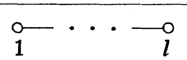
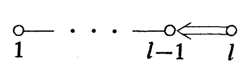
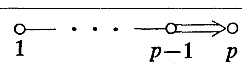
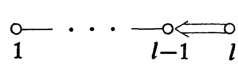
(6.7) We consider a pair $(\mathfrak{g}, \mathfrak{h})$ of Type $(\mathfrak{k}_\varepsilon)$ in the sense of (1.12). In this case, as is noted in (2.16) (3), the restricted root system of $(\mathfrak{g}, \mathfrak{h})$ coincides with that of the Riemannian symmetric pair $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$. It is also noted there that the signature of each restricted root of the system is easily determined (cf. (2.16) (3)).

(6.8) Next we consider a symmetric pair in Example (1.9.4) of Section 1. Let $(\mathfrak{g}, \mathfrak{h})$ be such a pair. In this case, there is a real semisimple Lie algebra $\mathfrak{g}'$ such that $\mathfrak{g}=\mathfrak{g}'\oplus\mathfrak{g}'$ and $\mathfrak{h}\simeq\mathfrak{g}'$. Let $\mathfrak{g}'=\mathfrak{k}'+\mathfrak{p}'$ be a Cartan decomposition of $\mathfrak{g}'$ and let θ' be the corresponding Cartan involution of $\mathfrak{g}'$. Then putting $\mathfrak{k}=\mathfrak{k}'\oplus\mathfrak{k}'$ and $\mathfrak{p}=\mathfrak{p}'\oplus\mathfrak{p}'$, we have a Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ of $\mathfrak{g}$. If $\mathfrak{a}_{\mathfrak{p}'}$ is a maximal abelian subspace of $\mathfrak{p}'$, then $\mathfrak{a}=\{(X, -X); X\in\mathfrak{a}_{\mathfrak{p}'}\}$ is that of $\mathfrak{p}\cap\mathfrak{q}$. It is clear that the restricted root system $\Sigma(\mathfrak{a})$ of the pair $(\mathfrak{g}, \mathfrak{h})$ coincides with the restricted root system $\Sigma(\mathfrak{a}_{\mathfrak{p}'})$ of $\mathfrak{g}'$. For any root λ of $\Sigma(\mathfrak{a}_{\mathfrak{p}'})$, we denote by $\mathfrak{g}'(\mathfrak{a}_{\mathfrak{p}'}; \lambda)$ the root space of λ in $\mathfrak{g}'$. Then it follows that $\mathfrak{g}^{\pm}(\mathfrak{a}; \lambda)=\{(X, \pm\theta'X); X\in\mathfrak{g}'(\mathfrak{a}_{\mathfrak{p}'}; \lambda)\}$. Hence we find that if $m'(\lambda)=\dim_{\boldsymbol{R}}\mathfrak{g}'(\mathfrak{a}_{\mathfrak{p}'}; \lambda)$, then $m^+(\lambda)=m^-(\lambda)=m'(\lambda)$. As is noted before, $(\mathfrak{g}'_C, \mathfrak{k}'_C)$ is dual to $(\mathfrak{g}, \mathfrak{h})$. Hence Lemma (2.15.1) implies that the restricted root system of $(\mathfrak{g}'_C, \mathfrak{k}'_C)$ coincides with that of $(\mathfrak{g}, \mathfrak{h})$.

(6.9) We collect in Table V the restricted root systems of all the irreducible symmetric pairs such that they are neither of the compact type nor of Type $(\mathfrak{k}_\varepsilon)$. The arguments in (3.12) play a fundamental role in the course of the determination of the restricted root system of a given symmetric pair. In Table V, we also collect the signatures of the simple roots and those of their multiples. Let $(\mathfrak{g}, \mathfrak{h})$ be an irreducible symmetric pair and let $\Sigma(\mathfrak{a})$ be its restricted root system. Then as is already remarked in (6.6.1), there exist a fundamental system $\Psi(\mathfrak{a})$ for $\Sigma(\mathfrak{a})$ and a simple root $\lambda \in \Psi(\mathfrak{a})$ such that $m^+(\mu) \geq m^-(\mu)$ for any $\mu \in \Psi(\mathfrak{a}) - \{\lambda\}$. In Table V, we take such $\Psi(\mathfrak{a})$ and λ. The choices of $\Psi(\mathfrak{a})$ and λ are not unique (cf. [O–S, Appendix] and (6.6)). The results of Sections 4 and 5 play fundamental roles in the course of the determination of the signatures.

We give here some remarks on Table V. It follows from Lemma (2.15.1) that for a given symmetric pair $(\mathfrak{g}, \mathfrak{h})$, the restricted root system of $(\mathfrak{g}, \mathfrak{h})^d$ coincides with that of $(\mathfrak{g}, \mathfrak{h})$. Hence we set them in the same row in Table V. It is useful to know $\mathfrak{h}^a$ from $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}, \mathfrak{h})^d$ (cf. Table (2.5.2)). In some cases, one of $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}, \mathfrak{h})^d$ is self-associated. In this case we always write the self-associated pair in the lower part of the frame. On the other hand, in some cases, each of $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}, \mathfrak{h})^d$ is not self-associated. It is preferable to give an information on $\mathfrak{h}^a$ in these cases. First if $(\mathfrak{g}, \mathfrak{h})$ is of Type $(\mathrm{C}, \boldsymbol{R})$, then $\mathfrak{h}^a$ is a complexification of a maximal compact

Table V

$(\mathfrak{g}, \mathfrak{h})$ $(\mathfrak{g}^d, \mathfrak{h}^d)$	$\Psi(\mathfrak{a})$	$\begin{pmatrix} m^+(\lambda_i) & m^+(2\lambda_i) \\ m^-(\lambda_i) & m^-(2\lambda_i) \end{pmatrix}$
$(\mathfrak{sl}(2l+2, \boldsymbol{R}), \mathfrak{sp}(l+1, \boldsymbol{R}))$ † $(\mathfrak{su}^*(2l+2), \mathfrak{so}^*(2l+2))$		$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$
$(\mathfrak{sl}(2l, \boldsymbol{R}), \mathfrak{sl}(l, \boldsymbol{C}) + \sqrt{-1}\boldsymbol{R})$ † $(\mathfrak{su}(l, l), \mathfrak{so}^*(2l))$		$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $(i<l)$ $(i=l)$
$(\mathfrak{sl}(2l, \boldsymbol{R}), \mathfrak{sl}(l, \boldsymbol{R}) + \mathfrak{sl}(l, \boldsymbol{R}) + \boldsymbol{R})$ $(\mathfrak{su}(l, l), \mathfrak{so}(l, l))$		$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $(i<l)$ $(i=l)$
$(\mathfrak{sl}(l+1, \boldsymbol{R}), \mathfrak{sl}(p, \boldsymbol{R})$ $+\mathfrak{sl}(l-p+1, \boldsymbol{R}) + \boldsymbol{R})$ $(\mathfrak{su}(p, l-p+1), \mathfrak{so}(p, l-p+1))$	 $(p<l/2)$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} l-2p & 0 \\ l-2p & 1 \end{pmatrix}$ $(i<p)$ $(i=p)$
$(\mathfrak{su}^*(4l), \mathfrak{su}^*(2l) + \mathfrak{su}^*(2l) + \boldsymbol{R})$ $(\mathfrak{su}(2l, 2l), \mathfrak{sp}(l, l))$		$\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$ $\begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix}$ $(i<l)$ $(i=l)$
$(\mathfrak{su}^*(4l), \mathfrak{sl}(2l, \boldsymbol{C}) + \sqrt{-1}\boldsymbol{R})$ † $(\mathfrak{su}(2l, 2l), \mathfrak{sp}(2l, \boldsymbol{R}))$		$\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$ $(i<l)$ $(i=l)$

(Continued from Table V)

$(\mathfrak{g}, \mathfrak{h})$ $(\mathfrak{g}^d, \mathfrak{h}^d)$	$\Psi(\mathfrak{a})$	$\begin{pmatrix} m^+(\lambda_i) & m^+(2\lambda_i) \\ m^-(\lambda_i) & m^-(2\lambda_i) \end{pmatrix}$
$(\mathfrak{su}^*(2l), \mathfrak{su}^*(2p)+\mathfrak{su}^*(2l-2p)+\boldsymbol{R})$ $(\mathfrak{su}(2p, 2l-2p), \mathfrak{sp}(p, l-p))$	$\underset{1}{\circ}-\cdots-\underset{p-1}{\circ}\Rightarrow\underset{p}{\circ}$ $(p<l/2)$	$\begin{pmatrix}4&0\\4&0\end{pmatrix}$ $\begin{pmatrix}4(l-2p)&3\\4(l-2p)&1\end{pmatrix}$ $(i<p)$ $(i=p)$
$(\mathfrak{su}^*(4l+2), \mathfrak{sl}(2l+1, \boldsymbol{C})+\sqrt{-1}\boldsymbol{R})$ † $(\mathfrak{su}(2l+1, 2l+1), \mathfrak{sp}(2l+1, \boldsymbol{R}))$	$\underset{1}{\circ}-\cdots-\underset{l-1}{\circ}\Rightarrow\underset{l}{\circ}$	$\begin{pmatrix}4&0\\4&0\end{pmatrix}$ $\begin{pmatrix}4&1\\4&3\end{pmatrix}$ $(i<l)$ $(i=l)$
$(\mathfrak{su}(l, l), \mathfrak{su}(p)+\mathfrak{su}(l, l-p)$ $+\sqrt{-1}\boldsymbol{R})$ $(1\leqq p<l)$ # $(\mathfrak{su}(p, 2l-p), \mathfrak{su}(l)+\mathfrak{su}(p, l-p)$ $+\sqrt{-1}\boldsymbol{R})$	$\underset{1}{\circ}-\cdots-\underset{p-1}{\circ}\Rightarrow\underset{p}{\circ}$	$\begin{pmatrix}2&0\\0&0\end{pmatrix}$ $\begin{pmatrix}2(l-p)&1\\2(l-p)&0\end{pmatrix}$ $(i<p)$ $(i=p)$
$(\mathfrak{su}(l, l), \mathfrak{su}(k, p-k)$ $+\mathfrak{su}(l-k, l-p+k)+\sqrt{-1}\boldsymbol{R})$ # $(\mathfrak{su}(p, 2l-p), \mathfrak{su}(k, l-k)$ $+\mathfrak{su}(p-k, l-p+k)+\sqrt{-1}\boldsymbol{R})$		$\begin{pmatrix}2&0\\0&0\end{pmatrix}$ $\begin{pmatrix}0&0\\2&0\end{pmatrix}$ $\begin{pmatrix}2(l-p)&1\\2(l-p)&0\end{pmatrix}$ $(i<p, i\neq k)$ $(i=k)$ $(i=p)$
$(\mathfrak{su}(r, p+q-r), \mathfrak{su}(p)+\mathfrak{su}(r, q-r)$ $+\sqrt{-1}\boldsymbol{R})$ # $(\mathfrak{su}(p, q), \mathfrak{su}(r)+\mathfrak{su}(p, q-r)$ $+\sqrt{-1}\boldsymbol{R})$	$\underset{1}{\circ}-\cdots-\underset{r-1}{\circ}\Rightarrow\underset{r}{\circ}$ $(0<k<r<q<p)$	$\begin{pmatrix}2&0\\0&0\end{pmatrix}$ $\begin{pmatrix}2(p-r)&1\\2(q-r)&0\end{pmatrix}$ $(i<r)$ $(i=r)$
$(\mathfrak{su}(r, p+q-r), \mathfrak{su}(k, p-k)$ $+\mathfrak{su}(r-k, q-r+k)+\sqrt{-1}\boldsymbol{R})$ $(\mathfrak{su}(p, q), \mathfrak{su}(k, r-k)$ $+\mathfrak{su}(p-k, q-r+k)+\sqrt{-1}\boldsymbol{R})$ #		$\begin{pmatrix}2&0\\0&0\end{pmatrix}$ $\begin{pmatrix}0&0\\2&0\end{pmatrix}$ $\begin{pmatrix}2(p-r)&1\\2(q-r)&0\end{pmatrix}$ $(i<r, i\neq k)$ $(i=k)$ $(i=r)$
$(\mathfrak{su}(r, p+q-r), \mathfrak{su}(r, p-r)$ $+\mathfrak{su}(q)+\sqrt{-1}\boldsymbol{R})$ # $(\mathfrak{su}(p, q), \mathfrak{su}(r)+\mathfrak{su}(p-r, q)$ $+\sqrt{-1}\boldsymbol{R})$		$\begin{pmatrix}2&0\\0&0\end{pmatrix}$ $\begin{pmatrix}2(q-r)&1\\2(p-r)&0\end{pmatrix}$ $(i<r)$ $(i=r)$
$(\mathfrak{sp}(l, l), \mathfrak{su}^*(2l)+\boldsymbol{R})$ $(\mathfrak{sp}(2l, \boldsymbol{R}), \mathfrak{sp}(l, \boldsymbol{C}))$	$\underset{1}{\circ}-\cdots-\underset{l-1}{\circ}\Leftarrow\underset{l}{\circ}$	$\begin{pmatrix}2&0\\2&0\end{pmatrix}$ $\begin{pmatrix}2&0\\1&0\end{pmatrix}$ $(i<l)$ $(i=l)$
$(\mathfrak{sp}(2l, \boldsymbol{R}), \mathfrak{sp}(l, \boldsymbol{R})+\mathfrak{sp}(l, \boldsymbol{R}))$ $(\mathfrak{sp}(l, l), \mathfrak{su}(l, l)+\sqrt{-1}\boldsymbol{R})$		$\begin{pmatrix}2&0\\2&0\end{pmatrix}$ $\begin{pmatrix}1&0\\2&0\end{pmatrix}$ $(i<l)$ $(i=l)$
$(\mathfrak{sp}(l, \boldsymbol{R}), \mathfrak{sp}(p, \boldsymbol{R})+\mathfrak{sp}(l-p, \boldsymbol{R}))$ $(\mathfrak{sp}(p, l-p), \mathfrak{su}(p, l-p)+\sqrt{-1}\boldsymbol{R})$	$\underset{1}{\circ}-\cdots-\underset{p-1}{\circ}\Rightarrow\underset{p}{\circ}$ $(p<(l+1)/2)$	$\begin{pmatrix}2&0\\2&0\end{pmatrix}$ $\begin{pmatrix}2(l-2p)&1\\2(l-2p)&2\end{pmatrix}$ $(i<p)$ $(i=p)$
$(\mathfrak{sp}(l, l), \mathfrak{sp}(p)+\mathfrak{sp}(l, l-p))$ # $(\mathfrak{sp}(p, 2l-p), \mathfrak{sp}(l)+\mathfrak{sp}(p, l-p))$	$\underset{1}{\circ}-\cdots-\underset{p-1}{\circ}\Rightarrow\underset{p}{\circ}$	$\begin{pmatrix}4&0\\0&0\end{pmatrix}$ $\begin{pmatrix}4(l-p)&3\\4(l-p)&0\end{pmatrix}$ $(i<p)$ $(i=p)$
$(\mathfrak{sp}(l, l), \mathfrak{sp}(k, p-k)$ $+\mathfrak{sp}(l-k, l-p+k))$ # $(\mathfrak{sp}(p, 2l-p), \mathfrak{sp}(k, l-k)$ $+\mathfrak{sp}(p-k, l-p+k))$		$\begin{pmatrix}4&0\\0&0\end{pmatrix}$ $\begin{pmatrix}0&0\\4&0\end{pmatrix}$ $\begin{pmatrix}4(l-p)&3\\4(l-p)&0\end{pmatrix}$ $(i<p, i\neq k)$ $(i=k)$ $(i=p)$

(Continued from Table V)

