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Toda Lattice Hierarchy

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Introduction

In the last decade the theory of completely integrable non linear systems, the so called "soliton theory", has made remarkable progress, in which intensive researches have been done by many physicists and mathematicians. Among them the Toda lattice [36] has always been, together with the Korteweg-de Vries (*KdV*) equation, one of the most classical and important objects to be investigated from various points of view, both physical and mathematical.

Several varieties of methods have been developed to reveal the profound mathematical structure in the Toda lattice: Inverse scattering method, spectral theory, Bäcklund transform [5, 7, 9, 18, 27, 37], algebro-geometric method [3, 5, 6, 7, 8, 13, 28, 29, 30], Hirota's method [10, 11, 19], orbit method, group representation theory [2, 3, 4, 14, 15, 16, 17, 30, 31, 32, 35].

In the present paper, inspired by the recent developments in the study on the Kadomtsev-Petviashvili (*KP*) hierarchies [20–25, 34], a hierarchy (a series of mutually commutative higher evolutions) for the two dimensional infinite Toda lattice is introduced. Its algebraic structure, the linearization, the bilinearization in terms of the τ function, the reductions and the special solutions are investigated in detail. Also its analogues of the B and C types and the multi-component type are considered. Our method, which is closely related with those used in [12, 20–26, 33, 34], has the advantage of making the treatment of the infinite lattice extremely clear and algebraic.

Our investigation in the present paper is motivated by the following observations:

The two dimensional infinite Toda lattice (hereafter we shall call it simply the "Toda lattice" (*TL*)) is, by definition, the non linear wave equation

$$\partial_{x_1}\partial_{y_1}u(s)=e^{u(s)-u(s-1)}-e^{u(s+1)-u(s)}, \tag{0.1}$$

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where $u(s)=u(s;x_1,y_1)$, $\partial_{x_1}=\partial/\partial x_1$, $\partial_{y_1}=\partial/\partial y_1$ and s runs over Z, the totality of integers. Notice that (0.1) is subholonomic in the sense that the general solutions depend on arbitrary functions of two variables.

(0.1) is represented in the form

$$\partial_{y_1}B_1-\partial_{x_1}C_1+[B_1, C_1]=0, \tag{0.2}$$

where the symbol [,] denotes the commutator and B_1, C_1 are the matrices (of size $Z\times Z$)

$$B_1=(\delta_{i,j-1})_{i,j\in Z}+(\partial_{y_1}u(i)\delta_{i,j})_{i,j\in Z}$$

$$=\left[\begin{array}{cc|ccc} \ddots & \ddots & & & \\ & \partial_{y_1}u(-1) & 1 & & \\ \hline & & \partial_{y_1}u(0) & 1 & \\ & & & \ddots & \ddots \\ & & & \partial_{y_1}u(1) & \\ & & & & \ddots \end{array}\right],$$

$$C_1=(e^{u(s)-u(s-1)}\delta_{i,j+1})_{i,j\in Z}$$

$$=\left[\begin{array}{cc|ccc} \ddots & \ddots & & & \\ & 0 & & & \\ \hline e^{u(0)-u(-1)} & & 0 & & \\ & & e^{u(1)-u(0)} & 0 & \\ & & \ddots & \ddots & \end{array}\right].$$

If a τ function $\tau(s)=\tau(s;x_1,y_1)$ is introduced by

$$\partial_{x_1}u(s)=\partial_{x_1}\log\frac{\tau(s+1)}{\tau(s)}, \qquad e^{u(s)-u(s+1)}=\frac{\tau(s+1)\tau(s-1)}{\tau(s)^2},$$

(0.1) is transformed into the bilinear equation of the Hirota type

$$\tfrac{1}{2}D_{x_1}D_{y_1}\tau(s)\cdot\tau(s)+\tau(s+1)\tau(s-1)=0, \tag{0.3}$$

where $D_{x_1}D_{y_1}$ is one of Hirota's D-operators [10] which are defined for linear differential operators $F(\partial_t)$ by

$$F(D_t)f(t)\cdot g(t)=F(\partial_{t'})f(t+t')g(t-t')|_{t'=0}. \tag{0.4}$$

Introducing another τ function $\tau'(s)=e^{x_1y_1}\tau(s)$, we can rewrite (0.3) into Hirota's original form [11]

$$\frac{1}{2}D_{x_1}D_{y_1}\tau'(s)\cdot\tau'(s)+\tau'(s+1)\tau'(s-1)-\tau'(s)^2=0. \tag{0.5}$$

The N soliton solution to (0.5) was obtained in [11]. A parametrization of $\tau'(s)$ in terms of the Clifford operators was discussed in [19].

Starting from these observations, we shall develope our consideration.

The plan of the present paper is as follows.

In Chapter 1 a hierarchy for (0.1) is investigated. In Section 1 our hierarchy is defined by the equations of the Lax type

$$\begin{aligned} &\partial_{x_n}L=[B_n, L], \quad \partial_{y_n}L=[C_n, L], \\ &\partial_{x_n}M=[B_n, M], \quad \partial_{y_n}M=[C_n, M], \quad n=1, 2, \cdots, \end{aligned} \tag{0.6}$$

or equivalently by the equations of the Zakharov-Shabat type

$$\begin{aligned} &\partial_{x_n}B_m-\partial_{x_m}B_n+[B_m, B_n]=0, \\ &\partial_{y_n}C_m-\partial_{y_m}C_n+[C_m, C_n]=0, \\ &\partial_{y_n}B_m-\partial_{x_m}C_n+[B_m, C_n]=0, \qquad m, n=1, 2, \cdots \end{aligned} \tag{0.7}$$

which contain (0.2) as a special one. Here $x=(x_1, x_2, \cdots)$ and $y=(y_1, y_2, \cdots)$ are independent variables, while L, M, B_n and C_n are matrices of infinite size in certain algebraic relations stated in Section 1, and serve as unknown dependent variables. In Section 2 the linearization is achieved by the linear equations

$$\begin{aligned} &LW=W\Lambda, \quad MW=W\Lambda^{-1}, \quad \Lambda^{\pm 1}=(\delta_{i,j\mp 1})_{i,j\in \mathbf{Z}}, \\ &\partial_{x_n}W=B_nW, \quad \partial_{y_n}W=C_nW, \quad n=1, 2, \cdots. \end{aligned} \tag{0.8}$$

Two types of matrix-solutions $W^{(\infty)}$ and $W^{(0)}$ of infinite size are constructed and called "wave matrices" as analogues of the wave functions in the classical inverse scattering theory. They are characterized by the bilinear equation

$$W^{(\infty)}(x', y')W^{(\infty)}(x, y)^{-1}=W^{(0)}(x', y')W^{(0)}(x, y)^{-1}. \tag{0.9}$$

In Section 3 the τ functions $\tau(s; x, y)$ and $\tau'(s; x, y)$ are consistently introduced, and the hierarchy is transformed into an infinite number of bilinear equations of the Hirota type. Also a close relation with the two component *KP* hierarchy is revealed. Finally in Section 4 the reductions to the periodic lattice and the hierarchy in the one dimensional sector are discussed.

In Chapter 2 the hierarchies of the B and C types are investigated. In Section 1 the Lie algebras $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ and their subalgebras $\mathfrak{o}(\infty)_l$, $\mathfrak{sp}(\infty)_l$, which were introduced in [23] in the study of the *KP* hierarchies of the B and C types, are reviewed. In Section 2 and Section 3 the Toda lattice hierarchies of the B and C types are introduced in the "odd sector" $\{x_{2n}=y_{2n}=0, n=1, 2, \cdots\}$ by imposing the conditions B_n, $C_n \in \mathfrak{o}(\infty)$ for $n=1, 3, 5, \cdots$ (B type), B_n, $C_n \in \mathfrak{sp}(\infty)$ for $n=1, 3, 5, \cdots$ (C type) respectively. Also the linearization, the τ functions and the periodic reductions associated with $\mathfrak{o}(\infty)_l$ and $\mathfrak{sp}(\infty)_l$ are discussed. In Section 4 another definition of the τ functions is remarked.

In Chapter 3 the multi-component hierarchy is considered. In Section 1 the hierarchy is formulated as an analogue of the multicomponent KP hierarchy [22, 34]. The non abelian Toda lattice is recovered in a special sector of the dependent and independent variables. In Section 2 the linearization, the characterization of wave matrices and a close connection with the multicomponent KP hierarchy are discussed. In Section 3 a generalization of the $AKNS$ hierarchy [1] is derived as a reduction.

In Chapter 4 special solutions are constructed by two algebraic methods. In Section 1 the aspect of the Riemann-Hilbert problem is applied to the Toda lattice hierachies. Actually (0.7) implies

$$W^{(0)}(x, y)=W^{(\infty)}(x, y)A, \qquad A \in GL(\infty), \tag{0.10}$$

which is regarded as an analogue of the Riemann-Hilbert problem. In this way the soliton solutions are recovered. Also a class of the polynomial τ functions of the KP hierarchy is constructed in the same way. In Section 2 another algebraic method is discussed, which originates in the construction of rational solutions [33] to the KP hierarchy.

In Appendix the recent results [12, 20–25, 33, 34] in the study of the KP hierarchies are briefly summarized for the reader's convenience.

In the recent preprint [40] we announced the results of Chapter 1. In the present paper we shall discuss more fully the derivations and further developments of these results.

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1. The Toda Lattice Hierarchy

1.1. Notations and preliminaries

First of all we fix notations to be used throughout this chapter, and explain some elementary facts about the formal Lie algebra $\mathfrak{gl}((\infty))$.

Let Λ^j be the j-th shift matrix, $\Lambda^j=(\delta_{\mu+j,\nu})_{\mu,\nu\in Z}$, and E_{ij} be the (i,j)-matrix unit, $E_{ij}=(\delta_{\mu i}\delta_{\nu j})_{\mu,\nu\in Z}$. Let $\mathfrak{gl}((\infty))$ be the formal Lie algebra consisting of all $Z\times Z$ matrices;

$$\mathfrak{gl}((\infty))=\{\sum_{i,j\in Z} a_{ij}E_{ij}\,|\,a_{ij}\in C\}.$$

A matrix $A\in\mathfrak{gl}((\infty))$ is written in a convenient form as

$$A=\sum_{j\in Z}\operatorname{diag}[a_j(s)]\Lambda^j \tag{1.1.1}$$

where $\operatorname{diag}[a_j(s)]$ denotes a diagonal matrix $\operatorname{diag}(\cdots a_j(-1),\ a_j(0), a_j(1),\ \cdots)$, and $\operatorname{diag}[a_j(s)]\Lambda^j$ is defined as the usual product of matrices. Namely the expression (1.1.1) indicates

$$A=\begin{pmatrix} \ddots & & \ddots & & \\ & a_0(-1) & a_1(-1) & & \\ \ddots & a_{-1}(0) & a_0(0) & a_1(0) & \\ & & a_{-1}(1) & a_0(1) & \ddots \\ & & & \ddots & \ddots \end{pmatrix}\rightarrow.$$

We call $\operatorname{diag}[a_j(s)]$ the j-th coefficient of A.

A matrix $A\in\mathfrak{gl}((\infty))$ is said to be a (strictly) lower triangular matrix if $a_j(s)=0$ for $j\geqq 0$ (resp. $j>0$), while it is said to be a (strictly) upper triangular matrix if $a_j(s)=0$ for $j\leqq 0$ (resp. $j<0$). We define the $(\pm)$ part of a matrix A by

$$(A)_+=\sum_{0\leqq j<+\infty}\operatorname{diag}[a_j(s)]\Lambda^j,\quad (A)_-=\sum_{-\infty<j<0}\operatorname{diag}[a_j(s)]\Lambda^j. \tag{1.1.2}$$

$\mathfrak{gl}((\infty))$ is equipped with two gradations with respect to the order of Λ^j: If $A=\sum_{-\infty<j\leqq m}\operatorname{diag}[a_j(s)]\Lambda^j$, it is said to be of order less than m, it being denoted by $\operatorname{ord} A\leqq m$. On the other hand, if $A=\sum_{m\leqq j<+\infty}\operatorname{diag}[a_j(s)]\Lambda^j$, it is said to be of order larger than m, and it being denoted by $\operatorname{ord} A\geqq m$. In particular, if $A=\sum_{n\leqq j\leqq m}\operatorname{diag}[a_j(s)]\Lambda^j$, it is called bounded.

When matrices A and B are of order less (or larger) than m, the product AB is well-defined. We remark further that $A=\sum_{-\infty<j\leqq m}\operatorname{diag}[a_j(s)]\Lambda^j$ (resp. $A=\sum_{m\leqq j<+\infty}\operatorname{diag}[a_j(s)]\Lambda^j$) with non-zero leading entries (i.e. $a_m(s)\neq 0$ for any s) has an inverse matrix of a form such as $\sum_{-\infty<j\leqq -m}\operatorname{diag}[b_j(s)]\Lambda^j$ (resp. $\sum_{-m\leqq j<+\infty}\operatorname{diag}[b_j(s)]\Lambda^j$).

A matrix A (1.1.1) naturally corresponds to a difference operator

$$\mathscr{A}(s; e^{\partial_s}) = \sum_{j \in Z} a_j(s) e^{j\partial_s}, \tag{1.1.3}$$

where the action of the operator $e^{j\partial_s}$ is defined by

$$e^{j\partial_s} f(s) = f(s+j) \qquad \text{for any } s.$$

The $(\pm)$ part of $\mathscr{A}(s, e^{\partial_s})$ is defined in a similar fashion as (1.1.2).

Throughout this article, the differentiation will be denoted by ∂_{x_1}, etc., namely, $\partial_{x_1} B = \partial B / \partial x_1$, and so on.

1.2. Definition of the Toda lattice hierarchy

Set two copies of time flows $x = (x_1, x_2, \cdots)$, $y = (y_1, y_2, \cdots)$. Let $L, M, B_n, C_n \in \mathfrak{gl}((\infty))$ be

$$\begin{aligned}
&L = \sum_{-\infty < j \leqq 1} \operatorname{diag}[b_j(s)] \Lambda^j && \text{with } b_1(s) = 1 \text{ for any } s, \\
&M = \sum_{-1 \leqq j < +\infty} \operatorname{diag}[c_j(s)] \Lambda^j && \text{with } c_{-1}(s) \neq 0 \text{ for any } s, \\
&B_n = (L^n)_+, \qquad C_n = (M^n)_- .
\end{aligned} \tag{1.2.1}$$

Each entry of L, M is a function in x, y i.e. $b_j(s) = b(s; x, y)$, $c_j(s) = c_j(s; x, y)$, and plays the role of unknown functions to be solved in our scheme. Since L and M are assumed to have non-zero leading entries, they are invertible.

The Toda lattice (hereafter we will abbreviate it to *TL*) hierarchy is formulated as a system of infinitely many equations of the Lax-type

$$\begin{aligned}
&\partial_{x_n} L = [B_n, L], \quad \partial_{x_n} M = [B_n, M], \\
&\partial_{y_n} L = [C_n, L], \quad \partial_{y_n} M = [C_n, M] \quad n = 1, 2, \cdots.
\end{aligned} \tag{1.2.2}$$

Since B_n, C_n are bounded, and ord $L \leqq 1$, ord $M \geqq -1$, the Lie brackets above are well-defined.

The following theorem states that our system (1.2.2) is consistent, namely, that the flows induced by this system mutually commute.

Theorem 1.1 (cf. [12, 20, 33]). *The TL hierarchy* (1.2.2) *is equivalent to a system of equations of the Zakharov-Shabat type,*

$$\begin{aligned}
&\partial_{x_n} B_m - \partial_{x_m} B_n + [B_m, B_n] = 0, \\
&\partial_{y_n} C_m - \partial_{y_m} C_n + [C_m, C_n] = 0, \\
&\partial_{y_n} B_m - \partial_{x_m} C_n + [B_m, C_n] = 0, \quad m, n = 1, 2, \cdots.
\end{aligned} \tag{1.2.3}$$

Proof. First we show that (1.2.2) reduces to (1.2.3). Let us introduce 1-forms ω, ξ, Ω, Ξ, etc., by

$$\omega=\sum_{n=1}^{\infty} L^n dx_n, \quad \xi=\sum_{n=1}^{\infty} M^n dy_n,$$
$$\Omega=(\omega)_+, \quad \Omega_c=-(\omega)_-, \quad \Xi=-(\xi)_+, \quad \Xi_c=(\xi)_-.$$

Note that

$$\partial_{x_n} L^p=[B_n, L^p]$$

follows from (1.2.2) for any positive integer p. Hence the first equations in (1.2.2) are encapsulated into the Pfaffian system,

$$d_x\omega=[\Omega, \omega]^+ \quad (=\Omega_\wedge\omega+\omega_\wedge\Omega),$$

where d_x (resp. d_y) stands for the exterior differentiation with respect to x (resp. y). (Henceforth we will abbreviate the symbol of the exterior product.) Since $[\omega, L^p]=0$ for any p, the above equation reduces to

$$d_x\Omega-d_x\Omega_c=\Omega^2-\Omega_c^2.$$

Since $d_x\Omega$, Ω^2 are upper triangular while $d_x\Omega_c$, Ω_c^2 are strictly lower triangular, the above equation breaks up into

$$d_x\Omega=\Omega^2, \quad d_x\Omega_c=\Omega_c^2.$$

The former equation yields the first one in (1.2.3).

Likewise one obtains

$$d_y\Xi=\Xi^2, \quad d_y\Xi_c=\Xi_c^2.$$

The latter yields the second equation in (1.2.3).

Next we deduce the third equation in (1.2.3) from (1.2.2). The second and third ones among (1.2.2) are rewritten as

$$d_y\omega=[\Xi_c, \omega]^+, \quad d_x\xi=[\Omega, \xi]^+$$

which further leads to

(1.2.4) $$d_y\Omega-[\Xi_c, \Omega]^+=d_y\Omega_c-[\Xi_c, \Omega_c]^+,$$

(1.2.5) $$d_x\Xi_c-[\Omega, \Xi_c]^+=d_x\Xi-[\Omega, \Xi]^+.$$

Using these equations, one sees that

$$\begin{aligned} d_y\Omega+d_x\Xi_c-[\Xi_c, \Omega]^+ &= -d_x\Xi_c+d_y\Omega_c-[\Xi_c, \Omega_c]^+ \quad &&\text{(by (1.2.4))}\\ &= -d_y\Omega-d_x\Xi-[\Xi, \Omega]^+. &&\text{(by (1.2.5))}. \end{aligned}$$

All the matrices in the second line above are strictly lower triangular, while those in the third line are upper triangular. Hence

$$d_y\Omega + d_x\Xi_c - [\Xi_c, \Omega]^+ = 0,$$

from which the third equation in (1.2.3) is derived.

Now we show the converse way. Note that the first equation in (1.2.3) reads

$$\partial_{x_n} L^m - [B_n, L^m] = \partial_{x_n}(L^m)_- + \partial_{x_m} B_n - [B_n, (L^m)_-].$$

Since all the matrices in the right-hand side are of order less than $n-1$, the order of the left-hand side should be bounded for fixed n;

(1.2.6) $\quad \operatorname{ord}(\partial_{x_n} L^m - [B_n, L^m]) \leqq n-1 \qquad \text{for any } m \geqq 0.$

Suppose $\partial_{x_n} L - [B_n, L] \neq 0$. Then it is easy to see that

$$\lim_{m\to\infty} \operatorname{ord}(\partial_{x_n} L^m - [B_n, L^m]) = +\infty,$$

which contradicts (1.2.6). Thus we have proved the first equation in (1.2.2). Other ones among (1.2.2) can be obtained in the same manner as above. Q.E.D.

The third equation with $m=n=1$ in (1.2.3) is the two-dimensional Toda lattice, and this is the reason why we call (1.2.2) (or (1.2.3)) the *TL hierarchy*.

Equations (1.2.2) and (1.2.3) arise as the compatibility condition for the linear problem

(1.2.7) $\quad LW^{(\infty)}(x, y) = W^{(\infty)}(x, y)\Lambda, \quad MW^{(0)}(x, y) = W^{(0)}(x, y)\Lambda^{-1},$

(1.2.8) $\quad \partial_{x_n} W(x, y) = B_n W(x, y), \quad \partial_{y_n} W(x, y) = C_n W(x, y) \quad n = 1, 2, \cdots,$

where $W(x, y) = W^{(\infty)}(x, y)$ and $W^{(0)}(x, y)$. (Hereafter we will often use an abbreviated notation, $W^{\binom{0}{\infty}}(x, y)$ instead of $W(x, y)$.) This linear system may be regarded as an analogue of the simultaneous eigenvalue problem in the *KP* theory [20, 22, 23, 34] (see also the appendix in this article).

We have the following theorem on an explicit expression of solution matrices to the linear problem. The method of our proof is based upon the ideas explored in Kashiwara's lecture note [12].

Theorem 1.2. *Suppose that L, M (1.2.1) are solutions to the TL hierarchy. Then there exist solution matrices $W^{(\infty)}(x, y)$, $W^{(0)}(x, y)$ to the* *linear problem* (1.2.7), (1.2.8) *such that*

$$W^{(\infty)}(x, y)=\hat{W}^{(\infty)}(x, y) \exp \xi(x, \Lambda),$$
$$W^{(0)}(x, y)=\hat{W}^{(0)}(x, y) \exp \xi(y, \Lambda^{-1}), \tag{1.2.9}$$

and

$$\hat{W}^{\binom{0}{\infty}}(x, y)=\sum_{j=0}^{\infty} \operatorname{diag}[\hat{w}_j^{\binom{0}{\infty}}(s; x, y)]\Lambda^{\pm j}$$
$$\hat{W}^{\binom{0}{\infty}}(x, y)^{-1}=\sum_{j=0}^{\infty} \Lambda^{\pm j} \operatorname{diag}[\hat{w}_j^{\binom{0}{\infty}*}(s+1; x, y)] \tag{1.2.10}$$

with $\hat{w}_0^{(\infty)}(s; x, y)=\hat{w}_0^{(\infty)*}(s; x, y)=1$ *and* $\hat{w}_0^{(0)}(s; x, y)\not\equiv 0$ *for any* s. *Here we have set* $\xi(x, \Lambda^{\pm 1})=\sum_{n=1}^{\infty} x_n\Lambda^{\pm n}$.

The solution matrix of such forms will be called wave matrices.

Wave matrices are uniquely determined up to arbitrariness

$$W^{\binom{0}{\infty}}(x, y) \longmapsto W^{\binom{0}{\infty}}(x, y)f^{\binom{0}{\infty}}(\Lambda), \tag{1.2.11}$$

where $f^{\binom{0}{\infty}}(\lambda)=\sum_{j=0}^{\infty} f_j^{\binom{0}{\infty}}\lambda^{\pm j}$ are formal power series in λ with constant scalar coefficients.

Proof. We proceed in steps. First of all we prepare the following lemma.

Lemma 1.3. *The TL hierarchy* (1.2.3) *is equivalent to*

$$\partial_{x_n}(L^m)_- -\partial_{x_m}(L^n)_- +[(L^n)_-, (L^m)_-]=0,$$
$$-\partial_{y_n}(M^m)_- +\partial_{y_m}(M^n)_- +[(M^n)_-, (M^m)_-]=0, \tag{1.2.12}$$
$$\partial_{x_n}(M^m)_- +\partial_{y_m}(L^n)_- +[(L^n)_-, (M^m)_-]=0$$

or

$$-\partial_{x_n}(L^m)_+ +\partial_{x_m}(L^n)_+ +[(L^n)_+, (L^m)_+]=0,$$
$$\partial_{y_n}(M^m)_+ -\partial_{y_m}(M^n)_+ +[(M^n)_+, (M^m)_+]=0, \tag{1.2.13}$$
$$\partial_{x_n}(M^m)_+ +\partial_{y_m}(L^n)_+ -[(L^n)_+, (M^m)_+]=0.$$

Proof. We only show that the first equation in (1.2.12) is derived from the *TL* hierarchy. Since the first equation in (1.2.2) reads as $[\partial_{x_n}+(L^n)_-, L^m]=0$, the first one in (1.2.3) implies

$$\begin{aligned}0&=[\partial_{x_n}-L^n+(L^n)_-, \partial_{x_m}-L^m+(L^m)_-]\\&=[\partial_{x_n}+(L^n)_-, \partial_{x_m}+(L^m)_-]-[\partial_{x_n}+(L^n)_-, L^m]-[L^n, \partial_{x_m}+(L^m)_-]\\&=[\partial_{x_n}+(L^n)_-, \partial_{x_m}+(L^m)_-].\end{aligned}$$

Thus the first equation in (1.2.12) is obtained. Other equations among (1.2.12) or (1.2.13) can be similarly verified. Q.E.D.

Applying this lemma we deduce the following proposition.

Proposition 1.4. *Let L, M (1.2.1) be solutions to the TL hierarchy. Then there exist matrices $\hat{W}^{(\infty)}(x,y)$, $\hat{W}^{(0)}(x,y)$ of the form* (1.2.10) *satisfying the following equations;*

$$L=\hat{W}^{(\infty)}(x,y)\Lambda\hat{W}^{(\infty)}(x,y)^{-} \qquad M=\hat{W}^{(0)}(x,y)\Lambda^{-1}\hat{W}^{(0)}(x,y)^{-1}, \tag{1.2.14}$$

and

$$\begin{aligned}&\partial_{x_n}\hat{W}^{(\infty)}(x,y)+(L^n)_-\hat{W}^{(\infty)}(x,y)=0,\\ &\partial_{y_n}\hat{W}^{(\infty)}(x,y)-(M^n)_-\hat{W}^{(\infty)}(x,y)=0, \quad n=1,2,\cdots,\end{aligned} \tag{1.2.15}$$

$$\begin{aligned}&\partial_{x_n}\hat{W}^{(0)}(x,y)-(L^n)_+\hat{W}^{(0)}(x,y)=0,\\ &\partial_{y_n}\hat{W}^{(0)}(x,y)+(M^n)_+\hat{W}^{(0)}(x,y)=0, \quad n=1,2,\cdots.\end{aligned} \tag{1.2.16}$$

Proof. Thanks to Lemma 1.3, both (1.2.15) and (1.2.16) are compatible systems. Hence the Cauchy problems for them have unique solutions. We observe that there exist $\hat{W}_0^{\binom{0}{\infty}}(x,y)$ of the form (1.2.10) satisfying

$$L=\hat{W}_0^{(\infty)}(x,y)\Lambda\hat{W}_0^{(\infty)}(x,y)^{-1}, \quad M=\hat{W}_0^{(0)}(x,y)\Lambda^{-1}\hat{W}_0^{(0)}(x,y)^{-1}.$$

Let us consider the Cauchy problems for (1.2.15) and (1.2.16) with initial conditions $\hat{W}^{(\infty)}(x,y)|_{x=y=0}=\hat{W}_0^{(\infty)}(x,y)|_{x=y=0}$ and $\hat{W}^{(0)}(x,y)|_{x=y=0}=W_0^{(0)}(x,y)|_{x=y=0}$. The previous remark assures that these problems have unique solutions of the form (1.2.10). Then, by making use of (1.2.2) and (1.2.15), one sees that

$$\begin{aligned}&\partial_{x_n}(L\hat{W}^{(\infty)}-\hat{W}^{(\infty)}\Lambda)\\ &\quad=[B_n,L]\hat{W}^{(\infty)}-L(L^n)_-\hat{W}^{(\infty)}+(L^n)_-\hat{W}^{(\infty)}\Lambda\\ &\quad=-[(L^n)_-,L]\hat{W}^{(\infty)}-L(L^n)_-\hat{W}^{(\infty)}+(L^n)_-\hat{W}^{(\infty)}\Lambda\\ &\quad=-(L^n)_-(L\hat{W}^{(\infty)}-\hat{W}^{(\infty)}\Lambda),\end{aligned}$$

and also that

$$\begin{aligned}&\partial_{y_n}(L\hat{W}^{(\infty)}-\hat{W}^{(\infty)}\Lambda)\\ &\quad=[(M^n)_-,L]\hat{W}^{(\infty)}+L\partial_{y_n}\hat{W}^{(\infty)}-\partial_{y_n}\hat{W}^{(\infty)}\Lambda\\ &\quad=L(\partial_{y_n}\hat{W}^{(\infty)}-(M^n)_-\hat{W}^{(\infty)})+(M^n)_-L\hat{W}^{(\infty)}\\ &\qquad\qquad-(\partial_{y_n}\hat{W}^{(\infty)}\cdot\hat{W}^{(\infty)-1})\hat{W}^{(\infty)}\Lambda\\ &\quad=(M^n)_-(L\hat{W}^{(\infty)}-\hat{W}^{(\infty)}\Lambda).\end{aligned}$$

Hence one finds $L\hat{W}^{(\infty)}-\hat{W}^{(\infty)}\Lambda$ to solve the Cauchy problem (1.2.15) with the initial condition

$$(L\hat{W}^{(\infty)}-\hat{W}^{(\infty)}\Lambda)|_{x=y=0}=(L\hat{W}_0^{(\infty)}-\hat{W}_0^{(\infty)}\Lambda)|_{x=y=0}=0.$$

The uniqueness of solutions shows it to be a null solution, i.e. $L\hat{W}^{(\infty)}-\hat{W}^{(\infty)}\Lambda=0$. Likewise one can prove $M\hat{W}^{(0)}-\hat{W}^{(0)}\Lambda^{-1}=0$. Q.E.D.

We proceed to the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\hat{W}^{\binom{0}{\infty}}(x, y)$ be the solutions to (1.2.14–16) in Proposition 1.4. For them, we set $W^{\binom{0}{\infty}}(x, y)$ as (1.2.9). Since Λ and $\xi(x, \Lambda^{\pm})$ mutually commute, (1.2.7) obviously holds. Moreover, by making use of (1.2.7) and (1.2.15), one has

$$\begin{aligned}\partial_{x_n}W^{(\infty)}&=-(L^n)_-W^{(\infty)}+W^{(\infty)}\Lambda^n\\&=B_nW^{(\infty)}-L^nW^{(\infty)}+W^{(\infty)}\Lambda^n\\&=B_nW^{(\infty)}\quad(\text{since } L^n=W^{(\infty)}\Lambda^nW^{(\infty)-1}).\end{aligned}$$

The other equations are proved by the same argument. Q.E.D.

