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F-mild Hyperfunctions and Fuchsian Partial Differential Equations

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§ 0. Introduction

Non-characteristic boundary value problems were formulated for hyperfunctions by Komatsu-Kawai [9] and Schapira [12]. They defined the boundary values of hyperfunction solutions and proved the uniqueness of solutions of the boundary value problem. Solvability of the (local) boundary value problem was proved by Kaneko [2] under the assumption of semi-hyperbolicity.

Kataoka [6, 8] introduced the notion of mildness on the boundary for hyperfunctions. He studied non-characteristic boundary value problems in detail by using the theory of mild hyperfunctions (see [7, 8]).

Let P be a linear partial differential operator of order m with analytic coefficients defined on an open subset M of $\boldsymbol{R}^n \ni x=(x_1, x')$, and set $\operatorname{int} M_+=\{x \in M; x_1>0\}$ and $N=\{x \in M; x_1=0\}$. Suppose that N is non-characteristic with respect to P. Then any hyperfunction $u(x)$ defined on $\operatorname{int} M_+$ satisfying $Pu(x)=0$ becomes mild on N, and the boundary value $v_j(x')=(\partial/\partial x_1)^j u(+0, x')$ is defined as a hyperfunction on N for any integer $j\geqq 0$. Moreover if $v_0(x'), \cdots, v_{m-1}(x')$ vanish, then $u(x)$ vanishes near N.

However, if N is characteristic with respect to P, then $u(x)$ is not mild in general. In this paper, we define the F-mildness for hyperfunctions defined on $\operatorname{int} M_+$. The notion of F-mildness is a generalization of that of mildness. If $u(x)$ is F-mild on N, we can define the boundary value $v_j(x')=(\partial/\partial x_1)^j u(+0, x')$ for any integer $j\geqq 0$ as a hyperfunction on N in a natural way.

Using F-mild hyperfunctions, we formulate boundary value problems for Fuchsian partial differential operators and prove the uniqueness of solutions of the boundary value problem. Let P be a Fuchsian partial differential operator of weight $m-k$ with respect to x_1 in the sense of Baouendi-Goulaouic [1] and let $u(x)$ be a hyperfunction on $\operatorname{int} M_+$ satisfying $Pu(x)=0$. Assume that the characteristic exponents of P avoid certain integral values. Under these assumptions, if $u(x)$ is F-mild on N

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and $(\partial/\partial x_1)^j u(+0, x')$ vanishes for any $0 \leqq j \leqq m-k-1$, then $u(x)$ vanishes near N (Corollary of Theorem 2). Moreover, if P is hyperbolic with respect to the x_1-direction on int M_+, then this boundary value problem is locally solvable. (Solvability will be treated in a forthcoming paper.) We should remark that Kashiwara-Oshima [4] formulated boundary value problems for all hyperfunction solutions of $Pu(x)=0$ on int M_+. They defined m 'boundary values' of a solution $u(x)$ and proved that these 'boundary values' determine $u(x)$ near N. However, as stated above, $(m-k)$ boundary values determine $u(x)$ if $u(x)$ is F-mild.

Cauchy problems for Fuchsian partial differential operators were studied by Tahara [13] and Ôaku [11] in the category of hyperfunctions with a real analytic parameter x_1: Well-posedness of the Cauchy problem for Fuchsian hyperbolic operators was proved in [13], and the uniqueness of solutions of the Cauchy problem was proved in [11] for general Fuchsian operators. Micro-local Cauchy problems for Fuchsian pseudo-differential operators were also treated in [10, 11]. In this and forthcoming papers, we shall extend these results to boundary value problems.

In Section 1, we develop the theory of F-mild hyperfunctions by using the curvilinear wave expansions (Radon transformations) for holomorphic functions (cf. Kataoka [5] and Kaneko [3]). Our main result in Section 1 is the edge of the wedge theorem for F-mild hyperfunctions (Theorem 1).

In Section 2, we formulate the boundary value problem for Fuchsian partial differential operators and state a micro-local uniqueness theorem (in other words, propagation of regularity from the boundary) (Theorem 2).

In Section 3, we give a characterization of the singular spectrum of a mild hyperfunction using the carrier of an analytic functional. Combining this characterization and the method of [1] (Cauchy problems for analytic functionals), we prove the micro-local uniqueness theorem in Section 4.

§ 1. Theory of F-mild hyperfunctions

Kataoka introduced the notion of mild hyperfunctions in his theory of micro-local boundary value problems ([6, 8]). Let M be a real analytic manifold and M_+ be a closed subset of M with real analytic boundary N. When N is non-characteristic for a linear partial differential operator P with analytic coefficients, each hyperfunction u on int M_+ (the interior of M_+) satisfying $Pu=0$ becomes mild on N and has boundary values (as hyperfunctions) on N in a natural way. Thus the notion of mildness is sufficient for non-characteristic boundary value problems. However, when N is characteristic for P, u is not mild on N generally.

We shall consider a wider class of hyperfunctions on int M_+ which have boundary values in a natural way.

First let us recall the notion of mild hyperfunctions in accordance with [6]. Since mildness is a local property and invariant under local coordinate transformations, we set $M=\boldsymbol{R}^n \ni x=(x_1, x')$ with $x'=(x_2, \cdots, x_n)$, $M_+=\{x \in M; x_1 \geqq 0\}$, and $N=\{x \in M; x_1=0\}$. We consider hyperfunctions defined on int M_+ locally on a neighborhood of a point of N. More precisely, let j: int $M_+ \to M$ be the natural embedding and consider the sheaf $\mathscr{B}_{N|M+}=(j_* j^{-1}\mathscr{B}_M)|_N$, where $\mathscr{B}_M$ denotes the sheaf of hyperfunctions on M. By the flabbiness of $\mathscr{B}_M$, we can also write $\mathscr{B}_{N|M+}=\Gamma_{M+}(\mathscr{B}_M)/\Gamma_N(\mathscr{B}_M)$; here for a subset S of M and a sheaf $\mathscr{F}$ on M, $\Gamma_S(\mathscr{F})$ denotes the sheaf of sections of $\mathscr{F}$ whose supports are contained in S.

Let $\mathring{x}=(0, \mathring{x}')$ be a point of N and $u(x)$ be a germ of $\mathscr{B}_{N|M+}$ at $\mathring{x}$. Then $u(x)$ is said to be mild at $\mathring{x}$ if and only if $u(x)$ can be expressed on $\{x \in \text{int}\, M_+; |x-\mathring{x}|<\varepsilon\}$ as a sum of boundary values of holomorphic functions

$$u(x)=\sum_{j=1}^{J} F_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0),$$

where J is a positive integer, ε is a positive number, Γ_j are open convex cones (whose vertices are 0) in $\boldsymbol{R}^{n-1}$, and $F_j(z)$ is a holomorphic function defined on

$$D(\mathring{x}, \Gamma_j, \varepsilon)=\{z=(z_1, z') \in \boldsymbol{C}^n; |z-\mathring{x}|<\varepsilon,$$
$$\sqrt{(\text{Im}\, z_1)^2+(-\text{Re}\, z_1)_+^2}<\varepsilon|\text{Im}\, z'|, \text{Im}\, z' \in \Gamma_j\}.$$

Here we set $(t)_+=\max(0, t)$ for $t \in \boldsymbol{R}$. The subsheaf of $\mathscr{B}_{N|M+}$ consisting of sectinos of $\mathscr{B}_{N|M+}$ which are mild on N is denoted by $\mathring{\mathscr{B}}_{N|M+}$, and its section is called a mild hyperfunction.

Now we define the notion of F-mildness which is a generalization of that of mildness.