$(\mathfrak{g}, \mathfrak{h})$ $(\mathfrak{g}^d, \mathfrak{h}^d)$	$\Psi(\mathfrak{a})$	$\begin{pmatrix} m^+(\lambda_i) & m^+(2\lambda_i) \\ m^-(\lambda_i) & m^-(2\lambda_i) \end{pmatrix}$
$(\mathfrak{sp}(r, p+q-r), \mathfrak{sp}(p)+\mathfrak{sp}(r, q-r))$ # $(\mathfrak{sp}(p, q), \mathfrak{sp}(r)+\mathfrak{sp}(p, q-r))$		$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 4(p-r) & 3 \\ 4(q-r) & 0 \end{pmatrix}$ $(i<r)$ $(i=r)$
$(\mathfrak{sp}(r, p+q-r), \mathfrak{sp}(k, p-k)$ $+\mathfrak{sp}(r-k, q-r+k))$ # $(\mathfrak{sp}(p, q), \mathfrak{sp}(k, r-k)$ $+\mathfrak{sp}(p-k, q-r+k))$	$\underset{1}{\circ}-\cdots-\underset{r-1}{\circ}\Rightarrow\underset{r}{\circ}$	$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$ $\begin{pmatrix} 4(p-r) & 3 \\ 4(q-r) & 0 \end{pmatrix}$ $(i<r, i\neq k)$ $(i=k)$ $(i=r)$
$(\mathfrak{sp}(r, p+q-r), \mathfrak{sp}(r, p-r)$ $+\mathfrak{sp}(q))$ # $(\mathfrak{sp}(p, q), \mathfrak{sp}(r)+\mathfrak{sp}(p-r, q))$	$(0<k<r<q<p)$	$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 4(q-r) & 3 \\ 4(p-r) & 0 \end{pmatrix}$ $(i<r)$ $(i=r)$
$(\mathfrak{so}(l, l), \mathfrak{so}(p)+\mathfrak{so}(l-p, l))$ # $(\mathfrak{so}(p, 2l-p), \mathfrak{so}(p, l-p)+\mathfrak{so}(l))$	$\underset{1}{\circ}-\cdots-\underset{p-1}{\circ}\Rightarrow\underset{p}{\circ}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} l-p & 0 \\ l-p & 0 \end{pmatrix}$ $(i<p)$ $(i=p)$
$(\mathfrak{so}(l, l), \mathfrak{so}(k, p-k)$ $+\mathfrak{so}(l-k, l-p+k))$ # $(\mathfrak{so}(p, 2l-p), \mathfrak{so}(k, l-k)$ $+\mathfrak{so}(p-k, l-p+k))$	$(k<p<l)$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} l-p & 0 \\ l-p & 0 \end{pmatrix}$ $(i<p, i\neq k)$ $(i=k)$ $(i=p)$
$(\mathfrak{so}^*(4l), \mathfrak{so}^*(2l)+\mathfrak{so}^*(2l))$ $(\mathfrak{so}(2l, 2l), \mathfrak{su}(l, l)+\sqrt{-1}\boldsymbol{R})$	$\underset{1}{\circ}-\cdots-\underset{l-1}{\circ}\Leftarrow\underset{l}{\circ}$	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $(i<l)$ $(i=l)$
$(\mathfrak{so}(2l, 2l), \mathfrak{sl}(2l, \boldsymbol{R})+\boldsymbol{R})$ $(\mathfrak{so}^*(4l), \mathfrak{so}(2l, \boldsymbol{C}))$		$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $(i<l)$ $(i=l)$
$(\mathfrak{so}(2l+1, 2l+1), \mathfrak{sl}(2l+1, \boldsymbol{R})+\boldsymbol{R})$ $(\mathfrak{so}^*(4l+2), \mathfrak{so}(2l+1, \boldsymbol{C}))$	$\underset{1}{\circ}-\cdots-\underset{l-1}{\circ}\Rightarrow\underset{l}{\circ}$	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$ $(i<l)$ $(i=l)$
$(\mathfrak{so}^*(4l+2), \mathfrak{so}^*(2p)$ $+\mathfrak{so}^*(4l-2p+2))$ $(\mathfrak{so}(2p, 4l-2p+2),$ $\mathfrak{su}(p, 2l-p+1)+\sqrt{-1}\boldsymbol{R})$	$\underset{1}{\circ}-\cdots-\underset{p-1}{\circ}\Rightarrow\underset{p}{\circ}$	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 2(2l-2p+1) & 1 \\ 2(2l-2p+1) & 0 \end{pmatrix}$ $(i<p)$ $(i=p)$
$(\mathfrak{so}^*(4l), \mathfrak{so}^*(2p)+\mathfrak{so}^*(4l-2p))$ $(\mathfrak{so}(2p, 4l-2p), \mathfrak{su}(p, 2l-p)$ $+\sqrt{-1}\boldsymbol{R})$	$(p\leqq l)$	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 4(l-p) & 1 \\ 4(l-p) & 0 \end{pmatrix}$ $(i<p)$ $(i=p)$ $(p\neq l)$
$(\mathfrak{so}(r, p+q-r), \mathfrak{so}(p)+\mathfrak{so}(r, q-r))$ # $(\mathfrak{so}(p, q), \mathfrak{so}(r)+\mathfrak{so}(p, q-r))$		$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} p-r & 0 \\ q-r & 0 \end{pmatrix}$ $(i<r)$ $(i=r)$
$(\mathfrak{so}(r, p+q-r), \mathfrak{so}(k, p-k)$ $+\mathfrak{so}(r-k, q-r+k))$ # $(\mathfrak{so}(p, q), \mathfrak{so}(k, r-k)$ $+\mathfrak{so}(p-k, q-r+k))$	$\underset{1}{\circ}-\cdots-\underset{r-1}{\circ}\Rightarrow\underset{r}{\circ}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} p-r & 0 \\ q-r & 0 \end{pmatrix}$ $(i<r, i\neq k)$ $(i=k)$ $(i=r)$
$(\mathfrak{so}(r, p+q-r), \mathfrak{so}(r, p-r)+\mathfrak{so}(q))$ # $(\mathfrak{so}(p, q), \mathfrak{so}(r)+\mathfrak{so}(p-r, q))$	$(0<k<r<q<p)$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} q-r & 0 \\ p-r & 0 \end{pmatrix}$ $(i<r)$ $(i=r)$

(Continued from Table V)

$(\mathfrak{g}, \mathfrak{h})$ $(\mathfrak{g}^d, \mathfrak{h}^d)$	$\Psi(a)$	$\begin{pmatrix} m^+(\lambda_i) & m^+(2\lambda_i) \\ m^-(\lambda_i) & m^-(2\lambda_i) \end{pmatrix}$
$(\mathfrak{so}^*(4l+4), \mathfrak{su}(1,2l+1)+\sqrt{-1}\boldsymbol{R}$ ## self-dual	○— · · · —○⇒○ 1 $l-1$ l	$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 4 & 1 \\ 4 & 0 \end{pmatrix}$ $(i<l)$ $(i=l)$
$(\mathfrak{so}^*(4l+4), \mathfrak{su}(2p+1,2l-2p+1)$ $+\sqrt{-1}\boldsymbol{R})$ ## self-dual $(2p\leqq l)$		$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$ $\begin{pmatrix} 4 & 1 \\ 4 & 0 \end{pmatrix}$ $(i<l, i\neq p)$ $(i=p)$ $(i=l)$
$(\mathfrak{e}_{6(6)}, \mathfrak{su}^*(6)+\mathfrak{su}(2))$ † $(\mathfrak{e}_{6(2)}, \mathfrak{sp}(3,1))$	○—○⇒○—○ 1 2 3 4	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $(i=1,2)$ $(i=3,4)$
$(\mathfrak{e}_{6(6)}, \mathfrak{sl}(6,\boldsymbol{R})+\mathfrak{sl}(2,\boldsymbol{R}))$ $(\mathfrak{e}_{6(2)}, \mathfrak{sp}(4,\boldsymbol{R}))$		$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $(i=1)$ $(i=2)$ $(i=3,4)$
$(\mathfrak{e}_{6(6)}, \mathfrak{so}(5,5)+\boldsymbol{R})$ $(\mathfrak{e}_{6(-14)}, \mathfrak{sp}(2,2))$	○⇒○ 1 2	$\begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix}$ $\begin{pmatrix} 4 & 0 \\ 4 & 1 \end{pmatrix}$ $(i=1)$ $(i=2)$
$(\mathfrak{e}_{6(6)}, \mathfrak{f}_{4(4)})$ † $(\mathfrak{e}_{6(-26)}, \mathfrak{sp}(3,1))$	○—○ 1 2	$\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$
$(\mathfrak{e}_{6(-14)}, \mathfrak{su}(5,1)+\mathfrak{sl}(2,\boldsymbol{R}))$ $(\mathfrak{e}_{6(2)}, \mathfrak{so}^*(10)+\sqrt{-1}\boldsymbol{R})$	○⇒○ 1 2	$\begin{pmatrix} 4 & 0 \\ 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 4 & 1 \\ 4 & 0 \end{pmatrix}$ $(i=1)$ $(i=2)$
$(\mathfrak{e}_{6(2)}, \mathfrak{so}(6,4)+\sqrt{-1}\boldsymbol{R})$ $(\mathfrak{e}_{6(-14)}, \mathfrak{su}(4,2)+\mathfrak{su}(2))$		$\begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}$ $\begin{pmatrix} 4 & 1 \\ 4 & 0 \end{pmatrix}$ $(i=1)$ $(i=2)$
$(\mathfrak{e}_{6(2)}, \mathfrak{f}_{4(4)})$ † $(\mathfrak{e}_{6(-26)}, \mathfrak{su}^*(6)+\mathfrak{su}(2))$	○	$\begin{pmatrix} 8 & 3 \\ 8 & 5 \end{pmatrix}$
$(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9,1)+\boldsymbol{R})$ $(\mathfrak{e}_{6(-14)}, \mathfrak{f}_{4(-20)})$	○	$\begin{pmatrix} 8 & 7 \\ 8 & 1 \end{pmatrix}$
$(\mathfrak{e}_{7(7)}, \mathfrak{so}^*(12)+\mathfrak{su}(2))$ † $(\mathfrak{e}_{7(-5)}, \mathfrak{su}(6,2))$	○—○⇒○—○ 1 2 3 4	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(i=1,2)$ $(i=3,4)$
$(\mathfrak{e}_{7(7)}, \mathfrak{so}(6,6)+\mathfrak{sl}(2,\boldsymbol{R}))$ $(\mathfrak{e}_{7(-5)}, \mathfrak{su}(4,4))$		$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(i=1)$ $(i=2)$ $(i=3,4)$
$(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(2)}+\sqrt{-1}\boldsymbol{R})$ † $(\mathfrak{e}_{7(-25)}, \mathfrak{su}(6,2))$	○⇒○—○ 1 2 3	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$ $(i=1)$ $(i=2,3)$
$(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)}+\boldsymbol{R})$ $(\mathfrak{e}_{7(-25)}, \mathfrak{su}^*(8))$		$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$ $(i=1)$ $(i=2,3)$

(Continued from Table V)

$(\mathfrak{g}, \mathfrak{h})$ $(\mathfrak{g}^d, \mathfrak{h}^d)$	$\Psi(\mathfrak{a})$	$\begin{pmatrix} m^+(\lambda_i) & m^+(2\lambda_i) \\ m^-(\lambda_i) & m^-(2\lambda_i) \end{pmatrix}$
$(\mathfrak{e}_{7(-25)}, \mathfrak{so}(10,2)+\mathfrak{sl}(2, \boldsymbol{R}))$ $(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(-14)}+\sqrt{-1}\boldsymbol{R})$	$\underset{1}{\circ} \Rightarrow \underset{2}{\circ}$	$\begin{pmatrix} 6 & 0 \\ 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 8 & 1 \\ 8 & 0 \end{pmatrix}$ $(i=1)$ $(i=2)$
$(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(2)}+\sqrt{-1}\boldsymbol{R})$ † $(\mathfrak{e}_{7(-25)}, \mathfrak{so}^*(12)+\mathfrak{su}(2))$		$\begin{pmatrix} 2 & 0 \\ 6 & 0 \end{pmatrix}$ $\begin{pmatrix} 8 & 1 \\ 8 & 0 \end{pmatrix}$ $(i=1)$ $(i=2)$
$(\mathfrak{e}_{8(-24)}, \mathfrak{so}(12,4))$ $(\mathfrak{e}_{8(8)}, \mathfrak{e}_{7(-5)}+\mathfrak{su}(2))$	$\underset{1}{\circ} - \underset{2}{\circ} \Rightarrow \underset{3}{\circ} - \underset{4}{\circ}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$ $(i=1,2)$ $(i=3,4)$
$(\mathfrak{e}_{8(8)}, \mathfrak{e}_{7(7)}+\mathfrak{sl}(2, \boldsymbol{R}))$ $(\mathfrak{e}_{8(-24)}, \mathfrak{so}^*(16))$		$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$ $(i=1)$ $(i=2)$ $(i=3,4)$
$(\mathfrak{f}_{4(4)}, \mathfrak{so}(5,4))$ $(\mathfrak{f}_{4(-20)}, \mathfrak{sp}(2,1)+\mathfrak{su}(2))$	$\circ$	$\begin{pmatrix} 4 & 3 \\ 4 & 4 \end{pmatrix}$
$(\mathfrak{sl}(2l, \boldsymbol{C}), \mathfrak{su}^*(2l))$ self dual	$\underset{1}{\circ} - \cdots - \underset{l-1}{\circ} \Leftarrow \underset{l}{\circ}$	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $(i<l)$ $(i=l)$
$(\mathfrak{sl}(2l, \boldsymbol{C}), \mathfrak{sl}(2l, \boldsymbol{R}))$ self dual	$\underset{1}{\circ} - \cdots - \underset{l-1}{\circ} \Rightarrow \underset{l}{\circ}$	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $(i<l)$ $(i=l)$
$(\mathfrak{sl}(2l+1, \boldsymbol{C}), \mathfrak{sl}(2l+1, \boldsymbol{R}))$ self dual		$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$ $(i<l)$ $(i=l)$
$(\mathfrak{so}(2l+2, \boldsymbol{C}), \mathfrak{so}(2l+1, 1))$ self dual	$\underset{1}{\circ} - \cdots - \underset{l-1}{\circ} \Rightarrow \underset{l}{\circ}$	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(i<l)$ $(i=l)$
$(\mathfrak{so}(2l+2, \boldsymbol{C}), \mathfrak{so}(2p+1, 2l-2p+1))$ self dual		$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(i<l, i\neq p)$ $(i=p)$ $(i=l)$
$(\mathfrak{e}_6, \mathfrak{e}_{6(-26)})$ self dual	$\underset{1}{\circ} - \underset{2}{\circ} \Rightarrow \underset{3}{\circ} - \underset{4}{\circ}$	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(i=1,2)$ $(i=3,4)$
$(\mathfrak{e}_6, \mathfrak{e}_{6(6)})$ self dual		$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(i=1)$ $(i=2)$ $(i=3,4)$

subalgebra of $\mathfrak{h}$ (cf. (1.13)). The remaining cases are those treated in (1.12), (1.14)–(1.16). We give a mark ♯ (resp. ♯♯, †) in the first column for the case (1.14) (resp. (1.15), (1.16)). Hence we can determine the Lie algebra $\mathfrak{h}^a$ by referring to Table I, (1.14)–(1.16) in these cases.

(6.10) Finally we give a remark on the restricted root systems.

Let $\mathfrak{g}$ be a real semisimple Lie algebra of the non-compact type. Then its restricted root system and the multiplicity of a given restricted root are defined. Accordingly, for a given real semisimple Lie algebra, we can uniquely define a root system each root of which has a multiplicity. Moreover it is known that if $\mathfrak{g}$ and $\mathfrak{g}'$ are real semisimple Lie algebras whose restricted root systems coincide with the given one including their multiplicities of roots, then $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic.

However the claim analogous to the above one does not hold for the restricted root systems of the symmetric pairs. That is, for a given symmetric pair, we can define its restricted root system and each restricted root has the signature defined as in Definition (2.14). But these do not characterize the symmetric pairs. More precisely, it is clear from Lemma (2.15.1) that the restricted root system of a symmetric pair coincides with that of its dual pair including their signatures of roots. Moreover there exist symmetric pairs such that they are not isomorphic and are not dual to each other and that not only their restricted root systems but also the signatures of the corresponding restricted roots coincide. We give here such examples. For brevity, we only consider the symmetric pairs of split rank 1. By comparing the signatures of roots in Table IV, we find that the signatures of the roots of the pairs in (6.10.1) coincide.

$$
(6.10.1)\qquad
\begin{array}{ll}
(\mathfrak{so}(p+1,1),\ \mathfrak{so}(p+1,p)) & \multirow{2}{*}{$\Big\rangle$ dual} \\
(\mathfrak{so}(2p+1,1),\ \mathfrak{so}(p+1)+\mathfrak{so}(p,1)) & \\
(\mathfrak{so}(p+2,\boldsymbol{C}),\ \mathfrak{so}(p+1,\boldsymbol{C})) & \Big\rangle \text{dual} \\
(\mathfrak{so}(p+1,1)+\mathfrak{so}(p+1,1),\ \mathfrak{so}(p+1,1)). &
\end{array}
$$

The claim also holds for the pairs in (6.10.2).

$$
(6.10.2)\qquad
\begin{array}{ll}
(\mathfrak{su}(p+1,p+1),\ \mathfrak{su}(p+1,p)+\sqrt{-1}\boldsymbol{R}) & \Big\rangle \text{dual} \\
(\mathfrak{su}(2p+1,1),\ \mathfrak{su}(p+1)+\mathfrak{su}(p,1)+\sqrt{-1}\boldsymbol{R}) & \\
(\mathfrak{so}^*(2(p+2)),\ \mathfrak{so}^*(2(p+1))+\mathfrak{so}^*(2)) & \Big\rangle \text{dual} \\
(\mathfrak{so}(2(p+1),2),\ \mathfrak{su}(p+1,1)+\sqrt{-1}\boldsymbol{R}). &
\end{array}
$$

§ 7. The Weyl group of a symmetric pair

(7.1) We have introduced some root systems. We next study the Weyl groups of these root systems and in particular discuss on the relations between them. To begin with, we introduce some notation.