Now we deduce a bilinear relation which characterizes wave matrices of the *TL* hierarchy.

Let $W^{(\infty)}(x, y)$, $W^{(0)}(x, y)$ be wave matrices. Since

$$\partial_{x_n}W^{(\infty)}(x, y)\cdot W^{(\infty)}(x, y)^{-1}=\partial_{x_n}W^{(0)}(x, y)\cdot W^{(0)}(x, y)^{-1}\quad(=B_n),$$

and

$$\partial_{y_n}W^{(\infty)}(x, t)\cdot W^{(\infty)}(x, y)^{-1}=\partial_{y_n}W^{(0)}(x, y)\cdot W^{(0)}(x, y)^{-1}\quad(=C_n),$$

one can show by induction that

$$\partial_x^\alpha\partial_y^\beta W^{(\infty)}(x, y)\cdot W^{(\infty)}(x, y)^{-1}=\partial_x^\alpha\partial_y^\beta W^{(0)}(x, y)\cdot W^{(0)}(x, y)^{-1}\tag{1.2.17}$$

holds for any multi-indices $\alpha=(\alpha_1, \alpha_2, \cdots)$, $\beta=(\beta_1, \beta_2, \cdots)$, where $\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots$. Furthermore the infinitely many equations in (1.2.17) are encapsulated into a single expression

$$\begin{gathered}W^{(\infty)}(x, y)\cdot W^{(\infty)}(x', y')^{-1}=W^{(0)}(x, y)\cdot W^{(0)}(x', y')^{-1}\\ \text{for any } x, x' \text{ and } y, y'.\end{gathered}\tag{1.2.18}$$

In fact, considering the Taylor expansion of (1.2.18), one easily finds (1.2.18) to be a generating functional expression of (1.2.17). This bilinear relation will play the crucial role in our scheme.

The following theorem says that (1.2.18) completely characterizes wave matrices.

Theorem 1.5. *Let* $W^{(\infty)}(x, y)$, $W^{(0)}(x, y)$ *be matrices of the forms* (1.2.9), (1.2.10), *and suppose them to satisfy the bilinear relation* (1.2.18) *for any* x, x' *and* y, y'. *Then they are wave matrices of the TL hierarchy. That is, setting*

$$L = W^{(\infty)}(x, y)\Lambda W^{(\infty)}(x, y)^{-1}, \quad M = W^{(0)}(x, y)\Lambda^{-1} W^{(0)}(x, y)^{-1},$$

and $B_n = (L^n)_+$, $C_n = (M^n)_-$, *we then have* $\partial_{x_n} W^{\binom{0}{\infty}}(x, y) = B_n W^{\binom{0}{\infty}}(x, y)$, $\partial_{y_n} W^{\binom{0}{\infty}}(x, y) = C_n W^{\binom{0}{\infty}}(x, y)$.

Proof. Using (1.2.17) with $\alpha = (0, \cdots, \overset{n}{\breve{1}}, 0, \cdots)$, $\beta = 0$, one see that

$$\partial_{x_n} \hat{W}^{(\infty)} \cdot \hat{W}^{(\infty)-1} + \hat{W}^{(\infty)} \Lambda^n \hat{W}^{(\infty)-1} = \partial_{x_n} \hat{W}^{(0)} \cdot \hat{W}^{(0)-1}.$$

Since $\hat{W}^{(\infty)}$ is a lower triangular matrix with unit diagonal entries, $\partial_{x_n} \hat{W}^{(\infty)} \cdot \hat{W}^{(\infty)-1}$ is strictly lower triangular. Note also that $\partial_{x_n} \hat{W}^{(0)} \cdot \hat{W}^{(0)-1}$ is upper triangular. Consequently, taking the $(+)$ part of the above equation, one has

$$\begin{aligned}(\partial_{x_n} \hat{W}^{(0)} \cdot \hat{W}^{(0)-1})_+ &= \partial_{x_n} \hat{W}^{(0)} \cdot \hat{W}^{(0)-1} \\ &= (\hat{W}^{(\infty)} \Lambda^n \hat{W}^{(\infty)-1})_+ \\ &= (L^n)_+ \quad (\text{since } L = \hat{W}^{(\infty)} \Lambda \hat{W}^{(\infty)-1}).\end{aligned}$$

Thus $\partial_{x_n} W^{(\infty)} \cdot W^{(\infty)-1} = \partial_{x_n} W^{(0)} \cdot W^{(0)-1} = B_n$.

Now setting $\alpha = 0$, $\beta = (0, \cdots, \overset{n}{\breve{1}}, 0, \cdots)$ in (1.2.17), one sees that

$$\partial_{y_n} \hat{W}^{(\infty)} \cdot \hat{W}^{(\infty)-1} = \partial_{y_n} \hat{W}^{(0)} \cdot \hat{W}^{(0)-1} + \hat{W}^{(0)} \Lambda^{-n} \hat{W}^{(0)-1}.$$

The $(-)$ part above yields

$$\partial_{y_n} \hat{W}^{(\infty)} \cdot \hat{W}^{(\infty)-1} = (\hat{W}^{(0)} \Lambda^{-n} \hat{W}^{(0)-1})_- = C_n.$$

Thus $\partial_{y_n} W^{(\infty)} \cdot W^{(\infty)-1} = \partial_{y_n} W^{(0)} \cdot W^{(0)-1} = C_n$. Q.E.D.

Remark. If matrices $\hat{W}^{(\infty)}(x, y)^{\pm 1}$, $\hat{W}^{(0)}(x, y)^{\pm 1}$ such as (1.2.10) satisfy the bilinear relation (1.2.18), then $\hat{W}^{\binom{0}{\infty}}(x, y)^{-1}$ are automatically inverse matrices of $\hat{W}^{\binom{0}{\infty}}(x, y)$. This fact can be proved as follows: Setting $x = x'$, $y = y'$ in (1.2.18), one has

$$\hat{W}^{(\infty)}(x, y)\hat{W}^{(\infty)}(x, y)^{-1} = \hat{W}^{(0)}(x, y)\hat{W}^{(0)}(x, y)^{-1}.$$

But the left-hand side above is a lower triangular matrix all of whose diagonal entries are 1, while the right-hand side is a upper triangular matrix. Consequently the both sides should be the unit matrix.

The bilinear relation (1.2.18) can be considered as an analogue of the residue formula for the wave function of the *KP* hierarchy [22] (see also the appendix in this paper). To see this claim, let us define wave functions by

$$(1.2.19)\quad \begin{aligned} w^{(\infty)}(s; x, y; \lambda) &= \hat{w}^{(\infty)}(s; x, y; \lambda)\lambda^{s} \exp \xi(x, \lambda), \\ w^{(\infty)*}(s; x, y; \lambda) &= \hat{w}^{(\infty)*}(s; x, y; \lambda)\lambda^{-s} \exp \xi(-x, \lambda), \\ w^{(0)}(s; x, y; \lambda) &= \hat{w}^{(0)}(s; x, y; \lambda)\lambda^{s} \exp \xi(y, \lambda^{-1}), \\ w^{(0)*}(s; x, y; \lambda) &= \hat{w}^{(0)*}(s; x, y; \lambda)\lambda^{-s}\xi(-y, \lambda^{-1}). \end{aligned}$$

Here $\hat{w}^{(\infty)}(s; x, y; \lambda)$, etc. are introduced through the entries of the wave matrices as follows;

$$(1.2.20)\quad \begin{aligned} \hat{w}^{\binom{0}{\infty}}(s; x, y; \lambda) &= \sum_{j=0}^{\infty} \hat{w}^{\binom{0}{\infty}}(s; x, y)\lambda^{\pm j}, \\ \hat{w}^{\binom{0}{\infty}*}(s; x, y; \lambda) &= \sum_{j=0}^{\infty} \hat{w}^{\binom{0}{\infty}*}(s; x, y)\lambda^{\pm j}. \end{aligned}$$

$\xi(x, \lambda)$ is defined by $\xi(x, \lambda) = \sum_{n=1}^{\infty} x_n \lambda^n$.

By a direct calculation, we obtain the following formula.

Proposition 1.6. *The bilinear relation* (1.2.18) *is equivalent to the following residue formulae*;

$$(1.2.21)\quad \begin{aligned} &\oint w^{(\infty)}(s; x, y; \lambda)w^{(\infty)*}(s'; x', y'; \lambda) \frac{d\lambda}{2\pi i} \\ &\quad = \oint w^{(0)}(s; x, y; \lambda^{-1})w^{(0)*}(s'; x', y'; \lambda^{-1}) \frac{\lambda^{-2}d\lambda}{2\pi i} \end{aligned}$$

for any x, x', y, y' *and any integers* s, s'.

Here the integration contours are taken to be a small circle around $\lambda = \infty$.

At the end of this section, we give a brief comment concering a link between the linear problem of the *TL* hierarchy and that of the *KP* hierarchy. For the purpose, we rewrite our linear problem (1.2.7), (1.2.8) in terms of difference operators (§ 1.1).

It is easy to see that the first equation in (1.2.7) reads as

(1.2.21) $$L(s;e^{\partial_s})w^{(\infty)}(s;x,y;\lambda)=\lambda w^{(\infty)}(s;x,y;\lambda),$$

where the difference operator $L(s;e^{\partial_s})$ is introduced through the entries $b_j(s)$ of L (1.2.1) as follows;

$$L(s;e^{\partial_s})=\sum_{-\infty<j\leq 1} b_j(s)e^{j\partial_s}.$$

Set $B_n(s;e^{\partial_s})=(L(s;e^{\partial_s})^n)_+$. The first equation in (1.2.8) now reduces to

(1.2.22) $$\partial_{x_n}w^{\binom{0}{\infty}}(s;x,y;\lambda)=B_n(s;e^{\partial_s})w^{\binom{0}{\infty}}(s;x,y;\lambda).$$

There also exist difference operators $M(s;e^{\partial_s})$, $C_n(s;e^{\partial_s})$, which correspond to M and C_n, such that

(1.2.23) $$M(s;e^{\partial_s})w^{(0)}(s;x,y;\lambda)=\lambda^{-1}w^{(0)}(s;x,y;\lambda),$$

(1.2.24) $$\partial_{y_n}w^{\binom{0}{\infty}}(s;x,y;\lambda)=C_n(s;e^{\partial_s})w^{\binom{0}{\infty}}(s;x,y;\lambda).$$

Equations (1.2.21–24) constitute a difference operator version of the linear problem of the TL hierarchy.

By the way, (1.2.22) with $n=1$,

$$\partial_{x_1}w^{(\infty)}(s;x,y;\lambda)=(e^{\partial_s}+b_0(s))w^{(\infty)}(s;x,y;\lambda)$$

means that the action of the operator $e^{j\partial_s}$ on $w^{(\infty)}(s;x,y;\lambda)$ is identified with $(\partial_{x_1}-b_0(s+j-1))\cdots(\partial_{x_1}-b_0(s))$. Thus we find a differential operator $\check{B}_n(s;\partial_{x_1})$ of order n such that

$$\partial_{x_n}w^{(\infty)}(s;x,y;\lambda)=\check{B}_n(s;\partial_{x_1})w^{(\infty)}(s;x,y;\lambda),\quad n=2,3,\cdots.$$

This is just the linear problem for the KP hierarchy [20, 34], so the compatibility condition for this gives the KP hierarchy.

The relationship between the TL hierarchy and the KP hierarchy can be also described as follows: Let $y=y'$ and $s=s'$ in (1.2.21). Then we have

$$\oint w^{(\infty)}(s;x,y;\lambda)w^{(\infty)*}(s;x',y;\lambda)\frac{d\lambda}{2\pi i}=0,$$

which is nothing but the residue formula in the KP theory. Hence each $w^{(\infty)}(s;x,y;\lambda)$ (resp. $w^{(\infty)*}(s;x,y;\lambda)$) is, viewed as a function in x, a wave function (resp. a dual wave function) of the KP hierarchy [20] (see also the appendix 1 in this article).

1.3. τ functions and Hirota's bilinear equations

As was seen in the introduction, τ functions of the Toda lattice satisfy the Hirota's bilinear equations (0.3). In this section we will formulate τ functions for the hierarchy, and show the hierarchy to be bi-linearized by means of τ functions. The existence of τ functions for the *KP* hierarchy (or the multi-component *KP* hierarchy) was formulated in [20, 22], however any algebraic proof for this has not been presented.

Let $\hat{w}^{(\infty)}(s; x, y; \lambda)$, etc. be the formal power series defined by (1.2.20) for the wave matrices. The main theorem in this section is the following.

Theorem 1.7. *τ functions $\tau(s)=\tau(s; x, y)$ of the TL hierarchy are uniquely determined up to a constant multiple factor so that*

(1.3.1)
$$\hat{w}^{(\infty)}(s; x, y; \lambda)=\frac{\tau(s; x-\varepsilon(\lambda^{-1}), y)}{\tau(s; x, y)},$$
$$\hat{w}^{(\infty)*}(s; x, y; \lambda)=\frac{\tau(s; x+\varepsilon(\lambda^{-1}), y)}{\tau(s; x, y)},$$
$$\hat{w}^{(0)}(s; x, y; \lambda)=\frac{\tau(s+1; x, y-\varepsilon(\lambda))}{\tau(s; x, y)},$$
$$\hat{w}^{(0)*}(s; x, y; \lambda)=\frac{\tau(s-1; x, y+\varepsilon(\lambda))}{\tau(s; x, y)},$$

where $\varepsilon(\lambda)=(\lambda, \frac{1}{2}\lambda^2, \frac{1}{3}\lambda^3, \cdots)$.

The proof will proceed in steps. By virtue of the bilinear relation (1.2.18) and the identities

(1.3.2) $$\exp \xi(\varepsilon(\lambda^{-1}), \Lambda)=(1-\lambda^{-1}\Lambda)^{-1},$$

(1.3.3) $$(1-\lambda_1^{-1}\Lambda)^{-1}(1-\lambda_2^{-1}\Lambda)^{-1}=\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\{(1-\lambda_1^{-1}\Lambda)^{-1}-(1-\lambda_2^{-1}\Lambda)^{-1}\}\Lambda^{-1},$$

we deduce the following proposition.

Lemma 1.8. *For any $x, y, \lambda_1, \lambda_2$, we have*

(1.3.4)
$$\begin{aligned}&\hat{w}^{(\infty)}(s; x, y; \lambda_1)\hat{w}^{(\infty)*}(s+1; x-\varepsilon(\lambda_1^{-1}), y-\varepsilon(\lambda_2); \lambda_1)\\ &\quad=\hat{w}^{(0)}(s; x, y; \lambda_2)\hat{w}^{(0)*}(s+1; x-\varepsilon(\lambda_1^{-1}), y-\varepsilon(\lambda_2); \lambda_2),\end{aligned}$$

(1.3.5)
$$\begin{aligned}&\hat{w}^{(\infty)}(s; x, y; \lambda_1)\hat{w}^{(\infty)*}(s; x-\varepsilon(\lambda_1^{-1})-\varepsilon(\lambda_2^{-1}), y; \lambda_1)\\ &\quad=\hat{w}^{(\infty)}(s; x, y; \lambda_2)\hat{w}^{(\infty)*}(s; x-\varepsilon(\lambda_1^{-1})-\varepsilon(\lambda_2^{-1}), y; \lambda_2),\end{aligned}$$

$$(1.3.6)\qquad \begin{aligned}&\hat{w}^{(0)}(s; x, y; \lambda_1)\hat{w}^{(0)*}(s+2; x, y-\varepsilon(\lambda_1)-\varepsilon(\lambda_2); \lambda_1)\\ &\quad =\hat{w}^{(0)}(s; x, y; \lambda_2)\hat{w}^{(0)*}(s+2; x, y-\varepsilon(\lambda_1)-\varepsilon(\lambda_2); \lambda_2).\end{aligned}$$

Proof. Letting $x'=x-\varepsilon(\lambda_1^{-1})$, $y'=y-\varepsilon(\lambda_2)$ in (1.2.18), one has

$$(1.3.7)\qquad \begin{aligned}&\hat{W}^{(\infty)}(x, y) \exp \xi(\varepsilon(\lambda_1^{-1}), \Lambda)\hat{W}^{(\infty)}(x-\varepsilon(\lambda_1^{-1}), y-\varepsilon(\lambda_2))^{-1}\\ &\quad =\hat{W}^{(0)}(x, y) \exp \xi(\varepsilon(\lambda_2), \Lambda^{-1})\hat{W}^{(0)}(x-\varepsilon(\lambda_1^{-1}), y-\varepsilon(\lambda_2))^{-1}.\end{aligned}$$

Applying (1.3.2), one sees that

$$\begin{aligned}
&\text{the l.h.s. of (1.3.7)}\\
&=\hat{W}^{(\infty)}(x, y)(1-\lambda_1^{-1}\Lambda)^{-1}\hat{W}^{(\infty)}(x-\varepsilon(\lambda_1^{-1}), y-\varepsilon(\lambda_2))\\
&=\sum_{j=0}^{\infty} \mathrm{diag}[\hat{w}_j^{(\infty)}(s; x, y)]\Lambda^{-j} \sum_{k=0}^{\infty} (\lambda_1^{-1}\Lambda)^k\\
&\qquad \times\sum_{l=0}^{\infty} \Lambda^{-l}\, \mathrm{diag}[\hat{w}_l^{(\infty)*}(s+1; x-\varepsilon(\lambda_1^{-1}), y-\varepsilon(\lambda_2))]\\
&=\Big\{ \sum_{-\infty<n<0} \Big(\sum_{\substack{n=k-j\\ j,k\geqq 0}} \lambda_1^{-k}\, \mathrm{diag}[\hat{w}_j^{(\infty)}(s; x, y)]\Big)\Lambda^n\\
&\qquad + \sum_{0\leqq n<+\infty} \Big(\sum_{\substack{n=k-j\\ j,k\geqq 0}} \lambda_1^{-k}\, \mathrm{diag}[\hat{w}_j^{(\infty)}(s; x, y)]\Big)\Lambda^n\Big\}\\
&\qquad \times\sum_{m=0}^{\infty} \Lambda^{-m}\, \mathrm{diag}[\hat{w}_m^{(\infty)*}(s+1; x-\varepsilon(\lambda_1^{-1}), y-\varepsilon(\lambda_2))]\\
&=\Big\{ \sum_{-\infty<n<0} \Big(\sum_{\substack{n=k-j\\ j,k\geqq 0}} \lambda_1^{-k}\, \mathrm{diag}[\hat{w}_j^{(\infty)}(s; x, y)]\Big)\Lambda^n\\
&\qquad +\sum_{n=0}^{\infty} \lambda_1^{-n}\, \mathrm{diag}[\hat{w}^{(\infty)}(s; x, y; \lambda_1)]\Lambda^n\Big\}\\
&\qquad \times\sum_{m=0}^{\infty} \Lambda^{-m}\, \mathrm{diag}[\hat{w}_m^{(\infty)}(s+1; x-\varepsilon(\lambda_1^{-1}), y-\varepsilon(\lambda_2))].
\end{aligned}$$

Hence

$$\begin{aligned}&\text{the 0-th coefficient of the l.h.s. of (1.3.7)}\\ &\quad =\hat{w}^{(\infty)}(s; x, y; \lambda_1)\hat{w}^{(\infty)*}(s+1; x-\varepsilon(\lambda_1^{-1}), y-\varepsilon(\lambda_2); \lambda_1).\end{aligned}$$

Likewise one has

$$\begin{aligned}&\text{the 0-th coefficient of the r.h.s. of (1.3.7)}\\ &\quad =\hat{w}^{(0)}(s; x, y; \lambda_2)\hat{w}^{(0)*}(s+1; x-\varepsilon(\lambda_1^{-1}), y-\varepsilon(\lambda_2); \lambda_2).\end{aligned}$$

Thus we have proved (1.3.4).

Next we set $x'=x-\varepsilon(\lambda_1^{-1})-\varepsilon(\lambda_2^{-1})$, $y'=y$ in (1.2.18). Then

$$\hat{W}^{(\infty)}(x, y) \exp \xi(\varepsilon(\lambda_1^{-1})+\varepsilon(\lambda_2^{-1}), \Lambda)\hat{W}^{(\infty)}(x-\varepsilon(\lambda_1^{-1})-\varepsilon(\lambda_2^{-1}), y) \\ =\hat{W}^{(0)}(x, y)\hat{W}^{(0)}(x-\varepsilon(\lambda_1^{-1})-\varepsilon(\lambda_2^{-1}), y)^{-1}. \tag{1.3.8}$$

By means of (1.3.3), one has

the l.h.s. of (1.3.8)

$$=\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\sum_{j=0}^{\infty}\operatorname{diag}[\hat{w}_j^{(\infty)}(s; x, y)]\Lambda^{-j}\sum_{k=0}^{\infty}\{(\lambda_1^{-1}\Lambda)^k-(\lambda_2^{-1}\Lambda)^k\} \\ \times\sum_{l=0}^{\infty}\Lambda^{-l}\operatorname{diag}[\hat{w}_l^{(\infty)*}(s; x-\varepsilon(\lambda_1^{-1})-\varepsilon(\lambda_2^{-1}), y)]\Lambda^{-1}.$$

Consequently

the (-1)-th coefficient of the l.h.s. of (1.3.8)

$$=\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\{\hat{w}^{(\infty)}(s; x, y; \lambda_1)\hat{w}^{(\infty)*}(s; x-\varepsilon(\lambda_1^{-1})-\varepsilon(\lambda_2^{-1}), y; \lambda_1) \\ -\hat{w}^{(\infty)}(s; x, y; \lambda_2)\hat{w}^{(\infty)*}(s; x-\varepsilon(\lambda_1^{-1})-\varepsilon(\lambda_2^{-1}), y; \lambda_2)\}.$$

On the other hand, the (-1)-th coefficient of the r.h.s. of (1.3.8)$=0$. Thus we conclude (1.3.5). Equation (1.3.6) can be similarly verified.

Q.E.D.

Corollary 1.9. *For any x, y, λ, λ_1, λ_2, we have*

$$\hat{w}^{(\infty)}(s; x, y; \lambda)\hat{w}^{(\infty)*}(s+1; x-\varepsilon(\lambda^{-1}), y; \lambda) \\ =\hat{w}_0^{(0)}(s; x, y)\hat{w}_0^{(0)*}(s+1; x-\varepsilon(\lambda^{-1}), y), \tag{1.3.9}$$

$$\hat{w}^{(\infty)}(s; x, y; \lambda)\hat{w}^{(\infty)*}(s; x-\varepsilon(\lambda^{-1}), y; \lambda)=1, \tag{1.3.10}$$

$$\hat{w}^{(0)}(s; x, y; \lambda)\hat{w}^{(0)*}(s+1; x, y-\varepsilon(\lambda); \lambda)=1, \tag{1.3.11}$$

$$\hat{w}_0^{(0)*}(s+1; x+\varepsilon(\lambda_1^{-1}), y)\hat{w}^{(\infty)*}(s; x, y; \lambda_2) \\ \times\hat{w}^{(\infty)*}(s+1; x+\varepsilon(\lambda_2^{-1}), y; \lambda_1) \\ =\hat{w}^{(0)*}(s+1; x+\varepsilon(\lambda_2^{-1}), y)\hat{w}^{(\infty)*}(s; x, y; \lambda_1) \\ \times\hat{w}^{(\infty)*}(s+1; x+\varepsilon(\lambda_1^{-1}), y; \lambda_2), \tag{1.3.12}$$

$$\hat{w}^{(0)*}(s; x, y+\varepsilon(\lambda_1); \lambda_2)\hat{w}^{(0)*}(s+1; x, y; \lambda_1) \\ =\hat{w}^{(0)*}(s; x, y+\varepsilon(\lambda_2); \lambda_1)\hat{w}^{(0)*}(s+1; x, y; \lambda_2). \tag{1.3.13}$$

Proof. Equations (1.3.9) and (1.3.11) follow from (1.3.4) with $\lambda_1=\infty$ and $\lambda_2=0$, respectively. (1.3.10) is deduced from (1.3.5) with $\lambda_2=0$. By making use of (1.3.5) and (1.3.9), one sees that

$$\begin{aligned}\hat{w}_0^{(0)*}(s+1; x-\varepsilon(\lambda_1^{-1}), y)\hat{w}^{(\infty)*}(s; x-\varepsilon(\lambda_1^{-1})-\varepsilon(\lambda_2^{-1}), y; \lambda_1)&\\ \times w^{(\infty)*}(s+1; x-\varepsilon(\lambda_2^{-1}), y; \lambda_2)&\\ =\hat{w}_0^{(0)*}(s+1; x-\varepsilon(\lambda_2^{-1}), y)\hat{w}^{(\infty)*}(s; x-\varepsilon(\lambda_1^{-1})-\varepsilon(\lambda_2^{-1}), y; \lambda_2)&\\ \times \hat{w}^{(\infty)*}(s+1; x-\varepsilon(\lambda_1^{-1}), y; \lambda_1).&\end{aligned}$$

Replacing $x-\varepsilon(\lambda_1^{-1})-\varepsilon(\lambda_2^{-1})$ by x in the above, one obtains (1.3.11). One can show (1.3.12) in the same way. Q.E.D.

Set

$$\log \hat{w}^{(\infty)*}(s; x, y; \lambda)=\sum_{j=1}^{\infty} t_j^{(\infty)}(s)\lambda^{-j},$$

$$\log \hat{w}^{(0)*}(s; x, y; \lambda)=\sum_{j=0}^{\infty} t_j^{(0)}(s)\lambda^{j}.$$

We note that the action of the nonlocal operator $\exp(\xi(\tilde{\partial}_x, \lambda^{-1}))$ ($\tilde{\partial}_x=(\partial_{x_1}, \frac{1}{2}\partial_{x_2}, \frac{1}{3}\partial_{x_3}, \cdots)$) is given by

$$\exp(\xi(\tilde{\partial}_x, \lambda^{-1}))f(x)=f(x+\varepsilon(\lambda^{-1})).$$

Let $p_j(x)$ ($j=0, 1, \cdots$) be a polynomial introduced through

$$e^{\xi(x,\lambda)}=\sum_{j=0}^{\infty} p_j(x)\lambda^j. \tag{1.3.14}$$

More explicitly,

$$p_j(x)=\sum_{\nu_1+2\nu_2+\cdots+j\nu_j=j}\frac{x_1^{\nu_1}\cdots x_j^{\nu_j}}{\nu_1!\cdots\nu_j!}.$$

Now we are in position to prove Theorem 1.7.

Proof of Theorem 1.7. (1) First we show

$$p_j(\tilde{\partial}_x)t_0^{(0)}(s)=t_j^{(\infty)}(s-1)-t_j^{(\infty)}(s) \qquad \text{for } j\geqq 1, \tag{1.3.15}$$

$$p_j(\tilde{\partial}_x)t_k^{(\infty)}(s)=p_k(\tilde{\partial}_x)t_j^{(\infty)}(s) \qquad \text{for } j, k\geqq 1, \tag{1.3.16}$$

$$\partial_{x_1}\log \hat{w}^{(\infty)*}(s; x, y; \lambda)=(\exp(\xi(\tilde{\partial}_x, \lambda^{-1}))-1)t_1^{(\infty)}(s). \tag{1.3.17}$$

Taking the logarithm of the both sides of (1.3.12), one gets

$$\begin{aligned}\exp(\xi(\tilde{\partial}_x, \lambda_2^{-1}))\log \hat{w}_0^{(0)*}(s+1; x, y)+\log \hat{w}^{(\infty)*}(s; x, u; \lambda_1)&\\ +\exp(\xi(\tilde{\partial}_x, \lambda_1^{-1}))\log \hat{w}^{(\infty)*}(s+1; x, y; \lambda_2)&\\ =\exp(\xi(\tilde{\partial}_x, \lambda_1^{-1}))\log \hat{w}_0^{(0)*}(s+1; x, y)+\log \hat{w}^{(\infty)*}(s; x, y; \lambda_2)&\\ +\exp(\xi(\tilde{\partial}_x, \lambda_2^{-1}))\log \hat{w}^{(\infty)*}(s+1; x, y; \lambda_1).&\end{aligned}$$

Expanding the both sides into power series in λ_1 and λ_2, one sees that

$$\begin{aligned}(1.3.18)\qquad &\sum_{j=0}^{\infty} p_j(\tilde{\partial}_x)t_0^{(0)}(s+1)\lambda_2^{-j}+\sum_{j=1}^{\infty} t_j^{(\infty)}(s+1)\lambda_2^{-j}\\ &\qquad =t_0^{(0)}(s+1)+\sum_{j=1}^{\infty} t_j^{(\infty)}(s)\lambda_2^{-j},\end{aligned}$$

and

$$\begin{aligned}(1.3.19)\qquad &\sum_{j=0}^{\infty} t_j^{(\infty)}(s)\lambda_1^{-j}+\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} p_j(\tilde{\partial}_x)t_k^{(\infty)}(s+1)\lambda_1^{-j}\lambda_2^{-k}\\ &\qquad =\sum_{j=1}^{\infty} p_j(\tilde{\partial}_x)t_0^{(0)}(s+1)\lambda_1^{-j}+\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} p_k(\tilde{\partial}_x)t_j^{(\infty)}(s+1)\lambda_1^{-j}\lambda_2^{-k}.\end{aligned}$$

Equations (1.3.15) and (1.3.16) are derived from (1.3.18) and (1.3.19), respectively. Equation (1.3.17) is a generating functional expression for the special case of (1.3.16), $\partial_{x_1}t_j^{(\infty)}(s)=p_j(\tilde{\partial}_x)t_1^{(\infty)}(s)$ $(j\geqq 1)$.