Definition 1. Let $u(x)$ be a germ of $\mathscr{B}_{N|M+}$ at $\mathring{x}$. Then $u(x)$ is said to be F-mild at $\mathring{x}$ if and only if $u(x)$ has the expression

$$u(x)=\sum_{j=1}^{J} F_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0) \tag{1}$$

on $\{x \in \text{int}\, M_+; |x-\mathring{x}|<\varepsilon\}$, where J is a positive integer, ε is a positive number, Γ_j are open convex cones in $\boldsymbol{R}^{n-1}$, and $F_j(z)$ is a holomorphic function defined on a neighborhood (in $\boldsymbol{C}^n$) of

$$D'(\mathring{x}, \Gamma_j, \varepsilon)=\{z=(z_1, z') \in \boldsymbol{C}^n; |z-\mathring{x}|<\varepsilon, \text{Re}\, z_1 \geqq 0, \text{Im}\, z_1=0, \text{Im}\, z' \in \Gamma_j\}.$$

For an open subset U of N, we set

$$\overset{\circ}{\mathscr{B}}{}^F_{N|M_+}(U)=\{u \in \mathscr{B}_{N|M_+}(U);\ u \text{ is F-mild at each point of } U\}.$$

Then it is easy to see that $\overset{\circ}{\mathscr{B}}{}^F_{N|M_+}: U\mapsto \overset{\circ}{\mathscr{B}}{}^F_{N|M_+}(U)$ defines a sheaf on N, which we call the sheaf of F-mild hyperfunctions. By the definition, we have the inclusions

$$\overset{\circ}{\mathscr{B}}_{N|M_+}\subset \overset{\circ}{\mathscr{B}}{}^F_{N|M_+}\subset \mathscr{B}_{N|M_+}.$$

These sheaves are invariant under the action of a linear partial differential operator with analytic coefficients.

In the sequel we shall prove several properties of F-mild hyperfunctions.

Lemma 1. *Let Γ be an open convex cone of $\boldsymbol{R}^{n-1}$ and $F(z)$ be a holomorphic function defined on a neighborhood of $D'(0, \Gamma, \varepsilon)$ with $\varepsilon>0$. Then for any open subcone Γ' of Γ such that $\bar{\Gamma}'\cap S^{n-2}\subset\Gamma$ (here $S^{n-2}=\{x'\in \boldsymbol{R}^{n-1}; |x'|=1\}$), there exists $c>0$ such that $F(z)$ is holomorphic on*

$$\{z=x+\sqrt{-1}\,y\in \boldsymbol{C}^n;\ |z|<c,\ y_1^2<c|y'|^2(x_1+c|y'|^2),\ y'\in\Gamma'\}.$$

Proof. The function $F(w_1^2, z')$ is holomorphic on

$$\{(w_1, z')\in \boldsymbol{C}^n;\ |w_1|<\sqrt{(\varepsilon/2)},\ |z'|<\varepsilon/2,\ \operatorname{Im} w_1=0,\ \operatorname{Im} z'\in\Gamma\}.$$

Let Γ' be an open convex cone of $\boldsymbol{R}^{n-1}$ such that $\Gamma'\Subset\Gamma$ (i.e. $\bar{\Gamma}'\cap S^{n-2}\subset\Gamma$). Then by virtue of the local version of Bochner's tube theorem, there exists $\delta>0$ such that $F(w_1^2, z')$ is holomorphic on

$$\{(w_1, z')\in \boldsymbol{C}^n;\ |w_1|<\sqrt{\delta},\ |z'|<\delta,\ |\operatorname{Im} w_1|<\delta|\operatorname{Im} z'|,\ \operatorname{Im} z'\in\Gamma'\}.$$

Hence $F(z)$ is holomorphic on

$$\left\{z\in \boldsymbol{C}^n;\ |z_1|<\delta,\ |z'|<\delta,\ \operatorname{Im} z'\in\Gamma',\ \operatorname{Re} z_1>\left(\frac{|\operatorname{Im} z_1|}{2\delta|\operatorname{Im} z'|}\right)^2-(\delta|\operatorname{Im} z'|)^2\right\}.$$

(Since $F(z)$ is defined on a neighborhood of $D'(0, \Gamma, \varepsilon)$, it is also single-valued on the above set.) This completes the proof.

Proposition 1. *F-mildness is invariant under local coordinate transformations of M which preserve M_+ and N.*

We can easily verify this proposition by using Lemma 1.

Note that we can define the sheaf of F-mild hyperfunctions on the real analytic boundary of a real analytic manifold by virtue of this proposition. The following proposition is also an immediate consequence of Lemma 1:

Proposition 2. *Let $u(x)$ be an F-mild hyperfunction. Then for any intergr $q \geqq 2$, $u(x_1^q, x')$ is a mild hyperfunction.*

Proposition 3. *Let $u(x)$ be an F-mild hyperfunction defined on an open subset U of N. Then $u(x_1^2, x')$ is well-defined as a hyperfunction with a real analytic parameter x_1 on a neighborhood (in M) of U.*

Proof. Let $u(x)$ be defined by

$$u(x)=\sum_{j=1}^{J} F_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0)$$

at $\mathring{x} \in U$, where F_j is holomorphic on a neighborhood of $D'(\mathring{x}, \Gamma_j, \varepsilon)$ with open convex cones Γ_j in $\boldsymbol{R}^{n-1}$ and $\varepsilon > 0$. Then

$$u(x_1^2, x')=\sum_{j=1}^{J} F_j(x_1^2, x'+\sqrt{-1}\,\Gamma_j 0)$$

is a hyperfunction with a real analytic parameter x_1. On the other hand, $u(x_1^2, x')$ is well-defined on $\{x \in M; |x-\mathring{x}|<\varepsilon, x_1 \neq 0\}$ as a hyperfunction and coincides with the above definition there. Hence by Holmgren's uniqueness theorem, the above definition of $u(x_1^2, x')$ does not depend on the choice of defining functions. This completes the proof.

By this proposition we can define boundary values of F-mild hyperfunctions.

Definition 2. Let $u(x)$ be an F-mild hyperfunction defined on an open subset U of N. Then the boundary value $u(+0, x') \in \mathscr{B}_N(U)$ is defined as the restriction of $u(x_1^2, x')$ to N.

Remark. Let $u(x)$ be defined by (1). Then it is easy to see that

$$u(+0, x')=\sum_{j=1}^{J} F_j(0, x'+\sqrt{-1}\,\Gamma_j 0)$$

holds on $\{(0, x') \in N; |x'-\mathring{x}'|<\varepsilon\}$.

Now we define the ρ-singular spectra of F-mild hyperfunctions. Let $\sqrt{-1}\,S^*M=(\sqrt{-1}\,T^*M-M)/\boldsymbol{R}^+=M\times\sqrt{-1}\,S^{n-1}$ and $\sqrt{-1}\,S^*N=N\times\sqrt{-1}\,S^{n-2}$ be the purely imaginary cosphere bundles of M and N respectively and let $\pi_M: \sqrt{-1}\,S^*M \to M$ and $\pi_N: \sqrt{-1}\,S^*N \to N$ be the canonical projections. Let $\rho: \sqrt{-1}\,S^*M|_N - \sqrt{-1}\,S^*_N M \to \sqrt{-1}\,S^*N$ be the canonical map. We denote by $SS(f)$ the singular spectrum of a hyperfunction f.

Definition 3, Let $u(x)$ be an F-mild hyperfunction defined on a subset U of N. Then the ρ-singular spectrum ρ-$SS(u)$ of u is the closed subset of $\pi_N^{-1}(U) \subset \sqrt{-1}\,S^*N$ defined by

$$\rho - SS(u) = \rho(SS(u(x_1^2, x')) \cap (\sqrt{-1}\,S^*M|_N - \sqrt{-1}\,S^*_N M)).$$

It is easy to see that $SS(u(+0, x')) \subset \rho - SS(u(x))$ holds.