(7.1.1) *Notation.*

$W(\mathfrak{a}_\mathfrak{p})=N_K(\mathfrak{a}_\mathfrak{p})/Z_K(\mathfrak{a}_\mathfrak{p})$: The Weyl group of $\Sigma(\mathfrak{a}_\mathfrak{p})$.

$W(\mathfrak{a}_\mathfrak{p})_\sigma$ = The Weyl group of $\Sigma(\mathfrak{a}_\mathfrak{p})_\sigma$.
$W^\sigma(\mathfrak{a}_\mathfrak{p})=\{w \in W(\mathfrak{a}_\mathfrak{p});\ w(\mathfrak{a})=\mathfrak{a}\}$.
$W(\mathfrak{a}_\mathfrak{p};H)=N_{K\cap H}(\mathfrak{a}_\mathfrak{p})/Z_{K\cap H}(\mathfrak{a}_\mathfrak{p})$.
$W(\mathfrak{a})$: The Weyl group of $\Sigma(\mathfrak{a})$.
$W(\mathfrak{j})_\theta$: The Weyl group of $\Sigma(\mathfrak{j})_\theta$.
$W^\theta(\mathfrak{j})=\{w \in W(\mathfrak{j});\ w(\mathfrak{a})=\mathfrak{a}\}$.

Under the above notation, we obtain the following lemma.

Lemma (7.2).
(i) $W(\mathfrak{a}_\mathfrak{p})_\sigma \subseteqq W(\mathfrak{a}_\mathfrak{p};H) \subseteqq W^\sigma(\mathfrak{a}_\mathfrak{p})$.
(ii) $W^\sigma(\mathfrak{a}_\mathfrak{p})/W(\mathfrak{a}_\mathfrak{p})_\sigma \simeq W^\theta(\mathfrak{j})/W(\mathfrak{j})_\theta \simeq W(\mathfrak{a})$.

Proof. (i) We first show that $W(\mathfrak{a}_\mathfrak{p})_\sigma \subseteqq W(\mathfrak{a}_\mathfrak{p};H)$. Take an element λ of $\Sigma(\mathfrak{a}_\mathfrak{p})_\sigma$. Then it follows from [W, Lemma 1.1.3.9] that there exists an element X of $\mathfrak{g}(\mathfrak{a}_\mathfrak{p};\lambda)$ such that $\exp(X+\theta X)$ is contained in $N_K(\mathfrak{a}_\mathfrak{p})$ and it is a representative of the reflection s_λ with respect to λ. On the other hand, combining Lemma (2.7) with the assumption on λ, we find that $\mathfrak{g}(\mathfrak{a}_\mathfrak{p};\lambda)$ is contained in $\mathfrak{h}$. This implies that $s_\lambda \in W(\mathfrak{a}_\mathfrak{p};H)$.

Next we show that $W(\mathfrak{a}_\mathfrak{p};H)\subset W^\sigma(\mathfrak{a}_\mathfrak{p})$. Let w be an element of $W(\mathfrak{a}_\mathfrak{p};H)$. For any $X\in\mathfrak{a}$, we express $wX=X_1+X_2$ ($X_1\in\mathfrak{a}$, $X_2\in\mathfrak{a}_\mathfrak{p}\cap\mathfrak{h}$). Then we see that

$$\theta(wX)=w(\theta X)=-wX.$$
$$\theta(X_1+X_2)=-X_1+X_2.$$

These imply that $X_2=0$. Hence we conclude that $w\mathfrak{a}=\mathfrak{a}$.

(ii) It follows from Lemma (2.10) and [*W*, Prop. 1.1.2.1] that for any element λ of $\Sigma(\mathfrak{a}_\mathfrak{p})-\Sigma(\mathfrak{a}_\mathfrak{p})_\sigma$, we have one of the following conditions: (1) $\theta\sigma\lambda=\lambda$, (2) $\langle\theta\sigma\lambda,\lambda\rangle=0$, (3) $\theta\sigma\lambda+\lambda\in\Sigma(\mathfrak{a}_\mathfrak{p})$. If s_λ and $s_{\theta\sigma\lambda}$ represent the reflections of λ and $\theta\sigma\lambda$, respectively, we find that the reflection on $\mathfrak{a}$ with respect to $\mu=\lambda|\mathfrak{a}$ coincides with $s_\lambda|\mathfrak{a}$ (resp. $(s_\lambda s_{\theta\sigma\lambda})|\mathfrak{a}$, $s_{\lambda+\theta\sigma\lambda}|\mathfrak{a}$) in the case (1) (resp. the case (2), (3)). Hence it follows that the map $W^\sigma(\mathfrak{a}_\mathfrak{p})\to W(\mathfrak{a})$ defined by $w\to w|\mathfrak{a}$ is surjective. The kernel of this map is obviously $\{w\in W(\mathfrak{a}_\mathfrak{p});\ w|\mathfrak{a}=\mathrm{id}\}$. We can show that this set coincides with $W(\mathfrak{a}_\mathfrak{p})_\sigma$ by an argument similar to that of [W, Lemma 1.1.3.4]. q.e.d.

Lemma (7.3). *We assume that* $\Sigma(\mathfrak{a})$ *satisfies the following condition: For any* $\lambda\in\Sigma(\mathfrak{a})$ $(\frac{1}{2}\lambda\notin\Sigma(\mathfrak{a}))$, *we have* $m^+(\lambda)>0$ *or* $m^+(2\lambda)>0$. *Then* $W(\mathfrak{a}_\mathfrak{p};H)=W^\sigma(\mathfrak{a}_\mathfrak{p})$.

Proof. Any element of $W(\mathfrak{a})$ has a representative g in $N_{K\cap H}(\mathfrak{a})$ because the assumption implies that $W(\mathfrak{a})$ is the Weyl group of the root system of $(\mathfrak{h}^a,\mathfrak{a})$. Since $\mathrm{Ad}(g)(\mathfrak{a}_\mathfrak{p}\cap\mathfrak{h})$ is a maximal abelian subspace in

$Z_{\mathfrak{h}}(\mathfrak{a})\cap\mathfrak{p}$, there exists $g'\in Z_{K\cap H}(\mathfrak{a})$ with $\mathrm{Ad}(g'g)(\mathfrak{a}_{\mathfrak{p}}\cap\mathfrak{h})=\mathfrak{a}_{\mathfrak{p}}\cap\mathfrak{h}$. This implies that $W(\mathfrak{a}_{\mathfrak{p}};H)/W(\mathfrak{a}_{\mathfrak{p}})_{\sigma}\supset W(\mathfrak{a})$. Now the lemma follows from Lemma (7.2). q.e.d.

Remark (7.4). Let $(\mathfrak{g},\mathfrak{h})$ be an irreducible symmetric pair of split rank one and let λ be the positive simple root of $\Sigma(\mathfrak{a})$. Then it follows from Table II that if $m^{+}(\lambda)=m^{+}(2\lambda)=0$, $(\mathfrak{g},\mathfrak{h})$ is isomorphic to the pair $(\mathfrak{so}(l+1,1),\mathfrak{so}(l,1))$ for some l.

Lemma (7.5). *Let α be a root of $\Sigma(\tilde{\mathfrak{j}})$ and let X, $Y(\neq 0)$ be an element of $\mathfrak{g}_C(\tilde{\mathfrak{j}};\alpha)$ and that of $\mathfrak{g}_C(\tilde{\mathfrak{j}};-\alpha)$, respectively. Then for any $A\in\tilde{\mathfrak{j}}$, we have*

$$\mathrm{Ad}(\exp(X+Y))A=A+(\cosh C-l)H_{\alpha}-\frac{\sinh C}{C}\alpha(A)(X-Y).$$

Here H_α is the element of $\tilde{\mathfrak{j}}_C$ such that $\alpha(H)=\langle H_\alpha,H\rangle$ for any $H\in\tilde{\mathfrak{j}}_C$ and $C=(2\alpha(H_\alpha)\langle X,Y\rangle)^{1/2}$.

Proof. Easy. (Cf. [He 2, p. 286].)

Lemma (7.6). *Let α be an element of $\Sigma(\tilde{\mathfrak{j}})$ such that $\sigma\alpha=\alpha$ and $\lambda=\alpha\,|\,\mathfrak{a}_{\mathfrak{p}}\neq 0$. Then there exists an element g of $K\cap(G^{\sigma})_0$ such that*

$$\mathrm{Ad}(g)\tilde{\mathfrak{j}}=\tilde{\mathfrak{j}},\quad \mathrm{Ad}(g)\mathfrak{a}_{\mathfrak{p}}=\mathfrak{a}_{\mathfrak{p}},\quad \mathrm{Ad}(g)\mathfrak{j}=\mathfrak{j} \tag{7.6.1}$$

and that $\mathrm{Ad}(g)\,|\,\mathfrak{a}_{\mathfrak{p}}=s(\lambda;\mathfrak{a}_{\mathfrak{p}})$, where for any $\mu\in\Sigma(\mathfrak{a}_{\mathfrak{p}})$, $s(\mu;\mathfrak{a}_{\mathfrak{p}})$ denotes the reflection on $\mathfrak{a}_{\mathfrak{p}}$ with respect to μ.

Proof. We prove the lemma in the cases (i) $\langle\alpha,\theta\alpha\rangle<0$, (ii) $\langle\alpha,\theta\alpha\rangle=0$ and (iii) $\langle\alpha,\theta\alpha\rangle>0$, separately.

First consider the case (i). It follows that $\theta\alpha=-\alpha$. Hence $\mathfrak{g}(\tilde{\mathfrak{j}};\alpha)$ $(\neq 0)$ is contained in $\mathfrak{g}(\mathfrak{a}_{\mathfrak{p}};\lambda)$. We take an element X $(\neq 0)$ of $\mathfrak{g}(\tilde{\mathfrak{j}};\alpha)$ and put $g=\exp(X+\theta X)\in G$. Multiplying X by a non-zero constant if necessary, we may assume that $2\langle\alpha,\alpha\rangle\langle X,\theta X\rangle=-\pi^2$. Then in virtue of Lemma (7.5), we find that

$$\mathrm{Ad}(g)Y=Y-2\frac{\alpha(Y)}{\alpha(H_\alpha)}H_\alpha \qquad \text{for any } Y\in\tilde{\mathfrak{j}}. \tag{7.6.2}$$

It follows from (7.6.2) that $\mathrm{Ad}(g)\,|\,\mathfrak{a}_{\mathfrak{p}}=s(\lambda;\mathfrak{a}_{\mathfrak{p}})$. By definition, it is clear that $g\in K$. On the other hand, the assumption $\sigma\alpha=\alpha$ combined with Lemma (2.7) implies that $\sigma X=X$. Hence g is contained in $(G^{\sigma})_0$. Then we find that $\mathrm{Ad}(g)\mathfrak{j}=\mathfrak{j}$. We have thus proved the lemma in this case.

Next consider the case (ii). Then it follows from the condition (N_θ) that $\alpha \pm \theta\alpha \notin \Sigma(\tilde{\mathfrak{j}})$. In this case, we take non-zero $X \in \mathfrak{g}_C(\tilde{\mathfrak{j}}; \alpha)$ and $Y \in \mathfrak{g}_C(\tilde{\mathfrak{j}}; -\theta\alpha)$ such that $X+Y \in \mathfrak{g}$. Then by virtue of the remark above, we find that $[X, Y]=0$, $[X, \theta Y]=0$. Since

$$[X+Y, \theta X+\theta Y]=[X, \theta Y]-\theta[X, \theta Y],$$

we see that $\langle X+Y, \theta X+\theta Y\rangle=2\langle X, \theta Y\rangle$. Then $\langle X, \theta Y\rangle<0$. Noting this, we may take X and Y so that $-2\langle\alpha, \alpha\rangle\langle X, \theta Y\rangle=\pi^2$. Now we put $g=g'\theta(g')$, where $g'=\exp(X+\theta Y)$. Lemma (7.5) implies that $\mathrm{Ad}(g)\tilde{\mathfrak{j}}=\tilde{\mathfrak{j}}$ and $\mathrm{Ad}(g)|\tilde{\mathfrak{j}}=s(\alpha; \tilde{\mathfrak{j}})$. Since $g=\exp(X+Y+\theta X+\theta Y)$, it also follows from Lemma (7.5) that $\mathrm{Ad}(g)\mathfrak{a}_\mathfrak{p}=\mathfrak{a}_\mathfrak{p}$ and $\mathrm{Ad}(g)|\mathfrak{a}_\mathfrak{p}=s(\lambda; \mathfrak{a}_\mathfrak{p})$. On the other hand, the assumption $\sigma\alpha=\alpha$ and Lemma (2.7) imply that $\sigma(X+Y)=X+Y$. Then it is clear that $\mathrm{Ad}(g)\mathfrak{j}=\mathfrak{j}$. Hence the claim follows in this case.

Last we consider the case (iii). It follows from [W, Prop. 1.1.2.1] that $\beta=\alpha-\theta\alpha \in \Sigma(\tilde{\mathfrak{j}})$. Since $\sigma\beta=\beta$ and $\theta\beta=-\beta$, we reduce this case to (i) by replacing α with β. Hence the claim follows in this case. q.e.d.

Lemma (7.7). *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a symmetric pair of split rank* 1. *Then there exists* $g \in K$ *satisfying the following three conditions* (i)–(iii):

(i) *g normalizes* $\tilde{\mathfrak{j}}$, $\mathfrak{a}_\mathfrak{p}$ *and* $\mathfrak{j}$.

(ii) *Put* $w=\mathrm{Ad}(g)|\tilde{\mathfrak{j}} \in W(\tilde{\mathfrak{j}})$ *and*

$$M(\theta, \sigma)^+=\{\beta \in \Sigma(\tilde{\mathfrak{j}})^+; \theta\beta=\beta, \sigma\beta\neq\beta\}.$$

Then w leaves $M(\theta, \sigma)^+$ invariant.

(iii) $\mathrm{Ad}(g)|\mathfrak{a}$ *is the reflection on* $\mathfrak{a}$ *with respect to the simple root.* (*We note that* $\dim \mathfrak{a}=1$ *in this case.*)

Proof. We prove the lemma in the following five cases, separately. It should be noted here that for any symmetric pair of split rank 1, one of the following conditions occurs (cf. Table III).

Case (a): $\exists\, \alpha \in \Sigma(\tilde{\mathfrak{j}})^+$ s.t. $\theta\alpha=\sigma\alpha=-\alpha$.

Case (b): $\exists\, \alpha \in \Sigma(\tilde{\mathfrak{j}})^+$ s.t. $\theta\alpha=-\alpha$ and $\langle\alpha, \sigma\alpha\rangle=0$.

Case (c): $\exists\, \alpha \in \Sigma(\tilde{\mathfrak{j}})^+$ s.t. $\langle\alpha, \theta\alpha\rangle=0$ and $\theta\alpha=\sigma\alpha$.

Case (d): $\exists\, \alpha \in \Sigma(\tilde{\mathfrak{j}})^+$ s.t. $\langle\alpha, \theta\alpha\rangle=0$ and $\sigma\alpha=-\alpha$.

Case (e): $\exists\, \alpha \in \Sigma(\tilde{\mathfrak{j}})^+$ s.t. $\langle\alpha, \theta\alpha\rangle=\langle\alpha, \sigma\alpha\rangle=\langle\alpha, \theta\sigma\alpha\rangle=0$.

