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On a Calculation of Vertex Operators for $E_n^{(1)}$ ($n=6, 7, 8$)

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In this note we give explicit formulas of vertex operators for $E_n^{(1)}$ ($n=6, 7, 8$).

Realization of basic representations of Euclidean Lie algebras was initiated by Lepowsky-Wilson [5] for $A_1^{(1)}$. In their work, a differential operator of infinite order in infinitely many variables, called vertex operator, played an important role (vertex representation). Subsequently this construction was generalized to almost all Euclidean Lie algebras by Kac-Kazhdan-Lepowsky-Wilson [3].

Lepowsky-Wilson [6] used vertex representations to study Rogers-Ramanujan type identities from the viewpoint of the theory of Lie algebras.

Meanwhile through the work [2], it has been shown that representation theory of Euclidean Lie algebras are intimately related to the theory of solitons. In this connection explicit forms of vertex operators directly relate to the expressions of the so-called multi-soliton solutions of soliton equations.

Therefore explicit forms of vertex operators may be of some interest not only for the theory of Euclidean Lie algebras but also for the theory of solitons.

Procedure for calculating vertex operators are given in [3]. On the other hand, in [2], vertex operators for some of Euclidean Lie algebras (mainly of the classical type) are derived from those for $\mathfrak{gl}(\infty)$, $\mathfrak{go}(\infty)$ or $\mathfrak{go}(2\infty)$ by the process of "reduction". At present it is not clear whether vertex operators for Euclidean Lie algebras not appeared in [2] (mainly of the exceptional type) are also obtained from those for $\mathfrak{gl}(\infty)$, $\mathfrak{go}(\infty)$ or $\mathfrak{go}(2\infty)$, or not.

In this note we describe a procedure for calculating vertex operators, which supplements the procedure given in [3]. This procedure can be applied to affine Lie algebras and makes use of the relations between the notion of Coxeter transformations and the notion of apposition of Cartan subalgebras studied by Kostant [4].

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1. First we recall the construction of Kac-Kazhdan-Lepowsky-Wilson. Let L be a finite dimensional simple Lie algebra of rank n and let E_j, F_j, H_j $(j=1, \cdots, n)$ be canonical generators of L:

$$[H_j, E_k]=a_{jk}E_j, \quad [H_j, F_k]=-a_{jk}F_j, \quad [E_j, F_k]=\delta_{jk}H_j,$$
$$(j, k=1, \cdots, n)$$

where (a_{jk}) is the Cartan matrix of L. Choose lowest (resp. highest) root vector E_0 (resp. F_0) in such a way that for $H_0=[E_0, F_0]$ the relations $[H_0, E_0]=2E_0$, $[H_0, F_0]=-2F_0$ hold.

Let h be the Coxeter number of L. We introduce a Z/hZ-gradation of L by defining $\deg H_j=0$, $\deg E_j=-\deg F_j=1, j=0, 1, \cdots, n$: $L=\oplus_{j\in Z/nZ}L_j$.

We put $E=\sum_{j=0}^{n} E_j$, then E is a cyclic element of L in the sense of Kostant [4]. Denote by S the centralizer of E in L. It is known that S is a Cartan subalgebra of L and that dimension of the space $S_j=S\cap L_j$ is equal to the multiplicity of j in the set of exponents m_k of L. Here we assume that the exponents are so ordered that $m_j\leq m_k$ $(j<k)$ hold.

The next procedure described in [3] is to choose n root vectors $A_1, \cdots, A_n$ with respect to S, corresponding to the roots $\beta_1, \cdots, \beta_n$, such that their projections on $L_0=\oplus_{j=1}^{n}CH_j$ form a basis of this space. Further choose a basis T_j $(j=1, \cdots, n)$ of S with the following properties: $T_j \in S_{m_j}$ and $\langle T_j, T_{n+1-k}\rangle=\delta_{jk}$ $j, k=1, \cdots, n$, where $\langle\, ,\rangle$ denotes the Killing form of L.