(2) A similar consideration for (1.3.13) as (1) enables us to obtain

$$(1.3.20)\qquad p_j(\tilde{\partial}_y)t_0^{(0)}(s)=t_j^{(0)}(s)-t_j^{(0)}(s+1)\qquad \text{for } j\geqq 1,$$

$$(1.3.21)\qquad p_j(\tilde{\partial}_y)t_k^{(0)}(s)=p_k(\tilde{\partial}_y)t_j^{(0)}(s)\qquad \text{for } j,k\geqq 1,$$

$$(1.3.22)\qquad \partial_{y_1}\left(\log\frac{\hat{w}^{(0)*}(s;x,y;\lambda)}{\hat{w}_0^{(0)*}(s;x,y)}\right)=(\exp(\xi(\tilde{\partial}_y,\lambda))-1)t_1^{(0)}(s).$$

(3) We wish to prove

$$(1.3.23)\qquad p_k(\tilde{\partial}_y)t_j^{(\infty)}(s)=p_j(\tilde{\partial}_x)t_k^{(0)}(s+1)\qquad \text{for } j,k\geqq 1.$$

Notice that (1.3.4) leads to

$$\begin{aligned}(1.3.24)\qquad &\hat{w}^{(\infty)}(s;x+\varepsilon(\lambda_1^{-1}),y+\varepsilon(\lambda_2);\lambda_1)\hat{w}^{(\infty)*}(s+1;x,y;\lambda_1)\\ &\qquad =\hat{w}^{(0)}(s;x+\varepsilon(\lambda_1^{-1}),y+\varepsilon(\lambda_2),\lambda_2)\hat{w}^{(0)*}(s+1;x,y;\lambda_2).\end{aligned}$$

On the other hand, replacing x,y by $x-\varepsilon(\lambda_1^{-1})$, $y+\varepsilon(\lambda_2)$ in (1.3.10) (resp. by $x+\varepsilon(\lambda_1^{-1})$, $y-\varepsilon(\lambda_2)$ in (1.3.11)), one gets

$$\begin{aligned}&\hat{w}^{(\infty)}(s;x+\varepsilon(\lambda_1^{-1}),y+\varepsilon(\lambda_2);\lambda_1)=\hat{w}^{(\infty)*}(s;x,y+\varepsilon(\lambda_2);\lambda_1)^{-1},\\ &\hat{w}^{(0)}(s;x+\varepsilon(\lambda_1^{-1}),y+\varepsilon(\lambda_2);\lambda_2)=\hat{w}^{(0)*}(s+1;x+\varepsilon(\lambda_1^{-1})y;\lambda_2)^{-1}.\end{aligned}$$

Substituting these into (1.3.24) and taking the logarithm of the both sides, one sees that

$$\begin{aligned}&-\exp(\xi(\tilde{\partial}_y,\lambda_2))\log\hat{w}^{(\infty)*}(s;x,y;\lambda_1)+\log\hat{w}^{(\infty)*}(s+1;x,y;\lambda_1)\\ &\qquad =-\exp(\xi(\tilde{\partial}_x,\lambda_1^{-1}))\log\hat{w}^{(0)*}(s+1;x,y;\lambda_2)+\log\hat{w}^{(0)*}(s+1;x,y;\lambda_2).\end{aligned}$$

Compairing the coefficients of $\lambda_1^{-j}\lambda_2^k$ $(j, k\geqq 1)$ in the Laurent expansions of the both sides, we conclude (1.3.23).

(4) Consider the following equations;

$$\log \hat{w}^{(\infty)*}(s; x, y; \lambda)=(\exp(\xi(\tilde{\partial}_x, \lambda^{-1}))-1)\log\tau(s; x, y),$$
$$\log\frac{\hat{w}^{(0)*}(s; x, y; \lambda)}{\hat{w}_0^{(0)*}(s; x, y)}=(\exp(\xi(\tilde{\partial}_y, \lambda))-1)\log\tau(s-1; x, y),$$
$$\hat{w}_0^{(0)*}(s; x, y)=\frac{\tau(s-1; x, y)}{\tau(s; x, y)}, \quad s\in \boldsymbol{Z}.$$

Equations (1.3.15–17) and (1.3.20–23) constitute the compatibility condition for the above equations to be solved. (We should observe that $p_j(\tilde{\partial}_x)$, $p_j(\tilde{\partial}_y)$ $(j=1, 2, \cdots)$ form generators of the ring of differential operators, $\boldsymbol{C}[\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{y_1}, \partial_{y_2}, \cdots]$.) Consequently the solutions $\{\tau(s; x, y)\}_{s\in \boldsymbol{Z}}$ are uniquely determined up to a constant multiple factor. Then we have

$$\hat{w}^{(\infty)*}(s; x, y; \lambda)=\frac{\tau(s; x+\varepsilon(\lambda^{-1}), y)}{\tau(s; x, y)},$$
$$\hat{w}^{(0)*}(s; x, y; \lambda)=\frac{\tau(s; x, y+\varepsilon(\lambda))}{\tau(s; x, y)}.$$

Substituting these into (1.3.10) and (1.3.11), we obtain the rest of equations among (1.3.1). This completes the proof. Q.E.D.

Remark 1. Theorem 1.7 can be also proved by means of the residue formula (1.2.21).

Remark 2. The arbitrariness (1.2.11) of the wave matrices corresponds to modifying τ functions as

$$\tau(s; x, y)\longmapsto a^s \exp\left(b+\sum_{n=1}^{\infty}(c_n x_n+d_n y_n)\right)\tau(s; x, y),$$

where a, b, c_n and d_n are constants independent of s.

Now let us discuss the bilinear equations of the Hirota-type satisfied by τ functions of the TL hierarchy. We prepare a lemma.

Lemma 1.10. *Let $a=(a_1, a_2, \cdots)$ be indeterminates, and $p_j(x)$ be as in* (1.3.14). *Then*

$$\text{(1.3.25)}\qquad \sum_{j=0}^{k} p_j(\tilde{\partial}_a)u(x-a)\cdot p_{k-j}(\tilde{\partial}_a)v(x+a)=p_k(\tilde{\partial}_a)\{u(x-a)v(x+a)\}$$

holds for any integer $k \geqq 0$.

Proof. One sees that

$$\begin{aligned}\text{the l.h.s.} &= \oint \lambda^{k-1} u(x-a-\varepsilon(\lambda^{-1}))v(x+a+\varepsilon(\lambda^{-1}))\frac{d\lambda}{2\pi i}\\ &= \oint \lambda^{k-1} \exp(\xi(\tilde{\partial}_a, \lambda^{-1}))\{u(x-a)v(x+a)\}\frac{d\lambda}{2\pi i}\\ &= \oint \lambda^{k-1} \sum_{l=0}^{\infty} p_l(\tilde{\partial}_a)\lambda^{-l}\{u(x-a)v(x+a)\}\frac{d\lambda}{2\pi i}\\ &= \text{the r.h.s.}\end{aligned}$$

Here the integration contour is a small circle around $\lambda=\infty$. Q.E.D.

Theorem 1.11. *Let* $a=(a_1, a_2, \cdots)$, $b=(b_1, b_2, \cdots)$ *be indeterminates.* τ *functions of TL hierarchy solve the following Hirota's bilinear equations*

$$\begin{aligned}&\sum_{j=0}^{\infty} p_{m+j}(-2a)p_j(\tilde{D}_x)\exp(\langle a, D_x\rangle+\langle b, D_y\rangle)\tau(s+m+1)\cdot\tau(s)\\ &= \sum_{j=0}^{\infty} p_{-m+j}(-2b)p_j(\tilde{D}_y)\exp(\langle a, D_x\rangle+\langle b, D_y\rangle)\tau(s+m)\cdot\tau(s+1)\end{aligned} \tag{1.3.26}$$

for $s, m \in Z$,

where $\tilde{D}_x=(D_{x_1}, \frac{1}{2}D_{x_2}, \cdots)$ *are Hirota's operators, and*

$$\langle a, D_x\rangle=\sum_{n=1}^{\infty} a_n D_{x_n}.$$

Proof. Letting $x\mapsto x-a$, $x'\mapsto x+a$, $y\mapsto y-b$, $y'\mapsto y+b$ in the bilinear relation (1.2.18), it reduces to

$$\begin{aligned}&W^{(\infty)}(x-a, y-b)W^{(\infty)}(x+a, y+b)^{-1}\\ &\qquad= W^{(0)}(x-a, y-b)W^{(0)}(x+a, y+b)^{-1}.\end{aligned} \tag{1.3.27}$$

Substituting (1.3.1) into the above, one has

$$\begin{aligned}&\text{the l.h.s. of (1.3.27)}\\ &= \hat{W}^{(\infty)}(x-a, y-b)\exp(\xi(-2a, \Lambda))\hat{W}^{(\infty)}(x+a, y+b)^{-1}\\ &= \sum_{i=0}^{\infty} \operatorname{diag}\left[\frac{p_i(\tilde{\partial}_a)\tau(s; x-a, y-b)}{\tau(s; x-a, y-b)}\right]\Lambda^{-i}\times\sum_{j=0}^{\infty} p_j(-2a)\Lambda^j\\ &\quad\times\sum_{k=0}^{\infty}\Lambda^{-k}\operatorname{diag}\left[\frac{p_k(\tilde{\partial}_a)\tau(s+1; x+a, y+b)}{\tau(s+1; x+a, y+b)}\right]\end{aligned}$$

$$=\sum_{i,j,k\geq 0}\operatorname{diag}\left[\frac{p_j(-2a)\cdot p_i(\tilde{\partial}_a)\tau(s;x-a,y-b)\cdot p_k(\tilde{\partial}_a)\tau(s+j-i-k+1)}{\tau(s;x-a,y-b)\tau(s+j-i-k+1;x+a,y+b)}\right]$$
$$\times\Lambda^{j-i-k}.$$

Set

$$(*)=\{\tau(s;x-a,y-b)\tau(s+m+1,x+a,y+b)\}^{-1}.$$

Then, applying (1.3.25), one gets

the m-th coefficient of the l.h.s. of (1.3.27)

$$\begin{aligned}&=(*)\cdot\sum_{k=0}^{\infty}\sum_{i+j=k}p_{m+k}(-2a)p_i(\tilde{\partial}_a)\tau(s;x-a,y-b)\cdot p_j(\tilde{\partial}_a)\\&\qquad\qquad\times\tau(s+m+1;x+a,y+b)\\&=(*)\cdot\sum_{k=0}^{\infty}p_{m+k}(-2a)p_k(\tilde{\partial}_a)\{\tau(s;x-a,y-b)\tau(s+m+1;x+a,y+b)\}\\&=(*)\cdot\sum_{k=0}^{\infty}p_{m+k}(-2a)p_k(\tilde{\partial}_c)\exp(\langle a,\partial_c\rangle+\langle b,\partial_d\rangle)\\&\qquad\qquad\times\{\tau(s;x-c,y-d)\tau(s+m+1;x+c,y+d)\}|_{c=d=0}\\&=(*)\cdot\sum_{k=0}^{\infty}p_{m+k}(-2a)p_k(\tilde{D}_x)\exp(\langle a,D_x\rangle+\langle b,D_y\rangle)\\&\qquad\qquad\times\tau(s+m+1;x,y)\cdot\tau(s;x,y),\end{aligned}$$

Similarly one has

the m-th coefficient of the r.h.s. of (1.3.27)

$$\begin{aligned}&=(*)\cdot\sum_{k=0}^{\infty}\sum_{i+j=k}p_{-m+k}(-2b)p_i(\tilde{\partial}_b)\tau(s+1;x-a,y-b)p_j(\tilde{\partial}_b)\\&\qquad\qquad\times\tau(s+m;x+a,y+b)\\&=(*)\cdot\sum_{k=0}^{\infty}p_{-m+k}(-2b)p_k(\tilde{D}_y)\exp(\langle a,D_x\rangle+\langle b,D_y\rangle)\\&\qquad\qquad\times\tau(s+m;x,y)\tau(s+1;x,y).\end{aligned}$$

This concludes the desirous result. Q.E.D.

Equation (1.3.26) means a generating functional expression of the bilinear equations of the Hirota-type satisfied by τ functions. For instance, non-trivial equations among (1.3.26) are

$$D_{x_1}D_{y_1}\tau(s)\cdot\tau(s)+2\tau(s-1)\cdot\tau(s+1)=0,\tag{1.3.28}$$

$$p_k(\tilde{D}_x)\tau(s-k+1)\cdot\tau(s)=0,\qquad p_k(\tilde{D}_y)\tau(s+k-1)\cdot\tau(s)=0\quad\text{for } k=2,3,\cdots,\tag{1.3.29}$$

$$\begin{aligned}&\sum_{j=0}^{\infty} p_j(-2a)p_{j+1}(\tilde{D}_x)\exp(\langle a, D_x\rangle)\tau(s)\cdot\tau(s)=0,\\ &\sum_{j=0}^{\infty} p_j(-2b)p_{j+1}(\tilde{D}_y)\exp(\langle b, D_y\rangle)\tau(s)\cdot\tau(s)=0.\end{aligned} \tag{1.3.30}$$

Equation (1.3.28) is nothing but the Hirota equation for the Toda lattice, and (1.3.29) means the condition for $W^{\binom{0}{\infty}}(x, y)^{-1}$ to be the inverse matrices of $W^{\binom{0}{\infty}}(x, y)$ (see Remark after Theorem 1.5). Equation (1.3.30) shows that our τ functions become those of the KP hierarchy (see the discussion after Proposition 1.6) [22].

We give a decisive result concering a relationship between the TL hierarchy and the 2-components KP hierarchy. Let $\tau_{s,-s}(x^{(1)}, x^{(2)})$ be the τ functions of the 2-component KP hierarchy introduced in [22]. Then we deduce;

Theorem 1.12. *Our τ function $\tau(s; x, y)$ given in Theorem 1.7 coincides with $\tau_{s,-s}(x^{(1)}, x^{(2)})$ except for a simple factor;*

$$\tau(s; x, y)=(-)^{s(s-1)/2}\tau_{s,-s}(x^{(1)}, x^{(2)}) \qquad \text{with } x=x^{(1)},\ y=y^{(1)}. \tag{1.3.31}$$

Proof. Hirota's bilinear equations satisfied by $\tau_{s,-s}(x^{(1)}, x^{(2)})$ [22] coincide with (1.3.26) by the above correspondence. Q.E.D.

This theorem asserts that the TL hierarchy can be embedded into the 2-components KP hierarchy. However we should observe that the totality of τ functions of the TL hierarchy does not exhaust that of τ functions of the 2-components KP hierarchy because a null τ function $\tau(s; x, y)\equiv 0$ should be excluded in our theory.

Finally we give another remark on τ functions.

As was mentioned in the introduction, we can slightly modify τ functions as follows;

$$\tau'(s; x, y)=\tau(s; x, y)\exp\left(\sum_{n=1}^{\infty} nx_n y_n\right). \tag{1.3.32}$$

This modification changes the expression of the wave matrices to

$$W^{\binom{0}{\infty}}(x, y)=\hat{V}^{\binom{0}{\infty}}(x, y)\exp(\xi(x, \Lambda)+\xi(y, \Lambda^{-1})), \tag{1.3.33}$$

where

$$\hat{V}^{\binom{0}{\infty}}(x, y)=\sum_{j=0}^{\infty}\operatorname{diag}[\hat{v}_j^{\binom{0}{\infty}}(s; x, y)]\Lambda^{\pm j}. \tag{1.3.34}$$

Set

$$\hat{v}^{(0)}_{(\infty)}(s;x,y;\lambda)=\sum_{j=0}^{\infty}\hat{v}^{(0)}_{(\infty)}(s;x,y)\lambda^{\pm j}.$$

Then they are represented by the new τ functions as

$$(1.3.35)\qquad \begin{aligned}\hat{v}^{(\infty)}(s;x,y;\lambda)&=\frac{\tau'(s;x-\varepsilon(\lambda^{-1}),y)}{\tau'(s;x,y)},\\ \hat{v}^{(0)}(s;x,y;\lambda)&=\frac{\tau'(s+1;x,y-\varepsilon(\lambda))}{\tau'(s;x,y)}.\end{aligned}$$

1.4 Periodic reduction of the Toda lattice hierarchy

Let l be a positive integer. The l-periodic Toda lattice ($(TL)_l$) is a subfamily of the Toda lattice with the constraint $u(s)=u(s+l)$ for any s (or it is obtained from the Zakharov-Shabat equation (0.2) by imposing therein the constraint $b(s)=b(s+l)$, $c(s)=c(s+l)$ for any s). That is, $(TL)_l$ is a system of differential equations given by

$$(1.4.1)\qquad \partial_{x_1}\partial_{y_1}u(s)=e^{u(s)-u(s-1)}-e^{u(s+1)-u(s)},\qquad s=0,\cdots,l-1 \text{ with } u(-1)=u(l).$$

We can impose a further constraint $\sum_{s=0}^{l-1}u(s)=0$ without loss of generality.

The one-dimensional Toda lattice is defined as

$$(1.4.2)\qquad \frac{1}{4}\partial_{t_1}^2 u(s)=e^{u(s)-u(s-1)}-e^{u(s+1)-u(s)},\ s\in Z,$$

where $u(s)=u(s;t_1)$. These subfamilies are subholonomic systems in the sense that their general solutions have arbitrariness of one-variable functions.

In this section we will study the hierarchies attached to these subfamilies. Our main interest is how the hierarchies are reduced from the original one.

To describe the $(TL)_l$ hierarchy, we need some preliminaries about Lie algebras.

Let us denote by $\mathfrak{gl}(\infty)$ the Lie algebra defined by

$$\mathfrak{gl}(\infty)=\{\sum_{i,j\in Z}a_{ij}E_{ij}\,|\,a_{ij}=0\qquad \text{for } |i-j|\gg 0\}.$$

Let $\mathfrak{gl}(\infty)_l$ (resp. $\mathfrak{gl}((\infty))_l$) be the subalgebra (resp. formal subalgebra) of $\mathfrak{gl}(\infty)$ (resp. $\mathfrak{gl}((\infty))$) given by

$$\mathfrak{gl}((\infty))_l = \{ \sum_{i,j \in Z} a_{ij} E_{ij} \in \mathfrak{gl}((\infty)) \mid a_{ij} = a_{i+l,j+l} \text{ for any } i, j \},$$

$$\mathfrak{gl}(\infty)_l = \mathfrak{gl}((\infty))_l \cap \mathfrak{gl}(\infty).$$

It is well known that the map,

$$\begin{array}{ccc} \mathfrak{gl}(l, \boldsymbol{C}[[\zeta, \zeta^{-1}]]) & \longrightarrow & \mathfrak{gl}((\infty)) \\ & & \ni \\ A(\zeta) = \sum_{n \in Z} A_n \zeta^n & \longmapsto & \begin{bmatrix} \ddots & \ddots & & \\ \ddots & A_0 & A_1 & \\ & A_{-1} & A_0 & A_1 \\ & & A_{-1} & A_0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \end{array}$$

defines the Lie algebra isomorphisms $\mathfrak{gl}(l, \boldsymbol{C}[[\zeta, \zeta^{-1}]]) \rightleftarrows \mathfrak{gl}((\infty))_l$ and $\mathfrak{gl}(l, \boldsymbol{C}[\zeta, \zeta^{-1}]) \rightleftarrows \mathfrak{gl}(\infty)_l$. This isomorphisms can be interpreted also in the following manner: Set

$$\Lambda_l(\zeta) = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \zeta & & & 0 \end{bmatrix} \in \mathfrak{gl}(l, \boldsymbol{C}[\zeta, \zeta^{-1}]).$$

Then (1.4.3) reads as

(1.4.4)
$$\begin{aligned} A(\zeta) &= \sum_{j \in Z} \operatorname{diag}(a_j(0), \cdots, a_j(l-1)) \Lambda_l(\zeta)^j \\ &\longmapsto A = \sum_{j \in Z} \operatorname{diag}(a_j(0), \cdots, a_j(l-1))^{(l)} \Lambda^j \end{aligned}$$

where $\operatorname{diag}(a_j(0), \cdots, a_j(l-1))^{(l)}$ stands for an l-periodic diagonal matrix $\operatorname{diag}(\cdots a_j(0), \cdots, a_j(l-1), a_j(0), \cdots, a_j(l-1), \cdots) \in \mathfrak{gl}(\infty)$.

Now we define the $(TL)_l$ hierarchy. We impose on the TL hierarchy the additional constraint

(1.4.5)
$$L^l = \Lambda^l, \qquad M^l = \Lambda^{-l}.$$

The system of nonlinear differential equations (1.2.2) with this constraint constitutes a subfamily of the TL hierarchy, and is said to be the l-periodic TL ($(TL)_l$) hierarchy. This is a subholonomic hierarchy.

The l-periodic condition (1.4.5) may be regarded as an analogue of the l-reduced condition of the KP hierarchy [25]. In fact we have the following proposition.

Proposition 1.13. *Let L, M be solutions to the $(TL)_l$ hierarchy, and $W^{(\infty)}(x, y)$, $W^{(0)}(x, y)$ be the corresponding wave matrices given in Theorem 1.2. Then L, M, $\hat{W}^{\binom{0}{\infty}}(x, y) \in \mathfrak{gl}((\infty))_l$, and*

$$(1.4.6)\qquad \partial_{x_n} W^{\binom{0}{\infty}}(x, y) = \Lambda^n W^{\binom{0}{\infty}}(x, y), \qquad \partial_{y_n} W^{\binom{0}{\infty}}(x, y) = \Lambda^{-n} W^{\binom{0}{\infty}}(x, y)$$
$$\text{for } n \equiv 0 \bmod l,$$

and

$$(1.4.7)\qquad \partial_{x_n} L = \partial_{x_n} M = 0, \quad \partial_{y_n} L = \partial_{y_n} M = 0 \qquad \text{for } n \equiv 0 \bmod l.$$

Hence the $(TL)_l$ hierarchy involves the l-periodic Toda lattice, and its solutions are independent of the variables x_n, $y_n (n \equiv 0 \bmod l)$.

Proof. Since $L = \hat{W}^{(\infty)} \Lambda \hat{W}^{(\infty)-1}$, $M = \hat{W}^{(0)} \Lambda^{-1} \hat{W}^{(0)-1}$, the l-periodic condition (1.4.5) implies

$$[\Lambda^n, \hat{W}^{(\infty)}] = [\Lambda^n, \hat{W}^{(0)}] = 0 \qquad \text{for } n \equiv 0 \bmod l,$$

so that $W^{\binom{0}{\infty}}$, L, $M \in \mathfrak{gl}((\infty))_l$. Thus the first assertion is proved. By the definition of B_n, C_n and (1.4.5), one sees that $B_n = \Lambda^n$, $C_n = \Lambda^{-n}$ for $n \equiv 0 \bmod l$, from which (1.4.6), (1.4.7) follows at once. Q.E.D.

Let us interpret the periodic condition (1.4.5) in terms of τ functions. Let $\tau'(s; x, y)$ be τ functions as in (1.3.32). Taking into account the arbitrariness of the wave matrices (1.2.11), we deduce the following corollary to Proposition 1.13.

Corollary 1.14 (cf. [25]). *Suppose L, M to be solutions to the $(TL)_l$ hierarchy. Then there exist a suitable wave matrices such that the corresponding τ functions are subject to the following conditions;*

$$(1.4.8)\qquad \tau'(s; x, y) = \tau'(s + l; x, y),$$

$$(1.4.9)\qquad \partial_{x_n} \tau'(s; x, y) = \partial_{y_n} \tau'(s; x, y) \qquad \text{for } n \equiv 0 \bmod l.$$

Conversely, if τ functions satisfy the above conditions, the corresponding L, M solve the $(TL)_l$ hierarchy.

Proof. First of all recall Remark 2 before Lemma 1.10. From the periodic condition, it follows that

$$p_j(\tilde{\partial}_x)\log(\tau(s)/\tau(s+l))=p_j(\tilde{\partial}_y)\log(\tau(s)/\tau(s+l))=0$$

for any s. Hence an appropriate modification such as $\tau'(s)\mapsto a^s\tau'(s)$ makes τ functions satisfy (1.4.8). Thus we may assume (1.4.8) without loss of generality. Set

$$W^{\binom{0}{\infty}}=\hat{V}^{\binom{0}{\infty}}\exp(\xi(x,\Lambda)+\xi(y,\Lambda^{-1}))\quad\text{(see (1.3.33)).}$$

From (1.4.6) one obtains $\partial_{x_n}\hat{V}^{\binom{0}{\infty}}=\partial_{y_n}\hat{V}^{\binom{0}{\infty}}=0$ for $n\equiv 0 \bmod l$. Therefore

$$\begin{aligned}&\partial_{x_n}\log\tau'(s)=\text{Cte.}\quad(=c_n),\\&\partial_{y_n}\log\tau'(s)=\text{Cte.}\quad(=d_n)\quad\text{for } n\equiv 0 \bmod l.\end{aligned}$$

Since the constants c_n, d_n are independent of s, the modified τ functions

$$\exp\Big(-\sum_{\substack{n\equiv 0\\ \bmod l}} c_nx_n+d_ny_n\Big)\tau'(s)$$

satisfy the both conditions. Q.E.D.

We investigate more explicitly the linear problem for the $(TL)_l$ hierarchy. Proposition 1.13 allows us to identify L, M, $W^{\binom{0}{\infty}}(x,y)$ with $L(\zeta)$, $M(\zeta)$, $W^{\binom{0}{\infty}}(x,y;\zeta)$ under the isomorphism (1.4.4). They take the following form;

$$\begin{aligned}L(\zeta)&=W^{(\infty)}(x,y;\zeta)\Lambda_l(\zeta)W^{(\infty)}(x,y;\zeta)^{-1},\\M(\zeta)&=W^{(0)}(x,y;\zeta)\Lambda_l(\zeta)^{-1}W^{(0)}(x,y;\zeta)^{-1},\end{aligned}\tag{1.4.10}$$

and

$$\begin{aligned}W^{(\infty)}(x,y;\zeta)&=\hat{W}^{(\infty)}(x,y;\zeta)\exp\xi(x,\Lambda_l(\zeta)),\\W^{(0)}(x,y;\zeta)&=\hat{W}^{(0)}(x,y;\zeta)\exp\xi(y,\Lambda_l(\zeta)^{-1}),\\\hat{W}^{\binom{0}{\infty}}(x,y;\zeta)&=\sum_{j=0}^{\infty}\mathrm{diag}[\hat{w}_j^{\binom{0}{\infty}}(0),\cdots,\hat{w}_j^{\binom{0}{\infty}}(l-1)]^{(l)}\Lambda_l(\zeta)^{\pm j},\end{aligned}\tag{1.4.11}$$

where $\exp\xi(x,\Lambda_l(\zeta))=\sum_{n=1}^{\infty}x_n\Lambda_l(\zeta)^n$, and $\hat{w}_j^{(\infty)}(s)=\hat{w}_j^{(\infty)}(s;x,y)$ are the entries of wave matrices.

We have the following proposition.

Proposition 1.15. (1) *$W^{(\infty)}(x,y;\zeta)$ and $W^{(0)}(x,y;\zeta)$ solve the linear problem*

(1.4.12) $$\partial_{x_n} W(\zeta) = B_n(\zeta) W(\zeta), \qquad \partial_{y_n} W(\zeta) = C_n(\zeta) W(\zeta),$$

where $B_n(\zeta) = (L(\zeta)^n)_+$, $C_n(\zeta) = (M(\zeta)^n)_-$. *The symbols* $(\cdot)_\pm$ *stand for the nonnegative power part and the strictly negative power part with respect to* $\Lambda_l(\zeta)$. *The compatibility condition for* (1.4.12),

$$\begin{aligned}
&\partial_{x_m} B_n(\zeta) - \partial_{x_n} B_m(\zeta) + [B_n(\zeta), B_m(\zeta)] = 0, \\
&\partial_{y_m} C_n(\zeta) - \partial_{y_n} C_m(\zeta) + [C_n(\zeta), C_m(\zeta)] = 0, \\
&\partial_{y_m} B_n(\zeta) - \partial_{x_n} C_m(\zeta) + [B_n(\zeta), C_m(\zeta)] = 0,
\end{aligned}$$

gives the $(TL)_l$ *hierarchy.*

(2) $B_n(\zeta)$, $C_n(\zeta) \in \mathfrak{sl}(l, C[\zeta, \zeta^{-1}])$.

Proof. (1) The assertion is clear because $B_n(\zeta)$, $C_n(\zeta)$ are identified with $B_n = (L^n)_+$, $C_n = (M^n)_-$ under the isomorphism (1.4.4).

(2) From (1.4.10) and trace $\Lambda_l(\zeta)^{\pm 1} = 0$, we obtain (2). Q.E.D.

In particular, the proposition gives us the Zakharov-Shabat representation for the l-periodic Toda lattice (1.4.1) [17];

$$\partial_{y_1} B_1(\zeta) - \partial_{x_1} C_1(\zeta) + [B_1(\zeta), C_1(\zeta)] = 0,$$

where

$$B_1(\zeta) = \begin{bmatrix} b(0) & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \zeta & & & b(l-1) \end{bmatrix}, \qquad C_1(\zeta) = \begin{bmatrix} 0 & & & \zeta^{-1} c(0) \\ c(1) & \ddots & & \\ & \ddots & \ddots & \\ & & c(l-1) & 0 \end{bmatrix}.$$

The one-dimensional Toda lattice (TL) hierarchy is defined as the TL hierarchy with the additional constraint

(1.4.13) $$L + L^{-1} = M + M^{-1}.$$

First we show the following lemma.

Lemma 1.16. *The condition* (1.4.13) *is equivalent to*

$$(\partial_{x_n} + \partial_{y_n}) W^{\binom{0}{\infty}}(x, y) = W^{\binom{0}{\infty}}(\Lambda^n + \Lambda^{-n}).$$

Proof. Suppose (1.4.13). Then $L^n + L^{-n} = M^n + M^{-n}$ holds for $n \geqq 1$. Considering the $(\pm)$ part of the both sides, one easily gets $B_n + C_n = L^n + L^{-n} = M^n + M^{-n}$. Hence one sees that

$$
\begin{aligned}
(\partial_{x_n}+\partial_{y_n})W^{(\infty)} &= (B_n+C_n)W^{(\infty)} \\
&= (L^n+L^{-n})W^{(\infty)} \\
&= W^{(\infty)}(\Lambda^n+\Lambda^{-n}),
\end{aligned}
$$

and also that $(\partial_{x_n}+\partial_{y_n})W^{(0)}=W^{(0)}(\Lambda^n+\Lambda^{-n})$. Thus (1.4.14) is proved. Next we verify the converse. By making use of (1.4.14) with $n=1$, one derives

$$
\begin{aligned}
(\partial_{x_1}+\partial_{y_1})W^{(\infty)} &= W^{(\infty)}(\Lambda+\Lambda^{-1}) \\
&= (L+L^{-1})W^{(\infty)},
\end{aligned}
$$

which yields $B_1+C_1=L+L^{-1}$. Likewise one has $B_1+C_1=M+M^{-1}$. This completes the proof. Q.E.D.