In the sequel, we shall characterize the ρ-singular spectrum by defining functions. For $\xi' = (\xi_2, \cdots, \xi_n) \in S^{n-2}$ and $z' = (z_2, \cdots, z_n) \in \boldsymbol{C}^{n-1}$, we put

$$W(z', \xi') = \frac{(n-2)!}{(-2\pi\sqrt{-1})^{n-1}} \times \frac{(1-\sqrt{-1}\langle z', \xi'\rangle)^{n-3}\{1-\sqrt{-1}\langle z', \xi'\rangle - (z'^2 - \langle z', \xi'\rangle^2)\}}{\{\langle z', \xi'\rangle + \sqrt{-1}(z'^2 - \langle z', \xi'\rangle^2)\}^{n-1}},$$

where $\langle z', \xi'\rangle = z_2\xi_2 + \cdots + z_n\xi_n$ and $z'^2 = z_2^2 + \cdots + z_n^2$. In the sequel, we use fundamental properties of curvilinear wave expansions (Radon transformations) for holomorphic functions proved in Section 1 of [5]. See also Chapter 2, Section 3 and Chapter 3, Section 3 of [3].

Lemma 2. *Let K be a compact subset of N with C^2-boundary and let $u(x)$ be an F-mild hyperfunction on K defined by*

$$u(x) = \sum_{j=1}^{J} F_j(x_1, x' + \sqrt{-1}\,\Gamma_j 0),$$

where F_j is holomorphic on a neighborhood of $D'(K, \Gamma_j, \varepsilon) = \bigcup_{x \in K} D'(x, \Gamma, \varepsilon)$ with open convex cones Γ_j in $\boldsymbol{R}^{n-1}$ and $\varepsilon > 0$. Let $\mathring{x}$ be a point of int K *and $\mathring{\xi}'$ be a point of S^{n-2}. Then the point $(\mathring{x}, \sqrt{-1}\,\mathring{\xi}') \in \sqrt{-1}\,S^*N$ is not contained in $\rho - SS(u)$ if and only if*

$$F(z; \xi') = \sum_{j=1}^{J} \int_{K + \sqrt{-1}a_j} F_j(z_1, w') W(z' - w', \xi') dw'$$

is analytic at $(z, \xi') = (\mathring{x}, \mathring{\xi}')$ for any sufficiently small $a_j \in \Gamma_j$.

Proof. First assume that $(\mathring{x}, \sqrt{-1}\,\mathring{\xi}') \notin \rho - SS(u)$. Then there exist holomorphic functions G_ν $(\nu = 1, \cdots, l)$ defined on a neighborhood of $\tilde{D}'(\mathring{x}, V_\nu, \varepsilon) = \{z \in \boldsymbol{C}^n; |z - \mathring{x}| < \varepsilon, \operatorname{Im} z_1 = 0, \operatorname{Im} z' \in V_\nu\}$ such that

$$u(x_1^2, x')=\sum_{\nu=1}^{l} G_\nu(x_1, x'+\sqrt{-1}\,V_\nu 0)$$

on $\{x \in M; |x-\mathring{x}|<\varepsilon\}$; here V_ν are open convex cones in $\boldsymbol{R}^{n-1}$ whose polar sets $V_\nu^0=\{\xi' \in \boldsymbol{R}^{n-1}; \langle y', \xi'\rangle\geqq 0$ for any $y' \in V_\nu\}$ do not contain $\mathring{\xi}'$. Set $D=\{x' \in \boldsymbol{R}^{n-1}; |x'-\mathring{x}'|\leqq\varepsilon/2\}$ and let $b_\nu \in V_\nu$ be small enough. Then by the edge of the wedge theorem for hyperfunctions, we know that

$$F(z_1^2, z'; \xi')-\sum_{\nu=1}^{l}\int_{D+\sqrt{-1}b_\nu} G_\nu(z_1, w')W(z'-w', \xi')dw'$$

is analytic in (z, ξ') on a neighborhood of $\{\mathring{x}\}\times S^{n-2}$ if $b_\nu \in V_\nu$ are small enough. On the other hand,

$$\int_{D+\sqrt{-1}b_\nu} G_\nu(z_1, w')W(z'-w', \xi')dw'$$

is analytic on a neighborhood of $\{\mathring{x}\}\times(S^{n-2}-V_\nu^0)$. Since $\mathring{\xi}'$ is not contained in $V_1^0, \cdots, V_l^0$, the function $F(z_1^2, z'; \xi')$ is analytic at $(z, \xi')=(\mathring{x}, \mathring{\xi}')$. Noting that $F(z; \xi')$ is analytic (and single-valued) on a neighborhood of $D'(\mathring{x}, \Delta, \varepsilon)\times\{\mathring{\xi}'\}$ with some $\varepsilon>0$ for any open cone $\Delta\Subset\{y' \in \boldsymbol{R}^{n-1}; \langle y', \mathring{\xi}'\rangle>0\}$, we know that $F(z; \xi')$ is analytic at $(\mathring{x}, \mathring{\xi}')$.

Now assume that $F(z; \xi')$ is analytic at $(\mathring{x}, \mathring{\xi}')$. Let Δ_k $(k=0, 1, \cdots, J')$ be closed proper convex cones in $\boldsymbol{R}^{n-1}$ such that $\bigcup_{0\leq k\leq J'}\Delta_k=S^{n-2}$, $\mathring{\xi}' \in \operatorname{int}\Delta_0$, the measure of $\Delta_j\cap\Delta_k$ is zero, and that $F(z; \xi')$ is analytic in (z, ξ') on a neighborhood of $\{\mathring{x}\}\times(\Delta_0\cap S^{n-2})$. Put

$$F_{jk}(z)=\int_{\Delta_k\cap S^{n-2}} d\sigma(\xi')\int_{K+\sqrt{-1}a_j} F_j(z_1, w')W(z'-w', \xi')dw'$$

for $k=0, 1, \cdots, J'$, where $d\sigma(\xi')$ denotes the volume element on S^{n-2}. Then $F_{jk}(z)$ is holomorphic on a neighborhood of $D'(\mathring{x}, V_{jk}, \varepsilon)$ with some $\varepsilon>0$ for any open convex cone $V_{jk}\Subset\Delta_k^0+\Gamma_j$, and we have

$$F_j(z)=F_{j0}(z)+\sum_{k=1}^{J'} F_{jk}(z)$$

on $D'(\mathring{x}, \Gamma_j, \varepsilon)$. By the above assumption, $F_{1,0}+\cdots+F_{J,0}$ is analytic at $z=\mathring{x}$. Since

$$\begin{aligned}\pi_N^{-1}(\{\mathring{x}\})\cap\rho-SS(F_{jk}(x_1, x'+\sqrt{-1}\,V_{jk}0))&\subset\{\mathring{x}\}\times\sqrt{-1}\,(\Delta_k^0+\Gamma_j)^0\\&\subset\{\mathring{x}\}\times\sqrt{-1}\,\Delta_k,\end{aligned}$$

we have $(\mathring{x}, \sqrt{-1}\,\mathring{\xi}')\notin\rho-SS(u(x))$. This completes the proof.