In the subsequent discussions, we use the results in Section 6 without any comments. If μ is a linear form on $\mathfrak{a}$, we define $Y_\mu \in \mathfrak{a}$ by $\langle Y_\mu, Y\rangle=\mu(Y)$ for any $Y \in \mathfrak{a}$.

Case (a). A symmetric pair satisfying this condition is contained in one of the following classes.

$$\mathrm{I}_1\ (p+q\colon \text{odd}),\quad \mathrm{I}_1^d\ (p+q\colon \text{odd}),\quad \mathrm{I}_i,\ \mathrm{I}_i^d\ (i=2,3),\ I_4,\ I_4^2,$$
$$\mathrm{II}_i,\ \mathrm{II}_i^d\ (i=1,2,3),\quad \mathrm{IV}_1,\ \mathrm{IV}_1^d.$$

Let $\alpha\in\Sigma(\tilde{\mathfrak{j}})$ be a root satisfying the condition. We put $\mu=\alpha|\mathfrak{a}$. In this case $\mathfrak{g}(\tilde{\mathfrak{j}};\alpha)$ $(\neq 0)$ is contained in $\mathfrak{g}(\mathfrak{a};\mu)$. We take $X\in\mathfrak{g}(\tilde{\mathfrak{j}};\alpha)$ and put $g=\exp(X+\theta X)\in K$. We may assume that $\langle\alpha,\alpha\rangle\langle X,\theta X\rangle=-\pi^2$. Then it follows from Lemma (7.5) that $\mathrm{Ad}(g)\tilde{\mathfrak{j}}=\tilde{\mathfrak{j}}$. In particular $\mathrm{Ad}(g)|\tilde{\mathfrak{j}}=s_\alpha$, the reflection on $\tilde{\mathfrak{j}}$ with respect to α. It also follows that $\mathrm{Ad}(g)\mathfrak{a}_\mathfrak{p}=\mathfrak{a}_\mathfrak{p}$. We now show that $\mathrm{Ad}(g)\mathfrak{j}=\mathfrak{j}$. By definition, we find that $\theta\sigma X=\pm X$. If $\theta\sigma X=X$, then $\sigma(g)=g$ and we have nothing to prove. On the other hand, if $\theta\sigma X=-X$, then $\sigma(g)=g^{-1}$. But in this case, $\mathrm{Ad}(g^2)$ clearly centralizes $\tilde{\mathfrak{j}}$ and therefore we conclude that $\mathrm{Ad}(g)\mathfrak{j}=\mathfrak{j}$. Hence (i) is proved. Next we show (ii). Take $\beta\in M(\theta,\sigma)^+$. Since $\theta\alpha=-\alpha$ and $\theta\beta=\beta$, it follows that $\langle\alpha,\beta\rangle=0$. Hence $s_\alpha(\beta)=\beta$ and therefore (ii) follows. The claim (iii) is clear in this case.

Case (b). A symmetric pair satisfying this condition but not treated in Case (a) is contained in one of the following classes.

$$\mathrm{I}_1\ (p+q\colon \text{even},\ p,q>0),\quad \mathrm{III}_1^d\ (m\colon \text{odd}),\quad \mathrm{III}_i^d\ (i=2,3,4),$$
$$\mathrm{IV}_2^d,\ \mathrm{IV}_3^d,\ \mathrm{V}_2^d,\ \mathrm{V}_3^d.$$

As in Case (a), we define an element $g'=\exp(X+\theta X)\in K$ for some $X\in\mathfrak{g}(\tilde{\mathfrak{j}};\alpha)$ satisfying that $\mathrm{Ad}(g')|\tilde{\mathfrak{j}}$ is the reflection with respect to α. We put $g'=g'\sigma(g')$. In this case, it follows that $\alpha\pm\sigma\alpha\notin\Sigma(\tilde{\mathfrak{j}})$. This implies that g' and $\sigma(g')$ commute with each other. Hence we see that g normalizes both $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{j}$. We can also prove that both s_α and $s_{\theta\alpha}$ leave each element of $M(\theta,\sigma)^+$ invariant by the same reason as in Case (a). Hence (ii) follows. Last we show (iii). We put $\mu=(\alpha-\sigma\alpha)/2$. It is contained in $\Sigma(\mathfrak{a})$. Then it follows that $\langle\mu,\mu\rangle=\frac{1}{2}\langle\alpha,\alpha\rangle$. For any $Y\in\mathfrak{a}$, we find that

$$\begin{aligned}
s_\alpha s_{\sigma\alpha}(Y)&=Y-2\frac{\alpha(Y)}{\langle\alpha,\alpha\rangle}H_\alpha-2\frac{\sigma\alpha(Y)}{\langle\alpha,\alpha\rangle}H_{\sigma\alpha}\\
&=Y-2\frac{\alpha(Y)}{\langle\alpha,\alpha\rangle}(H_\alpha-H_{\sigma\alpha})\\
&=Y-2\frac{\mu(Y)}{\langle\mu,\mu\rangle}Y_\mu.
\end{aligned}$$

Hence $\mathrm{Ad}(g)|\mathfrak{a}=s_\alpha s_{\sigma\alpha}$ is a reflection on $\mathfrak{a}$ with respect to μ. This proves (iii).

Case (c). A symmetric pair satisfying this condition but not treated in Cases (a)–(b) is contained in one of the following classes.

$$\mathrm{I}_1\,(p+q\colon \text{even and } p=0 \text{ or } q=0),\quad \mathrm{I}_1^d\,(p+q\colon \text{even and } p\neq q),$$
$$\mathrm{IV}_2,\ \mathrm{IV}_3,\ \mathrm{V}_i\ (i=1, 2, 3).$$

We now prove in this case. Let $X\in\mathfrak{g}_C(\tilde{\mathfrak{j}};\alpha)$ $(X\neq 0)$ and $Y\in\mathfrak{g}_C(\tilde{\mathfrak{j}};-\theta\alpha)$ $(Y\neq 0)$ be so taken that $X+Y\in\mathfrak{g}$ and that $\theta\sigma(X+Y)=\pm(X+Y)$. It follows from the assumption and (N_θ) that $\alpha\pm\theta\alpha\notin\Sigma(\tilde{\mathfrak{j}})$. This implies that $[X, Y]=[X, \theta X]=[Y, \theta Y]=0$. In this case, we find that

$$\langle X+Y, \theta X+\theta Y\rangle=2\langle X, \theta Y\rangle.$$

Hence $\langle X, \theta Y\rangle<0$. We put $g'=\exp(X+\theta Y)$. Multiplying $X+Y$ by a constant if necessary, we may take X and Y so that $\mathrm{Ad}(g')\,|\,\tilde{\mathfrak{j}}$ is a reflection with respect to α. It follows that g' and $\theta(g')$ commute with each other. We now put $g=g'\theta(g')$. Then it is clear that $g\in K$ and $\mathrm{Ad}(g)\mathfrak{a}_\mathfrak{p}=\mathfrak{a}_\mathfrak{p}$. Since $\theta\sigma(X+Y)=\pm(X+Y)$ and since $\mathrm{Ad}(g^{-1})\,|\,\tilde{\mathfrak{j}}=\mathrm{Ad}(g)\,|\,\tilde{\mathfrak{j}}$, it follows that $\mathrm{Ad}(g)\mathfrak{j}=\mathfrak{j}$. Hence (i) is shown. The proof of (ii) is based on the classification. In the cases $\mathrm{I}_1(p=0$ or $q=0)$, IV_3, $\mathrm{V}_i(i=1, 2, 3)$, we easily see that $\theta\alpha\neq\alpha$ for any $\alpha\in\Sigma(\tilde{\mathfrak{j}})$ satisfying $\sigma\alpha\neq\alpha$. This implies the $M(\theta, \sigma)^+=\emptyset$. Hence in these cases, we have nothing to prove. On the other hand, we will give in Lemma (A. 2) of Appendix A a proof of the existence of $\alpha\in\Sigma(\tilde{\mathfrak{j}})$ satisfying both (ii) and the assumption for a pair contained in the classes $\mathrm{I}_1^d(p+q\colon$ even and $p\neq q)$ and IV_2. Last we show (iii). Put $\mu=(\alpha-\theta\alpha)/2$. Clearly μ is contained in $\Sigma(\mathfrak{a})$. Then $\langle\mu, \mu\rangle=\frac{1}{2}\langle\alpha, \alpha\rangle$. As in Case (b), we see that $s_\alpha s_{\theta\alpha}(Y)=Y-2(\mu(Y)/\langle\mu, \mu\rangle)Y_\mu$ for $Y\in\mathfrak{a}$. Hence we conclude that $\mathrm{Ad}(g)\,|\,\mathfrak{a}=s_\alpha s_{\theta\alpha}\,|\,\mathfrak{a}$ is the reflection on $\mathfrak{a}$ with respect to μ and therefore (iii) is proved.

Case (d). A symmetric pair satisfying this condition but not contained in Cases (a)–(c) is contained in one of the following classes.

$$\mathrm{I}_1^d\,(p=q),\quad \mathrm{III}_1\,(m\colon \text{odd}),\quad \mathrm{III}_i\ (i=2, 3, 4).$$

As in Case (c), we take $X\in\mathfrak{g}_C(\tilde{\mathfrak{j}};\alpha)$ and $Y\in\mathfrak{g}_C(\tilde{\mathfrak{j}};-\theta\alpha)$ so that $X+Y\in\mathfrak{g}$. Due to Lemma (5.6), we may assume that $\theta\sigma(X+Y)=X+Y$. Then it follows that $Y=\theta\sigma X$. Put $g'=\exp(X+\sigma X)$. Multiplying X by a constant if necessary, we may also assume that $\mathrm{Ad}(g')\,|\,\tilde{\mathfrak{j}}$ is the reflection with respect to α. On the other hand, it follows from the assumption $\langle\alpha, \theta\alpha\rangle=0$ that g' and $\theta(g')$ commute with each other. We put $g==g'\theta(g')$. Then it follows that $g\in K$ and that g normalizes $\tilde{\mathfrak{j}}$, $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{j}$. We have thus shown (i). Next we prove (ii). In the case $\mathrm{III}_1\,(m\colon$ odd), $\mathrm{III}_i\ (i=2, 3, 4)$, it follows that $\theta\alpha\neq\alpha$ for any $\alpha\in\Sigma(\tilde{\mathfrak{j}})$. Hence we have nothing to prove. In the case $\mathrm{I}_1^d(p=q)$, we will give a proof of (ii) in Lemma (A. 2) of Appendix. On the other hand, we can show (iii) by an argument similar to that in Case (c).

Case (e). A symmetric pair satisfying this condition but not treated in Cases (a)–(d) is contained in one of the following classes.

$$\mathrm{III}_1\,(m\colon \text{even}),\quad \mathrm{III}_1^d\,(m\colon \text{even}).$$

From now on we restrict our attention to the pairs contained in these classes. As in Case (c), we take $X\in\mathfrak{g}_C(\bar{\mathfrak{j}};\alpha)$ and $Y\in\mathfrak{g}_C(\bar{\mathfrak{j}};-\theta\alpha)$ so that $X+Y\in\mathfrak{g}$. Put $g'=\exp(X+\theta Y)$. Multiplying $X+Y$ by a constant if necessary, we may assume that $\mathrm{Ad}(g')|\bar{\mathfrak{j}}$ is the reflection with respect to α. Under the assumption, we find that $\alpha\pm\theta\alpha,\ \alpha\pm\sigma\alpha,\ \alpha\pm\theta\sigma\alpha\notin\Sigma(\bar{\mathfrak{j}})$. This implies that any two of g', $\theta(g')$, $\sigma(g')$ and $\theta\sigma(g')$ commute with each other. We put $g=g'\theta(g')\sigma(g')\theta\sigma(g')$. Then it is easy to see that $g\in K$ and g normalizes $\bar{\mathfrak{j}}$, $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{j}$. Hence (i) is shown. Next we prove (ii). In the case $\mathrm{III}\,(m\colon\text{even})$, it is clear that $\theta\alpha\neq\alpha$ for any $\alpha\in\Sigma(\bar{\mathfrak{j}})$. This implies that $\Sigma(\bar{\mathfrak{j}})_\theta=\emptyset$ and therefore we have nothing to prove. In the case $\mathrm{III}_1^d\,(m\colon\text{even})$, we will give a proof of (ii) in Lemma (A. 2) of Appendix. Last we show (iii). Put $\mu=\frac{1}{4}(\alpha-\theta\alpha-\sigma\alpha+\theta\sigma\alpha)\in\Sigma(\mathfrak{a})$. Then $\langle\mu,\mu\rangle=\frac{1}{4}\langle\alpha,\alpha\rangle$. Noting this, we see from the assumption that $s_\alpha s_{\theta\alpha}s_{\sigma\alpha}s_{\theta\sigma\alpha}(Y)=Y-2(\mu(Y)/\langle\mu,\mu\rangle)Y_\mu$ for any $Y\in\mathfrak{a}$. Hence $\mathrm{Ad}(g)|\mathfrak{a}=s_\alpha s_{\theta\alpha}s_{\sigma\alpha}s_{\theta\sigma\alpha}|\mathfrak{a}$ is the reflection on $\mathfrak{a}$ with respect to μ.

Let $(\mathfrak{g},\mathfrak{h})$ be an irreducible symmetric pair of split rank 1. Then by the classification given in Section 6, we find that $(\mathfrak{g},\mathfrak{h})$ is contained in one of the classes given in Cases (a)–(e). Hence the lemma is completely proved.

(7.8). Let $(\mathfrak{g},\mathfrak{h})$ be a symmetric pair of general rank. Let $\Psi(\mathfrak{a})$ be a fundamental system of roots in $\Sigma(\mathfrak{a})$. For each fundamental root $\lambda\in\Psi(\mathfrak{a})$, we consider the symmetric pair $(\mathfrak{g}(\lambda),\mathfrak{h}(\lambda))$ (cf. § 4). This is of split rank 1. Then it follows from Lemma (7.6) that there exists an element $g\in K$ such that $\mathrm{Ad}(g)$ normalizes $\bar{\mathfrak{j}}(\lambda)$, $\mathfrak{a}_{\mathfrak{p}}(\lambda)$ and $\mathfrak{j}(\lambda)$, that $\mathrm{Ad}(g)|\mathfrak{a}(\lambda)$ is the reflection with respect to λ and that $\mathrm{Ad}(g)|\bar{\mathfrak{j}}(\lambda)$ leaves $M_\lambda(\theta,\sigma)^+$ invariant. Here $M_\lambda(\theta,\sigma)^+$ is the set defined for the pair $(\mathfrak{g}(\lambda),\mathfrak{h}(\lambda))$ similar to $M(\theta,\sigma)^+$ for $(\mathfrak{g},\mathfrak{h})$. For this $g\in K$, we have the following lemma.

Lemma (7.8.1). (i) $\mathrm{Ad}(g)$ *normalizes* $\bar{\mathfrak{j}}$, $\mathfrak{a}_{\mathfrak{p}}$ *and* $\mathfrak{j}$.
(ii) $\mathrm{Ad}(g)|\mathfrak{a}$ *is the reflection with respect to* λ.
(iii) $\mathrm{Ad}(g)|\mathfrak{j}$ *leaves the set* $\Sigma(\mathfrak{j})_\theta^+=\Sigma(\mathfrak{j})_\theta\cap\Sigma(\mathfrak{j})^+$ *invariant.*

Proof. (i) and (ii) follow from the remark before the lemma. In fact, for example, we show that $\mathrm{Ad}(g)\bar{\mathfrak{j}}=\bar{\mathfrak{j}}$. Let $\bar{\mathfrak{j}}(\lambda)^\perp$ be the orthogonal complement of $\bar{\mathfrak{j}}(\lambda)$ in $\bar{\mathfrak{j}}$. Then it follows from the definition of g that $\mathrm{Ad}(g)$ leaves each element of $\bar{\mathfrak{j}}(\lambda)^{\perp}$ invariant. Hence $\mathrm{Ad}(g)\bar{\mathfrak{j}}=\bar{\mathfrak{j}}$.