To show that the one-dimensional *TL* hierarchy actually contains the one-dimensional *TL*, we investigate the linear problem for the hierarchy.

Let us express $W^{\binom{0}{\infty}}(x, y)$ as (1.3.33). Then from (1.4.14) it follows that

(1.4.15) $$(\partial_{x_n}+\partial_{y_n})\hat{V}^{\binom{0}{\infty}}(x, y)=0 \qquad \text{for } n\geqq 1,$$

so that $\hat{V}^{\binom{0}{\infty}}(x, y)$ depend on only $t=(t_1, t_2, \cdots)=(\frac{1}{2}(x_1-y_1), \frac{1}{2}(x_2-y_2), \cdots)$. Namely, $\hat{V}^{(\infty)}(x, y)=\hat{V}(t)$. We set

$$V(t)=\hat{V}(t)\exp(\xi(t, \Lambda)+\xi(-t, \Lambda^{-1})).$$

Proposition 1.17. (1) *If L and M solve the one-dimensional TL hierarchy, then they depend on only t.*

(2) *Under the same assumption as above, $V(t)$ solves the linear problem*

(1.4.16) $$\begin{aligned} &(B_1+C_1)V(t)=V(t)(\Lambda+\Lambda^{-1}), \\ &\partial_{t_n}V(t)=(B_n-C_n)V(t), \quad n=1, 2, \cdots. \end{aligned}$$

The compatibility condition of this system amounts to the one-dimensional TL hierarchy

(1.4.17) $$\partial_{t_n}(B_1+C_1)=[B_n-C_n, B_1+C_1], \quad n=1, 2, \cdots.$$

Proof. (1) From (1.4.15) and $L=\hat{V}^{(\infty)}(x, y)\Lambda\hat{V}^{(\infty)}(x, y)^{-1}$, $M=\hat{V}^{(0)}(x, y)\Lambda^{-1}\hat{V}^{(0)}(x, y)^{-1}$, the statement is evident.

(2) It is sufficient to prove the second equation in (1.4.16). Since

$$V(t)=W^{(\infty)}(x, y)\exp\left(\xi\left(\frac{-1}{2}(x+y), \Lambda\right)+\xi\left(\frac{-1}{2}(x+y), \Lambda^{-1}\right)\right),$$

and $\partial_{t_n}=\partial_{x_n}-\partial_{y_n}$, we have

$$\begin{aligned}\partial_{t_n}V(t)&=\{(\partial_{x_n}-\partial_{y_n})W^{(\infty)}(x, y)\}\\ &\quad\times\exp\left(\xi\left(\frac{-1}{2}(x+y), \Lambda\right)+\xi\left(\frac{-1}{2}(x+y), \Lambda^{-1}\right)\right)\\ &=(B_n-C_n)V(t).\end{aligned}$$

Q.E.D.

The Lax representation [9, 27] of the one-dimensional TL is derived from (1.4.17) with $n=1$;

$$\partial_{t_1}(B_1+C_1)=[B_1-C_1, B_1+C_1],$$

where

$$B_1\pm C_1=\begin{pmatrix}\ddots & 1 & & & \\ \ddots & b(-1) & 1 & & \\ & \pm c(0) & b(0) & 1 & \\ & & \pm c(1) & b(1) & \ddots \\ & & & \ddots & \ddots\end{pmatrix}.$$

The entries of $B_1\pm C_1$ are related to the unknown functions in (1.4.2) through $b(s)=\frac{1}{2}\partial_{t_1}u(s)$, $c(s)=e^{u(s)-u(s-1)}$.

Finally we remark that the condition (1.4.13) are interpreted in terms of τ functions as follows: Let $\tau'(s; x, y)$ be as in (1.3.32). The condition (1.4.13) is true if and only if $\tau'(s; x, y)$ can be chosen so that they satisfy

(1.4.18) $$(\partial_{x_n}+\partial_{y_n})\tau'(s; x, y)=0 \qquad \text{for } n\geqq 1.$$

2. The Toda Lattice Hierarchies of B-type and C-type

2.1. Generalized Toda lattices and orthogonal Lie algebra $\mathfrak{o}(\infty)$ and symplectic Lie algebra $\mathfrak{sp}(\infty)$

First of all we will give a brief account of the generalized periodic Toda lattices studied by Bogoyavlensky [4] and Mikhailov, Olshanetsky and Perelomov [17].

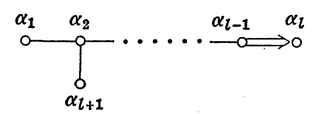
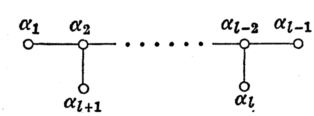
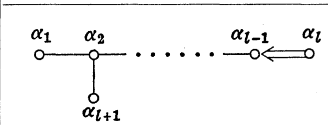
Let $\{\alpha_1, \cdots, \alpha_l\}$ $(\alpha_k=(\alpha_k^{(1)}, \cdots, \alpha_k^{(l)})\in \boldsymbol{R}^l)$ be a simple root system attached to the extended Dynkin diagram of Euclidean Lie algebras of the types $B_l^{(1)}$, $C_l^{(1)}$, $A_{2l}^{(2)}$, $D_{l+1}^{(2)}$, $A_{l+1}^{(2)}$, etc. For the simple root system of $A_l^{(1)}$, $\{e_i - e_{i+1}\ (i=1, \cdots, l),\ e_1 - e_{l+1}\}$ should be taken, where $\{e_i\}_{1 \leqq i \leqq l}$ is the standard basis of $\boldsymbol{R}^{l+1}$.

Let $u(s) = u(s; x_1, y_1)$ $(s=1, \cdots, l)$, and set $\langle \alpha_k, u \rangle = \sum_{s=1}^{l} \alpha_k^{(s)} u(s)$. The generalized periodic Toda lattice associated with the Euclidean Lie algebras are defined by

$$\partial_{x_1} \partial_{y_1} u(s) = \sum_{k=1}^{l+1} \alpha_k^{(s)} \exp \langle \alpha_k, u \rangle. \tag{2.1.1}$$

and will be denoted by $(TL)_{B_l^{(1)}}$, and so on. The l-periodic Toda lattice $(TL)_l$ is just $(TL)_{A_{l-1}^{(1)}}$ in this notation.

Table

Lie algebra	Dynkin diagram	simple root vectors
$B_l^{(1)}$		$\alpha_i = e_i - e_{i+1}$ $(i=1, \cdots, l-1)$ $\alpha_l = e_l$, $\alpha_{l+1} = -e_1 - e_2$
$C_l^{(1)}$		$\alpha_i = e_i - e_{i+1}$ $(i=1, \cdots, l-1)$ $\alpha_l = 2e_l$, $\alpha_{l+1} = -2e_1$
$D_l^{(1)}$		$\alpha_i = e_i - e_{i+1}$ $(i=1, \cdots, l-1)$ $\alpha_l = e_{l-1} + e_l$, $\alpha_{l+1} = -e_1 - e_2$
$A_{2l-1}^{(2)}$		$\alpha_i = e_i - e_{i+1}$ $(i=1, \cdots, l-1)$ $\alpha_l = 2e_l$, $\alpha_{l+1} = -e_1 - e_2$
$A_{2l}^{(2)}$		$\alpha_i = e_i - e_{i+1}$ $(i=1, \cdots, l-1)$ $\alpha_l = e_l$, $\alpha_{l+1} = -2e_1$
$D_{l+1}^{(2)}$		$\alpha_i = e_i - e_{i+1}$ $(i=1, \cdots, l-1)$ $\alpha_l = e_l$, $\alpha_{l+1} = -e_1$

For instance, $(TL)_{D^{(2)}_{l+1}}$, $(TL)_{A^{(2)}_{2l}}$, $(TL)_{C^{(1)}_{l}}$ are as follows:

$$(2.1.2)\quad (TL)_{D^{(2)}_{l+1}}\begin{cases}\partial_{x_1}\partial_{y_1}u(1)=e^{u(1)-u(2)}-e^{-u(1)},\\ \partial_{x_1}\partial_{y_1}u(s)=-e^{u(s-1)-u(s)}+e^{u(s)-u(s+1)} & (2\leqq s\leqq l-1),\\ \partial_{x_1}\partial_{y_1}u(l)=-e^{u(l-1)-u(l)}+e^{u(l)},\end{cases}$$

$$(2.1.3)\quad (TL)_{A^{(2)}_{2l}}\begin{cases}\partial_{x_1}\partial_{y_1}u(1)=e^{u(1)-u(2)}-2e^{-2u(1)},\\ \partial_{x_1}\partial_{y_1}u(s)=-e^{u(s-1)-u(s)}+e^{u(s)-u(s+1)} & (2\leqq s\leqq l-1),\\ \partial_{x_1}\partial_{y_1}u(l)=-e^{u(l-1)-u(l)}+e^{u(l)},\end{cases}$$

$$(2.1.4)\quad (TL)_{C^{(1)}_{l}}\begin{cases}\partial_{x_1}\partial_{y_1}u(1)=e^{u(1)-u(2)}-2e^{-2u(1)},\\ \partial_{x_1}\partial_{y_1}u(s)=-e^{u(s-1)-u(s)}+e^{u(s)-u(s+1)} & (2\leqq s\leqq l-1),\\ \partial_{x_1}\partial_{y_1}u(l)=-e^{u(l-1)-u(l)}+2e^{2u(l)}.\end{cases}$$

In particular, $(TL)_{A^{(2)}_{2}}$

$$\partial_{x_1}\partial_{y_1}u=e^{u}-2e^{-2u}$$

is referred to as the Bullough-Dodd equation [17].

Now we will explain the Lie algebras $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$, and their l-reduced subalgebras $\mathfrak{o}(\infty)_l$, $\mathfrak{sp}(\infty)_l$, which were discussed in [23, 25].

$\mathfrak{o}(\infty)$ is the orthogonal Lie algebra on $C^Z \simeq \{f(\lambda)=\sum f_i\lambda^i \in C[\lambda, \lambda^{-1}]\}$ equipped with the symmetric inner product

$$(f, g)_B=\sum_{i+j=0}(-)^j f_i g_j=\int f(\lambda)g(-\lambda)\underline{d\lambda}$$
$$\left(f(\lambda), g(\lambda)\in C^Z,\ \underline{d\lambda}=\frac{d\lambda}{2\pi i}\right).$$

Namely it is defined by

$$(2.1.5)\quad \begin{aligned}\mathfrak{o}(\infty)&=\{A\in\mathfrak{gl}(\infty)\,|\,JA+{}^tAJ=0\}\\ &=\{\sum_{i,j\in Z}a_{ij}E_{ij}\in\mathfrak{gl}(\infty)\,|\,a_{ij}=(-)^{i+j+1}a_{-j-i}\text{ for any } i,j\},\end{aligned}$$

where $J=((-)^i\delta_{i,-j})_{i,j\in Z}$ is a symmetric matrix. The generators for $\mathfrak{o}(\infty)$ are given by

$$Z_{ij}=(-)^jE_{i,-j}-(-)^iE_{j,-i}.$$

Clearly $\Lambda^n\in\mathfrak{o}(\infty)$ for odd n. We observe that if $A\in\mathfrak{o}(\infty)$, then $JA^n+(-)^{n+1}\,{}^tA^nJ=0$, and $(A)_\pm\in\mathfrak{o}(\infty)$.

The l-reduced subalgebra $\mathfrak{o}(\infty)_l$ is defined as

$$\begin{aligned}
\mathfrak{o}(\infty)_l &= \mathfrak{o}(\infty) \cap \mathfrak{gl}(\infty)_l \\
&= \{ \sum_{i,j \in Z} a_{i,j} E_{ij} \in \mathfrak{gl}(\infty) \mid a_{ij} = (-)^{i+j+1} a_{-j,-i} = a_{i+l,j+l} \\
&\qquad\qquad\qquad \text{for any } i, j \}.
\end{aligned} \tag{2.1.6}$$

The formal Lie algebra $\mathfrak{o}((\infty))$ is defined by replacing $\mathfrak{gl}(\infty)$ by $\mathfrak{gl}((\infty))$ in (2.1.5), and $\mathfrak{o}((\infty))_l = \mathfrak{o}((\infty)) \cap \mathfrak{gl}((\infty))_l$.

For the l-reduced subalgebra, we have the following Lie algebra isomorphism.

Lemma 3.1 [25].

$$\begin{aligned}
\mathfrak{o}(\infty)_l &\simeq \{A(\zeta) \in \mathfrak{gl}(l, C[\zeta, \zeta^{-1}]) \mid J_l(\zeta) A(\zeta) + {}^t A(-\zeta) J_l(\zeta) = 0\} \\
&\qquad\qquad \textit{for odd } l, \\
&\simeq \{A(\zeta) \in \mathfrak{sl}(l, C[\zeta, \zeta^{-1}]) \mid J_l(\zeta) A(\zeta) + {}^t A(\zeta) J_l(\zeta) = 0\} \\
&\qquad\qquad \textit{for even } l,
\end{aligned} \tag{2.1.7}$$

where

$$J_l(\zeta) = \begin{bmatrix} 1 & & & \\ & & & (-)^{l-1}\zeta^{-1} \\ & & \cdot^{\cdot^{\cdot}} & \\ & \zeta^{-1} & & \\ -\zeta^{-1} & & & \end{bmatrix}.$$

Proof. Define a bilinear form $\langle\ ,\ \rangle_l$; $C^Z \times C^Z \to C[\zeta, \zeta^{-1}]$ by

$$\begin{aligned}
\langle f, g \rangle_l &= \sum_{\nu \in Z} (\Lambda^{-\nu l} f, g) \zeta^\nu \\
&= \sum_{\nu \in Z} \int \lambda^{-\nu l} f(\lambda) g(-\lambda) \underline{d\lambda} \zeta^\nu.
\end{aligned}$$

$\mathfrak{o}(\infty)_l$ is the invariant Lie algebra for the bilinear form. In fact, $A \in \mathfrak{gl}(\infty)$ leaves it invariant if and only if

$${}^t A \Lambda^{\nu l} J + \Lambda^{\nu l} J A = 0$$

holds for any $\nu \in Z$. Letting $\nu = 0, 1$, we see ${}^t A J + J A = 0$, and $[A, \Lambda^l] = 0$. Hence $A \in \mathfrak{o}(\infty)_l$. The converse assertion is evident.

It is easy to see that $\langle f, g \rangle_l(\zeta) = \langle g, f \rangle_l(\zeta)$ for even l, and that $\langle f, g \rangle_l(\zeta) = \langle g, f \rangle_l(-\zeta)$ for odd l.

Note that C^Z are identifiable with $C^l \otimes C[\zeta, \zeta^{-1}] = \{f(\zeta) = \sum_j \vec{f}_j \zeta^j;\ {}^t\vec{f}_j = (f_{j,0}, \cdots, f_{j,l-1}) \in C^l\}$ by the correspondence

(2.1.8) $$f(\zeta)=\sum_j \vec{f}_j\zeta^j \longmapsto \sum_j \sum_{n=0}^{l-1} f_{j,n}\lambda^{lj+n}=\hat{f}(\lambda).$$

For $f(\zeta)$, $g(\zeta) \in C^l \otimes C[\zeta, \zeta^{-1}]$, we introduce a bilinear form by

$$(f, g)_l(\zeta)=\begin{cases} {}^t f(\zeta)J_l(\zeta)g(\zeta) & \text{for even } l, \\ {}^t f(\zeta)J_l(\zeta)g(-\zeta) & \text{for odd } l. \end{cases}$$

The left-hand side of (2.1.7) defines an invariant Lie algebra for this bilinear form. Therefore, in order to prove the lemma, it is sufficient to show that

(2.1.9) $$(f, f)_l(\zeta)=\langle \hat{f}, \hat{f}\rangle_l(\zeta),$$

holds under the isomorphism (2.1.8). Let l be even, and set $f(\zeta)=\vec{f}\zeta^j$ (${}^t\vec{f}=(f_0, \cdots, f_{l-1})$). Then one sees that

$$(f, f)_l(\zeta)=\left(f_0^2+\sum_{n=0}^{l-1}(-)^{l-n}f_n f_{l-n}\zeta^{-1}\right)\zeta^{2j},$$

and that

$$\begin{aligned}\langle f, f\rangle_l(\zeta)&=\sum_\nu f_0^2\int \lambda^{2jl-\nu l}\underline{d\lambda}\,\zeta^\nu+\sum_{n=0}^{l-1}(-)^n f_n f_{l-n}\sum_\nu \int \lambda^{(2j-1)-\nu l}\underline{d\lambda}\,\zeta^\nu \\ &=(f, f)_l(\zeta).\end{aligned}$$

Thus (2.1.9) is proved for even l. For odd l, the proof can be done in a similar way as above. Q.E.D.

$\mathfrak{sp}(\infty)$ is the symplectic Lie algebra on C^Z equipped with the skew-symmetric inner product

$$(f, g)_C=\sum_{i+j=-1}(-)^j f_i g_j=\int \lambda f(\lambda)g(-\lambda)\underline{d\lambda}.$$

That is to say,

(2.1.10) $$\begin{aligned}\mathfrak{sp}(\infty)&=\{A \in \mathfrak{gl}(\infty)\,|\,KA+{}^tAK=0\} \\ &=\{\sum_{i,j\in Z} a_{ij}E_{ij} \in \mathfrak{gl}(\infty)\,|\,a_{ij}=(-)^{i+j+1}a_{-j-1,-i-1}\},\end{aligned}$$

where $K=\Lambda J$ is skew symmetric. The generators for $\mathfrak{sp}(\infty)$ are given by

$$Z_{ij}=(-)^j E_{i,-j-1}-(-)^{i+1}E_{j,-i-1}.$$

$\Lambda^n \in \mathfrak{sp}(\infty)$ for odd n as in the case of the orthogonal algebra. We note also that if $A \in \mathfrak{sp}(\infty)$, $KA^n+(-)^{n+1}A^nK=0$, and $(A)_\pm \in \mathfrak{sp}(\infty)$.

The l-reduced subalgebra $\mathfrak{sp}(\infty)_l$ is defined as

$$\begin{aligned}(2.1.11)\qquad \mathfrak{sp}(\infty)_l &= \mathfrak{sp}(\infty)\cap\mathfrak{gl}(\infty)_l \\ &= \{\sum_{i,j\in Z} a_{ij}E_{ij}\in\mathfrak{gl}(\infty)\,|\,a_{ij}=(-)^{i+j+1}a_{-j-1,-i-1}=a_{i+l,j+l}\}.\end{aligned}$$

The formal Lie algebra $\mathfrak{sp}((\infty))$ and $\mathfrak{sp}((\infty))_l$ are defined as in the orthogonal case. The following is an analogue of Lemma 2.1.

Lemma 2.2. *We have the following isomorphism;*

$$\begin{aligned}(2.1.12)\qquad \mathfrak{sp}(\infty)_l &\simeq \{A(\zeta)\in\mathfrak{gl}(l, C[\zeta,\zeta^{-1}])\,|\,K_l(\zeta)A(\zeta)+{}^tA(-\zeta)K_l(\zeta)=0\} \\ &\qquad\qquad \text{for odd } l, \\ &\simeq \{A(\zeta)\in\mathfrak{sl}(l, C[\zeta,\zeta^{-1}])\,|\,K_l(\zeta)A(\zeta)+{}^tA(\zeta)K_l(\zeta)=0\} \\ &\qquad\qquad \text{for even } l,\end{aligned}$$

where

$$K_l(\zeta)=\begin{pmatrix} & -1 & & & \\ 1 & & & & \\ & & & & (-)^{l-1}\zeta^{-1} \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & -\zeta^{-1} & & \\ & \zeta^{-1} & & & \end{pmatrix}.$$

The orthogonal "group" $O(\infty)$ and the symplectic "group" $Sp(\infty)$ are defined as

$$(2.1.13)\qquad O(\infty)=\{W\in GL(\infty);\ J^{-1}\,{}^tWJ=W^{-1}\},$$

$$(2.1.14)\qquad Sp(\infty)=\{W\in GL(\infty);\ K^{-1}\,{}^tWK=W^{-1}\}.$$

Remark 1. The odd-reduced subalgebra $\mathfrak{o}(\infty)_{2l+1}$ and $\mathfrak{sp}(\infty)_{2l+1}$ are isomorphic to each other under the outer automorphism of $\mathfrak{gl}(2l+1, C[\zeta,\zeta^{-1}])$,

$$A(\zeta)\longmapsto\tilde{J}(\Lambda_{2l+1}(-\zeta^{-1}))^{-l+1}A(\zeta^{-1})(\Lambda_{2l+1}(-\zeta^{-1}))^{l-1}\tilde{J},$$

where $\Lambda_{2l+1}(\zeta)$ was given in §1.4, and $\tilde{J}=\begin{pmatrix} & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$. This fact is proved as follows: Let $A(\zeta)\in\mathfrak{o}(\infty)_{2l+1}$ i.e. ${}^tA(-\zeta)J_{2l+1}(\zeta)+J_{2l+1}(\zeta)A(\zeta)=0$. Set

$$\tilde{J}_{2l+1}(\zeta)=({}^t\Lambda_{2l+1}(\zeta))^{l-1}J_{2l+1}(\zeta)(\Lambda_{2l+1}(-\zeta))^{l-1},$$
$$\tilde{A}(\zeta)=(\Lambda_{2l+1}(-\zeta))^{-l+1}A(\zeta)(\Lambda_{2l+1}(-\zeta))^{l-1}.$$

Then it is seen that ${}^t\tilde{A}(-\zeta)\tilde{J}_{2l+1}(\zeta)+\tilde{J}_{2l+1}(\zeta)\tilde{A}(\zeta)=0$. Notice that

$$\tilde{J}\,{}^t\tilde{J}_{2l+1}(\zeta)\tilde{J}=(-)^l\zeta^{-1}K_{2l+1}(\zeta).$$

Hence one has

$${}^t\hat{A}(-\zeta)K_{2l+1}(-\zeta^{-1})+K_{2l+1}(-\zeta^{-1})\hat{A}(\zeta)=0,$$

where $\hat{A}(\zeta)=\tilde{J}\tilde{A}(-\zeta)\tilde{J}$. Thus $\hat{A}(-\zeta^{-1})\in\mathfrak{sp}(\infty)_{2l+1}$. Q.E.D.

Remark 2 ([25, 42]). Set

$$\begin{aligned}
&\mathfrak{o}(2l+2, \boldsymbol{C}[\zeta,\zeta^{-1}])\\
&\quad=\{A(\zeta)\in\mathfrak{sl}(2l+2, \boldsymbol{C}[\zeta,\zeta^{-1}])\,|\,J_{2l+2}(\zeta)A(\zeta)+{}^tA(\zeta)J_{2l+2}(\zeta)=0\},\\
&\mathfrak{su}(2l+1, \boldsymbol{C}[\zeta,\zeta^{-1}])\\
&\quad=\{A(\zeta)\in\mathfrak{sl}(2l+1, \boldsymbol{C}[\zeta,\zeta^{-1}])\,|\,J_{2l+1}(\zeta)A(\zeta)+{}^tA(-\zeta)J_{2l+1}(\zeta)=0\},\\
&\mathfrak{sp}(l, \boldsymbol{C}[\zeta,\zeta^{-1}])\\
&\quad=\{A(\zeta)\in\mathfrak{sl}(2l, \boldsymbol{C}[\zeta,\zeta^{-1}])\,|\,K_{2l}(\zeta)A(\zeta)+{}^tA(\zeta)K_{2l}(\zeta)=0\}.
\end{aligned}$$

The Euclidean Lie algebras attached to the extended Dynkin diagrams $D_{l+1}^{(2)}$, $A_{2l}^{(2)}$, $C_l^{(1)}$ are realized as the one-dimensional central extension of the Lie algebras $\mathfrak{o}(2l+2, \boldsymbol{C}[\zeta,\zeta^{-1}])$, $\mathfrak{su}(2l+1, \boldsymbol{C}[\zeta,\zeta^{-1}])$, $\mathfrak{sp}(l, \boldsymbol{C}[\zeta,\zeta^{-1}])$, respectively.

2.2. The Toda lattices of B-type and C-type

In the Zakharov-Shabat equation (0.2) for the Toda lattice, we impose the following constraint on B_1, C_1; $B_1, C_1\in\mathfrak{o}(\infty)$, or equivalently,

$$(2.2.1)\qquad b(s)=-b(-s),\quad c(s)=c(-s+1)\quad\text{for any } s.$$

The resulting equation is referred to as the Toda lattice of the B-type (BTL). Namely BTL amounts to the difference-differential equations

$$(2.2.2)\qquad \begin{aligned}&\partial_{x_1}c(1)=c(1)b(1),\quad \partial_{x_1}c(s)=c(s)(b(s)-b(s-1))\quad (s\geqq 2),\\ &\partial_{y_1}b(s)=c(s)-c(s+1)\quad (s\geqq 1).\end{aligned}$$

The Toda lattice of the C-type (CTL) is now defined by imposing the following constraint; $B_1, C_1\in\mathfrak{sp}(\infty)$, or

$$(2.2.3)\qquad b(s)=-b(-s-1),\quad c(s)=c(-s)\qquad\text{for any } s.$$

Hence it becomes the difference-differential equations

$$(2.2.4)\quad \begin{aligned}&\partial_{x_1}c(0)=2c(0)b_0(0),\quad \partial_{x_1}c(s)=c(s)(b(s)-b(s-1))\quad (s\geqq 1),\\ &\partial_{y_1}b(s)=c(s)-c(s+1)\quad (s\geqq 0).\end{aligned}$$

Both *BTL* and *CTL* are sub-subholonomic in the sense of the introduction.

For *BTL*, introducing τ functions $\tau(s)=\tau(s; x_1, y_1)$ $(s\geqq 1)$ through

$$b_0(s)=\partial_{x_1}\log(\tau(s+1)/\tau(s)),\qquad c_{-1}(s)=\tau(s+1)\tau(s-1)/\tau(s)^2,$$
$$\text{with } \tau(1)=\tau(0),$$

we obtain Hirota's bilinear equations

$$(2.2.5)\quad D_{x_1}D_{y_1}\tau(s)\cdot\tau(s)+2\tau(s+1)\tau(s-1)=0,\quad (s\geqq 1,\ \tau(1)=\tau(0)).$$

The τ functions $\tau(s)$ $(s\geqq 0)$ of *CTL* are introduced through

$$b_0(s)=\partial_{x_1}\log(\tau(s+1)/\tau(s)),\quad c_{-1}(s)=\tau(s+1)\tau(s-1)/\tau(s)^2,$$
$$\text{with } \tau(1)=\tau(-1),$$

and *CTL* is transformed into

$$(2.2.6)\quad D_{x_1}D_{y_1}\tau(s)\cdot\tau(s)+2\tau(s+1)\cdot\tau(s-1)=0,\quad (s\geqq 0,\ \tau(1)=\tau(-1)).$$

We remark that the Hirota equations of the Toda lattice reduces to (2.2.5) and (2.2.6) by imposing on the τ functions the following symmetries with respect to the discrete parameter s;

$$(2.2.7)\qquad \tau(s+1; x_1, y_1)=\tau(-s; x_1, y_1)\qquad \text{for } BTL,$$

$$(2.2.8)\qquad \tau(s; x_1, y_1)=\tau(-s; x_1, y_1)\qquad \text{for } CTL.$$

2.3. The Toda lattice hierarchies of *B*-type and *C*-type and their *l*-periodic reduction

Let us recall the fundamentals about the *TL* hierarchy: The *TL* hierarchy arises as the compatibility condition of the linear problem

$$(2.3.1)\quad \begin{aligned}&L=W^{(\infty)}(x,y)\Lambda W^{(\infty)}(x,y)^{-1},\quad M=W^{(0)}(x,y)\Lambda^{-1}W^{(0)}(x,y)^{-1},\\ &\partial_{x_n}W^{\binom{0}{\infty}}(x,y)=B_nW^{\binom{0}{\infty}}(x,y),\quad \partial_{y_n}W^{\binom{0}{\infty}}(x,y)=C_nW^{\binom{0}{\infty}}(x,y),\end{aligned}$$

where $B_n=(L^n)_+$, $C_n(M^n)_-$. The wave matrices take the form

$$W^{(\infty)}(x,y)=\hat{W}^{(\infty)}(x,y)\exp\xi(x,\Lambda),$$

$$\hat{W}^{(\infty)}(x,y)=\sum_{j=0}^{\infty}\operatorname{diag}[\hat{w}_j^{(\infty)}(s;x,y)]\Lambda^{-j}\quad \text{with } \hat{w}_0^{(\infty)}(s;x,y)\equiv 1,$$

$$W^{(0)}(x, y)=\hat{W}^{(0)}(x, y)\exp\xi(y, \Lambda^{-1}),$$
$$\hat{W}^{(0)}(x, y)=\sum_{j=0}^{\infty}\operatorname{diag}[\hat{w}_j^{(0)}(s; x, y)]\Lambda^j \quad \text{with } \hat{w}_0^{(0)}(s; x, y)\neq 0.$$

They are not uniquely determined, but have the arbitrariness

$$\text{(2.3.2)} \qquad \begin{aligned} W^{(\infty)}(x, y) &\longmapsto W^{(\infty)}(x, y)f^{(\infty)}(\Lambda), \\ W^{(0)}(x, y) &\longmapsto W^{(0)}(x, y)f^{(0)}(\Lambda). \end{aligned}$$

Here $f^{\binom{0}{\infty}}(\lambda)=\sum_{n=0}^{\infty} f_n^{\binom{0}{\infty}}\lambda^{\pm n}$ $(f_0^{(\infty)}=1, f_0^{(0)}\neq 0)$ are formal Laurent series with constant scalar coefficients.

We fix the notations to be used throughout this and the subsequent sections.