Proposition 4. *Let $u(x)$ be an F-mild hyperfunction defined on a neighborhood of $\mathring{x} \in N$. Then $(\mathring{x}, \sqrt{-1}\mathring{\xi}') \in \sqrt{-1}S^*N$ is not contained in $\rho-SS(u)$ if and only if there exist holomorphic functions F_j $(j=1, \cdots, J)$ defined on a neighborhood of $D'(\mathring{x}, \Gamma_j, \varepsilon)$ with $\varepsilon>0$ and open convex cones $\Gamma_j \subset \boldsymbol{R}^{n-1}$ whose polar sets do not contain $\mathring{\xi}'$ such that*

$$u(x)=\sum_{j=1}^{J} F_j(x_1, x'+\sqrt{-1}\Gamma_j 0). \tag{2}$$

Proof. It is obvious that $(\mathring{x}, \sqrt{-1}\mathring{\xi}') \notin \rho-SS(u)$ if u has the above expression. Assume $(\mathring{x}, \sqrt{-1}\mathring{\xi}') \notin \rho-SS(u)$. Let $u(x)$ be defined by

$$u(x)=\sum_{j=1}^{J'} G_j(x_1, x'+\sqrt{-1}V_j 0),$$

where G_j is holomorphic on a neighborhood of $D'(\mathring{x}, V_j, \varepsilon)$ with $\varepsilon>0$ and open convex cones $V_j \subset \boldsymbol{R}^{n-1}$. Then by Lemma 2,

$$G(z;\xi')=\sum_{j=1}^{J'} \int_{D+\sqrt{-1}b_j} G_j(z_1, w')W(z'-w', \xi')dw'$$

is analytic at $(z, \xi')=(\mathring{x}, \mathring{\xi}')$ if $b_j \in V_j$ are sufficiently small. Here we set $D=\{x' \in \boldsymbol{R}^{n-1}; |x'-\mathring{x}'|<\varepsilon/2\}$. Then by the argument of the latter part of the proof of Lemma 2, u has the expression (2) with $\Gamma_j^0 \not\ni \mathring{\xi}'$. This completes the proof.

In view of this proposition, we know that $\rho-SS(u(x))$ coincides with $\iota-SS(u(x))$ defined in [6] for a mild hyperfunction $u(x)$ (see Definition 2.2.1 of [6]).

Proposition 5. *Let $u(x)$ be an F-mild hyperfunction defined on an open subset U of N. Then we have*

$$\rho-SS(u(x))=\rho-SS(u(x_1^q, x'))$$

for any positive integer q.

Proof. Let $\mathring{x}$ be a point of U and let K be a compact subset of U with C^2-boundary such that $\mathring{x} \in \operatorname{int} K$. Let $u(x)$ be defined by (1) with F_j holomorphic on a neighborhood of $D'(K, \Gamma_j, \varepsilon)$. Set

$$F(z;\xi')=\sum_{j=1}^{J} \int_{K+\sqrt{-1}a_j} F_j(z_1, w')W(z'-w', \xi')dw'$$

with sufficiently small $a_j \in \Gamma_j$. Let $\mathring{\xi}' \in S^{n-2}$ and set

$$\Gamma=\{y' \in \boldsymbol{R}^{n-1}; \langle y', \mathring{\xi}'\rangle>\sqrt{|y'|^2-\langle y', \mathring{\xi}'\rangle^2}\}.$$

Since $F(z;\xi')$ is analytic on a neighborhood of $D'(\mathring{x}, \Gamma, \delta)\times\{\mathring{\xi}'\}$ with some $\delta>0$, $F(z_1^q, z'; \xi')$ is analytic on a neighborhood of $(z, \xi')=(\mathring{x}, \mathring{\xi}')$ if and only if $F(z;\xi')$ is analytic on a neighborhood of $(\mathring{x}, \mathring{\xi}')$. This completes the proof in view of Lemma 2.

Proposition 6. *Let K be a compact set of N and Γ be an open convex cone of $\boldsymbol{R}^{n-1}$. Let $u(x)$ be an F-mild hyperfunction defined on a neighborhood of K such that $\rho-SS(u)\subset K\times\sqrt{-1}\,\Gamma^0$. Then, for any open convex cone $\Gamma'\Subset\Gamma$, there exists a unique holomorphic function $F(z)$ defined on a neighborhood of $D'(K, \Gamma', \varepsilon)$ for some $\varepsilon>0$ such that*

$$u(x)=F(x_1, x'+\sqrt{-1}\,\Gamma'0)$$

holds on $\{x\in M;\ \mathrm{dis}\,(x, K)<\varepsilon,\ x_1>0\}$ (dis *denotes the distance*).

Proof. There exists a holomorphic function G defined on a neighborhood of $\tilde{D}'(K, \Gamma', \varepsilon)=\bigcup_{x\in K}\tilde{D}'(x, \Gamma', \varepsilon)$ with some $\varepsilon>0$ such that

$$u(x_1^2, x')=G(x_1, x'+\sqrt{-1}\,\Gamma'0)$$

on $\{x\in M;\ \mathrm{dis}\,(x, K)<\varepsilon\}$. Since

$$G(x_1, x'+\sqrt{-1}\,\Gamma'0)=G(-x_1, x'+\sqrt{-1}\,\Gamma'0),$$

we have $G(z_1, z')=G(-z_1, z')$. Hence there exists a holomorphic function $F(z)$ defined on a neighborhood of $D'(K, \Gamma', \delta)$ with some $\delta>0$ such that $G(z)=F(z_1^2, z')$. Then we have

$$u(x_1^2, x')=F(x_1^2, x'+\sqrt{-1}\,\Gamma'0)$$

on $\{x\in M;\ \mathrm{dis}\,(x, K)<\delta\}$. Hence we have

$$u(x)=F(x_1, x'+\sqrt{-1}\,\Gamma'0)$$

on $\{x\in M;\ \mathrm{dis}\,(x, K)<\delta,\ x_1>0\}$. This completes the proof.

Theorem 1 (*Edge of the wedge theorem for F-mild hyperfunctions*). *Let K be a compact subset of N and $\Gamma_1, \cdots, \Gamma_J$ be open convex cones in $\boldsymbol{R}^{n-1}$. Let $F_j(z)$ be a holomorphic function defined on a neighborhood of $D'(K, \Gamma_j, \varepsilon)$ with $\varepsilon>0$ such that*

$$\sum_{j=1}^{J}F_j(x_1, x'+\sqrt{-1}\,\Gamma_j0)=0$$

on $\{x\in M;\ x_1>0,\ \mathrm{dis}\,(x, K)<\varepsilon\}$. *Then for any subcone $\Gamma'_j\Subset\Gamma_j$, there exist holomorphic functions $F_{jk}(z)$ defined on a neighborhood of $D'(K, \Gamma'_j+\Gamma'_k, \delta)$ with some $\delta>0$ such that*

$$F_j(z)=\sum_{k=1}^{J} F_{jk}(z) \quad (j=1, \cdots, J)$$

and $F_{jk}(z)=-F_{kj}(z)$ *for* $1\leqq j, k\leqq J$.

Proof. We shall prove this theorem by induction on J. First set $J=2$. Note that

$$F_1(x_1^2, x'+\sqrt{-1}\Gamma_1 0)=-F_2(x_1^2, x'+\sqrt{-1}\Gamma_2 0)$$

holds on a neighborhood of K in M by virtue of Holmgren's uniqueness theorem. Hence by the usual edge of the wedge theorem for hyperfunctions, $F_1(z_1^2, z')=-F_2(z_1^2, z')$ is holomorphic on a neighborhood of $D'(K, \Gamma_1'+\Gamma_2', \varepsilon')$ with some $\varepsilon'>0$. Setting $F_{12}=F_1$ and $F_{21}=F_2$, we have proved this theorem for $J=2$.