Then vertex operators for the affine Lie algebra $L^{(1)}$ are given by the following formulas

$$Y^{(j)}=\exp\Big(\sum_{k=1}^{\infty}\lambda_{j,k'}\sqrt{\frac{\gamma}{b_k}}\,x_k\Big)\exp\Big(-\sum_{k=1}^{\infty}\lambda_{j,(n+1-k)'}\sqrt{\frac{\gamma}{b_k}}\,\frac{\partial}{\partial x_k}\Big).$$

Here $\lambda_{j,k}=\beta_j(T_k)$, $\lambda_{j,k'}=\lambda_{j,k(\mathrm{mod}\,h)}$ and γ is given in Table γ of [3] and $b_1, b_2, \cdots$ denote the sequence

$$m_j+kh, \quad j=1, \cdots, n, \quad k=0, 1, 2, \cdots$$

arranged in nondecreasing order.

We note that only the quantities $\lambda_{j,k}\lambda_{j,n+1-k}$ are invariant under the choice of T_j. It is also noted in p. 108 of [3] that general formula for the quantities $\lambda_{j,k}$ seems not to be known.

2. Now we supplement this procedure.

First we recall some of the result of Kostant [4]. A Cartan subalgebra $\mathfrak{h}'$ of L is said to be in apposition to a Cartan subalgebra $\mathfrak{h}$ of L with respect to a principal element P (for the definition, see [4, 6.7]) of the adjoint group G of L, if the following two conditions are satisfied: 1) $\mathfrak{h}$ is the set of fixed elements in L under the adjoint action of P, 2) $\mathfrak{h}'$ is stable under P and the set of eigenvalues of $P|_{\mathfrak{h}'}$ includes a primitive h ($=$the Coxeter number of L)-th root of unity.

Then, in our case, according to Theorem 6.7 of [4], S is in apposition to $\mathfrak{h}=CH_1+\cdots+CH_n=L_0$ with respect to a principal element P_0 (for the definition, cf. [4, 6. 7]). Also Corollary 8.6 of [4] states that $P_0|_S$ defines a Coxeter transformation of S.

It is known that eigenvalues of a Coxeter transformation are $\exp(2\pi i m_j/h)$ ($m_j=$the exponents of L) and that corresponding eigenvectors form a basis of a Cartan subalgebra.

In the present situation these eigenvectors have degrees m_j (cf. [4, Th. 6. 7]).

In this way, noting that Coxeter transformations form a single conjugate class in the Weyl group with respect to a Cartan subalgebra, we can reduce the determination of T_j, $j=1, \cdots, n$, to the determination of eigenvectors of a Coxeter transformation.

In other words, we can calculate vertex operators only from the knowledge of a root system of L.

3. In this section we explain this procedure in detail by taking $E_8^{(1)}$ as an example.

According to the table in [1], we realize the root system of E_8 in an eight-dimensional Euclidean space V. We denote by $\langle\ ,\ \rangle$ the inner product on V and by e_j ($j=1, \cdots, 8$) an orthonormal basis of V with respect to $\langle\ ,\ \rangle$. The set of roots of E_8 is given by

$$\pm e_j \pm e_k \quad (j<k), \quad \frac{1}{2}\sum_{j=1}^{8}(-1)^{\nu(j)}e_j\left(\sum_{j=1}^{8}\nu(j)=\text{even}\right)$$

and simple roots are given by

$$\alpha_1=(e_1+e_8)-(e_2+e_3+e_4+e_5+e_6+e_7)$$

$$\alpha_2=e_1+e_2, \quad \alpha_3=e_2-e_1, \quad \alpha_4=e_3-e_2, \quad \alpha_5=e_4-e_3$$

$$\alpha_6=e_5-e_4, \quad \alpha_7=e_6-e_5, \quad \alpha_8=e_7-e_6.$$

Then a Coxeter transformation $C=S_{\alpha_1}\cdot\cdots\cdot S_{\alpha_8}$ on V is given by the matrix

$$C=\begin{pmatrix} -3 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -3 & 1 \\ -1 & 3 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 3 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 3 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 3 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & 3 \end{pmatrix}$$

with respect to the basis (e_j).