Set $\tilde{x}=(x_1, x_2, \cdots)$, $\tilde{y}=(y_1, y_3, \cdots)$. We abbreviate $x_2=x_4=\cdots=y_2=y_4=\cdots=0$ to $x_e=y_e=0$. Let

$$\tilde{L}=L|_{x_e=y_e=0}, \qquad \tilde{M}=M|_{x_e=y_e=0},$$
$$\tilde{B}_n=B_n|_{x_e=y_e=0}, \qquad \tilde{C}_n=C_n|_{x_e=y_e=0},$$

and let

$$W^{(\pm)}(\tilde{x}, \tilde{y})=W^{\binom{0}{\infty}}(x, y)|_{x_e=y_e=0}, \quad \hat{W}^{(\pm)}(\tilde{x}, \tilde{y})=\hat{W}^{\binom{0}{\infty}}(x, y)|_{x_e=y_e=0}.$$

Note that the wave matrices $W^{(\pm)}(\tilde{x}, \tilde{y})$ take the form,

$$\text{(2.3.3)} \qquad \begin{aligned} W^{(-)}(\tilde{x}, \tilde{y}) &=\hat{W}^{(-)}(\tilde{x}, \tilde{y})\exp\tilde{\xi}(\tilde{x}, \Lambda), \\ W^{(+)}(\tilde{x}, \tilde{y}) &=\hat{W}^{(+)}(\tilde{x}, \tilde{y})\exp\tilde{\xi}(\tilde{y}, \Lambda^{-1}), \end{aligned}$$

where $\tilde{\xi}(\tilde{x}, \Lambda)=\sum_{n:\mathrm{odd}} x_n\Lambda^n$. Furthermore we set

$$\hat{w}^{(\pm)}(s; \tilde{x}, \tilde{y}; \lambda)=\sum_{j=0}^{\infty}\hat{w}_j^{(\pm)}(s; \tilde{x}, \tilde{y})\lambda^{\pm j},$$
$$\hat{w}^{(\pm)*}(s; \tilde{x}, \tilde{y}; \lambda)=\sum_{j=0}^{\infty}\hat{w}_j^{(\pm)*}(s; \tilde{x}, \tilde{y})\lambda^{\pm j},$$

where the coefficients are given by

$$\hat{W}^{(\pm)}(\tilde{x}, \tilde{y})=\sum_{j=0}^{\infty}\operatorname{diag}[\hat{w}_j^{(\pm)}(s; \tilde{x}, \tilde{y})]\Lambda^{\pm j},$$
$$\hat{W}^{(\pm)}(\tilde{x}, \tilde{y})^{-1}=\sum_{j=0}^{\infty}\Lambda^{\pm j}\operatorname{diag}[\hat{w}_j^{(\pm)*}(s+1; \tilde{x}, \tilde{y})].$$

The Toda lattice hierarchy of the B-type (the BTL hierarchy) is a specialization of the *TL* hierarchy in the sense that we impose the constraints

$$\tilde{L},\ \tilde{M} \in \mathfrak{o}((\infty)) \tag{2.3.4}$$

on (2.3.1) at the expense of freezing the even time flows. Note that (2.3.4) implies $\tilde{L}^n$, $\tilde{M}^n \in \mathfrak{o}((\infty))$ for odd n, so that $\tilde{B}_n$, $\tilde{C}_n \in \mathfrak{o}(\infty)$ for odd n. Thus the *BTL* hierarchy is a set of nonlinear differential equations given by

$$\begin{aligned} &\partial_{x_n}\tilde{B}_m - \partial_{x_m}\tilde{B}_n + [\tilde{B}_m, \tilde{B}_n] = 0, \quad &&\partial_{y_n}\tilde{C}_m - \partial_{y_m}\tilde{C}_n + [\tilde{C}_m, \tilde{C}_n] = 0, \\ &\partial_{y_n}\tilde{B}_m - \partial_{x_m}\tilde{C}_n + [\tilde{B}_m, \tilde{C}_n] = 0, \quad &&n,\ m;\ \text{odd}. \end{aligned} \tag{2.3.5}$$

The third equation above with $n=m=1$ is nothing but the *BTL*.

We will further deduce the following proposition on the wave matrices, which is analogous to Proposition 1 in [23].

Proposition 2.4. *Assume* (2.3.4). *Then* $W^{(\pm)}(\tilde{x}, \tilde{y}) \in O(\infty)$ *under a suitable choice of* $f^{(\infty)}(\lambda), f^{(0)}(\lambda)$ *in* (2.3.2).

Proof. We will only show $W^{(-)}(\tilde{x}, \tilde{y}) \in O(\infty)$.

Since $\tilde{B}_n$, $\tilde{C}_n \in \mathfrak{o}(\infty)$, it follows that

$$\partial_{x_n}(J^{-1}\,{}^tW^{(-)}JW^{(-)}) = \partial_{y_n}(J^{-1}\,{}^tW^{(-)}JW^{(-)}) = 0$$

for odd n. On the other hand, $[\Lambda, J^{-1}\,{}^tW^{(-)}JW^{(-)}]=0$ because

$$\tilde{L} = W^{(-)}\Lambda W^{(-)-1} \in \mathfrak{o}((\infty)).$$

Combining these facts, one sees that

$$J^{-1}\,{}^tW^{(-)}JW^{(-)} = \sum_{n=0}^{\infty} g_n \Lambda^{-n} \qquad \text{with } g_0 = 1, \tag{2.3.6}$$

where g_n is a constant scalar. Taking into account ${}^tJ=J$, one has also ${}^tW^{(-)}JW^{(-)}J^{-1} = \sum_{n=0}^{\infty} g_n \Lambda^n$. Since $J^{-1}\Lambda^n J = (-)^n \Lambda^{-n}$, one further sees that

$$J^{-1}\,{}^tW^{(-)}JW^{(-)} = \sum_{n=0}^{\infty} (-)^n g_n \Lambda^{-n}. \tag{2.3.7}$$

Comparing (2.3.6) with (2.3.7), one concludes that $g_n = 0$ for odd n.

Moreover let us modify $W^{(-)}$ to $W^{(-)}f^{(\infty)}(\Lambda)$, then

$$J^{-1}\,{}^tW^{(-)}JW^{(-)} = \left(\sum_{n=0}^{\infty} (-)^n f_n^{(\infty)} \Lambda^{-n}\right)^{-1} \left(\sum_{n:\,\text{even}} g_n \Lambda^{-n}\right) \left(\sum_{n=0}^{\infty} f_n^{(\infty)} \Lambda^{-n}\right)^{-1}.$$

It is evident that a suitable choice of $f^{(\infty)}(\Lambda)$ makes the left-hand side be 1. Thus $W^{(-)} \in O(\infty)$. Q.E.D.

The Toda lattice hierarchy of C-type (the CTL hierarchy) is also defined as a specialization of the TL hierarchy with the constraints

$$\tilde{L}, \tilde{M} \in \mathfrak{sp}((\infty)) \tag{2.3.8}$$

in (2.3.1) for which the even time flows are freezed. Then $B_n, C_n \in \mathfrak{sp}(\infty)$ for odd n, and the CTL hierarchy is a set of nonlinear differential equations such as (2.3.5). As in the orthogonal case, wave matrices $W^{(\pm)}(\tilde{x}, \tilde{y})$ belong to $Sp(\infty)$ under an appropriate choice of $f^{\binom{0}{\infty}}(\lambda)$.

Now we will describe the orthogonal or symplectic conditions in terms of τ functions. Though such conditions has been considered in [26], the authors have also obtained an algebraic proof for them, independently of [26].

Theorem 2.5. *Suppose that* $W^{\binom{0}{\infty}}(x, y)|_{x_e=y_e=0} \in O(\infty)$. *Then the corresponding* τ *functions satisfy*

$$\tau(s+1; x, y) = \tau(-s; \iota(x), \iota(y)) \tag{2.3.9}$$

for any s, *where we have set* $\iota(x) = (x_1, -x_2, x_3, -x_4, \cdots)$. *Conversely, if* τ *functions are subject to* (2.3.9), *the corresponding hierarchy is of the* B-*type.*

For the proof, we start with the following proposition.

Proposition 2.6. *The symmetry* (2. 3. 9) *is equivalent to that of wave matrices such as*

$$J^{-1}\,{}^t\hat{W}^{\binom{0}{\infty}}(\iota(x), \iota(y))J = \hat{W}^{\binom{0}{\infty}}(x, y)^{-1}. \tag{2.3.10}$$

Proof. First we show (2.3.9) to be deduced from (2.3.10). In view of $J^{-1}\Lambda^{-j}J = (-)^j\Lambda^j$, from (2.3.10) one obtains

$$J^{-1}\hat{W}^{(\infty)}(\iota(x), \iota(y))J = \sum_{j=0}^{\infty} (-)^j \Lambda^{-j} \operatorname{diag}[\hat{w}_j^{(\infty)}(-s; \iota(x), \iota(y))],$$

which further leads to

$$(-)^j \hat{w}_j^{(\infty)}(-s; \iota(x), \iota(y)) = \hat{w}_j^{(\infty)*}(s+1; x, y). \tag{2.3.11}$$

One has similarly

(2.3.12) $$(-)^j \hat{w}_j^{(0)}(-s; \iota(x), \iota(y)) = \hat{w}_j^{(0)*}(s+1; x, y).$$

By the way, notice that

(2.3.13) $$(-)^j p_j(-\tilde{\partial}_x) f(\iota(x)) = p_j(\tilde{\partial}_x) f(x)$$

holds for any j. This follows from

$$\exp \xi(-\tilde{\partial}_x, -\lambda) f(x)|_{x \to \iota(x)} = \exp \xi(\tilde{\partial}_x, \lambda) f(x).$$

Applying (2.3.13) to (2.3.11) and (2.3.12), one finds

(2.3.14) $$\begin{aligned} &p_j(\tilde{\partial}_x) \log (\tau(-s; \iota(x), \iota(y))/\tau(s+1; x, y)) \\ &\quad = p_j(\tilde{\partial}_y) \log (\tau(-s; \iota(x), \iota(y))/\tau(s+1; x, y)) = 0 \quad \text{for } j \geqq 1, \end{aligned}$$

and

(2.3.15) $$\left. \frac{\tau(-s+1; \iota(x), \iota(y))}{\tau(s; x, y)} \right|_{x_e = y_e = 0} = \left. \frac{\tau(-s; \iota(x), \iota(y))}{\tau(s+1; x, y)} \right|_{x_e = y_e = 0}.$$

(2.3.14) implies that

$$\text{Cte.} \times \tau(s+1; x, y) = \tau(-s; \iota(x), \iota(y)),$$

and (2.3.15) assures that the above constant factor is independent of s. Setting $s=1$ in (2.3.15), one sees that $(\tau(0; \iota(x), \iota(y))/\tau(1; x, y))|_{x_e = y_e = 0} = 1$. Hence one obtains (2.3.9).

The converse statement is evident. Q.E.D.

By virtue of Proposition 2.6, the proof of Theorem 2.5 reduces to that of (2.3.10). To show this symmetry, some lemmata are required.

Lemma 2.7. *If* $J^{-1}PJ = (-)^m\, {}^tP$, $J^{-1}QJ = (-)^n\, {}^tQ$ $(m, n \in Z)$, *then* $J^{-1}[P, Q]J = (-)^{m+n+1}[P, Q]$.

Proof. Straightforward. Q.E.D.

Set $|\alpha| = \sum_{j=1}^{\infty} \alpha_j$, $\|\alpha\| = \sum_{j=1}^{\infty} (j+1)\alpha_j$ for a multi-index α $(\alpha_j \geqq 0, \alpha_j = 0$ for $j \gg 0)$, and define f^ι for a function f by $f^\iota(x, y) = f(\iota(x), \iota(y))$.

Lemma 2.8. *For any multi-indices* α, β, *we have*

(2.3.16) $$J^{-1}\, {}^t(\partial_x^\alpha \partial_y^\beta B_k|_{x_e = y_e = 0}) J = (-)^{\|\alpha\| + \|\beta\| + k} \partial_x^\alpha \partial_y^\beta B_k|_{x_e = y_e = 0},$$

(2.3.17) $$\partial_x^\alpha \partial_y^\beta B_k^\iota|_{x_e = y_e = 0} = (-)^{\|\alpha\| + \|\beta\|} \partial_x^\alpha \partial_y^\beta B_k|_{x_e = y_e = 0}.$$

The equations which is obtained by replacing B_k by C_k in the above also hold.

Proof. By induction for $|\alpha|$, we will show (2.3.16) in the case of $\beta=0$;

$$J^{-1}(\partial_x^\alpha B_k|_{x_e=y_e=0})J=(-)^{\|\alpha\|+k}\partial_x^\alpha B_k|_{x_e=y_e=0}. \tag{2.3.16$'$}$$

Since $J^{-1}\,{}^t(L^k|_{x_e=y_e=0})J=(-)^k L^k|_{x_e=y_e=0}$, one sees, considering the $(+)$ part of the both sides, that (2.3.16)$'$ holds for $\alpha=0$. Next assume (2.3.16)$'$ to be true for $|\alpha|\leqq M$. Let $\partial_x^\alpha=\partial_{x_{i_1}}\cdots\partial_{x_{i_{M+1}}}$. Since $\partial_x L^k=[B_n, L^k]$, it follows that

$$\begin{aligned}\partial_x^\alpha L^k=&[\partial_{x_{i_1}}\cdots\partial_{x_{i_M}}B_{i_{M+1}}, L^k]+\sum_{\alpha=1}^{M}[\partial_{x_{i_1}}\underset{\hat{\alpha}}{\cdots}\partial_{x_{i_M}}B_{i_{M+1}}, [B_{i_\alpha}, L^k]]\\&+\cdots+[B_{i_{M+1}}, [B_{i_M}, [\cdots[B_{i_1}, L^k]]\cdots]].\end{aligned} \tag{2.3.18}$$

Here $\partial_{x_{i_1}}\underset{\hat{\alpha}}{\cdots}\partial_{x_{i_M}}$ indicates excluding $\partial_{x_{i_\alpha}}$ from $\partial_{x_{i_1}}\cdots\partial_{x_{i_M}}$. Thanks to Lemma 2.7 and the assumption of induction, the right-hand side of (2.3.18) restricted to $x_e=y_e=0$ satisfies an identity such as $J^{-1}\,{}^tPJ=(-)^{\|\alpha\|+k}P$. Hence

$$J^{-1}\,{}^t(\partial_x^\alpha L^k|_{x_e=y_e=0})J=(-)^{\|\alpha\|+k}(\partial_x^\alpha L^k|_{x_e=y_e=0}).$$

Considering the $(+)$ part above, one finds (2.3.16)$'$ to persist for $|\alpha|=M+1$. Thus it is proved.

(2.3.16) in a general case can be verified in the same fashion as above. The second identity (2.3.17) is obvious. Q.E.D.

Proof of Theorem 2.5. Set $Y=J^{-1}\,{}^t\hat{W}^{(\infty)}(x, y)^{-1}J$, and

$$Z=\hat{W}^{(\infty)}(\iota(x), \iota(y))^{-1}.$$

We wish to show $Y=Z$. For this purpose, we prove

$$\partial_x^\alpha Y|_{x_e=y_e=0}=\partial_x^\alpha Z|_{x_e=y_e=0} \tag{2.3.19}$$

by induction on $|\alpha|$. (2.3.19) is obviously true for $\alpha=0$. Next assume (2.3.19) to be true for $|\alpha|\leqq M$. Let $|\alpha|=M$. Since Y, Z solve the equations

$$\begin{aligned}&\partial_{x_n}Y=(-)^{n+1}\{(-)^n(J^{-1}\,{}^tB_nJ)Y-Y\Lambda^n\},\\&\partial_{x_n}Z=(-)^{n+1}\{B_n^\iota Z-Z\Lambda^n\},\end{aligned}$$

they also satisfy

$$\partial_{x_n}\partial_x^\alpha Y = (-)^{n+1}\{(-)^n \sum_{\beta+\gamma=\alpha} \partial_x^\beta(J^{-1}\,{}^tB_nJ)\cdot\partial_x^\gamma Y - \partial_x^\alpha Y\cdot\Lambda^n\},$$

$$\partial_{x_n}\partial_x^\alpha Z = (-)^{n+1}\{\sum_{\beta+\gamma=\alpha} \partial_x^\beta B_n^\iota\cdot\partial_x^\gamma Z - \partial_x^\gamma Z\cdot\Lambda^n\}.$$

Therefore the assumption of induction and (2.3.16) in Lemma 2.8 yield $\partial_{x_n}\partial_x^\alpha Y|_{x_e=y_e=0} = \partial_{x_n}\partial_x^\alpha Z|_{x_e=y_e=0}$. Thus (2.3.19) holds for any multi-index α. More generally one can show

$$\partial_x^\alpha\partial_y^\beta Y|_{x_e=y_e=0} = \partial_x^\alpha\partial_y^\beta Z|_{x_e=y_e=0}$$

for any multi-indices α, β. One can also obtain the equation obtained by replacing $\hat{W}^{(\infty)}$ by $\hat{W}^{(0)}$ in the above. Therefore one concludes (2.3.10).

Q.E.D.

We can deduce a similar statement as in Theorem 2.5 also for the symplectic case.

Theorem 2.9. (1) *Suppose that* $W^{\binom{0}{\infty}}(x, y)|_{x_e=y_e=0} \in Sp(\infty)$. *Then the corresponding* τ *functions satisfy*

(2.3.20) $$\tau(s; x, y) = \tau(-s; \iota(x), \iota(y))$$

for any s. *Conversely, if* τ *functions are subject to* (2.3.20), *the corresponding hierarchy is of the C-type.*

(2) *The symmetry* (2.3.20) *is equivalent to that of wave matrices as*

(2.3.21) $$K^{-1}\,{}^t\hat{W}^{\binom{0}{\infty}}(\iota(x), \iota(y))K = \hat{W}^{\binom{0}{\infty}}(x, y)^{-1}.$$

Remark. It is worthy to note that τ functions with the symmetry (2.3.9) (resp. (2.3.20)) are those of the 2-components *BKP* (resp. *CKP*) hierarchy [23]. In particular, when the time evolution of y is freezed, our τ functions belong to the (one-component) *BKP* (resp. *CKP*) hierarchy.

Now let us discuss the l-periodic *BTL*, *CTL* hierarchies. We will denote them by $(BTL)_l$, $(CTL)_l$. They are subfamilies of the *BTL*, *CTL* hierarchies with the l-periodic constraint

(2.3.22) $$L^l = \Lambda^l, \quad M^l = \Lambda^{-l},$$

besides (2.3.4), (2.3.8). As was considered in Proposition 1.13, (2.3.22) means $\tilde{L}$, $\tilde{M} \in \mathfrak{o}((\infty))_l$ for the $(BTL)_l$ hierarchy (resp. $\tilde{L}$, $\tilde{M} \in \mathfrak{sp}((\infty))_l$ for the $(CTL)_l$ hierarchy). Consequently $\tilde{B}_n$, $\tilde{C}_n \in \mathfrak{o}(\infty)_l$ for odd n (resp. $\tilde{B}_n$, $\tilde{C}_n \in \mathfrak{sp}(\infty)_l$ for odd n). Furthermore

$$\partial_{x_n}\tilde{L} = \partial_{x_n}\tilde{M} = 0, \quad \partial_{y_n}\tilde{L} = \partial_{y_n}\tilde{M} = 0 \quad \text{for odd } n \equiv 0 \bmod l.$$

Namely, the unknown functions of the l-periodic hierarchies are independent of the variables x_n, y_n for odd $n \equiv 0 \bmod l$.

Denote the images of $\tilde{B}_n$, $\tilde{C}_n$ under the isomorphisms (2.1.7), (2.1.12) by $\tilde{B}_n(\zeta)$, $\tilde{C}_n(\zeta)$, which turn out to be tracefree by the same argument as in Proposition 1.14. Then the $(BTL)_l$ and $(CTL)_l$ hierarchies amount to a system of the Zakharov-Shabat equations,

$$\begin{aligned} &\partial_{x_n}\tilde{B}_m(\zeta)-\partial_{x_m}\tilde{B}_n(\zeta)+[\tilde{B}_m(\zeta), \tilde{B}_n(\zeta)]=0,\\ &\partial_{y_n}\tilde{C}_m(\zeta)-\partial_{y_m}\tilde{C}_n(\zeta)+[\tilde{C}_m(\zeta), \tilde{C}_n(\zeta)]=0,\\ &\partial_{y_n}\tilde{B}_m(\zeta)-\partial_{x_m}\tilde{C}_n(\zeta)+[\tilde{B}_m(\zeta), \tilde{C}_n(\zeta)]=0,\\ &\qquad\qquad\qquad\qquad \text{for odd } n, m \not\equiv 0 \bmod l. \end{aligned} \tag{2.3.23}$$

Now we will describe the characterization of τ functions for the $(BTL)_l$, $(CTL)_l$ hierarchies. τ functions must be l-periodic with respect to the discrete parameter s (see § 1.4). Thus, combining this fact with Theorems 2.5, 2.9, we lead to the following characterization:

$$(BTL)_l;\ \begin{cases}\tau(-s; \iota(x), \iota(y))=\tau(s+1; x, y),\\ \tau(s+l; x, y)=\tau(s; x, y), \quad \text{for any } s,\end{cases} \tag{2.3.24}$$

$$(CTL)_l;\ \begin{cases}\tau(-s; \iota(x), \iota(y))=\tau(s; x, y),\\ \tau(s+l; x, y)=\tau(s; x, y) \quad \text{for any } s.\end{cases} \tag{2.3.25}$$

(If we consider $\tau'(s; x, y)$ instead of $\tau(s; x, y)$, we may assume further $\partial_{x_n}\tau'(s; x, y)=\partial_{y_n}\tau'(s; x, y)=0$ for $n \equiv 0 \bmod l$, besides (2.3.24), (2.3.25) (see § 1.4).

We obtain the following claim (cf. [26]).

Proposition 2.10. *If l is odd, $(BTL)_l$ is identifiable with $(CTL)_l$ (see also Remark* 1 *in* § 2.1).

Proof. We will show this proposition by considering an example. Let $l=5$. By virtue of the periodicity, a set of τ functions $\{\tau(1), \cdots, \tau(5)\}$ completely prescribes the $(BTL)_5$ hierarchy. From (2.3.24) it follows that this set reduces to

$$\{\tau(1), \tau(2), \tau(3)=\tau'(3)=\tau'(2), \tau'(1)\}$$

($\tau'(s; x, y)=\tau(s; \iota(x), \iota(y))$). On the other hand, (2.3.25) shows that a set of τ functions

$$\{\tau(-2), \tau(-1), \tau(0)=\tau'(0), \tau'(-1), \tau'(-2)\}$$

perfectly characterizes the $(CTL)_5$ hierarchy. Comparing these two sets enables us to obtain the claim. Q.E.D.

As was remarked in the previous section, $\mathfrak{o}(\infty)_{2l+2}$, $\mathfrak{o}(\infty)_{2l+1}=\mathfrak{sp}(\infty)_{2l+1}$, $\mathfrak{sp}(\infty)_{2l}$ relate to the Euclidean Lie algebras $D_{l+1}^{(2)}$, $A_{2l}^{(2)}$, $C_l^{(1)}$, respectively. We will show that $(BTL)_{2l+2}$, $(BTL)_{2l+1}$, $(CTL)_{2l}$ give $(TL)_{D_{l+1}^{(2)}}$, $(TL)_{A_{2l}^{(2)}}$, $(TL)_{C_l^{(1)}}$, by writing down the Zakharov-Shabat equation [17]

$$\partial_{y_1}\tilde{B}_1(\zeta)-\partial_{x_1}\tilde{C}_1(\zeta)+[\tilde{B}_1(\zeta),\tilde{C}_1(\zeta)]=0. \tag{2.3.26}$$

$(TL)_{D_{l+1}^{(2)}}$

$$\tilde{B}_1(\zeta)=\begin{pmatrix} 0 & 1 & & & & & \\ & b_1 & 1 & & & & \\ & & \ddots & \ddots & & & \\ & & & b_l & \ddots & & \\ & & & & 0 & \ddots & \\ & & & & & -b_l & \ddots \\ & & & & & & \ddots & 1 \\ \zeta & & & & & & & -b_1 \end{pmatrix},$$

$$\tilde{C}_1(\zeta)=\begin{pmatrix} 0 & & & & & \zeta^{-1}c_1 \\ c_1 & 0 & & & & \\ & \ddots & \ddots & & & \\ & & c_{l+1} & \ddots & & \\ & & & c_{l+1} & \ddots & \\ & & & & \ddots & \ddots \\ & & & & c_2 & 0 \end{pmatrix}.$$

Then (2.3.26) reads as

$$\begin{aligned} &\partial_{y_1}b_s=c_s-c_{s+1}, \quad (1\leqq s\leqq l), \\ &\partial_{x_1}c_1=c_1b_1, \quad \partial_{x_1}c_s=c_s(b_s-b_{s-1}), \quad (2\leqq s\leqq l-1), \\ &\partial_{x_1}c_{l+1}=-c_{l+1}b_l. \end{aligned} \tag{2.3.27}$$

Introducing $u(s)$ through $b_s=\partial_{x_1}u(s)$ $(1\leqq s\leqq l)$, $c_1=e^{u(1)}$, $c_s=e^{u(s)-u(s-1)}$, $(2\leqq s\leqq l-1)$, $c_{l+1}=e^{-u(l)}$, (2.3.27) turns out to be $(TL)_{D_{l+1}^{(2)}}$.

$(TL)_{A_{2l}^{(2)}}$

$$\tilde{B}_1(\zeta)=\begin{pmatrix} 0 & 1 & & & & \\ & b_1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & b_l & \ddots & \\ & & & & -b_l & \ddots \\ & & & & & \ddots & 1 \\ -\zeta & & & & & & -b_1 \end{pmatrix},$$

$$\tilde{C}_1(\zeta)=\begin{pmatrix} 0 & & & & & & -\zeta^{-1}c_1 \\ c_1 & 0 & & & & & \\ & \ddots & \ddots & & & & \\ & & c_l & \ddots & & & \\ & & & c_{l+1} & \ddots & & \\ & & & & c_l & \ddots & \\ & & & & & \ddots & \ddots \\ & & & & & c_2 & 0 \end{pmatrix}.$$

Then we obtain

$$\begin{aligned} &\partial_{y_1}b_s=c_s-c_{s+1}, \quad (1\leqq s\leqq l), \\ (2.3.28) \qquad &\partial_{x_1}c_1=c_1b_1, \quad \partial_{x_1}c_s=c_s(b_s-b_{s-1}), \quad (2\leqq s\leqq l), \\ &\partial_{x_1}c_{l+1}=-2c_{l+1}b_l. \end{aligned}$$

Setting $b_s=\partial_{x_1}u(s)$, $(1\leqq s\leqq l)$, $c_1=e^{u(1)}$, $c_s=e^{u(s)-u(s-1)}$, $(2\leqq s\leqq l)$, $c_{l+1}=2e^{-2u(l)}$, (2.3.28) becomes $(TL)_{A_{2l}^{(2)}}$.

$(TL)_{C_l^{(1)}}$

$$\tilde{B}_1(\zeta)=\begin{pmatrix} -b_0 & 1 & & & & & \\ & b_0 & 1 & & & & \\ & & b_1 & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & b_{l-1} & \ddots & \\ & & & & & b_{l-1} & \ddots \\ & & & & & & \ddots \quad 1 \\ \zeta & & & & & & -b_1 \end{pmatrix},$$

$$\tilde{C}_1(\zeta)=\begin{pmatrix} 0 & & & & & & & -\zeta^{-1}c_1 \\ c_0 & 0 & & & & & & \\ & c_1 & \ddots & & & & & \\ & & \ddots & \ddots & & & & \\ & & & c_{l-1} & \ddots & & & \\ & & & & c_l & \ddots & & \\ & & & & & c_{l-1} & \ddots & \\ & & & & & & \ddots & \ddots \\ & & & & & & c_2 & 0 \end{pmatrix}.$$

Then we obtain

$$\begin{aligned}(2.3.29)\qquad &\partial_{y_1} b_s = c_s - c_{s+1}, \quad (1 \leqq s \leqq l-1),\\ &\partial_{x_1} c_0 = 2c_0 b_0, \quad \partial_{x_1} c_s = c_s(b_s - b_{s-1}), \quad (1 \leqq s \leqq l-1),\\ &\partial_{x_1} c_l = -2c_l b_{l-1}.\end{aligned}$$

Setting $b_s = \partial_{x_1} u(s)$, $(0 \leqq s \leqq l-1)$, $c_0 = e^{2u(0)}$, $c_s = e^{u(s)-u(s-1)}$ $(1 \leqq s \leqq l-1)$, $c_l = 2e^{-2u(l-1)}$, (2.3.29) becomes $(TL)_{C_l^{(1)}}$.

2.4. Remarks on τ functions of the *BTL*, *CTL* hierarchies

In this section we will briefly describe another definition of τ functions of the *BTL*, *CTL* hierarchies. We will keep the notations in the preceding section.

For the *BTL*, *CTL* hierarchies, the bilinear relation (1.2.18) reads as

$$(2.4.1)\qquad W^{(-)}(\tilde{x}, \tilde{y}) W^{(-)}(\tilde{x}', \tilde{y}')^{-1} = W^{(+)}(\tilde{x}, \tilde{y}) W^{(+)}(\tilde{x}', \tilde{y}')^{-1}$$

for any $\tilde{x}, \tilde{x}', \tilde{y}, \tilde{y}'$. From this we deduce the following proposition valid for both the *BTL* and the *CTL* hierarchies.