Now we assume that the theorem has been proved for J. Let $F_1, \cdots, F_{J+1}$, etc. satisfy the assumptions of the theorem with J replaced by $J+1$. Let Γ_j'' be an open convex cone of $\boldsymbol{R}^{n-1}$ such that $\Gamma_j'\Subset\Gamma_j''\Subset\Gamma_j$. Let $\Delta_1, \cdots, \Delta_J$ be closed convex cones in $\boldsymbol{R}^{n-1}$ such that

$$\Delta_j\Subset(\Gamma_j''+\Gamma_{J+1}'')^0 \quad (j=1, \cdots, J),$$
$$\bigcup_{1\leq j\leq J} \Delta_j\supset(\Gamma_1^0\cup\cdots\cup\Gamma_J^0)\cap\Gamma_{J+1}^0$$

and that the measure of $\Delta_j\cap\Delta_k$ is zero if $j\neq k$. Let D be a compact set of N with C^2-boundary such that $K\Subset D\subset\{x\in\boldsymbol{R}^n; \operatorname{dis}(x, K)<\varepsilon/2\}$ and set

$$G_j(z)=\int_{\Delta_j\cap S^{n-2}} d\sigma(\xi')\int_{D+\sqrt{-1}a} F_{J+1}(z_1, w')W(z'-w', \xi')dw'$$

for $j=1, \cdots, J$, where $a\in\Gamma_{J+1}$ is small enough. Since the ρ-singular spectrum of

$$F_{J+1}(x_1, x'+\sqrt{-1}\Gamma_{J+1}0)=-\sum_{j=1}^{J} F_j(x_1, x'+\sqrt{-1}\Gamma_j 0)$$

is contained in $K\times\sqrt{-1}(\Gamma_{J+1}^0\cap(\Gamma_1^0\cup\cdots\cup\Gamma_J^0))$, the function

$$G_0(z)=F_{J+1}(z)-\sum_{j=1}^{J} G_j(z)$$

becomes holomorphic on a neighborhood of K by virtue of Lemma 2. Note that $G_j(z)$ is holomorphic on a neighborhood of $D'(K, \Gamma_j''+\Gamma_{J+1}'', \delta)$

with some $\delta>0$ and that $F_{J+1}=G_0+G_1+\cdots+G_J$ on $D'(K, \Gamma''_{J+1}, \delta)$. Applying the induction hypothesis to the functions $F_1+G_0+G_1$, F_2+G_2, $\cdots$, F_J+G_J, we can find holomorphic functions G_{jk} $(j, k=1, \cdots, J)$ defined on a neighborhood of $D'(K, \Gamma'_j+\Gamma'_k, \delta)$ with $\delta>0$ such that $G_{jk}+G_{kj}=0$ and

$$F_1+G_0+G_1=\sum_{k=1}^{J} G_{1,k},$$

$$F_j+G_j=\sum_{k=1}^{J} G_{jk} \quad (j=2, \cdots, J).$$

Set $F_{jk}=G_{jk}$ for $1\leqq j, k\leqq J$ and $F_{1,J+1}=-F_{J+1,1}=-G_0-G_1$, $F_{j,J+1}=-F_{J+1,j}=-G_j$ for $j=2, \cdots, J$. Then we have

$$F_j=\sum_{k=1}^{J+1} F_{jk}, \qquad F_{jk}+F_{kj}=0,$$

and F_{jk} is holomorphic on a neighborhood of $D'(K, \Gamma'_j+\Gamma'_k, \delta)$. This completes the proof.

Before ending this section, we give an example of F-mild hyperfunctions which are not mild.

Example. *Let α be a real number such that $1<\alpha<2$ and set*

$$F(z_1, z_2)=(z_1-e^{\pi\sqrt{-1}(2-\alpha)/2}z_2^{\alpha})^{-1};$$

here we take the branch of z_2^{α} such that $z_2^{\alpha}>0$ for $z_2>0$. Then $F(z_1, z_2)$ is holomorphic on

$$\{(z_1, z_2)\in \boldsymbol{C}^2; |z_1|<|z_2|^{\alpha}, \operatorname{Im} z_2>0\}$$
$$\cup \left\{(z_1, z_2)\in \boldsymbol{C}^2; |\arg z_1|<\left(1-\frac{\alpha}{2}\right)\pi, \operatorname{Im} z_2>0\right\}.$$

Thus $u(x)=F(x_1, x'+\sqrt{-1}0)$ defines an F-mild hyperfunction on $N=\{(x_1, x_2)\in \boldsymbol{R}^2; x_1=0\}$. On the other hand, $F(z_1, z_2)$ is singular on the set $\{(-t^{\alpha}, \sqrt{-1}t); t\geqq 0\}$. By virtue of Proposition 6 and the corresponding result for mild hyperfunctions (Proposition 2.1.21 of [6]), we know that $u(x)$ is not mild at $(0, 0)$.

§ 2. Fuchsian partial differential equations

We use the notation $D=(D_1, D')$ and $D'=(D_2, \cdots, D_n)$ with $D_j=\partial/\partial x_j$. Let P be a linear partial differential operator with real analytic coefficients defined on a neighborhood of $\mathring{x}=(0, \mathring{x}')\in N$. In accordance with Baouendi-Goulaouic [1], we call P a Fuchsian partial differential operator of weight $m-k$ with respect to x_1 if P can be written in the form

$$P=x_1^k D_1^m+A_1(x, D')x_1^{k-1}D_1^{m-1}+\cdots+A_k(x, D')D_1^{m-k}$$
$$+A_{k+1}(x, D')D_1^{m-k-1}+\cdots+A_m(x, D'),$$

where

(i) $k, m \in \boldsymbol{Z}, 0\leqq k\leqq m$,

(ii) the order of $A_j(x, D')$ is at most j for $1\leqq j\leqq m$,

(iii) the order of $A_j(0, x', D')$ is at most 0 for $1\leqq j\leqq k$.

Setting $A_j(0, x', D')=a_j(x')$ for $1\leqq j\leqq k$, we define the characteristic polynomial $e(\lambda, x')$ of P by

$$e(\lambda, x')=\lambda(\lambda-1)\cdots(\lambda-m+1)+a_1(x')\lambda(\lambda-1)\cdots(\lambda-m+2)$$
$$+\cdots+a_k(x')\lambda(\lambda-1)\cdots(\lambda-m+k+1).$$

Let $\lambda=0, \cdots, m-k-1, \lambda_1, \cdots, \lambda_k$ be the roots of the equation $e(\lambda, \mathring{x}')=0$ (they are called the characteristic exponents of P at $\mathring{x}=(0, \mathring{x}')$).

We consider boundary value problems for P in the framework of F-mild hyperfunctions and give a uniqueness theorem.

Theorem 2. *Assume $\lambda_j \notin \{\nu \in \boldsymbol{Z}; \nu\geqq m-k\}$ for $1\leqq j\leqq k$. Let $u(x)$ and $f(x)$ be F-mild hyperfunctions defined on a neighborhood of $\mathring{x}$ such that $Pu=f$. Let $\mathring{\xi}'$ be a point of S^{n-2} and suppose $(\mathring{x}, \sqrt{-1}\mathring{\xi}') \notin \rho-SS(f(x))$ and $(\mathring{x}, \sqrt{-1}\mathring{\xi}') \notin SS(D_1^j u(+0, x'))$ for $0\leqq j\leqq m-k-1$. Then we have $(\mathring{x}, \sqrt{-1}\mathring{\xi}') \notin \rho-SS(u(x))$.*

We shall give the proof of this theorem in Section 4.

Corollary. *Assume $\lambda_j \notin \{\nu \in \boldsymbol{Z}; \nu\geqq m-k\}$ for $1\leqq j\leqq k$. Let $u(x)$ be an F-mild hyperfunction defined on a neighborhood of $\mathring{x}$ satisfying $Pu(x)=0$ and $D_1^j u(+0, x')=0$ for $0\leqq j\leqq m-k-1$. Then $u(x)=0$ holds on a neighborhood of $\mathring{x}$.*

Proof. By Theorem 2, we know that $\rho-SS(u(x))\cap\pi_N^{-1}(\mathring{x})=\phi$. In view of Lemma 2, this implies that $u(x)$ is real analytic on a neighborhood of $\mathring{x}$. Hence we have $u(x)=0$ on a neighborhood of $\mathring{x}$ by virtue of the Cauchy-Kowalevsky type theorem for Fuchsian partial differential equations (see [1]).