We are going to prove (iii). Let $\mu \in \Sigma(\mathfrak{j})^+_\theta$ and put

$$M=\left\{\beta \in \Sigma(\bar{\mathfrak{j}});\ \frac{\beta-\sigma\beta}{2}=\mu\right\}.$$

Since $\theta\mu=\mu$, $\theta(\beta-\sigma\beta)=\beta-\sigma\beta$ for any $\beta \in M$. This combined with the condition (C) implies that $\theta\beta=\beta$. Hence $M\subset\Sigma(\bar{\mathfrak{j}})^+_\theta$. Let $R_i (1\leqq i\leqq p)$ be the connected components of $\Sigma(\bar{\mathfrak{j}})_\theta$ which intersect with M. On the other hand, let R_i $(p<i\leqq r)$ be the connected components of $\Sigma(\bar{\mathfrak{j}})_\theta$ which do not intersect with M. Put $\Sigma_1=\bigcup_{i=1}^p R_i$ and $\Sigma_2=\bigcup_{i>p} R_i$. Then it follows from the definition that $\Sigma_1\cap\Sigma(\bar{\mathfrak{j}};\lambda)=\emptyset$ or $\Sigma_1\subset\Sigma(\bar{\mathfrak{j}};\lambda)$, First assume that $\Sigma_1\cap\Sigma(\bar{\mathfrak{j}};\lambda)=\emptyset$. Then it is clear that $w(\mu)=\mu$. Here $w=\mathrm{Ad}(g)\,|\,\mathfrak{j}$. On the other hand if $\Sigma_1\subset\Sigma(\bar{\mathfrak{j}};\lambda)$, it follows that $M\subset M_\lambda(\theta,\sigma)^+$. Hence we see from the discussion before the lemma that $\tilde{w}(\beta) \in M_\lambda(\theta,\sigma)^+$ for any $\beta \in M$. Here $\tilde{w}=\mathrm{Ad}(g)\,|\,\bar{\mathfrak{j}}$. This implies that $w(\mu) \in \Sigma(\mathfrak{j})^+_\theta$. We have thus proved that $w(\Sigma(\mathfrak{j})^+_\theta)=\Sigma(\mathfrak{j})^+_\theta$. q.e.d.

Let $\Psi(\mathfrak{a})=\{\lambda_1,\cdots,\lambda_l\}$ and for each $i\,(1\leqq i\leqq l)$, we take an element $g_i \in K$ satisfying the conditions (i)–(iii) in Lemma (7.8.1) for $\lambda=\lambda_i$. We now consider the subgroup $\tilde{W}(\mathfrak{a})$ of K generated by $g_1,\cdots,g_l$. Then it is clear that $\tilde{W}(\mathfrak{a})$ is a finite group. Moreover we put

$$Z(\mathfrak{a})=\{g \in \tilde{W}(\mathfrak{a});\ \mathrm{Ad}(g)\,|\,\mathfrak{a}=\mathrm{id}\}.$$

Then it follows that $\tilde{W}(\mathfrak{a})/Z(\mathfrak{a})\simeq W(\mathfrak{a})$.

We next consider the group $W(\mathfrak{a}_\mathfrak{p})_\sigma$. Clearly $W(\mathfrak{a}_\mathfrak{p})_\sigma$ is generated by the reflections with respect to the roots of $\Sigma(\mathfrak{a}_\mathfrak{p})_\sigma$. Let $\{\mu_1,\cdots,\mu_p\}$ be a fundamental system of $\Sigma(\mathfrak{a}_\mathfrak{p})_\sigma$. Then by definition, $\sigma\mu_i=\mu_i$ $(1\leqq i\leqq p)$. For any μ_i, there exists $\beta_i \in \Sigma(\bar{\mathfrak{j}})$ such that $\sigma\beta_i=\beta_i$ and $\beta_i-\theta\beta_i=2\mu_i$. Then it follows from Lemma (7.6) that there exists $h_i \in K\cap(G^\sigma)_0$ satisfying the conditions described there. Let $\tilde{W}(\mathfrak{a}_\mathfrak{p})_\sigma$ be the subgroup of K generated by $h_1,\cdots,h_p$. Then $\tilde{W}(\mathfrak{a}_\mathfrak{p})_\sigma$ is clearly contained in H. We put $Z(\mathfrak{a}_\mathfrak{p})_\sigma=\{g \in \tilde{W}(\mathfrak{a}_\mathfrak{p})_\sigma;\ \mathrm{Ad}(g)\,|\,\mathfrak{a}_\mathfrak{p}=\mathrm{id}\}$. Then $\tilde{W}(\mathfrak{a}_\mathfrak{p})_\sigma/Z(\mathfrak{a}_\mathfrak{p})_\sigma\simeq W(\mathfrak{a}_\mathfrak{p})_\sigma$.

Theorem (7.9). *For any $w \in W^\sigma(\mathfrak{a}_\mathfrak{p})$, there exist $g \in \tilde{W}(\mathfrak{a})$ and $h \in \tilde{W}(\mathfrak{a}_\mathfrak{p})_\sigma$ such that* $\mathrm{Ad}(hg)\,|\,\mathfrak{a}_\mathfrak{p}=w$.

Proof. By definition, w normalizes $\mathfrak{a}$. Hence $w\,|\,\mathfrak{a} \in W(\mathfrak{a})$. Then there exists $g \in \tilde{W}(\mathfrak{a})$ such that $\mathrm{Ad}(g)|\mathfrak{a}=w|\mathfrak{a}$. We now put $w'=\mathrm{Ad}(g)\,|\,\mathfrak{a}_\mathfrak{p}$. Clearly w' is contained in $W(\mathfrak{a}_\mathfrak{p})$. Since ww'^{-1} leaves each element of $\mathfrak{a}$ fixed, there exists $h \in \tilde{W}(\mathfrak{a}_\mathfrak{p})_\sigma$ such that $\mathrm{Ad}(h)\,|\,\mathfrak{a}_\mathfrak{p}=ww'^{-1}$. Then $\mathrm{Ad}(hg)\,|\,\mathfrak{a}_\mathfrak{p}=w$ and the theorem is proved. q.e.d.

Corollary (7.10). *Let $w_1=e, w_2, \cdots, w_r$ form a complete system of the representatives of the coset $W(\mathfrak{a}_\mathfrak{p}; H)\backslash W^\sigma(\mathfrak{a}_\mathfrak{p})$. Here we put*

$$r=[W^\sigma(\mathfrak{a}_\mathfrak{p}): W(\mathfrak{a}_\mathfrak{p}; H)].$$

Then for any i, there exists a representative $\bar{w}_i \in N_K(\mathfrak{a}_\mathfrak{p})$ of w_i such that the following conditions hold:

$$\mathrm{Ad}(\bar{w}_i)\tilde{\mathfrak{j}}=\tilde{\mathfrak{j}}, \quad \mathrm{Ad}(\bar{w}_i)\mathfrak{j}=\mathfrak{j}, \quad \bar{w}_i(\Sigma(\mathfrak{j})_\theta^+)=\Sigma(\mathfrak{j})_\theta^+.$$

This is a direct consequence of Theorem (7.9) and the definition of $W^\sigma(\mathfrak{a}_\mathfrak{p})$.

(7.11) In the course of the discussions in the paragraph (7.8), we have shown by considering the case where $\theta=\sigma$ the following claim which seems to be known.

Proposition (7.11.1). *There exists a finite subgroup $\tilde{W}$ of K satisfying the following conditions*:

(i) *Each element of $\tilde{W}$ normalizes both $\tilde{\mathfrak{j}}$ and $\mathfrak{a}_\mathfrak{p}$.*

(ii) *If $\tilde{Z}=\{g \in \tilde{W}; \mathrm{Ad}(g)\,|\,\mathfrak{a}_\mathfrak{p}=\mathrm{id}\}$, then $\tilde{W}/\tilde{Z}$ coincides with the Weyl group $W(\mathfrak{a}_\mathfrak{p})$.*

§ 8. A parabolic subalgebra connected with a symmetric pair

In this section, we introduce a standard parabolic subalgebra $\mathfrak{p}_\sigma$ of $\mathfrak{g}$ which plays a basic role in Fourier analysis on the symmetric space as a minimal parabolic subalgebra does a role in Fourier analysis on a Riemannian symmetric space.

(8.1) First we recall a minimal parabolic subalgebra of $\mathfrak{g}$. A standard one is given by $\mathfrak{m}+\mathfrak{a}_\mathfrak{p}+\mathfrak{n}$, where

$$\mathfrak{m}=Z_\mathfrak{k}(\mathfrak{a}_\mathfrak{p}), \qquad \mathfrak{n}=\sum_{\lambda\in\Sigma(\mathfrak{a}_\mathfrak{p})^+} \mathfrak{g}(\mathfrak{a}_\mathfrak{p}; \lambda).$$

In the study of a symmetric pair, we frequently need another parabolic subalgebra of $\mathfrak{g}$. A standard one is defined by $\mathfrak{p}_\sigma=Z_\mathfrak{g}(\mathfrak{a})+\mathfrak{n}_\sigma$, where we put $\mathfrak{n}_\sigma=\sum_{\lambda\in\Sigma(\mathfrak{a})^+}\mathfrak{g}(\mathfrak{a}; \lambda)$. It is clear that $\mathfrak{p}_\sigma$ is actually a parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{p}_\sigma=\mathfrak{m}_\sigma+\mathfrak{a}_\sigma+\mathfrak{n}_\sigma$ be a Langlands decomposition of $\mathfrak{p}_\sigma$. We may assume without loss of generality that $\mathfrak{a}_\sigma\subset\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{m}_\sigma$ is generated by $\mathfrak{m}$ and $\{\mathfrak{g}(\mathfrak{a}_\mathfrak{p}; \lambda); \lambda \in \Sigma(\mathfrak{a}_\mathfrak{p})_\sigma\}$.

In this section, we closely study the structure of the reductive subalgebra $\mathfrak{m}_\sigma$. In particular we will show in Theorem (8.8) that $[\mathfrak{m}_\sigma, \mathfrak{m}_\sigma]=\mathfrak{g}(\sigma)+\mathfrak{u}(\sigma)+\mathfrak{m}^\sigma$ is a direct sum decomposition, where $\mathfrak{g}(\sigma)$ is a semisimple Lie algebra of the non-compact type and $\mathfrak{u}(\sigma)$ and $\mathfrak{m}^\sigma$ are semisimple Lie algebras of the compact type with some additional conditions. Moreover we shall show in (8.9) that there is a duality between $\mathfrak{g}(\sigma)$ and $\mathfrak{u}(\sigma)$.

(8.2) We fix a (θ, σ)-order on $\Sigma(\tilde{\mathfrak{j}})$ and compatible orders on $\Sigma(\mathfrak{a}_{\mathfrak{p}})$, $\Sigma(\mathfrak{j})$ and $\Sigma(\mathfrak{a})$ as we introduced in Section 2. Let $\Psi(\tilde{\mathfrak{j}})$ be the (θ, σ)-fundamental system of $\Sigma(\tilde{\mathfrak{j}})$.

It follows from the argument in [W, p. 23] that $\Sigma(\tilde{\mathfrak{j}})_\sigma$ is a root system and $\Psi(\tilde{\mathfrak{j}}) \cap \Sigma(\tilde{\mathfrak{j}})_\sigma$ is a fundamental system of $\Sigma(\tilde{\mathfrak{j}})_\sigma$. It is clear that $\theta(\Sigma(\tilde{\mathfrak{j}})_\sigma) = \Sigma(\tilde{\mathfrak{j}})_\sigma$. Let $\Sigma_1, \cdots, \Sigma_r$ be the totality of the irreducible components of $\Sigma(\tilde{\mathfrak{j}})_\sigma$. We divide Σ_i $(1 \leqq i \leqq r)$ into two sets by the condition whether $\alpha | \mathfrak{a}_{\mathfrak{p}} \neq 0$ for some $\alpha \in \Sigma_i$ or not. For the sake of convenience, we may assume that if $1 \leqq i \leqq p$, then $\alpha | \mathfrak{a}_{\mathfrak{p}} \neq 0$ for some $\alpha \in \Sigma_i$ and if $p < i \leqq r$, then $\alpha | \mathfrak{a}_{\mathfrak{p}} = 0$ for any $\alpha \in \Sigma_i$. Then we put $\langle\theta\rangle = \bigcup_{i=1}^{p} \Sigma_i$ and $[\theta] = \Psi(\tilde{\mathfrak{j}}) \cap \langle\theta\rangle$. It is clear from the definition that $[\theta]$ is a fundamental system of the root system $\langle\theta\rangle$. We denote by $\mathfrak{g}(\langle\theta\rangle)_C$ the subalgebra of $\mathfrak{g}_C$ generated by $\{\mathfrak{g}_C(\tilde{\mathfrak{j}}; \alpha); \alpha \in \langle\theta\rangle\}$.

Lemma (8.3). *We put* $\mathfrak{g}(\sigma) = \mathfrak{g}(\langle\theta\rangle)_C \cap \mathfrak{g}$. *Then* $\mathfrak{g}(\sigma)$ *is generated by* $\{\mathfrak{g}(\mathfrak{a}_{\mathfrak{p}}; \lambda); \lambda \in \Sigma(\mathfrak{a}_{\mathfrak{p}})_\sigma\}$ *and is semisimple of the non-compact type. Furthermore* $\mathfrak{g}(\sigma)$ *is contained in* $\mathfrak{h}$.

Proof. First recall that

$$r_\theta^{-1}(\Sigma(\mathfrak{a}_{\mathfrak{p}})_\sigma) = \{\alpha \in \Sigma(\tilde{\mathfrak{j}}); \alpha | \mathfrak{a}_{\mathfrak{p}} \neq 0, \alpha | \mathfrak{a} = 0\}.$$

Then it follows from Lemma (2.8) that

$$r_\theta^{-1}(\Sigma(\mathfrak{a}_{\mathfrak{p}})_\sigma) = (\Sigma(\tilde{\mathfrak{j}}) - \Sigma(\tilde{\mathfrak{j}})_\theta) \cap \Sigma(\tilde{\mathfrak{j}})_\sigma. \tag{8.3.1}$$

Comparing this equality with the definition of $\langle\theta\rangle$, we conclude that $\mathfrak{g}(\sigma)$ contains the subalgebra $\mathfrak{g}(\sigma)'$ of $\mathfrak{g}$ generated by $\{\mathfrak{g}(\mathfrak{a}_{\mathfrak{p}}; \lambda); \lambda \in \Sigma(\mathfrak{a}_{\mathfrak{p}})_\sigma\}$. It follows from [W, Lemma 1.2.3.14] that $\mathfrak{g}(\sigma)'$ is semisimple of the non-compact type. Let $\mathfrak{g}(\sigma)'_C$ be the complexification of $\mathfrak{g}(\sigma)'$ in $\mathfrak{g}_C$. Then we find from the definition that the root system of $\mathfrak{g}(\sigma)'_C$ coincides with $\langle\theta\rangle$. Hence $\mathfrak{g}(\sigma)'$ must coincide with $\mathfrak{g}(\sigma)$. It follows from Lemma (2.7) and the equality (8.3.1) that $\mathfrak{g}(\sigma)$ is contained in $\mathfrak{h}$. Hence the lemma is completely proved.

(8.4) By exchanging the roles of $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{j}$, and those of θ and σ, we define $[\sigma]$, $\langle\sigma\rangle$ and $\mathfrak{g}(\langle\sigma\rangle)_C$ similar to $[\theta]$, $\langle\theta\rangle$ and $\mathfrak{g}(\langle\theta\rangle)_C$.

Lemma (8.5). (i) *The Lie algebra* $\mathfrak{g}(\langle\sigma\rangle)_C$ *is generated by*

$$\{\mathfrak{g}(\tilde{\mathfrak{j}}; \alpha); \alpha \in \Sigma(\tilde{\mathfrak{j}}), \alpha | \mathfrak{a}_{\mathfrak{p}} = 0, \alpha | \mathfrak{j} \neq 0\}.$$

(ii) *We put* $\mathfrak{u}(\sigma)=\mathfrak{g}(\langle\sigma\rangle)_C\cap\mathfrak{g}$. *Then* $\mathfrak{u}(\sigma)$ *is semisimple of the compact type and is contained in* $\mathfrak{m}$.

Proof. First remark that if α is a root of $\Sigma(\tilde{\mathfrak{j}})_\theta$, then $\mathfrak{g}_C(\tilde{\mathfrak{j}};\alpha)$ is contained in $\mathfrak{m}_C$. Noting this, we can prove the lemma by an argument quite similar to that of Lemma (8.3). Hence we do not enter into its proof. q.e.d.