Proposition 2.11. *For the wave matrices $W^{(\pm)}(\tilde{x}, \tilde{y})$, τ functions $\tilde{\tau}(s; \tilde{x}, \tilde{y})$ are uniquely determined up to a constant multiple factor so that*

$$(2.4.2)\qquad \begin{aligned} \hat{w}^{(-)}(s; \tilde{x}, \tilde{y}; \lambda) &= \frac{\tilde{\tau}(s; \tilde{x} - \tilde{\varepsilon}(\lambda^{-1}), \tilde{y})}{\tilde{\tau}(s; \tilde{x}, \tilde{y})},\\ \hat{w}^{(-)*}(s; \tilde{x}, \tilde{y}; \lambda) &= \frac{\tilde{\tau}(s; \tilde{x} + \tilde{\varepsilon}(\lambda^{-1}), \tilde{y})}{\tilde{\tau}(s; \tilde{x}, \tilde{y})},\\ \hat{w}^{(+)*}(s; \tilde{x}, \tilde{y}; \lambda) &= \frac{\tilde{\tau}(s+1; \tilde{x}, \tilde{y} - \tilde{\varepsilon}(\lambda))}{\tilde{\tau}(s; \tilde{x}, \tilde{y})},\\ \hat{w}^{(+)*}(s; \tilde{x}, \tilde{y}; \lambda) &= \frac{\tilde{\tau}(s-1; \tilde{x}, \tilde{y} + \tilde{\varepsilon}(\lambda))}{\tilde{\tau}(s; \tilde{x}, \tilde{y})}, \end{aligned}$$

where $\tilde{\varepsilon}(\lambda) = (2\lambda, \frac{2}{3}\lambda^3, \frac{2}{5}\lambda^5, \cdots)$. Furthermore they have the following symmetries;

$$(2.4.3)\qquad \tilde{\tau}(-s; \tilde{x}, \tilde{y}) = \tilde{\tau}(s+1; \tilde{x}, \tilde{y}) \qquad \textit{for the BTL hierarchy,}$$

$$(2.4.4)\qquad \tilde{\tau}(-s; \tilde{x}, \tilde{y}) = \tilde{\tau}(s; \tilde{x}, \tilde{y}) \qquad \textit{for the CTL hierarchy.}$$

Proof. We will only give a rough sketch of the proof. First of all, note that the following equalities hold;

$$(2.4.5)\qquad \exp \tilde{\xi}(\tilde{\varepsilon}(\lambda_1^{-1}), \Lambda) = (1 + \lambda_1^{-1}\Lambda)(1 - \lambda_1^{-1}\Lambda)^{-1},$$

$$\begin{aligned}
&\exp \tilde{\xi}(\tilde{\varepsilon}(\lambda_1^{-1})+\tilde{\varepsilon}(\lambda_2^{-1}), \Lambda) \\
(2.4.6) \qquad &\quad =\frac{2(\lambda_2+\lambda_1)}{\lambda_2-\lambda_1}\{(1-\lambda_1^{-1}\Lambda)^{-1}-(1-\lambda_2^{-1}\Lambda)^{-1}\}\Lambda^{-1}.
\end{aligned}$$

Let $\tilde{x}'=\tilde{x}-\tilde{\varepsilon}(\lambda_1^{-1})$, $\tilde{y}'=\tilde{y}-\tilde{\varepsilon}(\lambda_2^{-1})$ in (2.4.1). Applying (2.4.5), one gets the bilinear equation

$$\begin{aligned}
&\hat{w}^{(-)}(s;\tilde{x},\tilde{y};\lambda_1)\hat{w}^{(-)*}(s+1;\tilde{x}-\tilde{\varepsilon}(\lambda_1^{-1}),\tilde{y}-\tilde{\varepsilon}(\lambda_2);\lambda_1) \\
&\quad =\hat{w}^{(+)}(s;\tilde{x},\tilde{y};\lambda_2)\hat{w}^{(+)*}(s+1;\tilde{x}-\tilde{\varepsilon}(\lambda_1^{-1}),\tilde{y}-\tilde{\varepsilon}(\lambda_2);\lambda_2).
\end{aligned}$$

Next we set $\tilde{x}'=\tilde{x}-\tilde{\varepsilon}(\lambda_1^{-1})-\tilde{\varepsilon}(\lambda_2^{-1})$, $\tilde{y}'=\tilde{y}$ (resp. $\tilde{x}'=\tilde{x}$, $\tilde{y}'=\tilde{y}-\tilde{\varepsilon}(\lambda_1)-\tilde{\varepsilon}(\lambda_2)$). By making use of (2.4.6), one derives the following bilinear equations;

$$\begin{aligned}
&\hat{w}^{(-)}(s;\tilde{x},\tilde{y};\lambda_1)\hat{w}^{(-)*}(s;\tilde{x}-\tilde{\varepsilon}(\lambda_1^{-1})-\tilde{\varepsilon}(\lambda_2^{-1}),\tilde{y};\lambda_1) \\
&\quad =\hat{w}^{(-)}(s;\tilde{x},\tilde{y};\lambda_2)\hat{w}^{(-)*}(s;\tilde{x}-\tilde{\varepsilon}(\lambda_1^{-1})-\tilde{\varepsilon}(\lambda_2^{-1}),\tilde{y};\lambda_2), \\
&\hat{w}^{(+)}(s;\tilde{x},\tilde{y};\lambda_1)\hat{w}^{(+)*}(s+2;\tilde{x},\tilde{y}-\tilde{\varepsilon}(\lambda_1)-\tilde{\varepsilon}(\lambda_2);\lambda_1) \\
&\quad =\hat{w}^{(+)}(s;\tilde{x},\tilde{y};\lambda_2)\hat{w}^{(+)*}(s+2;\tilde{x},\tilde{y}-\tilde{\varepsilon}(\lambda_1)-\tilde{\varepsilon}(\lambda_2);\lambda_2).
\end{aligned}$$

Considering these equations, one can achieve the existence proof of the τ functions defined by (2.4.2), by the same discussion as in Theorem 1.7. A similar consideration as the proof of Proposition 2.6 leads us to (2.4.3), (2.4.4). Q.E.D.

Substituting (2.4.2) into (2.4.1), the *BTL* and *CTL* hierarchies are transformed into the infinitely many bilinear equations of the Hirota type,

$$\begin{aligned}
&\sum_{j=0}^{\infty}\tilde{p}_{m+j}(-2\tilde{a})\tilde{p}_j(2\tilde{D}_x)\exp(\langle\tilde{a},D_x\rangle+\langle\tilde{b},D_y\rangle)\tilde{\tau}(s+m+1)\cdot\tilde{\tau}(s) \\
(2.4.7) \qquad &=\sum_{j=0}^{\infty}\tilde{p}_{-m+j}(-2\tilde{b})\tilde{p}_j(2\tilde{D}_y)\exp(\langle\tilde{a},D_x\rangle+\langle\tilde{b},D_y\rangle)\tilde{\tau}(s+m)\cdot\tilde{\tau}(s), \\
&\qquad s, m\in \boldsymbol{Z}.
\end{aligned}$$

Here $\tilde{a}=(a_1, a_3, \cdots)$ is arbitrary, and $\tilde{D}_x=(D_{x_1}, \frac{1}{3}D_{x_3}, \cdots)$, $\langle\tilde{a}, D_x\rangle=\sum_{n:\mathrm{odd}} a_n D_{x_n}$, while $\tilde{p}_j(x)$ is defined by $\exp\tilde{\xi}(\tilde{x},\lambda)=\sum_{j=0}^{\infty}\tilde{p}_j(x)\lambda^j$.

Let us restrict our attention on the *BTL* hierarchy. Set $m=s=0$ in (2.4.7). Taking $\tilde{\tau}(1)=\tilde{\tau}(0)$ into account, we obtain

$$\begin{aligned}
&\sum_{j=1}^{\infty}\{\tilde{p}_j(-2\tilde{a})\tilde{p}_j(2\tilde{D}_x)-\tilde{p}_j(-2\tilde{b})\tilde{p}_j(2\tilde{D}_y)\} \\
&\qquad \times\exp(\langle\tilde{a},D_x\rangle+\langle\tilde{b},D_y\rangle)\tilde{\tau}(0)\cdot\tilde{\tau}(0)=0,
\end{aligned}$$

which is just the same as the equation to be satisfied by τ function of the 2-components *BKP* hierarchy [24]. That is to say, $\tilde{\tau}(0;\tilde{x},\tilde{y})$ may be thought of to be embedded into the 2-components *BKP* hierarchy.

Remark. In the reference [24], τ function for the *BKP* hierarchy, $\tau_{BKP}(\tilde{x})$, was introduced through

$$\tau_{BKP}(\tilde{x})^2 = \tau(x)\,|_{x_e=0}$$

where $\tau(x)$ is the corresponding τ function of the *KP* hierarchy. From the above discussion, it turns out that $\tilde{\tau}(0;\tilde{x};0)$ corresponds to $\tilde{\tau}_{BKP}(\tilde{x})$.

3. Multi-Component Theory

3.1. Formulation of the multi-component hierarchy

The multi-component theory of the *KP* hierarchy is established in [23, 34] (see Appendix 1). The multi-component theory is indispensable in the treatment of many concrete soliton equations as its specializations. In this sense it is desirable to generalize our theory developed in Chapter 1 to a multi-component analogue.

The so called non abelian Toda lattice [18, 31] is regarded as a multi-component version of the original Toda lattice. However, compared with the formalism of the multi-component *KP* hierarchy, the non abelian Toda lattice seems to be insufficient in the sense that its evolution is restricted to a special sector of the fully possible evolution (see Remark 3.2).

We shall proceed in the same way as the *KP* hierarchy was generalized to the multi-component case.

Remember that the r component *KP* hierarchy is formulated by use of matrix-(micro) differential operators of size $r\times r$, instead of scalar ones used in the one component hierarchy (see Appendix 1). On the other hand, as we noticed at the ends of Section 1.1 and Section 1.2, our hierarchy of the Toda lattice can be reformulated in terms of scalar difference operators. Hence the r component hierarchy of the Toda lattice must be realized by use of matrix-difference operators of size $r\times r$ or, equivalently, matrices of infinite size which consist of the blocks of size $r\times r$ indexed by $\boldsymbol{Z}\times\boldsymbol{Z}$.

According to the above observations, let us prepare some notations of matrices of infinite size. In the r component theory we need the matrices of infinite size acting on the tensor product $\boldsymbol{C}^{\boldsymbol{Z}}\otimes\boldsymbol{C}^r$. We shall often use the Kronecker product $P\otimes Q$ of a matrix P of size $\boldsymbol{Z}\times\boldsymbol{Z}$ and a matrix Q of size $r\times r$.

A matrix A acting on $\boldsymbol{C}^{\boldsymbol{Z}}\otimes\boldsymbol{C}^r$ is expressed in the form

$$A=\sum_{j\in Z}\operatorname{diag}[a^j(s)]\Lambda^j$$

(3.1.1)
$$=\left(\begin{array}{ccc|ccc} \ddots & & \ddots & \ddots & & \ddots \\ & a_0(-1) & & a_1(-1) & & \ddots \\ \hline & a_{-1}(0) & & a_0(0) & a_1(0) & \\ \ddots & & & a_{-1}(1) & a_0(1) & \\ & & \ddots & \ddots & & \ddots \end{array}\right), \quad \boldsymbol{\Lambda}^j=\Lambda^j\otimes 1_r,$$

where $a_j(s)$ is a matrix of size $r\times r$ and $\operatorname{diag}[a_j(s)]$ denotes the block-diagonal matrix $\operatorname{diag}(\cdots a_j(-1), a_j(0), a_j(1), \cdots)$. 1_r is the unit matrix of size $r\times r$. We define $(A)_\pm$ by

(3.1.2)
$$(A)_+=\sum_{j\geq 0}\operatorname{diag}[a_j(s)]\boldsymbol{\Lambda}^j,$$
$$(A)_-=\sum_{j<0}\operatorname{diag}[a_j(s)]\boldsymbol{\Lambda}^j.$$

The following notations are often used throughout this chapter.

(3.1.3)
$$\boldsymbol{E}_\alpha=1_Z\otimes E_\alpha,\quad E_\alpha=\begin{array}{c} \overset{\alpha}{\smile} \\ \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} \end{array} (\alpha,\qquad \alpha=1,\cdots,r.$$

Here 1_Z denotes the unit matrix of size $Z\times Z$. We notice that $\boldsymbol{E}_1,\cdots,\boldsymbol{E}_r$ and $\boldsymbol{\Lambda}$ commute each other, and that $\boldsymbol{E}_\alpha$ $(\alpha=1,\cdots,r)$ give partition of the unity

(3.1.4)
$$\sum_{\alpha=1}^r \boldsymbol{E}_\alpha=1_{Z\times r},\quad \boldsymbol{E}_\alpha\boldsymbol{E}_\beta=\delta_{\alpha\beta}\boldsymbol{E}_\beta,$$

where $1_{Z\times r}$ is the unit matrix in the whole space $\boldsymbol{C}^Z\otimes\boldsymbol{C}^r$.

Now let us introduce a discrete variable s, independent variables $x=(x^{(1)},\cdots,x^{(r)})$, $y=(y^{(1)},\cdots,y^{(r)})$ with $x^{(\alpha)}=(x_1^{(\alpha)}, x_2^{(\alpha)},\cdots)$, $y^{(\alpha)}=(y_1^{(\alpha)}, y_2^{(\alpha)},\cdots)$ and the matrices L, M, U_α, V_α of infinite size of the form

(3.1.5)
$$\begin{cases} L=\sum_{-\infty<j\leq 1}\operatorname{diag}[b_j(s)]\boldsymbol{\Lambda}^j, & b_1(s)=1_r, \\ M=\sum_{-1\leq j<\infty}\operatorname{diag}[c_j(s)]\boldsymbol{\Lambda}^j, & c_{-1}(s)=\hat{w}_0^{(0)}(s)\hat{w}_0^{(0)}(s-1)^{-1}, \end{cases}$$

$$\begin{cases} U_\alpha = \sum_{-\infty<j\le 0} \operatorname{diag}[u_{j,\alpha}(s)]\Lambda^j, & u_{0,\alpha}(s)=E_\alpha, \\ V_\alpha = \sum_{0\le j<\infty} \operatorname{diag}[v_{j,\alpha}(s)]\Lambda^j, & v_{0,\alpha}(s)=\hat{w}_0^{(0)}(s)E_\alpha\hat{w}_0^{(0)}(s)^{-1}, \end{cases}$$

where $b_j(s)$, $c_j(s)$, $u_{j,\alpha}(s)$, $v_{j,\alpha}(s)$ and $\hat{w}_0^{(0)}(s)$ are matrix-valued functions of (s, x, y) of $r\times r$ size, $b_j(s)=b_j(s; x, y)$, $\cdots$, $\hat{w}_0^{(0)}(s)=\hat{w}_0^{(0)}(s; x, y)$, and serve as unknown functions. $\hat{w}_0^{(0)}$ is assumed to be invertible. Furthermore we assume the following algebraic conditions.

$$(3.1.6)\qquad \begin{cases} [L, U_\alpha]=0, \quad [U_\alpha, U_\beta]=0, \\ \sum_{\alpha=1}^{r} U_\alpha = 1_{Z\times r}, \quad U_\alpha U_\beta=\delta_{\alpha\beta}U_\beta, \\ [M, V_\alpha]=0, \quad [V_\alpha, V_\beta]=0, \\ \sum_{\alpha=1}^{r} V_\alpha = 1_{Z\times r}, \quad V_\alpha V_\beta=\delta_{\alpha\beta}V_\beta, \quad \alpha, \beta=1, \cdots, r. \end{cases}$$

We set

$$(3.1.7)\qquad B_n^{(\alpha)}=(L^n U_\alpha)_+, \quad C_n^{(\alpha)}=(M^n V_\alpha)_-.$$

Then the hierarchy of the r-component Toda lattice is defined by the following system of the Lax-type equations.

$$(3.1.8)\qquad \begin{cases} \partial_{x_n^{(\alpha)}}L=[B_n^{(\alpha)}, L], \quad \partial_{x_n^{(\alpha)}}U_\beta=[B_n^{(\alpha)}, U_\beta], \\ \partial_{y_n^{(\alpha)}}L=[C_n^{(\alpha)}, L], \quad \partial_{y_n^{(\alpha)}}U_\beta=[C_n^{(\alpha)}, U_\beta], \\ \partial_{x_n^{(\alpha)}}M=[B_n^{(\alpha)}, M], \quad \partial_{x_n^{(\alpha)}}V_\beta=[B_n^{(\alpha)}, V_\beta], \\ \partial_{y_n^{(\alpha)}}M=[C_n^{(\alpha)}, M], \quad \partial_{y_n^{(\alpha)}}V_\beta=[C_n^{(\alpha)}, V_\beta], \\ \alpha, \beta=1, \cdots, r, \quad n=1, 2, \cdots. \end{cases}$$

Theorem 3.1. (3.1.8) *is equivalent to the system of the Zakharov-Shabat type equations*

$$(3.1.9)\qquad \begin{cases} \partial_{x_n^{(\beta)}}B_m^{(\alpha)}-\partial_{x_m^{(\alpha)}}B_n^{(\beta)}+[B_m^{(\alpha)}, B_n^{(\beta)}]=0, \\ \partial_{y_n^{(\beta)}}C_m^{(\alpha)}-\partial_{y_m^{(\alpha)}}C_n^{(\beta)}+[C_m^{(\alpha)}, C_n^{(\beta)}]=0, \\ \partial_{y_n^{(\beta)}}B_m^{(\alpha)}-\partial_{x_m^{(\alpha)}}C_n^{(\beta)}+[B_m^{(\alpha)}, C_n^{(\beta)}]=0, \\ \alpha, \beta=1, \cdots, r, \quad m, n=1, 2, \cdots. \end{cases}$$

Remark. It is obvious that in the case $r=1$ we recover the hierarchy discussed in Chapter 1

We can prove Theorem in the same way as in the one component case:

First we notice, in view of the algebraic conditions (3.1.6), that (3.1.8) is equivalent to

$$(3.1.8') \quad \begin{cases} \partial_{x_n^{(\alpha)}}(LU_\beta)=[B_n^{(\alpha)}, LU_\beta], & \partial_{y_n^{(\alpha)}}(LU_\beta)=[C_n^{(\alpha)}, LU_\beta], \\ \partial_{x_n^{(\alpha)}}(MV_\beta)=[B_n^{(\alpha)}, MV_\beta], & \partial_{y_n^{(\alpha)}}(MV_\beta)=[C_n^{(\alpha)}, MV_\beta], \\ \alpha, \beta=1, \cdots, r, \quad n=1, 2, \cdots. \end{cases}$$

On the other hand the equivalence of (3.1.8′) and (3.1.9) can be proved just in the same way as the proof of Theorem 1.1. We omit the detail.

Remark 3.2. The so called non abelian Toda lattice [18, 31] is recovered, together with its hierarchy, in the sector of independent variables

$$(3.1.10) \qquad x^{(1)}=\cdots=x^{(r)}\ (=x), \quad y^{(1)}=\cdots=y^{(r)}\ (=y),$$

as follows: Let us set

$$B_n=\sum_{\alpha=1}^{r} B_n^{(\alpha)}, \qquad C_n=\sum_{\alpha=1}^{r} C_n^{(\alpha)}$$

and consider the restriction of L, M, B_n and C_n to the sector (3.1.10). Then they satisfy, with respect to x and y, the systems of the Lax type and the Zakharov-Shabat type which are of the same form as (1.2.2) and (1.2.3). They give a hierarchy of the non abelian Toda lattice.

3.2. Linearization and characterization of wave matrices

Now we shall investigate the linearization of the r component hierarchy: Let us consider the following linear problem.

$$(3.2.1) \qquad \begin{cases} LW^{(\infty)}=W^{(\infty)}\Lambda, & MW^{(0)}=W^{(0)}\Lambda^{-1}, \\ U_\alpha W^{(\infty)}=W^{(\infty)}E_\alpha, & V_\alpha W^{(0)}=W^{(0)}E_\alpha, \end{cases}$$

$$(3.2.2) \qquad \partial_{x_n^{(\alpha)}}W=B_n^{(\alpha)}W, \quad \partial_{y_n^{(\alpha)}}W=C_n^{(\alpha)}W$$

for $W=W^{(\infty)}(x, y)$, $W^{(0)}(x, y)$, $\alpha=1, \cdots, r$ and $n=1, 2, \cdots$.

The following theorem implies that (3.2.1) and (3.2.2) serve as a suitable linearization of the r component hierarchy.

Theorem 3.3. (i) *Suppose that L, M, U and V are solutions to the r component hierarchy. Then there exist two solutions $W^{(\infty)}(x, y)$ and $W^{(0)}(x, y)$ to* (3.2.1) *and* (3.2.2) *of the form*

$$
(3.2.3) \qquad \begin{cases} W^{(\infty)}(x, y) = \hat{W}^{(\infty)}(x, y) \exp \sum_{\alpha=1}^{r} \xi(x^{(\alpha)}, \Lambda) E_\alpha, \\ W^{(0)}(x, y) = \hat{W}^{(0)}(x, y) \exp \sum_{\alpha=1}^{r} \xi(y^{(\alpha)}, \Lambda^{-1}) E_\alpha, \\ \hat{W}^{\binom{0}{\infty}}(x, y) = \sum_{j=0}^{\infty} \operatorname{diag} [\hat{w}_j^{\binom{0}{\infty}}(s; x, y)] \Lambda^{\pm j} \\ \text{with } \hat{w}_0^{(\infty)}(s; x, y) = 1_r. \end{cases}
$$

$W^{(\infty)}$ and $W^{(0)}$ are unique up to the arbitrariness

$$
W^{(\infty)} \longrightarrow W^{(\infty)} F, \qquad W^{(0)} \longrightarrow W^{(0)} G
$$

where $F = \sum_{j=0}^{\infty} \Lambda^{-j} \otimes f_j$, $G = \sum_{j=0}^{\infty} \Lambda^j \otimes g_j$, f_j and g_j are constant matrices of size $r \times r$, $f_0 = 1$ and g_0 is invertible.

(ii) *Conversely if there are two solutions $W^{(\infty)}$ and $W^{(0)}$ to* (3.2.2) *of the form* (3.2.3) *for certain matrices $B_n^{(\alpha)}$ and $C_n^{(\alpha)}$, then the matrices L, M, U_α and V_α defined by* (3.1.11), *or equivalently by*

$$
(3.2.4) \qquad \begin{cases} L = \hat{W}^{(\infty)} \Lambda \hat{W}^{(\infty)-1}, & M = \hat{W}^{(0)} \Lambda^{-1} \hat{W}^{(0)-1}, \\ U_\alpha = \hat{W}^{(\infty)} E_\alpha \hat{W}^{(\infty)-1}, & V_\alpha = \hat{W}^{(0)} E_\alpha \hat{W}^{(0)-1}, \end{cases}
$$

solve (3.1.8), *and also satisfy* (3.1.7).

Remark. Of course in the case $r=1$ we obtain the corresponding results for the hierarchy discussed in Chapter 1.

Proof. Let us prove (ii). (3.2.2) is equivalent to

$$
(3.2.5) \qquad \begin{cases} \partial_{x_n^{(\alpha)}} \hat{W}^{(\infty)} = B_n^{(\alpha)} \hat{W}^{(\infty)} - \hat{W}^{(\infty)} \Lambda^n E_\alpha, \\ \partial_{y_n^{(\alpha)}} \hat{W}^{(\infty)} = C_n^{(\alpha)} \hat{W}^{(\infty)}, \\ \partial_{x_n^{(\alpha)}} \hat{W}^{(0)} = B_n^{(\alpha)} \hat{W}^{(0)}, \\ \partial_{y_n^{(\alpha)}} \hat{W}^{(0)} = C_n^{(\alpha)} \hat{W}^{(0)} - \hat{W}^{(0)} \Lambda^{-n} E_\alpha. \end{cases}
$$

By a direct calculation (3.1.8) follows immediately from (3.2.5). On the other hand (3.2.5) leads to the following two expressions of $B_n^{(\alpha)}$.

$$
B_n^{(\alpha)} = \partial_{x_n^{(\alpha)}} \hat{W}^{(0)} \cdot \hat{W}^{(0)-1}, \quad B_n^{(\alpha)} = \partial_{x_n^{(\alpha)}} \hat{W}^{(\infty)} \cdot W^{(\infty)-1} + \hat{W}^{(\infty)} \Lambda^n E_\alpha \hat{W}^{(\infty)-1}.
$$

The first one implies that $B_n^{(\alpha)}$ is upper triangular relative to $r \times r$ blocks. Hence, if we take the $(\quad)_+$ part of the second one and remember that the diagonal blocks of $\hat{W}^{(\infty)}$ are constant, we have

$$
B^{(\alpha)} = (\hat{W}^{(\infty)} \Lambda^n E_\alpha \hat{W}^{(\infty)-1})_+ = (L^n U_\alpha)_+.
$$

Similarly we can prove the second equality in (3.1.7).

Next, let us prove (i). (3.2.2) is rewritten in the form

$$(3.2.6)\qquad \begin{cases}\partial_{x_n^{(\alpha)}}\hat{W}^{(\infty)}+(L^nU_\alpha)_-\hat{W}^{(\infty)}=0,\\ \partial_{y_n^{(\alpha)}}\hat{W}^{(\infty)}-(M^nV_\alpha)_-\hat{W}^{(\infty)}=0,\\ \partial_{x_n^{(\alpha)}}\hat{W}^{(0)}-(L^nU_\alpha)_+\hat{W}^{(0)}=0,\\ \partial_{y_n^{(\alpha)}}\hat{W}^{(0)}+(M^nV_\alpha)_+\hat{W}^{(0)}=0, \quad n=1,2,\cdots.\end{cases}$$

The integrability conditions of (3.2.6) are guaranteed by (3.1.8) and (3.1.9), as one can easily show in the same way as in the one component case (cf. the proof of Lemma 1.3). Hence the remaining problem is the choice of initial values (cf. the proof of Theorem 1.2). Thus we have only to prove that there exist some matrices $\hat{W}_0^{(\infty)}$ and $\hat{W}_0^{(0)}$ of the same form as $\hat{W}^{(\infty)}$ and $\hat{W}^{(0)}$ such that

$$L=\hat{W}_0^{(\infty)}\Lambda\hat{W}_0^{(\infty)-1},\quad M=\hat{W}_0^{(0)}\Lambda^{-1}\hat{W}_0^{(0)-1},$$
$$U_\alpha=\hat{W}_0^{(\infty)}E_\alpha\hat{W}_0^{(\infty)-1},\quad V_\alpha=\hat{W}_0^{(0)}E_\alpha\hat{W}_0^{(0)-1}.$$

In the following we shall show only the existence of $\hat{W}_0^{(0)}$. $\hat{W}_0^{(\infty)}$ can be treated just in the same way.

At first let us choose a matrix $W^{(1)}=\sum_{j=0}^\infty \operatorname{diag}[w_j^{(1)}(s;x,y)]\Lambda^{-j}$ such that

$$M=W^{(1)}\Lambda^{-1}W^{(1)-1},\quad w_0^{(1)}(s;x,y)=\hat{w}_0^{(0)}(s;x,y).$$

We can actually construct such a $W^{(1)}$, solving the linear equations for $w_j^{(1)}$ which are derived by comparing the both sides of $MW^{(1)}=W^{(1)}\Lambda^{-1}$. Now we set $V_\alpha^{(1)}=W^{(1)-1}V_\alpha W^{(1)}$ $(\alpha=1,\cdots,r)$. Then, from (3.1.6),

$$(3.2.7)\qquad \begin{cases}[V_\alpha^{(1)},\Lambda]=0,\quad [V_\alpha^{(1)},V_\beta^{(1)}]=0,\\ \sum_{\alpha=1}^r V_\alpha^{(1)}=1_{Z\times r},\quad V_\alpha^{(1)}V_\beta^{(1)}=\delta_{\alpha\beta}V_\beta^{(1)},\quad \alpha,\beta=1,\cdots,r.\end{cases}$$

From the first equality $V_\alpha^{(1)}$ takes the form

$$\begin{cases}V_\alpha^{(1)}=\sum_{j=0}^\infty \Lambda^j\otimes v_{\alpha,j}^{(1)},\\ v_{\alpha,0}^{(1)}=E_\alpha,\quad \alpha=1,\cdots,r,\end{cases}$$

where $v_{\alpha,j}^{(1)}$ is a matrix valued function of size $r\times r$.

Now we claim that, for the matrices $V_\alpha^{(1)}$ as above, there exist a matrix $W^{(2)}=\sum_{j=0}^\infty \Lambda^j\otimes w_j^{(2)}$ with $w_0^{(2)}=1_r$ and $w_j^{(2)}$ being a matrix-valued function of size $r\times r$ such that

(3.2.8) $$V_\alpha^{(1)} = W^{(2)} E_\alpha W^{(2)-1} \quad \text{for } \alpha = 1, \cdots, r.$$

If such a $W^{(2)}$ exists, then we have only to set $\hat{W}_0^{(0)} = W^{(1)} W^{(2)}$.

Let us construct such a $W^{(2)}$ by induction on r. The case $r=1$ is trivial. Suppose $r>1$: As we constructed $W^{(1)}$ for M, we can choose a matrix $W^{(3)} = \sum_{j=0}^{\infty} \Lambda^j \otimes w_j^{(3)}$ with $w_j^{(3)}$ being a matrix of size $r \times r$ such that

$$V_r^{(1)} = W^{(3)} E_r W^{(3)-1}, \quad w_0^{(3)} = 1_r.$$

If we set $V_\alpha^{(2)} = W^{(3)-1} V_\alpha^{(1)} W^{(3)}$ $(\alpha = 1, \cdots, r)$, then $V_\alpha^{(2)}$ $(\alpha = 1, \cdots, r)$ satisfy (3.2.7) in place of $V_\alpha^{(1)}$. In particular,

$$V_\alpha^{(2)} E_r = V_\alpha^{(2)} V_r^{(2)} = 0, \quad E_r V_\alpha^{(2)} = V_r^{(2)} V_\alpha^{(2)} = 0 \quad \text{for } \alpha = 1, \cdots, r-1.$$

Hence the r-th row and the r-th column of $V_\alpha^{(2)}$ vanish. Extracting the remaining $(r-1) \times (r-1)$ part of $V_\alpha^{(2)}$ for $\alpha = 1, \cdots, r-1$, we can reduce the problem to the case of size $(r-1) \times (r-1)$ instead of size $r \times r$.

Thus we have proved the existence of $W^{(2)}$, and hence that of $\hat{W}_0^{(0)}$.

The last statement of (ii) can be easily verified. This proves Theorem 3.3.

In the following, as in Section 1.2, we shall call $W^{(\infty)}$ and $W^{(0)}$ the wave matrices of the r component Toda lattice hierarchy.

A similar argument as we developed in Section 1.2 leads to a set of bilinear equations which characterize the wave matrices $W^{(\infty)}$ and $W^{(0)}$:

Theorem 3.4. *The wave matrices $W^{(\infty)}$ and $W^{(0)}$ of the r component hierarchy satisfy the bilinear equation*

(3.2.9) $$\partial_x^\alpha \partial_y^\beta W^{(\infty)}(x, y) \cdot W^{(\infty)}(x, y)^{-1} = \partial_x^\alpha \partial_y^\beta W^{(0)}(x, y) \cdot W^{(0)}(x, y)^{-1}$$

for any multi-indices α and β. Conversely if some matrices $W^{(\infty)}$ and $W^{(0)}$ of the form (3.2.3) *satisfy* (3.2.9) *for any α and β, then they are wave matrices of the r component hierarchy, i.e., they solve* (3.2.2).