On the other hand, we can solve the boundary value problem if P is hyperbolic with respect to the x_1-direction on int M_+. More precisely, let P be as above and assume that $p_m(x, \zeta_1, \sqrt{-1}\xi')$ never vanishes if $x \in \boldsymbol{R}^n$, $|x-\mathring{x}|<\varepsilon, x_1>0, \xi' \in \boldsymbol{R}^{n-1}$, Re $\zeta_1\neq 0$ with some $\varepsilon>0$; here p_m denotes the principal symbol of P. Assume moreover that $\lambda_j \notin \{\nu \in \boldsymbol{Z}; \nu\geqq m-k\}$ for

$1 \leqq j \leqq k$. Then for any F-mild hyperfunction $f(x)$ defined on a neighborhood of $\mathring{x}$ and for any hyperfunction $v_j(x')$ defined on a neighborhood of $\mathring{x}'$ $(0 \leqq j \leqq m-k-1)$, there exists a unique F-mild hyperfunction $u(x)$ defined on a neighborhood of $\mathring{x}$ such that $Pu=f$ and $D_1^j u(+0, x')=v_j(x')$ for $0 \leqq j \leqq m-k-1$. We shall give a proof of this statement elsewhere.

§ 3. Mild hyperfunctions and analytic functionals

We shall consider mild hyperfunctions with compact supports and regard them as analytic functional valued functions.

We fix $R>0$ and put $B_R=\{x' \in \boldsymbol{R}^{n-1}; |x'| \leqq R\}$. For a subset K of $\boldsymbol{C}^{n-1}$ and $s>0$, we set

$$K_s=\{z' \in \boldsymbol{C}^{n-1}; \operatorname{dis}(z', K)<s\},$$

where $\operatorname{dis}(z', K)=\inf\{|z'-w'|; w' \in K\}$. For an open bounded subset Ω of $\boldsymbol{C}^{n-1}$, we denote by $\mathcal{O}_c(\Omega)$ the space of the continuous functions on $\bar{\Omega}$ which are holomorphic on Ω. By the norm $\|f\|=\sup\{|f(z')|; z' \in \Omega\}$, $\mathcal{O}_c(\Omega)$ becomes a Banach space. We denote by $F(\Omega)$ the closure of $\mathcal{O}(\boldsymbol{C}^{n-1})$ (the space of all entire functions on $\boldsymbol{C}^{n-1}$) in $\mathcal{O}_c(\Omega)$, and by $F'(\Omega)$ its dual space (note that $F(\Omega)$ and $F'(\Omega)$ are Banach spaces). If $\Omega_1 \subset \Omega_2$ are two open connected bounded sets in $\boldsymbol{C}^{n-1}$, there are natural inclusions $F(\Omega_2) \subset F(\Omega_1)$ and $F'(\Omega_1) \subset F'(\Omega_2)$.

Let $f(x)$ be a mild hyperfunction defined on N whose support is contained in int B_R. Then $f(x)$ can be regarded as a hyperfunction defined on $\{x \in M; 0<x_1<\varepsilon\}$ whose support is contained in $(0, \varepsilon) \times \operatorname{int} B_R$ for some $\varepsilon>0$. Hence for any $\phi(x') \in \mathcal{A}(B_R)$ (where $\mathcal{A}$ denotes the sheaf of real analytic functions on $\boldsymbol{R}^{n-1}$),

$$\langle f(x_1, \cdot), \phi \rangle = \int_{\boldsymbol{R}^{n-1}} f(x_1, x')\phi(x')dx'$$

is well-defined as a real analytic function on $\{x_1 \in \boldsymbol{R}; 0<x_1<\varepsilon\}$. Moreover this function becomes analytic on a neighborhood of $x_1=0$:

Proposition 7. *Let $f(x)$ be a mild hyperfunction defined on N whose support is contained in* int B_R. *Then there exists $\delta>0$ such that for any $s>0$ the function $f(x_1, \cdot)$ can be regarded as an $F'((B_R)_s)$-valued function on $\{x_1 \in \boldsymbol{R}; 0<x_1 \leqq \delta\}$ by the pairing*

$$\langle f(x_1, \cdot), \phi \rangle = \int_{\boldsymbol{R}^{n-1}} f(x_1, x')\phi(x')dx'$$

for $\phi(z') \in F((B_R)_s)$. Moreover the function $f(x_1, \cdot)$ can be continued to an

$F'((B_R)_s)$-valued holomorphic function defined on a complex neighborhood of $\{x_1 \in \boldsymbol{R}; 0 \leqq x_1 \leqq \delta\}$. (This complex neighborhood depends on $s>0$).

Proof. Using the softness of the sheaf of mild microfunctions $\mathring{\mathscr{C}}_{N|M+}$ (see Theorem 2.1.12 of [6], where $\mathring{\mathscr{C}}_{N|M+}$ is denoted by $\hat{\mathscr{C}}_{N|M+}$) and Proposition 2.1.21 of [6], we can take holomorphic functions $F_j(z)$ ($j=1, \cdots, J$) defined on a neighborhood of $D'(B_R, \Gamma_j, 2\varepsilon)$ with open convex cones $\Gamma_j \subset \boldsymbol{R}^{n-1}$ and $\varepsilon>0$ such that

$$f(x)=\sum_{j=1}^{J} F_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0).$$

Let $a_j \in \Gamma_j$ be small enough and set

$$F(z; \xi')=\sum_{j=1}^{J}\int_{B_R+\varepsilon+\sqrt{-1}a_j} F_j(z_1, w')W(z'-w', \xi')dw'.$$

In view of Lemma 2, $F(z; \xi')$ is analytic on a neighborhood of $(z, \xi') \in \{0\}\times\partial B_R\times S^{n-2}$. Let Δ_k ($k=1, \cdots, n$) be closed proper convex cones in $\boldsymbol{R}^{n-1}$ such that $\Delta_1 \cup \cdots \cup \Delta_n=\boldsymbol{R}^{n-1}$ and the measure of $\Delta_j \cap \Delta_k$ is zero for $j \neq k$. We set

$$G_k(z)=\int_{\Delta_k \cap S^{n-2}} F(z; \xi')d\sigma(\xi').$$

Then for arbitrary open convex cones $V_k \Subset \Delta_k^0$, G_k is holomorphic on a neighborhood of

$$D'(B_R, V_k, \delta) \cup \{z \in \boldsymbol{C}^n; |z_1|<\delta, \text{dis}\,(z', \partial B_R)<\delta\}$$

with some $0<\delta<1$. Moreover we have

$$f(x)=\sum_{k=1}^{n} G_k(x_1, x'+\sqrt{-1}\,V_k 0)$$

on B_R. By Lemma 1, we may assume that each $G_k(z)$ is holomorphic on

$$\{z=x+\sqrt{-1}\,y \in \boldsymbol{C}^n; |z_1|<\delta, \text{dis}\,(z', B_R)<\delta, x_1>-\delta\,|y'|^2, \\ |y_1|<\delta|y'|^2, y' \in V_k\}.$$

We may assume $0<s<\delta/4$. Let $\varepsilon_k: B_R \to \boldsymbol{R}^{n-1}$ be a C^2 map such that

(i) $\varepsilon_k(x')=0$ for $x' \in \partial B_R$,

(ii) $\varepsilon_k(x') \in V_k$, $|\varepsilon_k(x')| \leqq s/2$ for $x' \in \text{int}\, B_R$,

(iii) $|\varepsilon_k(x')|=s/2$ for $x' \in B_R$, $\text{dis}\,(x', \partial B_R) \geqq \delta/2$,

and set $\gamma_k=\{x'+\sqrt{-1}\,\varepsilon_k(x'); x' \in B_R\}$. Then by the definition of the integration of a hyperfunction, we have

$$\int_{R^{n-1}} f(x_1, x')\phi(x')dx' = \sum_{k=1}^{n} \int_{\gamma k} G_k(x_1, w')\phi(w')dw'$$

for $\phi \in F((B_R)_s)$. Thus the function

$$\langle f(z_1, \cdot), \phi \rangle = \sum_{k=1}^{n} \int_{\gamma k} G_k(z_1, w')\phi(w')dw'$$

is holomorphic on

$$U_s = \{z_1 = x_1 + \sqrt{-1}\, y_1 \in C; |z_1| < \delta/4,\ x_1 > -\delta s^2/4,\ |y_1| < \delta s^2/4\}.$$

It is easy to see that $f(z_1, \cdot)$ is an $F'((B_R)_s)$-valued holomorphic function on U_s. This completes the proof.