(8.6) It is clear that $\Sigma(\tilde{\mathfrak{j}})_\theta\cap\Sigma(\tilde{\mathfrak{j}})_\sigma$ is a root system and $\Psi(\tilde{\mathfrak{j}})\cap\Sigma(\tilde{\mathfrak{j}})_\theta\cap\Sigma(\tilde{\mathfrak{j}})_\sigma$ is its fundamental system of roots. Let $\Sigma(\theta,\sigma)_1,\cdots,\Sigma(\theta,\sigma)_k$ be the irreducible components of $\Sigma(\tilde{\mathfrak{j}})_\theta\cap\Sigma(\tilde{\mathfrak{j}})_\sigma$. We may assume that $\Sigma(\theta,\sigma)_i$ $(1\leqq i\leqq j)$ are orthogonal to both $[\theta]$ and $[\sigma]$ but $\Sigma(\theta,\sigma)_i$ $(j<i\leqq k)$ are not to $[\theta]$ and $[\sigma]$. Then we put

$$\langle\theta,\sigma\rangle=\bigcup_{i=1}^{j}\Sigma(\theta,\sigma)_i,\qquad [\theta,\sigma]=\langle\theta,\sigma\rangle\cap\Psi(\tilde{\mathfrak{j}}).$$

It follows that $[\theta,\sigma]$ is a fundamental system of $\langle\theta,\sigma\rangle$. We denote by $(\mathfrak{m}^\sigma)_C$ the subalgebra of $\mathfrak{g}_C$ generated by $\{\mathfrak{g}_C(\tilde{\mathfrak{j}};\alpha);\alpha\in\langle\theta,\sigma\rangle\}$ and put $\mathfrak{m}^\sigma=\mathfrak{g}\cap(\mathfrak{m}^\sigma)_C$.

Lemma (8.7). $\mathfrak{m}^\sigma$ *is semisimple of the compact type and is contained in* $\mathfrak{m}\cap\mathfrak{h}$.

Proof. It is clear that $\mathfrak{m}^\sigma$ is contained in $\mathfrak{m}$ and is semisimple. Hence to prove the lemma, it suffices to show that $\mathfrak{m}^\sigma$ is contained in $\mathfrak{h}$. If $\alpha\in[\theta,\sigma]$, then $\alpha(\theta X)=\alpha(\sigma X)=\alpha(X)$ for any $X\in\tilde{\mathfrak{j}}$. This in particular implies that $\sigma(\mathfrak{g}_C(\tilde{\mathfrak{j}};\alpha))=\mathfrak{g}_C(\tilde{\mathfrak{j}};\alpha)$. If there exists an element $X(\neq 0)$ of $\mathfrak{g}_C(\tilde{\mathfrak{j}};\alpha)$ such that $\sigma X=-X$, then X is in $\mathfrak{q}_C$ and commutes with the maximal abelian subspace $\mathfrak{j}_C$ of $\mathfrak{q}_C$. This implies that $X\in\mathfrak{j}_C$, which is a contradiction. Since $\dim_C\mathfrak{g}_C(\tilde{\mathfrak{j}};\alpha)=1$, we find that $\sigma X=X$ for any $X\in\mathfrak{g}_C(\tilde{\mathfrak{j}};\alpha)$. Hence $\mathfrak{m}^\sigma\subset\mathfrak{h}$. q.e.d.

Theorem (8.8). (i) *If* $Z(\mathfrak{m}_\sigma)$ *is the center of* $\mathfrak{m}_\sigma$, *then*

$$Z(\mathfrak{m}_\sigma)=\{Y\in\tilde{\mathfrak{j}}\cap\mathfrak{k};\alpha(Y)=0\quad\textit{for any }\alpha\in\Sigma(\tilde{\mathfrak{j}})_{\theta,\sigma}\}.$$

(ii) $[\mathfrak{m}_\sigma,\mathfrak{m}_\sigma]=\mathfrak{g}(\sigma)+\mathfrak{u}(\sigma)+\mathfrak{m}^\sigma$ *is a direct sum decomposition and* $\mathfrak{g}(\sigma)$, $\mathfrak{u}(\sigma)$ *and* $\mathfrak{m}^\sigma$ *commute with each other.*

Proof. It is easy to check (i). We are now going to prove (ii). By definition, $Z_\mathfrak{g}(\mathfrak{a})$ is generated by $\tilde{\mathfrak{j}}$ and $\{\mathfrak{g}_C(\tilde{\mathfrak{j}};\alpha);\alpha\in\Sigma(\tilde{\mathfrak{j}})_{\theta,\sigma}\}$. Then due to the definition of $\mathfrak{g}(\sigma)$, $\mathfrak{u}(\sigma)$ and $\mathfrak{m}^\sigma$, we find that $(\mathfrak{m}_\sigma)_C=(\mathfrak{g}(\sigma)+\mathfrak{u}(\sigma)+\mathfrak{m}^\sigma+Z(\mathfrak{m}_\sigma))_C$. (Cf. Lemma (4.1.1) and the proof of Theorem (B.6).) Hence if we show that $\mathfrak{g}(\sigma)$, $\mathfrak{u}(\sigma)$ and $\mathfrak{m}^\sigma$ commute with each other, the claim in (ii) follows. By definition, it is clear that $\mathfrak{m}^\sigma$ commutes with both $\mathfrak{g}(\sigma)$ and $\mathfrak{u}(\sigma)$. We now prove that $\mathfrak{g}(\sigma)$ and $\mathfrak{u}(\sigma)$ commute with each other. It follows from Lemmas (8.3) and (8.5) that $\mathfrak{g}(\sigma)_C=\mathfrak{g}(\langle\theta\rangle)_C$ (resp. $\mathfrak{u}(\sigma)_C=\mathfrak{g}(\langle\sigma\rangle)_C$) is generated by $\{\mathfrak{g}_C(\tilde{\mathfrak{j}};\alpha);\alpha\in(\Sigma(\tilde{\mathfrak{j}})-\Sigma(\tilde{\mathfrak{j}})_\theta)\cap\Sigma(\tilde{\mathfrak{j}})_\sigma\}$ (resp. $\{\mathfrak{g}_C(\tilde{\mathfrak{j}};\alpha);\alpha\in(\Sigma(\tilde{\mathfrak{j}})-\Sigma(\tilde{\mathfrak{j}})_\sigma)\cap\Sigma(\tilde{\mathfrak{j}})_\theta\}$). This implies that if we show that

$$(\Sigma(\tilde{\mathfrak{j}})-\Sigma(\tilde{\mathfrak{j}})_\theta)\cap\Sigma(\tilde{\mathfrak{j}})_\sigma \quad \text{and} \quad (\Sigma(\tilde{\mathfrak{j}})-\Sigma(\tilde{\mathfrak{j}})_\sigma)\cap\Sigma(\tilde{\mathfrak{j}})_\theta$$

are orthogonal, then $[\mathfrak{g}(\sigma),\mathfrak{u}(\sigma)]=0$. If not so, there exist

$$\alpha\in(\Sigma(\tilde{\mathfrak{j}})-\Sigma(\tilde{\mathfrak{j}})_\theta)\cap\Sigma(\tilde{\mathfrak{j}})_\sigma \quad \text{and} \quad \beta\in(\Sigma(\tilde{\mathfrak{j}})-\Sigma(\tilde{\mathfrak{j}})_\sigma)\cap\Sigma(\tilde{\mathfrak{j}})_\theta$$

such that $\langle\alpha,\beta\rangle\neq 0$. We may assume that $\langle\alpha,\beta\rangle>0$. Then it follows from [W, Prop. 1.1.2.1] that $\alpha-\beta$ is a root of $\Sigma(\tilde{\mathfrak{j}})$. Moreover we have $(\alpha-\beta)\,|\,\mathfrak{a}_\mathfrak{p}=\alpha\,|\,\mathfrak{a}_\mathfrak{p}\neq 0$, $(\alpha-\beta)\,|\,\mathfrak{j}=-\beta\,|\,\mathfrak{j}\neq 0$, $(\alpha-\beta)\,|\,\mathfrak{a}=0$. This contradicts Lemma (4.1.1). Hence $(\Sigma(\tilde{\mathfrak{j}})-\Sigma(\tilde{\mathfrak{j}})_\theta)\cap\Sigma(\tilde{\mathfrak{j}})_\sigma$ and $(\Sigma(\tilde{\mathfrak{j}})-\Sigma(\tilde{\mathfrak{j}})_\sigma)\cap\Sigma(\tilde{\mathfrak{j}})_\theta$ are orthogonal and therefore $[\mathfrak{g}(\sigma),\mathfrak{u}(\sigma)]=0$. We have thus proved the theorem completely.

(8.9) We recall the symmetric pair $(\mathfrak{g}^d,\mathfrak{h}^d)$ dual to $(\mathfrak{g},\mathfrak{h})$. Then there is a kind of duality between $Z_\mathfrak{g}(\mathfrak{a})$ and $Z_{\mathfrak{g}^d}(\mathfrak{a})$. From now on, we explain this duality. First we put

$$\mathfrak{a}^\sigma=\mathfrak{a}_\sigma\cap\mathfrak{h},\quad \mathfrak{t}^\sigma=Z(\mathfrak{m}_\sigma)\cap\mathfrak{q},\quad \mathfrak{z}^\sigma=Z(\mathfrak{m}_\sigma)\cap\mathfrak{h}.$$

Then due to Theorem (8.8), we have a direct sum decomposition

$$Z_\mathfrak{g}(\mathfrak{a})=\mathfrak{g}(\sigma)+\mathfrak{u}(\sigma)+\mathfrak{m}^\sigma+\mathfrak{z}^\sigma+\mathfrak{t}^\sigma+\mathfrak{a}^\sigma+\mathfrak{a}. \tag{8.9.1}$$

Here we used that $\mathfrak{a}$ is contained in $\mathfrak{a}_\sigma$. In fact, it follows from the definition that $\mathfrak{a}_\sigma=\{Y\in\mathfrak{a}_\mathfrak{p};\lambda(Y)=0 \text{ for any } \lambda\in\Sigma(\mathfrak{a}_\mathfrak{p})_\sigma\}$. Putting

$$\begin{aligned}
\mathfrak{g}(\sigma)^d&=\mathfrak{u}(\sigma)_C\cap\mathfrak{g}^d\ (=\mathfrak{g}^d\cap\mathfrak{g}(\langle\theta\rangle)_C)\\
\mathfrak{u}(\sigma)^d&=\mathfrak{g}(\sigma)_C\cap\mathfrak{g}^d\ (=\mathfrak{g}^d\cap\mathfrak{g}(\langle\sigma\rangle)_C)\\
(\mathfrak{a}^\sigma)^d&=\sqrt{-1}\,\mathfrak{t}^\sigma\\
(\mathfrak{t}^\sigma)^d&=\sqrt{-1}\,\mathfrak{a}^\sigma,
\end{aligned}$$

we obtain a direct sum decomposition of $Z_{\mathfrak{g}^d}(\mathfrak{a})$:

$$Z_{\mathfrak{g}^d}(\mathfrak{a})=\mathfrak{g}(\sigma)^d+\mathfrak{u}(\sigma)^d+\mathfrak{m}^\sigma+\mathfrak{z}^\sigma+(\mathfrak{t}^\sigma)^d+(\mathfrak{a}^\sigma)^d+\mathfrak{a}. \tag{8.9.2}$$

Moreover we have that

(8.9.3) $\qquad\mathfrak{g}(\sigma)^d$ is of the non-compact type, (8.9.4) $\mathfrak{u}(\sigma)^d$ is compact,

(8.9.5) $(\mathfrak{m}^\sigma)_C \cap \mathfrak{g}^d = \mathfrak{m}^\sigma$,

(8.9.6) $(\mathfrak{z}^\sigma)_C \cap \mathfrak{g}^d = \mathfrak{z}^\sigma$.

There exists a duality between the decompositions (8.9.1) and (8.9.2). We now explain this duality.

Proposition (8.9.7). *If we decompose $Z_{\mathfrak{g}^d}(\mathfrak{a})$ as we did for $Z_{\mathfrak{g}}(\mathfrak{a})$ of the form* (8.9.1), *we obtain the decomposition* (8.9.2) *and the correspondence of the factors are given by*

$$\mathfrak{g}(\sigma) \to \mathfrak{g}(\sigma)^d, \quad \mathfrak{u}(\sigma) \to \mathfrak{u}(\sigma)^d, \quad \mathfrak{m}^\sigma \to \mathfrak{m}^\sigma,$$

$$\mathfrak{z}^\sigma \to \mathfrak{z}^\sigma, \quad \mathfrak{t}^\sigma \to (\mathfrak{t}^\sigma)^d, \quad \mathfrak{a}^\sigma \to (\mathfrak{a}^\sigma)^d, \quad \mathfrak{a} \to \mathfrak{a}.$$

Proof. It is easy to check the correspondence relations between $\mathfrak{g}(\sigma)$, $\mathfrak{g}(\sigma)^d$, $\mathfrak{u}(\sigma)$, $\mathfrak{u}(\sigma)^d$ and $\mathfrak{m}^\sigma$.

The center $\mathfrak{z}$ of $Z_{\mathfrak{g}}(\mathfrak{a})$ coincides with $Z(\mathfrak{m}_\sigma) + \mathfrak{a}_\sigma$. Since $\mathfrak{a}_\sigma$ is contained in $\mathfrak{p}$ and since it follows from Theorem (8.8) that $Z(\mathfrak{m}_\sigma) \subset \mathfrak{k}$, we find that $\mathfrak{z} \cap \mathfrak{k} \cap \mathfrak{h} = \mathfrak{z}^\sigma$, $\mathfrak{z} \cap \mathfrak{k} \cap \mathfrak{q} = \mathfrak{t}^\sigma$, $\mathfrak{z} \cap \mathfrak{h} \cap \mathfrak{p} = \mathfrak{a}^\sigma$, $\mathfrak{z} \cap \mathfrak{p} \cap \mathfrak{q} = \mathfrak{a}$. These imply the rest of the claim. q.e.d.

Owing to the duality between $\mathfrak{g}(\sigma)$ and $\mathfrak{u}(\sigma)$ given in Proposition (8.9.7) and the argument before in this section, we find the following.

Proposition (8.9.8). (i) *The restriction of θ to $\mathfrak{g}(\sigma)$ is a Cartan involution* of $\mathfrak{g}(\sigma)$.

(ii) *The restriction of σ to $\mathfrak{u}(\sigma)$ is non trivial on each simple factors of* $\mathfrak{u}(\sigma)$.

(iii) *The restrictions of θ and σ to $\mathfrak{m}^\sigma$ are trivial.*

Proof. The claim (i) follows from the definition of $\mathfrak{g}(\sigma)$ and [W, Lemma 1.2.3.14]. Due to the duality between $\mathfrak{g}(\sigma)$ and $\mathfrak{u}(\sigma)$, we see that (ii) is reduced to (i). The claim (iii) follows from Lemma (8.7). q.e.d.

(8.10) In Table VI, we collect all the informations on the subalgebras $\mathfrak{g}(\sigma)$, $\mathfrak{u}(\sigma)$, etc. for all the symmetric pairs of split rank 1. There we use the notation $\mathfrak{m}_r^\sigma = \mathfrak{m}^\sigma + \mathfrak{z}^\sigma$ for short. (In Table VI, we omit the symmetric pairs of Type $(\mathfrak{k}_\varepsilon)$.)