Of course the bilinear equations (3.2.9) for all α and β can be rewritten into the generating functional form,

(3.2.10) $$W^{(\infty)}(x, y) W^{(\infty)}(x', y')^{-1} = W^{(0)}(x, y) W^{(0)}(x', y')^{-1} \quad \text{for any } x \text{ and } x'.$$

Now we proceed to investigate the relation between our theory of the multi-component Toda lattice and the multi-component KP hierarchy.

We introduce the following matrices of formal Laurent series in λ of size $r\times r$, which we call the wave functions of the r component Toda lattice.

$$
(3.2.11)\qquad
\begin{cases}
w^{(\infty)}(s; x, y; \lambda)=\hat{w}^{(\infty)}(s; x, y; \lambda)\lambda^{s}\exp\xi(x, \lambda),\\
w^{(0)}(s; x, y; \lambda)=\hat{w}^{(0)}(s; x, y; \lambda)\lambda^{s}\exp\xi(y, \lambda^{-1}),\\
w^{*(\infty)}(s; x, y; \lambda)=\hat{w}^{*(\infty)}(s; x, y; \lambda)\lambda^{-s}\exp\xi(-x, \lambda),\\
w^{*(0)}(s; x, y; \lambda)=\hat{w}^{*(0)}(s; x, y; \lambda)\lambda^{-s}\exp\xi(-y, \lambda^{-1}),\\
\hat{w}^{\binom{0}{\infty}}(s; x, y; \lambda)=\sum_{j=0}^{\infty} w_j^{\binom{0}{\infty}}(s; x, y)^{\pm j},\\
\hat{w}^{*\binom{0}{\infty}}(s; x, y; \lambda)=\sum_{j=0}^{\infty} \hat{w}_j^{*\binom{0}{\infty}}(s; x, y)\lambda^{\pm j},
\end{cases}
$$

where $\hat{w}_j^{\binom{0}{\infty}}$ was defined in (3.2.3) and $\hat{w}_j^{*\binom{0}{\infty}}$ is defined by

$$
(3.2.12)\qquad \hat{W}^{\binom{0}{\infty}}(s; x, y)^{-1}=\sum_{j=0}^{\infty} \Lambda^{\pm j}\operatorname{diag}[\hat{w}_j^{*\binom{0}{\infty}}(s+1; x, y)].
$$

Then the bilinear equations (3.2.9) and (3.2.10) are rewritten respectively in the following integral forms.

$$
(3.2.13)\qquad
\begin{aligned}
&\oint (\partial_x^{\alpha}\partial_y^{\beta} w^{(\infty)}(s; x, y; \lambda))w^{*(\infty)}(s', x, y; \lambda)d\lambda\\
&\qquad =\oint (\partial_x^{\alpha}\partial_y^{\beta} w^{(0)}(s; x, y; \lambda^{-1}))w^{*(0)}(s'; x, y; \lambda^{-1})\lambda^{-2}d\lambda,
\end{aligned}
$$

$$
(3.2.14)\qquad
\begin{aligned}
&\oint w^{(\infty)}(s; x, y; \lambda)w^{*(\infty)}(s'; x', y'; \lambda)d\lambda\\
&\qquad =\oint w^{(0)}(s; x, y; \lambda^{-1})w^{*(0)}(s'; x', y'; \lambda^{-1})\lambda^{-2}d\lambda.
\end{aligned}
$$

They are analogous to the bilinear equations for the wave functions of the multi-component *KP* theory [22] (cf. Appendix). A direct comparison with them yields

Theorem 3.5. *Let us denote by*

$$
W_l(x^{(1)}, \cdots, x^{(2r)}; \lambda) \quad \textit{and} \quad W_l^*(x^{(1)}, \cdots, x^{(2r)}; \lambda)
$$

with $l=(l_1, \cdots, l_{2r})$, $\sum_{j=1}^{2r} l_j=0$, *the wave functions for the* $2r$ *component KP hierarchy introduced in* [22] (*cf. Appendix*, (*A*. 44)), *and define* $w_l^{(\infty)}$, $w_l^{(0)}$, $w_l^{*(\infty)}$, $w_l^{*(0)}$ *by*

$$(3.2.15)\quad \begin{cases} W_{l(s)}(x^{(1)}, \cdots, x^{(r)}, y^{(1)}, \cdots, y^{(r)}; \lambda)_{\alpha,\beta} \\ \qquad = \begin{cases} w_l^{(\infty)}(s; x, y; \lambda)_{\alpha,\beta}\lambda^{l_\beta} & \text{for } 1 \leq \beta \leq r, \\ w_l^{(0)}(s; x, y; \lambda^{-1})_{\alpha,\beta-r}\lambda^{l_\beta - 1} & \text{for } r+1 \leq \beta \leq 2r, \end{cases} \\ W^*_{l(s)}(x^{(1)}, \cdots, x^{(r)}, y^{(1)}, \cdots, y^{(r)}; \lambda)_{\alpha,\beta} \\ \qquad = \begin{cases} w_l^{*(\infty)}(s; x, y; \lambda)_{\alpha,\beta}\lambda^{-l_\beta} & \text{for } 1 \leq \beta \leq r, \\ w_l^{*(0)}(s; x, y; \lambda^{-1})_{\alpha,\beta-r}\lambda^{-l_\beta - 1} & \text{for } r+1 \leq \beta \leq 2r, \end{cases} \end{cases}$$

where $l(s) = l + (s, \cdots, s, -s, \cdots, -s)$ (s *for the first* r *components, and* $-s$ *for the second ones), and the subindices* (α, β) *and* $(\alpha, \beta - r)$ *indicate the matrix-components. Then, for each* l, $w_l^{(\infty)}$, $w_l^{(0)}$, $w_l^{*(\infty)}$ *and* $w_l^{*(0)}$ *satisfy* (3.2.13), (3.2.14). *Hence they are wave functions of the* r *component Toda lattice.*

Remark 3.6. Theorem 3.5 provides a class of solutions to the r component Toda lattice hierarchy parametrized by the vacuum expectation values τ_l, $l = (l_1, \cdots, l_{2r})$, $\sum_{\alpha=1}^{2r} l_\alpha = 0$ [22] (see Appendix 1). However, to make the statements of Theorem 3.5 more precise, we must add the following remark: It may happen that the $w_l^{(\infty)}$ and $w_l^{(0)}$ do not make sense for some $s \in \boldsymbol{Z}$. As far as we consider the infinite lattice they must be excluded as the wave functions of the Toda lattice hierarchy. For this reason the rational solutions and a class of soliton solutions to the multi-component *KP* hierarchy do not induce solutions to the infinite Toda lattice hierarchy.

At the end of this section we shall briefly comment on the linear equations for the wave functions $w^{(\infty)}$ and $w^{(0)}$:

Let us express $B_n^{(\alpha)}$ and $C_n^{(\alpha)}$ in the form

$$(3.2.16)\quad \begin{cases} B_n^{(\alpha)} = \sum_{j=0}^{n} \operatorname{diag}[b_{n,j}^{(\alpha)}(s, x, y)]\Lambda^{n-j}, \\ C_n^{(\alpha)} = \sum_{j=0}^{n-1} \operatorname{diag}[c_{n,j}^{(\alpha)}(s, x, y)]\Lambda^{j-n}. \end{cases}$$

Then (3.2.2) is equivalent to the following classical formulation of linearization

$$(3.2.17)\quad \begin{cases} \partial_{x_n^{(\alpha)}} w(s, x, y; \lambda) = \sum_{j=0}^{n} b_{n,j}^{(\alpha)}(s, x, y) w(s+n-j, x, y; \lambda), \\ \partial_{y_n^{(\alpha)}} w(s, x, y; \lambda) = \sum_{j=0}^{n-1} c_{n,j}^{(\alpha)}(s, x, y) w(s+j-n, x, y; \lambda), \\ \text{for} \quad w = w^{(\infty)}, w^{(0)} \quad \text{and} \quad n = 1, 2, \cdots. \end{cases}$$

Hence Theorem 3.3, (ii), implies that, if there are two solutions $w^{(\infty)}$ and $w^{(0)}$ of the form as indicated in (3.2.11), then the matrices $B_n^{(\alpha)}$ and $C_n^{(\alpha)}$ defined by (3.2.16) solve (3.1.9), while the matrices L, M, U_α and V_α defined by (3.2.4) solve (3.1.8). Of course in the case $r=1$ we obtain the corresponding result for the hierarchy discussed in Chapter 1.

In the construction of special solutions (e.g., soliton solutions, quasi-periodic solutions, etc.) the linearization (3.2.17) is often effectively used.

3.3. Reduction to a system of the Zakharov-Mikhailov type

Zakharov-Mikhailov [41] investigated the zero-curvature equation

$$\partial_\xi A(\xi, \eta; \lambda) - \partial_\eta B(\xi, \eta; \lambda) + [A(\xi, \eta; \lambda), B(\xi, \eta; \lambda)] = 0$$

and its linearization

$$\partial_\xi \Phi = A(\xi, \eta; \lambda)\Phi, \quad \partial_\eta \Phi = B(\xi, \eta; \lambda)\Phi,$$

in which A and B depend on λ rationally. In Chapters 1 and 2 we encountered some examples of the systems of this type in the periodic reductions. Among them the sine-Gordon equation is one of the most typical examples, and can be obtained, together with its hierarchy, as the 2-periodic reduction as in Section 1.4.

In this section we shall derive another type of examples in a reduction of the multi-component hierarchy. One of the typical examples is the Pohlmeyer-Lund-Regge equation.

Throughout this section we assume the reduction conditions

$$[W^{(\infty)}, \Lambda] = 0, \quad [W^{(0)}, \Lambda] = 0. \tag{3.3.1}$$

Proposition 3.7. *Each of the following conditions* (i), (ii) *and* (iii) *is equivalent to* (3.3.1).

(i) $L = \Lambda, \quad M = \Lambda^{-1}$.

(ii) $w(s; x, y; \lambda) = \lambda^s w(0; x, y; \lambda)$ *for* $w = w^{(\infty)}, w^{(0)}$.

(iii) $\sum_{\alpha=1}^{r} \partial_{x_n^{(\alpha)}} W = W\Lambda^n, \quad \sum_{\alpha=1}^{r} \partial_{y_n^{(\alpha)}} W = W\Lambda^{-n}$

for $W = W^{(\infty)}, W^{(0)}$ *and* $n = 1, 2, \cdots$.

This is an immediate consequence of (3.1.7), (3.2.1) and (3.2.2).

In the expression (3.1.1) of a matrix A, we notice that

$$[A, \Lambda] = 0 \Leftrightarrow a_j(s) \text{ is independent of } s \text{ for any } j.$$

In this case A is expressed in the form $A = \sum_{j \in Z} \Lambda^j \otimes a_j$, where a_j is a constant matrix of size $r\times r$. Also we notice that the correspondence

$$A=\sum_{j\in Z}\Lambda^j\otimes a_j \leftrightarrow A(\lambda)=\sum_{j\in Z}a_j\lambda^j \tag{3.3.2}$$

preserves sums, products and commutators. Here λ is used as a formal indeterminate.

Under (3.3.1) the matrices $\hat{W}^{(\infty)}$, $\hat{W}^{(0)}$, U_α, V_α, $B_n^{(\alpha)}$ and $C_n^{(\alpha)}$ commute with Λ, as one can show easily from (3.1.7), (3.2.4). Hence let us denote the matrices of size $r\times r$ corresponding to them through (3.3.2) by $\hat{W}^{(\infty)}(\lambda)$, $\hat{W}^{(0)}(\lambda)$, $U_\alpha(\lambda)$, $V_\alpha(\lambda)$, $B_n^{(\alpha)}(\lambda)$ and $C_n^{(\alpha)}(\lambda)$ (or, more precisely, by $\hat{W}^{(\infty)}(x, y; \lambda)$ etc., $\cdots$, if we indicate the (x, y) dependence explicitly.) Also denote $w^{(\infty)}(0; x, y; \lambda)$ and $w^{(0)}(0; x, y; \lambda)$ by $\Phi^{(\infty)}(x, y; \lambda)$ and $\Phi^{(0)}(x, y; \lambda)$. In other words,

$$\begin{cases}\Phi^{(\infty)}(x, y; \lambda)=\hat{w}^{(\infty)}(0; x, y; \lambda)\operatorname{diag}(e^{\xi(x^{(1)},\lambda)}, \cdots, e^{(x^{(r)},\lambda)}),\\ \Phi^{(0)}(x, y; \lambda)=\hat{w}^{(0)}(0; x, y; \lambda)\operatorname{diag}(e^{\xi(y^{(1)},\lambda^{-1})}, \cdots, e^{\xi(y^{(r)},\lambda^{-1})}).\end{cases} \tag{3.3.3}$$

Then we obtain a system of the Zakharov-Mikhailov type together with the Lax representation, the zero-curvature representation and the linearization as follows.

Theorem 3.8. (i) (3.1.8) *and* (3.1.4) *reduce to the following equations*

$$\begin{cases}\partial_{x_n^{(\alpha)}}U_\beta(\lambda)=[B_n^{(\alpha)}(\lambda), U_\beta(\lambda)], & \partial_{y_n^{(\alpha)}}U_\beta(\lambda)=[C_n^{(\alpha)}(\lambda), U_\beta(\lambda)],\\ \partial_{x_n^{(\alpha)}}V_\beta(\lambda)=[B_n^{(\alpha)}(\lambda), V_\beta(\lambda)], & \partial_{y_n^{(\alpha)}}V_\beta(\lambda)=[C_n^{(\alpha)}(\lambda), V_\beta(\lambda)],\end{cases} \tag{3.3.4}$$

$$\begin{cases}\partial_{x_n^{(\beta)}}B_m^{(\alpha)}(\lambda)-\partial_{x_m^{(\alpha)}}B_n^{(\beta)}(\lambda)+[B_m^{(\alpha)}(\lambda), B_n^{(\beta)}(\lambda)]=0,\\ \partial_{y_n^{(\beta)}}C_m^{(\alpha)}(\lambda)-\partial_{y_m^{(\alpha)}}C_n^{(\beta)}(\lambda)+[C_m^{(\alpha)}(\lambda), C_n^{(\beta)}(\lambda)]=0,\\ \partial_{y_n^{(\beta)}}B_m^{(\alpha)}(\lambda)-\partial_{x_m^{(\alpha)}}C_n^{(\beta)}(\lambda)+[B_m^{(\alpha)}(\lambda), C_n^{(\beta)}(\lambda)]=0,\end{cases} \tag{3.3.5}$$

for $\alpha, \beta=1, \cdots, r$ *and* $m, n=1, 2, \cdots$.

(3.3.4) *serves as the Lax representation, while* (3.3.5) *as the zero curvature representation.*

Furthermore if we expand $U_\alpha(\lambda)$ *and* $V_\alpha(\lambda)$ *in the form*

$$\begin{cases}U_\alpha(\lambda)=\sum_{j=0}^{\infty}b_{\alpha,j}\lambda^{-j} \quad \text{with } b_{\alpha,0}=E_\alpha,\\ V_\alpha(\lambda)=\sum_{j=0}^{\infty}c_{\alpha,j}\lambda^{j} \quad \text{with } c_{\alpha,0}=\hat{w}_0^{(0)}E_\alpha\hat{w}_0^{(0)-1},\end{cases} \tag{3.3.6}$$

then we have

(3.3.7) $$B_n^{(\alpha)}(\lambda)=\sum_{j=0}^{n} b_{\alpha,j}\lambda^{n-j}, \quad C_n^{(\alpha)}(\lambda)=\sum_{j=0}^{n-1} c_{\alpha,j}\lambda^{j-n},$$

(3.3.8) $$U_\alpha(\lambda)=\hat{W}^{(\infty)}(\lambda)E_\alpha\hat{W}^{(\infty)}(\lambda)^{-1}, \quad V_\alpha(\lambda)=\hat{W}^{(0)}(\lambda)E_\alpha\hat{W}^{(0)}(\lambda)^{-1},$$

(3.3.9) $$\begin{cases} U_\alpha(\lambda)U_\beta(\lambda)=\delta_{\alpha\beta}U_\beta(\lambda), & \sum_{\alpha=1}^{r} U_\alpha(\lambda)=1_r, \\ V_\alpha(\lambda)V_\beta(\lambda)=\delta_{\alpha\beta}V_\beta(\lambda), & \sum_{\alpha=1}^{r} V_\alpha(\lambda)=1_r, \quad \alpha, \beta=1, \cdots, r. \end{cases}$$

(ii) (3.2.2) *reduces to the linear system*

(3.3.10) $$\partial_{x_n^{(\alpha)}}\Phi=B_n^{(\alpha)}(\lambda)\Phi, \quad \partial_{y_n^{(\alpha)}}\Phi=C_n^{(\alpha)}(\lambda)\Phi \quad \textit{for } \Phi=\Phi^{(\infty)}, \Phi^{(0)},$$

which serves as a linearization of (3.3.4) *and* (3.3.5).

(iii) $\sum_{\alpha=1}^{r} \partial_{x_n^{(\alpha)}} U_\beta(\lambda)=0, \quad \sum_{\alpha=1}^{r} \partial_{y_n^{(\alpha)}} U_\beta(\lambda)=0$

for $\alpha, \beta=1, \cdots, r$, $n=1, 2, \cdots$. *Also the same equalities hold for* $V_\beta(\lambda)$, $B_m^{(\beta)}(\lambda)$, $C_m^{(\beta)}(\lambda)$.

This theorem is an immediate consequence of the contents of Section 3.1 and Section 3.2, Proposition 3.7 and the fact that (3.3.2) preserves sums, product and commutators. Hence we omit the proof.

Remark 3.9. We can develope, for the system of the Zakharov-Mikhailov type indicated above, similar arguments as we did in Section 3.1 and Section 3.2 for the r component Toda lattice hierarchy. For example; (3.3.4) and (3.3.5) are equivalent to each other under (3.3.6), (3.3.7) and (3.3.9); if (3.3.4) and (3.3.5) are satisfied, we can construct the solutions $\Phi^{(\infty)}$ and $\Phi^{(0)}$ to (3.3.10); etc. We omit the detail.

The system obtained in Theorem 3.8 can be regarded as a generalization of the so called *AKNS* systems [1]. In the rest of this section we shall investigate its structure a little bit further.

At first let us consider the case $r=2$. This is nothing but the *AKNS* case:

As we know (cf. Theorem 3.8, (iii)), the matrices $U_\alpha(\lambda)$, $V_\alpha(\lambda)$, $B_n^{(\alpha)}(\lambda)$ and $C_n^{(\alpha)}(\lambda)$ depend only on the differences $x^{(1)}-x^{(2)}$ and $y^{(1)}-y^{(2)}$. Hence we restrict the independent variables to the sector

(3.3.11) $$x^{(1)}=-x^{(2)}(=x), \quad y^{(1)}=-y^{(2)}(=y),$$

and set

(3.3.12) $$\begin{cases} B_n(\lambda)=B_n^{(1)}(\lambda)-B_n^{(2)}(\lambda), & C_n(\lambda)=C_n^{(1)}(\lambda)-C_n^{(2)}(\lambda), \\ U(\lambda)=U_1(\lambda)-U_2(\lambda), & V(\lambda)=V_1(\lambda)-V_2(\lambda). \end{cases}$$

Then (3.3.4) and (3.3.5) reduce to

$$\text{(3.3.13)}\qquad \begin{cases} \partial_{x_n}U(\lambda)=[B_n(\lambda),\, U(\lambda)], & \partial_{y_n}U(\lambda)=[C_n(\lambda),\, U(\lambda)],\\ \partial_{x_n}V(\lambda)=[B_n(\lambda),\, V(\lambda)], & \partial_{y_n}V(\lambda)=[C_n(\lambda),\, V(\lambda)],\end{cases}$$

$$\text{(3.3.14)}\qquad \begin{cases} \partial_{x_n}B_m(\lambda)-\partial_{x_m}B_n(\lambda)+[B_m(\lambda),\, B_n(\lambda)]=0,\\ \partial_{y_n}C_m(\lambda)-\partial_{y_m}C_n(\lambda)+[C_m(\lambda),\, C_n(\lambda)]=0,\\ \partial_{y_n}B_m(\lambda)-\partial_{x_m}C_n(\lambda)+[B_m(\lambda),\, C_n(\lambda)]=0,\end{cases}$$

for $m, n=1, 2, \cdots$. Furthermore if we express $U(\lambda)$ and $V(\lambda)$ in the form

$$U(\lambda)=\sum_{j=0}^{\infty} b_j\lambda^{-j}, \quad V(\lambda)=\sum_{j=0}^{\infty} c_j\lambda^{j},$$

then

$$\text{(3.3.15)}\qquad \begin{cases} b_0=J=\begin{bmatrix}1 & \\ & -1\end{bmatrix}, \quad c_0=\hat{w}_0^{(0)}J\hat{w}_0^{(0)-1},\\ B_n(\lambda)=\sum_{j=0}^{n} b_j\lambda^{n-j}, \quad C_n(\lambda)=\sum_{j=0}^{n-1} c_j\lambda^{j-n} \quad \text{for } n=1, 2, \cdots,\\ \text{trace } b_j=0, \ \text{trace } c_j=0 \ \text{for } j=0, 1, \cdots.\end{cases}$$

(3.3.14) is nothing but the *AKNS* hierarchy [1]. (3.3.13) serves as its Lax representation in terms of the formal power series $U(\lambda)$ and $V(\lambda)$ which are connected with $B_n(\lambda)$ and $C_n(\lambda)$ by (3.3.15). The last statements in (3.3.15) follows from

$$U(\lambda)=\hat{W}^{(\infty)}(\lambda)J\hat{W}^{(\infty)}(\lambda)^{-1}, \quad V(\lambda)=\hat{W}^{(0)}(\lambda)J\hat{W}^{(0)}(\lambda)^{-1}.$$

The following result, essentially stated in [1], is then recovered in our formulation. The proof given here is due to M. Sato:

Theorem 3.10. *For any j ($\geqq 1$) b_j are differential polynomials of b_1 with respect to x_1, and c_j differential polynomials of c_0 with respect to y_1. In particular* (3.3.14) *are regarded as non linear differential equations for the unknown functions b_1 and c_0.*

Proof. We shall prove the statement only for b_j. c_j can be treated just in the same way.

At first, from $U(\lambda)^2=1_2$, we have

$$b_0b_j+b_jb_0+\sum_{k=1}^{j-1} b_kb_{j-k}=0 \quad \text{for } j>0.$$

On the other hand, from $\partial_{x_1}U(\lambda)-[B_1(\lambda), U(\lambda)]=0$ in (3.3.14),

$$\partial b_{j-1}/\partial x_1-[b_0, b_j]-[b_1, b_{j-1}]=0 \quad \text{for } j>0.$$

Hence we have

$$2b_0b_j-\partial b_{j-1}/\partial x_1+[b_1, b_{j-1}]+\sum_{k=1}^{j-1} b_kb_{j-k}=0 \quad \text{for } j>0. \tag{3.3.16}$$

Since b_0 $(=J)$ is invertible, b_j are recursively determined by (3.3.10) as differential polynomial of b_1 with respect to x_1. This proves Theorem 3.10.

Now let us consider the general case $(r\geq 2)$: (3.3.9) implies

$$\begin{cases} \sum_{k=0}^{j} b_{\alpha,k}b_{\beta,j-k}=\delta_{\alpha\beta}b_{\beta,j}, & \sum_{k=0}^{j} c_{\alpha,k}c_{\beta,j-k}=\delta_{\alpha\beta}c_{\beta,j}, \\ \sum_{\alpha=1}^{r} b_{\alpha,j}=\delta_{j,0}, & \sum_{\alpha=1}^{r} c_{\alpha,j}=\delta_{j,0}, \end{cases} \tag{3.3.17}$$

while the equations $\partial_{x_1^{(\beta)}}U_j^{(\alpha)}(\lambda)-[U_1^{(\beta)}(\lambda), U_j^{(\alpha)}(\lambda)]=0$ and $V_{y_1^{(\beta)}}[V_j^{(\alpha)}(\lambda)-[V_1^{(\beta)}(\lambda), V_j^{(\alpha)}(\lambda)]=0$ yield

$$\begin{cases} \partial b_{\alpha,j-1}/\partial x_1^{(\beta)}-[b_{\beta,0}, b_{\alpha,j}]-[b_{\beta,1}, b_{\alpha,j-1}]=0, \\ \partial c_{\alpha,j-1}/\partial y_1^{(\beta)}-[c_{\beta,0}, c_{\alpha,j}]=0. \end{cases} \tag{3.3.18}$$

Hence $b_{\alpha,j}$ and $c_{\alpha,j}$ $(j\geq 1)$ are recursively determined by (3.3.17) and (3.3.18), and the components of $b_{\alpha,j}$ and $c_{\alpha,j}$ are differential polynomials (with respect to $x_1^{(1)}, \cdots, x_1^{(r)}, y_1^{(1)}, \cdots, y_1^{(r)}$) of the components of $b_{\beta,1}$ and $c_{\beta,0}$, $\beta=1, \cdots, r$. This is a generalization of Theorem 3.10 to the general case $(r\geq 2)$.

Theorem 3.8, (iii), implies that the evolution is trivial in a direction. In the case $r=2$, we have extracted the essential evolution by introducing new independent and dependent variables as indicated in (3.3.11) and (3.3.12).

In the general case let us consider an example of the choise of new variables:

Let $t^{(\alpha)}=(t_1^{(\alpha)}, t_2^{(\alpha)}, \cdots)$ and $\bar{t}^{(\alpha)}=(\bar{t}_1^{(\alpha)}, \bar{t}_2^{(\alpha)}, \cdots)$, $\alpha=1, \cdots, r$, be the reduced independent variables, and take the sector of the independent variables

$$\begin{aligned} x^{(\alpha)}&=-t^{(\alpha-1)}+t^{(\alpha)} \ (\alpha=2, \cdots, r-1), \quad x^{(1)}=t^{(1)}, \ x^{(r)}=-t^{(r-1)}, \\ y^{(\alpha)}&=-\bar{t}^{(\alpha-1)}+\bar{t}^{(\alpha)} \ (\alpha=2, \cdots, r-1), \quad y^{(1)}=\bar{t}^{(1)}, \ y^{(r)}=-\bar{t}^{(r-1)}. \end{aligned} \tag{3.3.19}$$

Let us introduce the following dependent variables,

$$B_{\alpha,n}(\lambda)=B_n^{(\alpha)}(\lambda)-B_n^{(\alpha+1)}(\lambda),\quad C_{\alpha,n}(\lambda)=C_n^{(\alpha)}(\lambda)-C_n^{(\alpha+1)}(\lambda),\quad \alpha=1,\cdots,r. \tag{3.3.20}$$

Notice that $B_{\alpha,n}(\lambda)$ and $C_{\alpha,n}(\lambda)$ are trace free.

$$\operatorname{trace} B_{\alpha,n}(\lambda)=0,\quad \operatorname{trace} C_{\alpha,n}(\lambda)=0. \tag{3.3.21}$$

The zero-curvature representation (3.3.5) reduces to

$$\begin{cases}\partial_{t_n^{(\beta)}}B_{\alpha,m}(\lambda)-\partial_{t_m^{(\alpha)}}B_{\beta,n}(\lambda)+[B_{\alpha,m}(\lambda),B_{\beta,n}(\lambda)]=0,\\ \partial_{\bar{t}_n^{(\beta)}}C_{\alpha,m}(\lambda)-\partial_{\bar{t}_m^{(\alpha)}}C_{\beta,n}(\lambda)+[C_{\alpha,m}(\lambda),C_{\beta,n}(\lambda)]=0,\\ \partial_{\bar{t}_n^{(\beta)}}B_{\alpha,m}(\lambda)-\partial_{t_m^{(\alpha)}}C_{\beta,n}(\lambda)+[B_{\alpha,m}(\lambda),C_{\beta,n}(\lambda)]=0,\\ \alpha,\beta=1,\cdots,r-1,\quad m,n=1,2,\cdots.\end{cases} \tag{3.3.22}$$

4. Examples of Exact Solutions

4.1. Applications of an infinite dimensional analogue of the Riemann-Hilbert problem

As is well known, the Riemann-Hilbert problem plays an important role to analyse the two-dimensional (or subholonomic) soliton equations. By means of this problem, the exact solutions of many classes have been constructed, and the infinitesimal transformation groups acting on the solution spaces have been discovered [18, 38, 39, 41].