In the sequel we shall characterize the ρ-singular spectra of mild hyperfunctions by using the carriers of analytic functionals. For

$$\xi' = (\xi_2, \cdots, \xi_n) \in S^{n-2} \quad \text{and} \quad z' = (z_2, \cdots, z_n) \in C^{n-1},$$

we set

$$\Phi(z', \xi') = \langle z', \xi' \rangle + \sqrt{-1}\,(z'^2 - \langle z', \xi' \rangle^2).$$

For $w' \in C^{n-1}$ and $\xi' \in S^{n-2}$ we define an open set $U(w', \xi')$ of C^{n-1} by

$$U(w', \xi') = \{z' \in C^{n-1};\ \mathrm{Im}\, \Phi(w' - z', \xi') > -(\mathrm{Re}\, \Phi(w' - z', \xi'))^2\}.$$

We fix $\mathring{\xi}' \in S^{n-2}$ and set

$$V(\mathring{\xi}', s, r) = (B_R)_s \cap \mathrm{int}\, \Big(\bigcap_{(w', \xi') \in K_r} U(w', \xi') \Big),$$

where $K_r = \{(w', \xi') \in C^{n-1} \times S^{n-2}; |w'| \leq r, |\xi' - \mathring{\xi}'| \leq r\}$ for $s, r > 0$. It is easy to see that $V(\mathring{\xi}', s, r)$ is a polynomially convex open subset of C^{n-1}. Note that there is a natural inclusion

$$F'(V(\mathring{\xi}', s, r)) \subset F'((B_R)_s).$$

Proposition 8. *Let $f(x)$ be a mild hyperfunction defined on N whose support is contained in* int B_R. *Then the following conditions* (i), (ii), (iii) *are equivalent*:

(i) $(0, \sqrt{-1}\,\mathring{\xi}') \notin \rho - SS(f)$.

(ii) *For any $s > 0$, there exists $r > 0$ such that $f(z_1, \cdot)$ is an $F'(V(\mathring{\xi}', s, r))$-valued holomorphic funciton on a neighborhood of $z_1 = 0$.*

(iii) *There exist $s, r > 0$ such that $f(z_1, \cdot)$ is an $F'(V(\mathring{\xi}', s, r))$-valued holomorphic function defined on a neighborhood of $z_1 = 0$.*

Proof. First let us show that (i) implies (ii). We inherit the notations in the proof of Proposition 7. We may assume $\mathring{\xi}' \in \operatorname{int} \Delta_1$ and $F(z;\xi')$ is analytic on a neighborhood of $\{z \in \boldsymbol{C}^n; |z_1| \leqq \varepsilon, |z'| \leqq \varepsilon\} \times (\Delta_1 \cap S^{n-2})$ with some $\varepsilon > 0$. Then $G_1(z)$ becomes holomorphic on $\{z \in \boldsymbol{C}^n; |z_1| \leqq \varepsilon, |z'| \leqq \varepsilon\}$. We may assume $V_k \Subset \{y' \in \boldsymbol{R}^{n-1}; \langle y', \mathring{\xi}' \rangle < 0\}$ for $k=2, \cdots, n$. We can modify ε_1 so that $\langle \varepsilon_1(x'), \mathring{\xi}' \rangle < 0$ if $|x'| \leqq \varepsilon/2$ and that $\varepsilon_1(x') \in V_1$ is small enough if $x' \in B_R - B_\varepsilon$. Since $U(w', \xi')$ contains

$$\{z' = x' + \sqrt{-1}\, y' \in \boldsymbol{C}^{n-1}; \langle v' - y', \xi' \rangle > ((v' - y')^2 - \langle v' - y', \xi' \rangle^2)\}$$

with $w' = u' + \sqrt{-1}\, v'$, we may assume $\gamma_k \subset V(\mathring{\xi}', s, r)$ for $k=2, \cdots, n$ and $x' + \sqrt{-1}\, \varepsilon_1(x') \in V(\mathring{\xi}', s, r)$ for $|x'| \leqq \varepsilon/2$ if $r > 0$ is small enough. On the other hand, since $B_R - B_\varepsilon$ is contained in $V(\mathring{\xi}', s, r)$ if $r < \varepsilon$, we may assume $\gamma_1 \subset V(\mathring{\xi}', s, r)$ by letting $\varepsilon_1(x') \in V_1$ be sufficiently small for $x' \in B_R - B_\varepsilon$. Hence we can take $\gamma_k \subset V(\mathring{\xi}', s, r)$ so that $G_k(z)$ is holomorphic on a neighborhood of $\{0\} \times \gamma_k$ for $k=1, \cdots, n$. This implies (ii).

Next let us show that (iii) implies (i). Suppose that $f(z_1, \cdot)$ is an $F'(V(\mathring{\xi}', s_0, r_0))$-valued holomorphic function on $\{|z_1| < \delta\}$ with some $s_0, r_0 > 0$. Note that

$$f(x_1^2, x') = \int_{S^{n-2}} d\sigma(\xi') \int_{\boldsymbol{R}^{n-1}} f(x_1^2, y') W(x' - y' + \sqrt{-1}\, 0, \xi') dy'$$

holds on $(-\delta^{1/2}, \delta^{1/2}) \times \boldsymbol{R}^{n-1}$. Since the singular spectrum of

$$g(x, \xi') = \int_{\boldsymbol{R}^{n-1}} f(x_1^2, y') W(x' - y' + \sqrt{-1}\, 0, \xi') dy'$$

is contained in

$$\{(x, \xi'; \sqrt{-1}\,(a dx_1 + \langle \xi', dx' \rangle)\infty) \in \sqrt{-1}\, S^*(\boldsymbol{R}^n \times S^{n-2}); a \in \boldsymbol{R}\},$$

it suffices to show that $g(x, \xi')$ is analytic at $(x, \xi') = (0, \mathring{\xi}')$. Set

$$G(x_1, z', \xi') = \int_{\boldsymbol{R}^{n-1}} f(x_1^2, u') W(z' - u', \xi') du'.$$

Then $G(x_1, z', \xi')$ is a real analytic function defined on

$$\{(x_1, z', \xi') \in \boldsymbol{R} \times \boldsymbol{C}^{n-1} \times S^{n-2}; |x_1| < \delta^{1/2}, \langle \operatorname{Im} z', \xi' \rangle > (\operatorname{Im} z')^2 - \langle \operatorname{Im} z', \xi' \rangle^2\}$$

with holomorphic parameters z', and we have

$$g(x, \xi') = G(x_1, x' + \sqrt{-1}\, \xi' 0, \xi').$$

It is easy to see that $\{V(\mathring{\xi}', s, r); s>s_0, 0<r<r_0\}$ constitutes a system of fundamental neighborhoods of the closure of $V(\mathring{\xi}, s_0, r_0)$ consisting of polynomially convex open sets. Thus, as a function of w', we can regard $W(z'-w', \xi')$ as an $F(V(\mathring{\xi}', s_0, r_0))$-valued holomorphic function defined on a complex neighborhood of $\{(z', \xi') \in C^{n-1}\times S^{n-2}; |z'|<r_0, |\xi'-\mathring{\xi}'|<r_0\}$. By the assumption (iii), the function

$$G(z, \xi')=\langle f(z_1^2, \cdot), W(z'-w', \xi')\rangle_{w'}$$

is holomorphic on a complex neighborhood of $(z, \xi')=(0, \mathring{\xi}')$. Hence $g(x, \xi')$ is real analytic at $(x, \xi')=(0, \mathring{\xi}')$. This completes the proof.