Since the exponents of E_8 are 1, 7, 11, 13, 17, 19, 23, and 29, the eigenvalues of C (or ${}^tC$) are primitive 30-th roots of unity and they are the roots of the equation

$$\lambda^8+\lambda^7-\lambda^5-\lambda^4-\lambda^3+\lambda+1=0$$

(of course we can directly calculate them by using the explicit form of C).

An eigenvector v_λ of ${}^tC$ corresponding to the eigenvalue λ is given by

$$\begin{pmatrix} -3 & -2\lambda & +2\lambda^2 & +2\lambda^3 & & +2\lambda^5 & -\lambda^6 & -4\lambda^7 \\ -1 & +2\lambda & & & & & -\lambda^6 & \\ -1 & & +2\lambda^2 & & & & -\lambda^6 & \\ -1 & & & +2\lambda^3 & & & -\lambda^6 & \\ -1 & & & & +2\lambda^4 & & -\lambda^6 & \\ -1 & & & & & +2\lambda^5 & -\lambda^6 & \\ -1 & & & & & & +\lambda^6 & \\ 1 & & & & & & +\lambda^6 & \end{pmatrix}$$

For the sake of notational simplicity, we put $v_j=v_{\omega^j}$, $\omega=\exp(2\pi i/30)$. We express vertex operators in the following form

$$\exp\Big(\sum_{j\in Z_+} a_j x_j\Big)\exp\Big(-\sum_{j\in Z_+} b_j/j\,\partial/\partial x_j\Big)$$

where Z_+ denotes the set of positive integers. Then according to the procedure described in **2**, for a root α of E_8, the coefficients a_j and b_j are given by

$$a_j=\begin{cases} 30\,\langle\alpha, v_j\rangle & j=1, 7, 11, 13, 17, 19, 23, 29 \pmod{30} \\ 0 & \text{otherwise,} \end{cases}$$

$$b_j=\begin{cases} \langle\alpha, v_j\rangle/\langle v_j, v_{30-j}\rangle & j=1, 7, 11, 13, 17, 19, 23, 29 \pmod{30} \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\langle v_1, v_{29}\rangle=4(8-4\omega-2\omega^2-\omega^3-2\omega^4+\omega^5+3\omega^6)$$
$$\langle v_7, v_{23}\rangle=4(4-2\omega-\omega^2+2\omega^3+4\omega^4+3\omega^5-\omega^6-5\omega^7)$$
$$\langle v_{11}, v_{19}\rangle=4(7+4\omega+2\omega^2+\quad+2\omega^4-\quad 3\omega^6)$$
$$\langle v_{13}, v_{17}\rangle=4(11+2\omega+\quad-2\omega^3-4\omega^4-3\omega^5+5\omega^7)$$

further their inverses are given by

$$\langle v_1, v_{29}\rangle^{-1}=(3+3\omega+3\omega^2+2\omega^3+\omega^4-\omega^5-2\omega^6-\omega^7)/60$$
$$\langle v_7, v_{23}\rangle^{-1}=(7+\omega-\omega^2-3\omega^3-3\omega^4-2\omega^5+\omega^6+5\omega^7)/60$$
$$\langle v_{11}, v_{19}\rangle^{-1}=(3-3\omega\quad+\omega^3-\omega^4+\omega^5+2\omega^6-2\omega^7)/60$$
$$\langle v_{13}, v_{17}\rangle^{-1}=(2-\omega-2\omega^2\quad+3\omega^4+2\omega^5-\omega^6-2\omega^7)/60.$$

Using these, the result can be expressed in the following form

$$a_j=\omega^{kj}(1-\omega^{15j})\left(\sum_{l=0}^{14} c_l\omega^{lj}\right)$$
$$b_j=\omega^{-kj}(1-\omega^{15j})\left(\sum_{l=0}^{14} d_l\omega^{lj}\right)/60, \qquad k=0, 1, \cdots, 29.$$

We put $c=(c_0, c_1, \cdots, c_{14})$, $d=(d_0, d_1, \cdots, d_{14})$. Then c and d are given by the following table. The following types correspond to the orbits of the Coxeter transformation C.