(8.11) We take a connected Lie group G and its closed subgroup H as we did in Section 1. We put $\bar{\mathfrak{n}} = \theta(\mathfrak{n})$. Let $A_\mathfrak{p}$, N and $\bar{N}$ be analytic subgroups of G corresponding to $\mathfrak{a}_\mathfrak{p}$, $\mathfrak{n}$ and $\bar{\mathfrak{n}}$, respectively. Moreover we take the maximal compact subgroup K of G whose Lie algebra is $\mathfrak{k}$. Then

Table VI

	$\mathfrak{g}(\sigma)$	$\mathfrak{u}(\sigma)$	$\mathfrak{m}_r^\sigma$	$\dim \mathfrak{t}^\sigma$	$\dim \mathfrak{a}^\sigma$
I_1	$\mathfrak{so}(p, q)$	0	0	0	0
I_1^d	0	$\mathfrak{so}(p+q)$	0	0	0
I_2	$\mathfrak{su}(p, q)$	0	$\sqrt{-1}\boldsymbol{R}$	0	0
I_2^d	0	$\mathfrak{su}(p+q)$	$\sqrt{-1}\boldsymbol{R}$	0	0
I_3	$\mathfrak{sp}(p, q)$	0	$\mathfrak{sp}(1)$	0	0
I_3^d	0	$\mathfrak{sp}(p+q)$	$\mathfrak{sp}(1)$	0	0
II_1	$\mathfrak{sl}(m, \boldsymbol{R})$	0	0	0	1
II_1^d	0	$\mathfrak{su}(m)$	0	1	0
II_2	$\mathfrak{sp}(m, \boldsymbol{R})+\mathfrak{sp}(1, \boldsymbol{R})$	0	0	0	0
II_2^d	0	$\mathfrak{sp}(m)+\mathfrak{sp}(1)$	0	0	0
II_3	$\mathfrak{so}(4, 3)$	0	0	0	0
II_3^d	0	$\mathfrak{so}(7)$	0	0	0
III_1	$\mathfrak{so}(m, \boldsymbol{C})$	0	0	1	0
III_1^d	0	$\mathfrak{so}(m)+\mathfrak{so}(m)$	0	0	1
III_2	$\mathfrak{sl}(m, \boldsymbol{C})$	0	$\sqrt{-1}\boldsymbol{R}$	1	1
III_2^d	0	$\mathfrak{su}(m)+\mathfrak{su}(m)$	$\sqrt{-1}\boldsymbol{R}$	1	1
III_3	$\mathfrak{sp}(m, \boldsymbol{C})+\mathfrak{sp}(1, \boldsymbol{C})$	0	0	1	0
III_3^d	0	$2\mathfrak{sp}(m)+2\mathfrak{sp}(1)$	0	0	1
III_4	$\mathfrak{so}(7, \boldsymbol{C})$	0	0	1	0
III_4^d	0	$\mathfrak{so}(7)+\mathfrak{so}(7)$	0	0	1
IV_1	$\mathfrak{so}^*(2m)$	$\mathfrak{su}(2)$	0	0	0
IV_1^d	$\mathfrak{sl}(2, \boldsymbol{R})$	$\mathfrak{so}(2m)$	0	0	0
IV_2	$\mathfrak{su}^*(2m)$	$\mathfrak{su}(2)+\mathfrak{su}(2)$	0	0	1
IV_2^d	$\mathfrak{sl}(2, \boldsymbol{C})$	$\mathfrak{su}(2m)$	0	1	0
IV_3	0	$\mathfrak{so}(8)$	0	0	1
IV_3^d	$\mathfrak{so}(7, 1)$	0	0	1	0
V_1	0	0	$\sqrt{-1}\boldsymbol{R}$	1	1
V_2	$\mathfrak{sl}(2, \boldsymbol{C})+\mathfrak{sl}(2, \boldsymbol{R})$	0	0	1	0
V_2^d	0	$3\mathfrak{su}(2)$	0	0	1
V_3	$\mathfrak{so}(5, 3)$	0	0	1	0
V_3^d	0	$\mathfrak{so}(8)$	0	0	1

$G=KA_{\mathfrak{p}}N$ is an Iwasawa decomposition of G. As usual, we put $M=Z_K(A_{\mathfrak{p}})$, $M^*=N_K(A_{\mathfrak{p}})$. Clearly $\mathfrak{m}$ is the Lie algebra of M. The $P=MA_{\mathfrak{p}}N$ is a minimal parabolic subgroup of G. Now we define a parabolic subgroup P_σ by

$$P_\sigma=\bigcup_{w\in W(\mathfrak{a}_{\mathfrak{p}})_\sigma} P\bar{w}P.$$

Here $\bar{w}$ denotes a representative of w in M^*. Let $P_\sigma=M_\sigma A_\sigma N_\sigma$ be the Langlands decomposition of P_σ with $A_\sigma\subset A_{\mathfrak{p}}$. It follows from the definition that $\mathfrak{p}_\sigma$, $\mathfrak{m}_\sigma$, $\mathfrak{a}_\sigma$ and $\mathfrak{n}_\sigma$ are the Lie algebras of P_σ, M_σ, A_σ and N_σ, respectively. We put $\bar{N}_\sigma=\theta(N_\sigma)$ and denote by $G(\sigma)$, $U(\sigma)_0$, M^σ, T^σ and Z^σ the analytic subgroups of G corresponding to $\mathfrak{g}(\sigma)$, $\mathfrak{u}(\sigma)$, $\mathfrak{m}^\sigma$, $\mathfrak{t}^\sigma$ and $\mathfrak{z}^\sigma$, respectively. Moreover we put $U(\sigma)=U(\sigma)_0(K\cap\exp(\sqrt{-1}\mathfrak{a}_{\mathfrak{p}}))$.

Lemma (8.12).
(i) $G(\sigma)\subseteqq H$, $U(\sigma)\subseteqq M$, $M^\sigma\subseteqq M$.
(ii) $M_\sigma=U(\sigma)G(\sigma)M^\sigma T^\sigma Z^\sigma$.

Proof. Since $Z(\mathfrak{m}_\sigma)$ is contained in $\mathfrak{m}$, the claim (i) follows from Lemma (8.3), Lemma (8.5) and Theorem (8.8), and (ii) does from (i), the definition of $U(\sigma)$ and [W, Lemma 1.2.4.5]. q.e.d.

Lemma (8.13). *For any $w\in W^\sigma(\mathfrak{a}_{\mathfrak{p}})$, we take an element $\bar{w}$ of M^* such that $w=\bar{w}M$. Then $H\bar{w}P_\sigma=H\bar{w}P$.*

Proof. It follows from the assumption that $\mathrm{Ad}(\bar{w})(\mathfrak{g}(\sigma))=\mathfrak{g}(\sigma)$. Hence due to Lemma (8.12), we find that

$$H\bar{w}P_\sigma\subseteqq H\bar{w}G(\sigma)P=HG(\sigma)\bar{w}P=H\bar{w}P.$$

The converse inclusion relation is obvious. q.e.d.

Remark (8.14). Since the set $H\bar{w}P_\sigma$ only depends on $w\in W^\sigma(\mathfrak{a}_{\mathfrak{p}})$, we frequently write HwP_σ or HwP instead of this set.

Lemma (8.15). *We take representatives $w_1,\cdots,w_r$ of the set*

$$W(\mathfrak{a}_{\mathfrak{p}};H)\backslash W^\sigma(\mathfrak{a}_{\mathfrak{p}})$$

as in Corollary (7.10). *Then for each i $(1\leqq i\leqq r)$, Hw_iP_σ is an open subset of G and*

$$Hw_iP_\sigma\cap Hw_jP_\sigma=\emptyset\quad(i\neq j).$$

Moreover the union $\bigcup_{i=1}^r Hw_iP_\sigma$ is dense in G.

Proof. This follows from Lemma (8.13) and [Ma, Prop. 1].

Lemma (8.16). *If* $A_H = \exp(\mathfrak{a}_{\mathfrak{p}} \cap \mathfrak{h})$, *we have*

$$\begin{aligned} H \cap P_\sigma &= (M_\sigma \cap H)(A_\sigma \cap H) \\ &= Z_{K\cap H}(\mathfrak{a}) A_H Z_N(\mathfrak{a}). \end{aligned}$$

Proof. By definition, we have that $H \cap P_\sigma = H \cap P_\sigma \cap \sigma(P_\sigma)$. On the other hand, $\sigma(M_\sigma) = M_\sigma$, $\sigma(A_\sigma) = A_\sigma$. Hence we find that

$$H \cap P_\sigma \cap \sigma(P_\sigma) = (M_\sigma \cap H)(A_\sigma \cap H)(N_\sigma \cap \sigma(N_\sigma) \cap H).$$

Since $\Sigma(\mathfrak{a}_{\mathfrak{p}})^+$ is $\theta\sigma$-compatible, it follows that $N_\sigma \cap \sigma(N_\sigma) = \{e\}$. Therefore we have

$$H \cap P_\sigma = (M_\sigma \cap H)(A_\sigma \cap H) = Z_G(\mathfrak{a}) \cap H.$$

Noting that $H = (K \cap H) A_H (N \cap H)$ is an Iwasawa decomposition of H, we find that

$$Z_G(\mathfrak{a}) \cap H = Z_{K\cap H}(\mathfrak{a}) A_H Z_{N\cap H}(\mathfrak{a}).$$

On the other hand, since Lemma (2.7) implies that $Z_{N\cap H}(\mathfrak{a}) = Z_N(\mathfrak{a})$, it follows that

$$H \cap P_\sigma = Z_{K\cap H}(\mathfrak{a}) A_H Z_N(\mathfrak{a}). \qquad \text{q.e.d.}$$

Proposition (8.17).

(i) $G = KAH = HAK$.

(ii) *Let* $k_i \in K$, $a_i \in A$, $h_i \in H$ $(i=1, 2)$ *and assume that* $k_1 a_1 h_1 = k_2 a_2 h_2$. *Then we have*

$$\begin{aligned} &k_1^{-1} k_2 = h_1 h_2^{-1} \in K \cap H, \\ &a_1 = (k_1^{-1} k_2) a_2 (k_1^{-1} k_2)^{-1}. \end{aligned}$$

Proof. (i) follows from [F–J] and (ii) is shown by an argument similar to that in [O–S].

Appendix A. A lemma on the root systems

In this Appendix, we show a lemma which is used in the proof of Lemma (7.7).

(A. 1) Let $(\mathfrak{g}, \mathfrak{h})$ be an irreducible symmetric pair. Retain the notation in the text. Let $\Sigma(\tilde{\mathfrak{j}})$ be the root system of $\mathfrak{g}$. We introduce a (θ, σ)-compatible order on $\Sigma(\tilde{\mathfrak{j}})$ and fix it. For any $\alpha \in \Sigma(\tilde{\mathfrak{j}})$, we denote by s_α the reflection with respect to α. We now put

$$M(\theta, \sigma)^+ = \{\beta \in \Sigma(\tilde{\mathfrak{j}})^+ ; \theta\beta = \beta, \sigma\beta \neq \beta\}$$

as in Section 7.

Lemma (A. 2). (i) *If* $(\mathfrak{g}, \mathfrak{h})$ *is contained in the class* I_1^d $(p+q$*: even,* $p \neq q$, $p, q > 0)$, *there exists* $\alpha \in \Sigma(\tilde{\mathfrak{j}})$ *satisfying the conditions*:

(i.1) $$\theta\alpha = \sigma\alpha, \quad \langle \alpha, \theta\alpha \rangle = 0.$$

(i.2) $$s_\alpha(M(\theta, \sigma)^+) = M(\theta, \sigma)^+.$$

(ii) *If* $(\mathfrak{g}, \mathfrak{h})$ *is contained in the class* I_1^d $(p=q)$, *there exists* $\alpha \in \Sigma(\tilde{\mathfrak{j}})$ *satisfying the conditions*:

(ii.1) $$\langle \alpha, \theta\alpha \rangle = 0, \quad \sigma\alpha = -\alpha.$$

(ii.2) $$s_\alpha s_{\theta\alpha}(M(\theta, \sigma)^+) = M(\theta, \sigma)^+.$$

(iii) *If* $(\mathfrak{g}, \mathfrak{h})$ *is contained in the class* III_1^d $(m$*:even*$)$, *there exists* $\alpha \in \Sigma(\tilde{\mathfrak{j}})$ *satisfying the conditions*:

(iii.1) $$\langle \alpha, \theta\alpha \rangle = \langle \alpha, \sigma\alpha \rangle = \langle \alpha, \theta\sigma\alpha \rangle = 0.$$

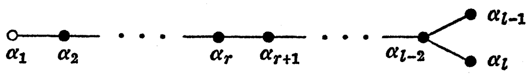
(iii.2) $$s_\alpha s_{\theta\alpha} s_{\sigma\alpha} s_{\theta\sigma\alpha}(M(\theta, \sigma)^+) = M(\theta, \sigma)^+.$$

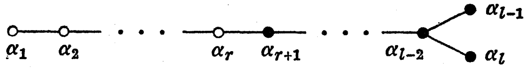
(iv) *If* $(\mathfrak{g}, \mathfrak{h})$ *is contained in the class* IV_2, *there exists* $\alpha \in \Sigma(\tilde{\mathfrak{j}})$ *satisfying the conditions*:

(iv.1) $$\theta\alpha = \sigma\alpha, \quad \langle \alpha, \theta\alpha \rangle = 0.$$

(iv.2) $$s_\alpha s_{\theta\alpha}(M(\theta, \sigma)^+) = M(\theta, \sigma)^+.$$

Proof. (i). In this case, $\mathfrak{g} = \mathfrak{so}(p+q+1, 1)$ and $\mathfrak{h} = \mathfrak{so}(p+1) + \mathfrak{so}(q, 1)$. Put $l = (p+q+2)/2$ and $r = \min(p, q) + 1$. By the assumption, $1 \leq r < l$. Then the root system $\Sigma(\tilde{\mathfrak{j}})$ is of type D_l. Let $\Psi = \{\alpha_1, \cdots, \alpha_l\}$ be a fundamental positive system of $\Sigma(\tilde{\mathfrak{j}})$. Let $\tilde{\mathfrak{j}}_0 = \sqrt{-1}(\mathfrak{k} \cap \tilde{\mathfrak{j}}) + \mathfrak{a}_\mathfrak{p}$. Then as was already remarked in (3.7), every root of $\Sigma(\tilde{\mathfrak{j}})$ is real-valued on $\tilde{\mathfrak{j}}_0$. By taking a suitable orthonormal basis $\{e_1, \cdots, e_l\}$ on the dual vector space $\tilde{\mathfrak{j}}_0^*$ of $\tilde{\mathfrak{j}}_0$, we may put $\alpha_i = e_i - e_{i+1}$ $(1 \leq i < l)$ and $\alpha_l = e_{l-1} + e_l$. We may assume that Ψ is a (θ, σ)-fundamental system. Then it is clear from the definition that the Satake diagram of $(\Sigma(\tilde{\mathfrak{j}}), (-\theta))$ and that of $(\Sigma(\tilde{\mathfrak{j}}), (-\sigma))$ are given by

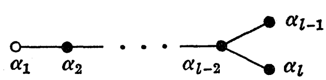
$(\Sigma(\tilde{\mathfrak{j}}), (-\theta))$: 

$(\Sigma(\tilde{\mathfrak{j}}), (-\sigma))$: 

Then it is clear that $M(\theta, \sigma)^+ = \{\pm(e_i \pm e_j);\ 1<i<j,\ i \leqq r\}$. In particular, if $r=1$, then $M(\theta, \sigma)^+ = \emptyset$ and therefore we have nothing to prove. Hence assume that $r>1$. We take $\alpha = e_1 - e_l$. Then it is clear that $\theta\alpha = \sigma\alpha = -e_1 - e_l$ and $\langle \alpha, \theta\alpha \rangle = 0$. Moreover, by direct computation we find that $s_\alpha(M(\theta, \sigma)^+) = M(\theta, \sigma)^+$.

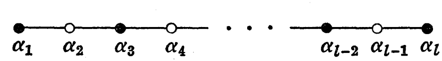
Proof of (ii). The proof of (ii) is quite similar to that of (i). Hence we omit it.

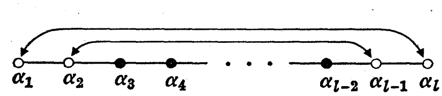
Proof of (iii). Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair contained in the class III_1^d (m: even). Then $\mathfrak{g} = \mathfrak{so}(m+1, 1) + \mathfrak{so}(m+1, 1)$ and $\mathfrak{h} = \mathfrak{so}(m+1, 1)$. In this case $\Sigma(\tilde{\mathfrak{j}})$ has two irreducible components and each of them is of type D_l, where $l = (m+2)/2$. Let Σ be one of the irreducible components of $\Sigma(\tilde{\mathfrak{j}})$. Retain the notation in the proof of (i). Let $\Psi = \{\alpha_1, \cdots, \alpha_l\}$ be a fundamental system for Σ. Then we may assume without loss of generality that $\Psi \cup \sigma\Psi$ is a (θ, σ)-fundamental system of $\Sigma(\tilde{\mathfrak{j}})$. If we denote the restriction of θ to Σ by the same letter, the Satake diagram of $(\Sigma, (-\theta))$ is given by



In this case, $M(\theta, \sigma)^+ = M \cup \sigma M$, where $M = \{\pm e_i \pm e_j;\ 1<i<j\}$. Put $\alpha = e_1 - e_l$. Then it is clear that $\langle \alpha, \theta\alpha \rangle = \langle \alpha, \sigma\alpha \rangle = \langle \alpha, \theta\sigma\alpha \rangle = 0$. On the other hand, we find by direct computation that $s_\alpha s_{\theta\alpha}(M) = M$ and this implies (iii.2).