§ 4. Proof of Theorem 2

We inherit the notation in Section 2. First let us show that it suffices to prove Theorem 2 when $k=m$. For this purpose we begin with the following lemma.

Lemma 3. *Let l be a positive integer and let $u(x)$ be an F-mild hyperfunction defined on a neighborhood of $\mathring{x} \in N$ such that $(\mathring{x}, \sqrt{-1}\,\mathring{\xi}')$ is not contained in the singular spectrum of $D_1^\nu u(+0, x')$ for $\nu=0, 1, \cdots, l-1$. Then there exist two F-mild hyperfunctions $v(x)$ and $u'(x)$ defined on a neighborhood of $\mathring{x}$ such that*

$$u(x)=x_1^l v(x)+u'(x)$$

and that $(\mathring{x}, \sqrt{-1}\,\mathring{\xi}') \notin \rho-SS(u'(x))$.

Proof. Let $u(x)$ be defined by

$$u(x)=\sum_{j=1}^{J} F_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0)$$

with F_j holomorphic on a neighborhood of $D'(\mathring{x}, \Gamma_j, \varepsilon)$, where $\Gamma_1, \cdots, \Gamma_J$ are open convex cones in R^{n-1} and $\varepsilon>0$. Put

$$G_j(z)=F_j(z)-\sum_{\nu=0}^{l-1}\frac{1}{\nu!}z_1^\nu\left(\frac{\partial}{\partial z_1}\right)^\nu F_j(0, z').$$

Then there is a holomorphic function $H_j(z)$ defined on a neighborhood of $D'(\mathring{x}, \Gamma_j, \varepsilon)$ such that $G_j(z)=z_1^l H_j(z)$. Set

$$v(x)=\sum_{j=1}^{J} H_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0),$$

$$u'(x)=\sum_{\nu=0}^{l-1}\frac{1}{\nu!}x_1^\nu D_1^\nu u(+0, x').$$

Then we have $u(x)=x_1^l v(x)+u'(x)$ and $(\mathring{x}, \sqrt{-1}\,\mathring{\xi}') \notin \rho-SS(u')$. This completes the proof.

Now let P, $u(x)$, $f(x)$ be as in Theorem 2 with $0\leqq k<m$. Then by Lemma 3, there exist F-mild hyperfunctions $v(x)$ and $u'(x)$ defined on a neighborhood of $\mathring{x}$ such that

$$u(x)=x_1^{m-k}v(x)+u'(x)$$

and $(\mathring{x}, \sqrt{-1}\,\mathring{\xi}') \notin \rho-SS(u')$. Then we have

$$Px_1^{m-k}v(x)=f(x)-Pu'(x),$$

and $(\mathring{x}, \sqrt{-1}\,\mathring{\xi}')$ is not contained in the ρ-singular spectrum of $f(x)-Pu'(x)$. It is easy to see that Px_1^{m-k} is a Fuchsian partial differential operator of weight 0 with respect to x_1 and its characteristic exponents are not contained in $\{\nu \in Z; \nu\geqq 0\}$. Hence in order to prove Theorem 2, we have only to show the following proposition.

Proposition 9. *Set*

$$P=(x_1D_1)^m-A_1(x, D')(x_1D_1)^{m-1}-\cdots-A_m(x, D'),$$

where $A_j(x, D')$ is a linear partial differential operator of order $\leqq j$ with analytic coefficients defined on a neighborhood of $x=0$ such that $A_j(0, x', D')$ is a function $a_j(x')$ for $j=1, \cdots, m$. Put

$$e(\lambda, x')=\lambda^m-a_1(x')\lambda^{m-1}-\cdots-a_m(x')$$

and assume $e(j, 0)\neq 0$ for any $j \in Z$ with $j\geqq 0$. Let $u(x)$ be an F-mild hyperfunction defined on a neighborhood of 0 such that $\rho-SS(Pu(x))$ does not contain $(0, \sqrt{-1}\,\mathring{\xi}')$. Under these assumptions, $\rho-SS(u(x))$ does not contain $(0, \sqrt{-1}\,\mathring{\xi}')$.

Proof. We can choose an integer $q\geqq 2$ such that $e(j/q, 0)\neq 0$ for any $j \in Z$ with $j\geqq 0$. Set $Pu(x)=f(x)$, $v(x)=u(x_1^q, x')$, and

$$Q=\left(\frac{1}{q}x_1D_1\right)^m-A_1(x_1^q, x', D')\left(\frac{1}{q}x_1D_1\right)^{m-1}-\cdots-A_m(x_1^q, x', D').$$

Note that Q is a Fuchsian partial differential operator of weight 0 with respect to x_1 whose characteristic exponents are not contained in $\{\nu \in Z; \nu\geqq 0\}$. We have

$$g(x)=Qv(x)=f(x_1^q, x')$$

and $(0, \sqrt{-1}\,\mathring{\xi}') \notin \rho-SS(g(x))$. Since $v(x)$ and $g(x)$ are mild hyperfunctions, we may assume that they are defined on N and their supports are contained in int B_R with sufficiently small $R>0$ by virtue of the softness of the sheaf of mild hyperfunctions (Corollary 2.1.22 of [6]). Thus, for any $s>0$, we can regard $v(z_1, \cdot)$ and $g(z_1, \cdot)$ as $F'((B_R)_s)$-valued holomoprhic functions on a neighborhood of $z_1=0$. By Proposition 8, $g(z_1, \cdot)$ is an $F'(V(\mathring{\xi}', s, r))$-valued holomorphic function on a neighborhood of $z_1=0$ for some $s, r>0$.

Since $V(\mathring{\xi}', 2s, r/2) \supset V(\mathring{\xi}', s, r)$, there exists $\varepsilon>0$ such that $V(\mathring{\xi}', 2s, r/2) \supset V(\mathring{\xi}', s, r)_\varepsilon$. Using the method of Baouendi-Goulaouic (Theorem 3 of [1]), we can show that there exists a unique $F'(V(\mathring{\xi}', s, r)_\varepsilon)$-valued holomorphic function $\tilde{v}(z_1, \cdot)$ defined on a neighborhood of $z_1=0$ such that $Q\tilde{v}(z_1, \cdot)=g(z_1, \cdot)$. (Though they assume $V(\mathring{\xi}', s, r) \subset \boldsymbol{R}^{n-1}$ in Theorem 3 of [1], their proof also applies to our case.) By the uniqueness of the solution w of $Qw=g$, we know that $v(z_1, \cdot)$ is an $F'(V(\mathring{\xi}', 2s, r/2))$-valued holomorphic function defined on a neighborhood of $z_1=0$. Thus by Proposition 8, we have $(0, \sqrt{-1}\,\mathring{\xi}') \notin \rho - SS(v(x))$. In view of Proposition 5, we get $(0, \sqrt{-1}\,\mathring{\xi}') \notin \rho - SS(u(x))$. This completes the proof.

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