type \ c	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
I	−12	6	−2	2	4	−18	18	−4	−2	2	−6	12	−2	−4	−2
II	−10	2	2	0	−2	−2	10	−8	12	−10	8	8	−10	12	−8
III	6	2	−4	4	−2	−6	16	−8	−4	14	−12	14	−4	−8	16
IV	−6	4	0	6	−14	0	14	−6	0	−4	6	10	−6	−6	10
V	−6	10	−14	6	0	−6	14	−10	6	6	0	4	4	0	6
VI	−8	6	4	−12	4	−4	12	−4	−6	8	4	6	−8	6	4
VII	−2	−8	4	2	−2	−4	8	2	4	2	−2	16	−2	2	4
VIII	−10	8	−2	0	2	−8	10	8	−12	10	2	2	10	−12	8

type \ d	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}
I	−1	0	0	0	−1	0	0	0	0	−1	1	0	0	0	0
II	−1	0	−1	0	−1	−1	0	−1	0	−1	0	0	0	0	0
III	0	−1	0	0	−1	0	−1	0	0	−1	0	0	0	0	0
IV	−1	−1	0	0	−1	−1	0	0	0	−1	0	1	0	0	0
V	−1	−1	−1	−1	−1	−1	−1	−1	0	−1	0	0	0	1	0
VI	−1	−1	−1	0	−1	−1	−1	0	0	−1	0	0	1	0	0
VII	−1	−1	−1	−1	−2	−1	−1	−1	−1	−1	0	0	0	0	1
VIII	−1	−1	0	−1	−1	−1	−1	0	−1	−1	0	0	0	0	0

In a similar way we have following result for E_6 ($\omega=\exp(2\pi i/12)$)

$$a_j=\omega^{kj}\Big(\sum_{l=0}^{11} c_l\omega^{lj}\Big)$$

$$b_j=\omega^{-kj}\Big(\sum_{l=0}^{11} d_l\omega^{lj}\Big)/48, \qquad k=0, 1, \cdots, 11$$

type \ c	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}
I	8	−2	10	−4	−10	−2	−8	−10	2	4	−2	14
II	10	−2	−4	2	−14	−8	2	−10	4	10	2	8
III	14	−6	4	6	−10	0	−2	−6	−4	6	−2	0
IV	4	−6	2	0	−14	6	−4	−6	10	0	2	6
VII	8	−4	4	−8	−4	−4	−8	4	−4	8	4	4
VIII	12	4	0	−4	−12	−8	−12	−4	0	4	12	8

type \ d	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}
I	1	2	−1	1	0	−1	−1	0	−1	−1	2	−1
II	2	0	0	2	0	−2	2	−2	−2	2	−2	0
III	4	−2	0	2	−2	0	0	0	−2	2	0	−2
IV	1	0	1	−1	2	−1	−1	2	−3	1	0	−1
V	2	0	0	2	−2	2	−2	0	0	−2	2	−2
VI	2	0	2	0	0	0	−2	0	−2	0	0	0

for E_7 ($\omega=\exp(2\pi i/18)$)

$$a_j=\omega^{kj}(1-\omega^{9j})\Big(\sum_{l=0}^{8} c_l\omega^{lj}\Big)$$

$$b_j=\omega^{-kj}(1-\omega^{9j})\Big(\sum_{l=0}^{8} d_l\omega^{lj}\Big)/108, \qquad k=0, 1, \cdots, 17$$

type \ c	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
I	4	8	−2	−4	10	−16	4	−10	−2
II	6	−6	6	−6	6	−6	−12	12	−12
III	16	−4	10	2	4	−10	−2	−4	−8
IV	12	6	−6	6	12	−12	−6	6	−6
V	6	12	6	−6	6	−6	−12	−6	−12
VI	18	0	0	0	0	0	−18	0	0
VII	10	2	4	−10	16	−4	−8	2	4

type \ d	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
I	1	0	2	−1	3	−2	1	0	−1
II	1	2	−2	2	1	−1	1	−1	1
III	3	1	1	0	2	−1	0	−2	1
IV	2	1	−1	1	2	−2	−1	1	−1
V	1	2	1	2	1	−1	1	−1	−2
VI	3	0	0	3	0	0	0	0	0
VII	2	−1	0	1	1	−3	2	−1	0

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