Proof of (iv). Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair contained in the class IV_2. Then $\mathfrak{g} = \mathfrak{su}^*(2(m+2))$ and $\mathfrak{h} = \mathfrak{su}^*(2(m+1)) + \mathfrak{su}^*(2) + \boldsymbol{R}$. In this case $\Sigma(\tilde{\mathfrak{j}})$ is of type A_l, where $l = 2m+3$. By taking a suitable choice of a basis, we may take $\Psi = \{\alpha_i = e_i - e_{i+1}\ (1 \leqq i \leqq l)\}$ as a fundamental system for $\Sigma(\tilde{\mathfrak{j}})$. We may assume that this is (θ, σ)-compatible. Then the Satake diagram of $(\Sigma(\tilde{\mathfrak{j}}), (-\theta))$ and that of $(\Sigma(\tilde{\mathfrak{j}}), (-\sigma))$ are given by

$(\Sigma(\tilde{\mathfrak{j}}), (-\theta))$: 

$(\Sigma(\tilde{\mathfrak{j}}), (-\sigma))$: 

From this, it is clear that $M(\theta, \sigma)^+=\{e_1-e_2, e_l-e_{l+1}\}$. We take $\alpha=e_1-e_l$. Since $\theta\alpha=\sigma\alpha=-e_2+e_{l+1}$, it follows that $\langle\alpha, \theta\alpha\rangle=0$. On the other hand, we see that $s_\alpha s_{\theta\alpha}(e_1-e_2)=e_l-e_{l+1}$. Hence $s_\alpha s_{\theta\alpha}(M(\theta, \sigma)^+)=M(\theta, \sigma)^+$.

We have thus proved the lemma completely.

Appendix B. A decomposition of the Levi part of a parabolic subalgebra

(B.1) Let $\mathfrak{g}$ be a semisimple Lie algebra. As usual, G denotes a connected linear semisimple Lie group with its Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}_\mathfrak{p}+\mathfrak{n}$ be its Iwasawa decomposition. Let θ be the Cartan involution of $\mathfrak{g}$ corresponding to $\mathfrak{k}$. In this appendix, we study a fine structure of the Levi part of a parabolic subalgebra of $\mathfrak{g}$. We already studied such a fine structure of the parabolic subalgebra $\mathfrak{p}_\sigma$ in Section 8. The result of this section is weaker than this but as a corollary, we obtain a procedure to determine the Satake diagram of the Levi part of an arbitrary parabolic subalgebra. The result of this appendix seems to be known (cf. [Mm]).

(B.2) Let $\tilde{\mathfrak{j}}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_\mathfrak{p}$. Let $\Sigma(\tilde{\mathfrak{j}})$ and $\Sigma(\mathfrak{a}_\mathfrak{p})$ be the root systems of $\tilde{\mathfrak{j}}$ and $\mathfrak{a}_\mathfrak{p}$, respectively. We fix compatible orders on $\Sigma(\tilde{\mathfrak{j}})$ and $\Sigma(\mathfrak{a}_\mathfrak{p})$ and denote by $\Sigma(\tilde{\mathfrak{j}})^+$ and $\Sigma(\mathfrak{a}_\mathfrak{p})^+$ the sets of positive roots with respect to these orders. Let $\Psi(\tilde{\mathfrak{j}})$ and $\Psi(\mathfrak{a}_\mathfrak{p})$ be the fundamental systems for $\Sigma(\tilde{\mathfrak{j}})$ and $\Sigma(\mathfrak{a}_\mathfrak{p})$, respectively. Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$.

(B.3) Let Θ be a subset of $\Psi(\mathfrak{a}_\mathfrak{p})$. We denote by W_Θ the subgroup of W generated by the reflections with respect to the roots in Θ. Let $\mathfrak{g}(\Theta)$ be the subalgebra of $\mathfrak{g}$ generated by $\{\mathfrak{g}(\mathfrak{a}_\mathfrak{p}; \lambda); \lambda\in\langle\Theta\rangle\}$, where

$$\langle\Theta\rangle=(\bigoplus_{\alpha\in\Theta} \boldsymbol{R}\alpha)\cap\Sigma(\mathfrak{a}_\mathfrak{p}).$$

It follows from [W, Lemma 1.12.3.14] that $\mathfrak{g}(\Theta)$ is semisimple. We note that $\langle\Theta\rangle$ is the root system of the pair $(\mathfrak{g}(\Theta), \mathfrak{a}_\mathfrak{p}\cap\mathfrak{g}(\Theta))$.

(B.4) We define

$$\begin{aligned}
\mathfrak{a}_\Theta&=\{Y\in\mathfrak{a}_\mathfrak{p}; \alpha(Y)=0 \text{ for any } \alpha\in\Theta\},\\
\mathfrak{m}_\Theta&=\mathfrak{g}(\Theta)+\mathfrak{m},\\
\mathfrak{n}_\Theta^+&=\sum_{\lambda\in\Sigma(\mathfrak{a}_\mathfrak{p})^+-\langle\Theta\rangle}\mathfrak{g}(\mathfrak{a}_\mathfrak{p}; \lambda).
\end{aligned}$$

Let A_Θ, $(M_\Theta)_0$, N_Θ^+ be the analytic subgroups of G corresponding to $\mathfrak{a}_\Theta$, $\mathfrak{m}_\Theta$, $\mathfrak{n}_\Theta^+$, respectively. Moreover put $M_\Theta=(M_\Theta)_0Z(\mathfrak{a}_\mathfrak{p})$, where $Z(\mathfrak{a}_\mathfrak{p})=\exp(\sqrt{-1}\,\mathfrak{a}_\mathfrak{p})\cap K$ (K is the maximal compact subgroup of G with its Lie algebra $\mathfrak{k}$). If $P_\Theta = PW_\Theta P$, where P is the parabolic subgroup of G with its Lie algebra $\mathfrak{m}+\mathfrak{a}_\mathfrak{p}+\mathfrak{n}^+$, then $P_\Theta = M_\Theta A_\Theta N_\Theta^+$ is its Langlands decomposition.

(B.5) We define

$$\Sigma(\tilde{\mathfrak{j}})_\theta = \{\mu \in \Sigma(\tilde{\mathfrak{j}});\ \mu\,|\,\mathfrak{a}_\mathfrak{p}=0\},$$
$$\Sigma(\tilde{\mathfrak{j}})_{\theta,\Theta} = \{\mu \in \Sigma(\tilde{\mathfrak{j}});\ \mu\,|\,\mathfrak{a}_\Theta=0\} = \{\mu \in \Sigma(\tilde{\mathfrak{j}});\ \mu\,|\,\mathfrak{a}_\mathfrak{p} \in \langle\Theta\rangle \cup \{0\}\},$$
$$\Sigma(\tilde{\mathfrak{j}};\Theta) = \{\mu \in \Sigma(\tilde{\mathfrak{j}})_{\theta,\Theta};\ \langle\lambda,\mu\rangle=0 \text{ for any } \lambda \in \Sigma(\tilde{\mathfrak{j}})_{\theta,\Theta}-\Sigma(\tilde{\mathfrak{j}})_\theta\}.$$

It is clear that $\Sigma(\tilde{\mathfrak{j}};\Theta)$ is a root system. We define subalgebras $\mathfrak{m}(\Theta)$ and $\mathfrak{z}_\Theta$ of $\mathfrak{g}$ by

$$\mathfrak{m}(\Theta) = \mathfrak{g} \cap \langle \sum_{\mu \in \Sigma(\tilde{\mathfrak{j}};\Theta)} \mathfrak{g}_C(\tilde{\mathfrak{j}};\mu)\rangle,$$
$$\mathfrak{z}_\Theta = \{Y \in \tilde{\mathfrak{j}} \cap \mathfrak{k};\ \mu(Y)=0 \text{ for any } \mu \in \Sigma(\tilde{\mathfrak{j}})_{\theta,\Theta}\}.$$

Theorem (B.6). (1) *$\mathfrak{z}_\Theta$ is the center of $\mathfrak{m}_\Theta$.*
(2) *$\mathfrak{m}_\Theta = \mathfrak{g}(\Theta)+\mathfrak{m}(\Theta)+\mathfrak{z}_\Theta$ is a direct sum decomposition.*

Proof. It is easy to see that $\mathfrak{z}_\Theta$ is the center of $\mathfrak{m}_\Theta$.
We are going to prove (2). It follows from the definition that

$$\mathfrak{m}_\Theta = \mathfrak{m} + \sum_{\lambda \in \langle\Theta\rangle} \mathfrak{g}(\mathfrak{a}_\mathfrak{p};\lambda) + \mathfrak{a}_\mathfrak{p} \cap \mathfrak{g}(\Theta).$$

Now let $\mu \in \Sigma(\tilde{\mathfrak{j}})_\theta$ and $\lambda \in \Sigma(\tilde{\mathfrak{j}})_{\theta,\Theta}-\Sigma(\tilde{\mathfrak{j}})_\theta$ be such that $\langle\lambda,\mu\rangle \neq 0$. We may assume that $\langle\lambda,\mu\rangle<0$ without loss of generality. Then it follows from [W, Prop. 1.1.2.1] that $\lambda+\mu \in \Sigma(\tilde{\mathfrak{j}})_{\theta,\Theta}$. Since $[\mathfrak{g}_C(\tilde{\mathfrak{j}};-\lambda), \mathfrak{g}_C(\tilde{\mathfrak{j}};\lambda+\mu)] = \mathfrak{g}_C(\tilde{\mathfrak{j}};\mu)$, we see from the definition that $\mathfrak{g}_C(\tilde{\mathfrak{j}};\mu)$ is contained in $\mathfrak{g}(\Theta)_C$, the complexification of $\mathfrak{g}(\Theta)$. This implies that

$$\sum_{\mu \in \Sigma(\tilde{\mathfrak{j}})_{\theta,\Theta}} \mathfrak{g}_C(\tilde{\mathfrak{j}};\mu) \subset \mathfrak{g}(\Theta)_C + \mathfrak{m}(\Theta)_C$$

and therefore that

$$(\mathfrak{m}_\Theta)_C = (\mathfrak{g}(\Theta)+\mathfrak{m}(\Theta)+\mathfrak{z}_\Theta)_C.$$

Hence to prove the theorem, it suffices to show that $\mathfrak{g}(\Theta)$ and $\mathfrak{m}(\Theta)$ commute with each other. For this purpose, take $\alpha \in \Sigma(\tilde{\mathfrak{j}};\Theta)$ and $\beta \in \Sigma(\tilde{\mathfrak{j}})_{\theta,\Theta} - \Sigma(\tilde{\mathfrak{j}})_\theta$. By definition, $\langle\alpha,\beta\rangle=0$. Assume now that $[\mathfrak{g}_C(\tilde{\mathfrak{j}};\alpha), \mathfrak{g}_C(\tilde{\mathfrak{j}};\beta)] \neq \{0\}$. Then $\alpha+\beta \in \Sigma(\tilde{\mathfrak{j}})_{\theta,\Theta}-\Sigma(\tilde{\mathfrak{j}})_\theta$ and therefore $\langle\alpha,\alpha+\beta\rangle = \langle\alpha,\alpha\rangle \neq 0$. This contradicts the definition of $\Sigma(\tilde{\mathfrak{j}};\Theta)$. Accordingly, $\mathfrak{g}(\Theta)_C$ and $\mathfrak{m}(\Theta)_C$ commute with each other. Therefore the theorem is completely proved.

(B.7) From now on, we discuss on the Satake diagram of

$$[\mathfrak{m}(\Theta), \mathfrak{m}(\Theta)] = \mathfrak{g}(\Theta) + \mathfrak{m}(\Theta)$$

and the dimension of $\mathfrak{z}_\Theta$. For this purpose, we give the indices of the simple roots in the following manner.

$$\Psi(\tilde{\mathfrak{j}}) = \{\alpha_1, \cdots, \alpha_R\},$$
$$\Psi(\tilde{\mathfrak{j}}) \cap \Sigma(\tilde{\mathfrak{j}})_\theta = \{\alpha_i \in \Psi(\tilde{\mathfrak{j}});\ R(\Theta) < i \leqq R\},$$
$$\Psi(\tilde{\mathfrak{j}}) \cap \Sigma(\tilde{\mathfrak{j}})_{\theta,\Theta} = \{\alpha_i \in \Psi(\tilde{\mathfrak{j}});\ R(\theta, \Theta) < i \leqq R\}.$$

Here $R(\Theta)$ are $R(\theta, \Theta)$ are certain numbers such that $R(\theta, \Theta) \leqq R(\Theta) \leqq R$. Then

$\Psi(\tilde{\mathfrak{j}}) \cap \Sigma(\tilde{\mathfrak{j}}; \Theta) = \{\alpha_i \in \Psi(\tilde{\mathfrak{j}});$ (1) $R(\Theta) < i \leqq R$,

(2) α_i is contained in the connected component of the Dynkin diagram of $\{\alpha_j; R(\theta, \Theta) < j \leqq R\}\}$.

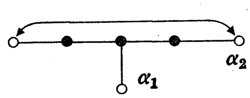
Let $S(\Psi(\tilde{\mathfrak{j}}); -\theta)$ be the Satake diagram of the $(-\theta)$-system of the roots $(\Sigma(\tilde{\mathfrak{j}}), (-\theta))$. We erase all the white circles corresponding to the roots $\alpha_i \in \Psi(\tilde{\mathfrak{j}})$ such that $\alpha_i|\mathfrak{a}_\mathfrak{p} \notin \Theta$ and also erase the lines and arrows connected with the vanished circles. Then we obtain a new Satake diagram. It is easy to prove the followings.

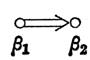
I. $\dim \mathfrak{z}_\Theta =$ The number of arrows which are erased in the procedure above.

II. The Satake diagram of the semisimple Lie algebra $[\mathfrak{m}(\Theta), \mathfrak{m}(\Theta)]$ is the one obtained in the procedure above.

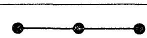
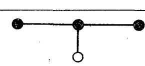
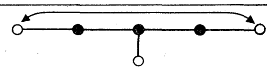
(B.8) We give here an example.

We consider the simple Lie algebra $\mathfrak{e}_{6(-14)}$. The Satake diagram and the Dynkin diagram for the restricted root system are given by

$S(\Psi(\tilde{\mathfrak{j}}); -\theta)$: 

$\Sigma(\mathfrak{a}_\mathfrak{p})$: 

Here $\beta_i = \alpha_i|\mathfrak{a}_\mathfrak{p}$ $(i=1, 2)$.

Θ	$\dim \mathfrak{z}_\Theta$	The Dynkin diagram of $[\mathfrak{m}_\Theta, \mathfrak{m}_\Theta]$	$[\mathfrak{m}_\Theta, \mathfrak{m}_\Theta]$
$\emptyset$	1		$\mathfrak{su}(4)$
$\{\beta_1\}$	1		$\mathfrak{so}(7, 1)$
$\{\beta_2\}$	0		$\mathfrak{su}(5, 1)$
$\{\beta_1, \beta_2\}$	0		$\mathfrak{e}_{6(-14)}$

Remark (B.9). Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair and let σ be the involution for it. Take a Cartan involution θ of $\mathfrak{g}$ commuting with σ and use the notation in the text without notice.

If we take $\Theta = \Sigma(\mathfrak{a}_\mathfrak{p})_\sigma \cap \Psi(\mathfrak{a}_\mathfrak{p})$, then $P_\sigma = P_\Theta$, $\Sigma(\mathfrak{a}_\mathfrak{p})_\sigma = \langle\Theta\rangle$, $\Sigma(\tilde{\mathfrak{j}})_{\theta,\sigma} = \Sigma(\tilde{\mathfrak{j}})_{\theta,\Theta}$, $\mathfrak{g}(\sigma) = \mathfrak{g}(\Theta)$, $\mathfrak{u}(\sigma) + \mathfrak{m}^\sigma = \mathfrak{m}(\Theta)$, $\mathfrak{z}^\sigma + \mathfrak{t}^\sigma = \mathfrak{z}_\Theta$. Needless to say, we find that in this case, Theorem (8.7) give a finer structure than Theorem (B.6).

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T. Oshima
Department of Mathematics
Faculty of Science
University of Tokyo
Tokyo 113, Japan

J. Sekiguchi
Department of Mathematics
Faculty of Science
Tokyo Metropolitan University
Tokyo 158, Japan