We will briefly explain how to apply the problem to the soliton equations. For example, let us consider the $SU(n)$ chiral field [38, 40],

$$\partial_\xi(g^{-1}\partial_\eta g)+\partial_\eta(g^{-1}\partial_\xi g)=0,$$

where $g=g(\xi,\eta)\in SU(n)$, and ξ, η are the light cone coordinates. Let

$$\Omega(\lambda)=\frac{\lambda A}{1-\lambda}d\xi-\frac{\lambda B}{1+\lambda}d\eta$$

be a one-form with $\mathfrak{su}(n)$-coefficients A, B. Then the equation is represented as the 0-curvature condition, $d\Omega=\Omega^2$. Hence the linear problem,

$$dY(\lambda)=\Omega(\lambda)Y(\lambda),$$

has a fundamental solution matrix $Y(\lambda)=Y(x,y;\lambda)$, such that

$$\det Y(\lambda)=1,\quad Y(\bar{\lambda})^\dagger Y(\lambda)=1,\quad Y(0)=1.$$

Here $\bar{\lambda}$ is the complex conjugate variable of λ, and $\dagger$ indicates the hermitian conjugate matrix. Let C be a circle with the centre at $\lambda=0$, and $C_+(C_-)$ be the inside (outside) of C. We assume that $Y(\lambda)$ is holomorphic in $C\cup C_+$. Let $u(\lambda)$ be an $n\times n$ matrix-valued function, which is independent of ξ, η, analytic on C, such that $u(\bar{\lambda})^{\dagger}u(\lambda)=1$, $\det u(\lambda)=1$. Then, setting

$$H(\lambda)=Y(\lambda)u(\lambda)Y(\lambda)^{-1},$$

we consider the Riemann-Hilbert problem to find matrices $\hat{V}^{\binom{0}{\infty}}(\lambda)$ such that

$$\hat{V}^{(\infty)}(\lambda)=\hat{V}^{(0)}(\lambda)H(\lambda),\quad \lambda\in C. \tag{4.1.1}$$

Here we assume $\hat{V}^{\binom{0}{\infty}}(\lambda)$ to be holomorphic in λ and invertible on $C\cup C_+$ (resp. $C\cup C_-$), and satisfy the normalization condition $\hat{V}^{(0)}(0)=1$. For the solution to the problem, we define $\tilde{Y}(\lambda)$ and $\tilde{\Omega}(\lambda)$ as follows;

$$\begin{aligned}
&\tilde{Y}(\lambda)=\hat{V}^{(0)}(\lambda)Y(\lambda)\ \text{in}\ C_+,\quad =\hat{V}^{(\infty)}(\lambda)Y(\lambda)u(\lambda)^{-1}\ \text{in}\ C_-,\\
&\tilde{\Omega}(\lambda)=\frac{\lambda\tilde{A}}{1-\lambda}d\xi-\frac{\lambda\tilde{B}}{1+\lambda}d\eta,\quad \text{where}\\
&\tilde{A}=A+\partial_\xi\dot{V}^{(0)}(0),\quad \tilde{B}=B-\partial_\eta\dot{V}^{(0)}(0).
\end{aligned}$$

The dot denotes the differentiation with respect to λ. Then we find;

(1) $\tilde{A}$, $\tilde{B}$ are $\mathfrak{su}(n)$-matrices.

(2) $\tilde{Y}(\lambda)$ is a fundamental solution matrix of the equation $d\tilde{Y}=\Omega\tilde{Y}$, and satisfies the same condition that $Y(\lambda)$ does.

These facts imply that $\tilde{\Omega}(\lambda)$ provides a new solution to the $SU(n)$ chiral field. In other words, the Riemann-Hilbert problem induces a transformation on the solution space.

As far as the authors know, there has not been any systematic approach to the construction of the exact solutions of the three dimensional (or, sub-subholonomic) soliton equations such as the KP equation in the framework of the Riemann-Hilbert problem.

The Riemann-Hilbert problem may be thought of to correspond to the Bruhat decomposition of Euclidean Lie groups. So generalizing the Bruhat decomposition to the category of $GL(\infty)$, we wish to construct the exact solutions, namely, the rational or soliton solutions to the KP or the TL hierarchy, etc.

To state our viewpoint more clearly, rewrite the bilinear relation (1.2.8) in a little formal fashion as follows;

$$W^{(\infty)}(x,y)^{-1}\cdot W^{(0)}(x,y)=W^{(\infty)}(x',y')^{-1}W^{(0)}(x',y').$$

This equation must hold for any x, x' and y, y', so that the both sides do not depend on these variables. Thus there exists a constant matrix $A \in GL(\infty)$ such that

$$W^{(0)}(x, y) = W^{(0)}(x, y)A.$$

This may be interpreted as the Bruhat decomposition of

$$H(x, y) = \exp(\xi(x, \Lambda) + \xi(y, \Lambda^{-1}))A \exp(\xi(-x, \Lambda) + \xi(-y, \Lambda^{-1})),$$

or an analogue of the Riemann-Hilbert problem. Of course, such a decomposition may be meaningless in a general case. However we adopt it as a fundamental setup. In other words, our strategy is to consider the decomposition

$$\hat{V}^{(0)}(x, y) = \hat{V}^{(\infty)}(x, y)H(x, y). \tag{4.1.2}$$

Further we assume that $\hat{V}^{\binom{0}{\infty}}(x, y)$ satisfy the following condition:

$$\begin{cases} \hat{V}^{(0)}(x, y) \text{ is an upper triangular, invertible} \\ \text{matrix, and } \hat{V}^{(\infty)}(x, y) \text{ is a lower triangular} \\ \text{matrix with unit diagonal entries.} \end{cases} \tag{4.1.3}$$

We will call (4.1.2) (with (4.1.3)) the *RH* decomposition. It should be noticed that the wave matrices are recovered as

$$W^{\binom{0}{\infty}}(x, y) = \hat{V}^{\binom{0}{\infty}}(x, y) \exp(\xi(x, \Lambda) + \xi(y, \Lambda^{-1})), \tag{4.1.4}$$

(as for $\hat{V}^{\binom{0}{\infty}}(x, y)$, see Section 1.3 (1.3.30)–(1.3.33)).

As was remarked above, the *RH* decomposition may fail to make a sense in a general situation. But, specifying the matrix A in various ways, we will actually carry out this decomposition.

The following theorem describes how the matrix A characterizes the wave matrices (4.1.4).

Theorem 4.1. *Suppose that the RH decomposition* (4.1.2) *with* (4.1.3) *is achieved*;

(1) *Then the decomposition is unique.*

(2) *If* $[A, \Lambda^l] = 0$, *the resulting wave matrices solve the* $(TL)_l$ *hierarchy.*

(3) *If* $[A, \Lambda + \Lambda^{-1}] = 0$, *the resulting wave matrices solve the one-dimensional TL hierarchy.*

(4) *If* $A \in O(\infty)$, *then* $\hat{W}^{\binom{0}{\infty}}(x, y)|_{x_e=y_e=0} \in O(\infty)$.
$(\hat{W}^{(\infty)}(x, y)=\hat{V}^{(\infty)}(x, y) \exp \xi(y, \Lambda^{-1})$, $\hat{W}^{(0)}(x, y)=\hat{V}^{(0)}(x, y) \exp \xi(x, \Lambda)$.)
Therefore the resulting hierarchy is of the B-type.

(5) *If* $A \in Sp(\infty)$, *then* $\hat{W}^{\binom{0}{\infty}}(x, y)|_{x_e=y_e=0} \in Sp(\infty)$.
Therefore the resulting hierarchy is of the C-type.

Proof. (1) Let $\hat{V}_i^{\binom{0}{\infty}}(i=1, 2)$ be two pairs of matrices to achieve the same decomposition. Then one sees that

$$\hat{V}_1^{(0)}\hat{V}_2^{(0)-1}=\hat{V}_1^{(\infty)}\hat{V}_2^{(\infty)-1}.$$

The right-hand side is a upper triangular matrix, while the left-hand side is a lower triangular matrix with unit diagonal entries. Hence the both sides must be the unit matrix, so that $\hat{V}_1^{\binom{0}{\infty}}=\hat{V}_2^{\binom{0}{\infty}}$.

(2) By the assumption, $\Lambda^{-l}\hat{V}^{\binom{0}{\infty}}\Lambda^l$ also give the *RH* decomposition. The uniqueness of the decomposition yields $\Lambda^{-l}\hat{V}^{\binom{0}{\infty}}\Lambda^l=\hat{V}^{\binom{0}{\infty}}$, so that $[W^{\binom{0}{\infty}}, \Lambda^l]=0$. Hence we have the desirous result (see Proposition 1.13).

(3) Let $L=W^{(\infty)}\Lambda W^{(\infty)-1}$, $M=W^{(0)}\Lambda^{-1}W^{(0)-1}$. It is easy to see that the assumption implies $L+L^{-1}=M+M^{-1}$. Thus the resulting hierarchy falls into the one-dimensional sector (see (1.4.13)).

(4) First we observe that, if the *RH* decomposition is achieved by $\hat{V}^{\binom{0}{\infty}}(x, y)$, then the decomposition problem

$$\hat{X}^{(0)}(\tilde{x}, \tilde{y})=\hat{X}^{(\infty)}(\tilde{x}, \tilde{y})H(x, y)|_{x_e=y_e=0} \tag{4.1.5}$$

($\tilde{x}=(x_1, x_3, \cdots)$, and $\hat{X}^{\binom{0}{\infty}}$ are assumed to be matrices of such form as (4.1.3)) has a unique pair of solutions, $\hat{V}^{\binom{0}{\infty}}(x, y)|_{x_e=y_e=0}$. From the assumption of (4), it follows that $H(x, y)|_{x_e=y_e=0} \in O(\infty)$, i.e.

$$J^{-1}({}^tH(x, y)|_{x_e=y_e=0})J=H(x, y)^{-1}|_{x_e=y_e=0}.$$

The uniqueness of the decomposition yields

$$J({}^t\hat{V}^{\binom{0}{\infty}}(x, y)^{-1}|_{x_e=y_e=0})J^{-1}=\hat{V}^{\binom{0}{\infty}}(x, y)|_{x_e=y_e=0},$$

from which one obtains the desirous result.

(5) The proof goes in the same manner as above. Q.E.D.

Let $y=0$ in the *RH* decomposition. Then the resulting wave matrices correspond to the *KP* hierarchy (see § 1.2). Furthermore, if $A \in O(\infty)$ (resp. $A \in Sp(\infty)$), they correspond to the *BKP* (resp. *CKP*) hierarchy because of the above theorem (4) (resp. (5)) and the remark in Section 2.3.

Motivated by this observation, we will construct polynomial τ functions of the *KP*, *BKP*, *CKP* hierarchies. Before proceeding to the construction, we state two lemmata, which are well-known, however fundamental in the following discussion. The first of them is concerned with linear algebra.

Lemma 4.2. (1) *Let* ${}^t a_i=(a_{i_1}, a_{i_2}, \cdots)$, ${}^t b_i=(b_{i_1}, b_{i_2}, \cdots) \in \boldsymbol{C}^{\boldsymbol{N}}$ $(i=1, \cdots, r)$ *($\boldsymbol{N}$ denotes the totality of natural numbers), and assume* ${}^t b_i a_j$ *converges for any* i, j. *Set*

$$\Delta=\det(\delta_{ij}+{}^t b_i a_j)_{1\leqq i,j\leqq r},$$

and assume $\Delta\neq 0$. *Then the* $\boldsymbol{N}\times\boldsymbol{N}$ *matrix* $(I+\sum_{i=1}^r a_i\,{}^t b_i)$ *has the invertible matrix;*

$$\left(I+\sum_{i=1}^{r} a_i\,{}^t b_i\right)^{-1}=1+\sum_{i,j=1}^{r} x_{ij} a_i\,{}^t b_j \tag{4.1.6}$$

where x_{ij} *is given by*

$$(x_{ij})_{1\leqq i,j\leqq N}=-\{(\delta_{ij}+{}^t b_i a_j)_{1\leqq i,j\leqq r}\}^{-1}. \tag{4.1.7}$$

(2) *Then the following expansion formula holds:*

$$\Delta=\sum_{l=0}^{N}\sum_{i_1<\cdots<i_l}\sum_{j_1,\cdots,j_l}\operatorname{sgn}\begin{pmatrix} i_1, & \cdots, & i_l\\ j_1, & \cdots, & j_l\end{pmatrix}({}^t b_{i_1} a_{j_1})\cdots({}^t b_{i_l} a_{j_l}). \tag{4.1.8}$$

Here $\operatorname{sgn}\begin{pmatrix} i_1, & \cdots, & i_l\\ j_1, & \cdots, & j_l\end{pmatrix}=0$ *unless* $\begin{pmatrix} i_1, & \cdots, & i_l\\ j_1, & \cdots, & j_l\end{pmatrix}$ *is a permutation.*

Proof. The formula (4.1.6), (4.1.7) is easily verified by considering the Neumann expansion of $(I+\sum_{i=1}^r a_i\,{}^t b_i)^{-1}$. We omit the details.

Q.E.D.

Remark. Let $X=(x_{ij})$ be a matrix of infinite size. Suppose that $|x_{ij}|\leqq a_i b_j M$, and $K=\Sigma a_i b_j<+\infty$. Then

$$\det(1+X)=1+\frac{1}{1!}\sum_i x_{ii}+\frac{1}{2!}\sum_{i,j}\det\begin{pmatrix} x_{ii} & x_{ij}\\ x_{ji} & x_{jj}\end{pmatrix}$$

$$+\frac{1}{3!}\sum_{i,j,k}\det\begin{pmatrix} x_{ii} & x_{ij} & x_{ik}\\ x_{ji} & x_{ij} & x_{jk}\\ x_{ki} & x_{kj} & x_{kk}\end{pmatrix}+\cdots$$

is well-defined, and is absolutely convergent (See [33]). Note that

$$\Delta=\det\left(I+\sum_{i=1}^{r} a_i\,{}^t b_i\right)$$

in this sense.

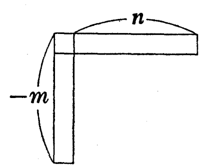
Lemma 4.3 [33, 34]. *Let $\chi_Y(x)$ be the Schur function corresponding to the Young tableau Y, and let Y^* be the conjugate tableau of Y, which is defined by converting Y with respect to the diagonal line. Then we have*

$$\chi_{Y^*}(x)=(-)^{\text{size of }Y}\chi_Y(-x).$$

In particular

$$\chi_{mn}(-x)=(-)^{n-m}\chi_{-n-1,-m-1}(x), \tag{4.1.9}$$

where $\chi_{mn}(x)=(-)^m\sum_{\nu\geqq 0}p_{\nu-m}(-x)p_{n-\nu}(x)$ $(m<0\leqq n)$ is the Schur function for the hook



Remark. If we further set

$\chi_{mm}(x)=1$ for any m,

$\chi_{nm}(x)=0$ for $n\geqq 0$, $m<0$ or, n, $m\geqq 0$, $n\neq m$, or n, $m>0$, $n\neq m$,

(4.1.9) persists for any integers n, m.

Let us consider the following *RH* decomposition:

$$\hat{V}^{(0)}(x)=\hat{V}^{(\infty)}(x)H(x), \tag{4.1.11}$$

where

$$H(x)=\exp\xi(x,\Lambda)\left(I+\sum_{i=1}^{r}a_iE_{m_in_i}\right)\exp\xi(-x,\Lambda)$$

(a_i is a scalor constant), and $\hat{V}^{\binom{0}{\infty}}(x)$ are subject to the condition (4.1.3), i.e., $(\hat{V}^{(0)}(x))_-=0$, $\hat{V}^{(\infty)}=I+Z$, $Z=(z_{ij})_{i,j\in Z}$ with $z_{ij}=0$ for $i<j$. This decomposition is carried out as follows: Taking the $(-)$ part of (4.1.10), we get

$$\left[Z\left\{I+\sum_{i=1}^{r} a_i \exp \xi(x, \Lambda) E_{m_i n_i} \exp \xi(-x, \Lambda)\right\}\right]_- = -\sum_{i=1}^{r} a_i [\exp \xi(x, \Lambda) E_{m_i n_i} \exp \xi(-x, \Lambda)]_-.$$

Define

$$p(m, s) = p(m; s; x) = (p_{m-k}(x))_{k<s} = \begin{pmatrix} \vdots \\ p_{m-s+2}(x) \\ p_{m-s+1}(x) \end{pmatrix}$$

$${}^t p^*(n; s) = {}^t p^*(n; s; x) = {}^t(p_{k-n}(-x))_{k<s} = (\cdots p_{s-2-n}(-x), p_{s-1-n}(-x)),$$

$${}^t z(s) = {}^t(z_{s,k})_{k<s} = (\cdots z_{s,s-2}, z_{s,s-1}),$$

where we set $p_j(x)=0$ for $j<0$. Then the above equation reads

$$\text{(4.1.12)} \quad {}^t z(s)\left\{I+\sum_{i=1}^{r} a_i p(m_i; s)\, {}^t p^*(n_i; s)\right\} = -\sum_{i=1}^{r} a_i p_{m_i - s}(x)\, {}^t p^*(n_i; s).$$

Let us apply Lemma 4.2 (1) to this equation. Set

$$\text{(4.1.13)} \quad \left\{I+\sum_{i=1}^{r} a_i p(m_i; s)\, {}^t p^*(n_i; s)\right\}^{-1} = 1 + \sum_{i,j=1}^{r} a_i x_{ij}(s) p(m_i: s)\, {}^t p^*(n_j; s).$$

By (4.1.7) together with Cramér's formula, $x_{ij}(s)$ is given by

$$x_{ij}(s) = -\tau(s)^{-1}$$

$$\text{(4.1.14)} \quad \times \det \begin{array}{c} \begin{pmatrix} 1+a_1\, {}^t p^*(n_1; s) p(m_1; s) & \cdots\cdots & & a_r\, {}^t p^*(n_1; s) p(m_r; s) \\ \vdots & & & \vdots \\ 0 & \cdots\cdots & 1 & \cdots\cdots \quad 0 \\ \vdots & & & \vdots \\ a_1\, {}^t p^*(n_r; s) p(m_1; s) & \cdots & \uparrow & \cdots\; 1+a_r\, {}^t p^*(n_r; s) p(m_r; s) \end{pmatrix} \\ \quad (i) \end{array} \leftarrow (j)$$

where

$$\begin{aligned} \tau(s) = \tau(s; x) &= \det(\delta_{ij} + a_j\, {}^t p^*(n_i; s) p(m_j; s))_{1 \leq i,j \leq r} \\ &= \det(\delta_{ij} + (-)^{s-m_j-1} a_j \chi_{s-m_j-1, s-n_i-1}(-x))_{1 \leq i,j \leq r} \\ \text{(4.1.15)} \qquad & \qquad \text{(by (4.1.10))} \\ &= \det(\delta_{ij} + (-)^{n_i - m_j} a_j \chi_{n_i - s, m_j - s}(x))_{1 \leq i,j \leq r} \\ & \qquad \text{(by (4.1.9))} \end{aligned}$$

(In the last equation above we should set $\chi_{mm}(x)=1$ for $m\leqq -1$, $\chi_{mm}(x)=0$ for $m\geqq 0$.) We observe that if a_j are very small, then $\tau(s;x)\neq 0$ for $|x|\ll 1$, so that the linear problem can be solved simulataneously for all s.

Set

$$\hat{w}^{(\infty)}(s;x;\lambda)=1+\sum_{j=1}^{\infty} z_{s,s-j}\lambda^{-j}. \tag{4.1.16}$$

$w^{(\infty)}(s;x;\lambda)=\hat{w}^{(\infty)}(s;x;\lambda)\lambda^s \exp \xi(x,\lambda)$ becomes the wave function for the *KP* hierarchy. Furthermore we obtain the following.

Proposition 4.4. *Let* $\tau(s;x)$ *and* $\hat{w}^{(\infty)}(s;x;\lambda)$ *be as in* (4.1.15), (4.1.16), *respectively. Then we have*

$$\hat{w}^{(\infty)}(s;x;\lambda)=\frac{\tau(s;x-\varepsilon(\lambda^{-1}))}{\tau(s;x)}, \tag{4.1.17}$$

where $\varepsilon(\lambda)=(\lambda, \frac{1}{2}\lambda^2, \frac{1}{3}\lambda^3, \cdots)$. *Hence* $\tau(s;x)$ (4.1.15) *is a* τ *function of the KP hierarchy.*

For the proof, the following lemma is needed.

Lemma 4.4. *We have*

$$p_j(x-\varepsilon(\lambda^{-1}))=p_j(x)-\lambda^{-1}p_{j-1}(x), \tag{4.1.18}$$

$$p_j(-x+\varepsilon(\lambda^{-1}))=\sum_{k=0}^{\infty} p_{j-k}(-x)\lambda^{-k}, \tag{4.1.19}$$

$$\begin{aligned} &{}^t p^*(n;s;x-\varepsilon(\lambda^{-1}))p(m;s;x-\varepsilon(\lambda^{-1})) \\ &\quad={}^t p^*(n;s;x)p(m;s;x)-\lambda^{-1}p_{m-s}(x)p_{s-1-n}(-x+\varepsilon(\lambda^{-1})). \end{aligned} \tag{4.1.20}$$

Proof. (4.1.18), (4.1.19) follow from

$$e^{\xi(\pm x\mp\varepsilon(\lambda^{-1}),\lambda_1)}=(1-\lambda_1/\lambda)^{\pm 1}e^{\xi(\pm x,\lambda_1)},$$

respectively. (4.1.20) is deduced from the former equalities. Q.E.D.

Proof of Proposition 4.4. From (4.1.12), (4.1.13), one sees that

$$\begin{aligned} {}^t z(s)&=-\sum_{i=1}^{r} a_i p_{m_i-s}(x)\,{}^t p^*(n_i;s)\Big\{I+\sum_{j,k=1}^{r} a_j x_{jk}(s)p(m_j;s)p^*(n_k;s)\Big\} \\ &=-\sum_{i,j=1}^{r} a_j p_{m_j-s}(x)\Big\{\delta_{ji}+\sum_{k=1}^{r} a_j\,{}^t p^*(n_j;s)p(m_k;s)x_{ki}(s)\Big\}\,{}^t p^*(n_i;s). \end{aligned}$$

Since the definition of $x_{ji}(s)$ reads as

$$\delta_{ji}+\sum_{k=1}^{r} a_j\, {}^t p^*(n_j;s)p(m_k;s)x_{ki}(s)=-x_{ji}(s),$$

one finds

$${}^t z(s)=\sum_{i,j=1}^{r} a_j p_{m_j-s}(x)x_{ji}(s)\, {}^t p^*(n_i;s).$$

Hence, by the definitions (4.1.14), (4.1.16) together with the above results, one sees that

$$\begin{aligned}
&\hat{w}^{(\infty)}(s;x;\lambda)\\
&\quad=\tau(s)^{-1}\Big\{\tau(s)-\sum_{i,j=1}^{r}\lambda^{-1}a_j p_{m_j-s}(x)x_{ji}(s)p_{s-1-n_i}(-x+\varepsilon(\lambda^{-1}))\Big\}\\
&\quad=\tau(s)^{-1}\Big\{\tau(s)-\sum_{i=1}^{r}\lambda^{-1}
\end{aligned}$$

$$\text{(4.1.21)}\qquad \times\det\begin{pmatrix} 1+a_1\,{}^tp^*(n_1;s)p(m_1;s) & \cdots\cdots & a_r\,{}^tp^*(n_1;s)p(m_r;s)\\ \vdots & & \vdots\\ a_1p_{m_1-s}(x) & \cdots\cdots\cdots\cdots & a_rp_{m_r-s}(x)\\ \vdots & & \vdots\\ a_1\,{}^tp^*(n_r;s)p(m_1;s) & \cdots\cdots & 1+a_r\,{}^tp^*(n_r;s)p(m_r;s)\end{pmatrix}\begin{array}{l} \\ \\ \leftarrow(i)\\ \\ \\ \end{array}$$

$$\times p_{s-1-n_i}(-x+\varepsilon(\lambda^{-1}))\Big\}.$$

On the other hand, applying (4.1.20) to (4.1.15), one easily finds

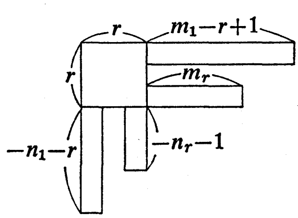
$$\text{(4.1.22)}\qquad \begin{aligned}\tau(s;x-\varepsilon(\lambda^{-1}))=\det\,(&\delta_{ij}+a_i\,{}^tp^*(n_i;s)p(m_j;s)\\ &-\lambda^{-1}p_{m_j-s}(x)p_{s-1-n_i}(-x+\varepsilon(\lambda^{-1}))).\end{aligned}$$

Compairing (4.1.21) and (4.1.22), one concludes (4.1.17) Q.E.D.

Corollary 4.6 [20, 33, 34]. *Let $n_1<\cdots<n_r<0\leqq m_r<\cdots<m_1$. Then*

$$\tau(x)=\det\,(\chi_{n_im_j}(x))_{1\leqq i,j\leqq r}$$

is a τ function of the KP hierarchy. This is the Schur function for the Young tableau,



Proof. Since the τ function has constant multiple arbitrariness,

$$\left(\prod_{l=1}^{r} a_l^{-1}\right)\tau(0; x)=\det\left(\left(-\right)^{n_i-m_j}a_j^{-1}\delta_{ij}+\chi_{n_i m_j}\right)_{1\leq i,j\leq r}$$

is also the τ function of the *KP* hierarchy. Letting $a_j\to\infty$, we obtain the corollary. Q.E.D.

Next we consider the *BKP*, *CKP* hierarchy. Recall that the generators of $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ are given respectively by (§ 2.1)

$$Z_{B,mn}=(-)^n E_{m,-n}-(-)^m E_{n,-m},$$
$$Z_{C,mn}=(-)^n E_{m,n-1}-(-)^{m+1}E_{n,-m-1}.$$

If we assume $m+n\neq 0$, $m, n\neq 0$ (resp. $m+n\neq 0$), $\exp(aZ_{B,mn})=1+aZ_{B,mn} \in O(\infty)$ (resp. $\exp(aZ_{C,mn})=1+aZ_{C,mn}\in Sp(\infty)$). Then applying Theorem 4.1 (4), (5) and Proposition 4.4 to this case, we obtain examples of the τ function of the *BKP*, *CKP* hierarchies;

$$\tau_B(s)=\det\begin{pmatrix}1+(-)^m a\chi_{-n-s,m-s} & (-)^{m+1}a\chi_{-n-s,n-s}\\ (-)^n a\chi_{-m-s,m-s} & 1+(-)^{n+1}a\chi_{-m-s,n-s}\end{pmatrix},$$
$$\tau_C(s)=\det\begin{pmatrix}1+(-)^{m+1}a\chi_{-n-1-s,m-s} & (-)^{m+1}a\chi_{-n-1-s,n-s}\\ (-)^{n+1}a\chi_{-m-1-s,m-s} & 1+(-)^{n+1}\chi_{-m-1-s,n-s}\end{pmatrix}. \tag{4.1.23}$$

To construct an N-soliton solution of the *TL* hierarchy, let us consider the following *RH* decomposition;

$$\hat{V}^{(0)}(x,y)=\hat{V}^{(\infty)}(x,y)H(x,y),$$
$$H(x,y)=\exp\left(\xi(x,\Lambda)+\xi(y,\Lambda^{-1})\right)\left(I+\sum_{j=1}^{N}X_{p_jq_j}\right)$$
$$\times\exp\left(\xi(-x,\Lambda)+\xi(-y,\Lambda^{-1})\right), \tag{4.1.24}$$

where $a_j>0$, and $0<q_N<\cdots<q_1<p_1<\cdots<p_N$, and X_{pq} is defined

$$X_{pq}=\sum_{m,n\in Z}p^m q^{-n}E_{mn}. \tag{4.1.25}$$

$\hat{V}^{\binom{0}{\infty}}(x,y)$ should satisfy the condition (4.1.3), that is,

$$\hat{V}^{(0)}=(v_{ij}^{(0)})\ (v_{ij}^{(0)}=0 \text{ for } i>j),\quad \hat{V}^{(\infty)}=I+Z,$$
$$Z=(z_{ij})\ (z_{ij}=0 \text{ for } i<j).$$

Define

$$e(p;s)=e(p;s;x,y)=(p^k e^{\eta(p)})_{k<s},$$
$${}^te^*(q;s)={}^te^*(q;s;x,y)={}^t(q^{-k}e^{-\eta(q)})_{k<s},$$

where $\eta(p)=\xi(x,p)+\xi(y,p^{-1})$. Set ${}^tz(s)=(z_{s,k})_{k<s}$. By the same argument as the preceding one, it turns out that (4.1.24) reduces to

$${}^tz(s)\Big(I+\sum_{i=1}^N a_i e(p_i;s)\,{}^te^*(q_i;s)\Big)=-\sum_{i=1}^N a_i p_i^s e^{\eta(p_i)}\,{}^te^*(q_i;s),$$

which further leads to

$$\text{(4.1.26)}\qquad {}^tz(s)=\sum_{i,j=1}^N a_i p_j^s e^{\eta(p_j)} x_{ji}(s)\,{}^te^*(q_i;s),$$

where $x_{ji}(s)$ is defined by

$$\text{(4.1.27)}\qquad \begin{aligned} x_{ji}(s)&=-(\tau'(s))^{-1}\\ &\times\det\begin{array}{cc}\begin{pmatrix} 1+a_1\,{}^te^*(q_1;s)e(p_1;s) & \cdots\cdots & & \cdots\cdots & a_N\,{}^te^*(q_1;s)e(p_N;s)\\ \vdots & & & & \vdots\\ 0 & \cdots\cdots & 1 & \cdots\cdots & 0\\ \vdots & & & & \vdots\\ a_1\,{}^te^*(q_N;s)e(p_1;s) & \cdots & & \cdots & 1+a_N\,{}^te^*(q_N;s)e(p_N;s)\end{pmatrix} & \begin{array}{c}\\ \\ \leftarrow(j)\\ \\ \\ \end{array}\\ \begin{array}{c}\uparrow\\ (i)\end{array} & \end{array}\end{aligned}$$

and

$$\text{(4.1.28)}\qquad \tau'(s)=\tau(s;x,y)=\det(\delta_{ij}+a_j\,{}^te^*(q_i;s)e(p_j;s))_{1\leq i,j\leq N}.$$

We will show below that the τ functions (4.1.28) are expressed as (4.1.35). From the assumption on a_j and p_j, q_j, it follows that $c_{ij}<0$ for $i<j$, and $a_i(s)<0$. Hence the τ functions are positive for real x, y, so that the above linear equations can be solved simultaneously for real x, y. (Of course, the τ functions (4.1.35) themselves is well-defined for mutually distinct p_j, q_j).

The following equalities will be useful later.

$$\text{(4.1.29)}\qquad {}^te^*(q;s)e(p;s)=\frac{q/p}{1-q/p}(p/q)^s e^{\eta(p)-\eta(q)},$$

$$\text{(4.1.30)}\qquad \begin{aligned}&{}^te^*(q;s;x-\varepsilon(\lambda^{-1}),y)e(p;s;x-\varepsilon(\lambda^{-1}),y)\\ &\quad=te^*(q;s)e(p;s)-\frac{q/\lambda}{1-q/\lambda}(p/q)^s e^{\eta(p)-\eta(q)},\end{aligned}$$

$$
(4.1.31)\qquad \begin{aligned} &{}^t e^*(q;s;x,y-\varepsilon(\lambda))e(p;s;x,y-\varepsilon(\lambda)) \\ &= {}^t e^*(q;s)e(p;s)+\frac{1}{1-\lambda/q}(p/q)^s e^{\eta(p)-\eta(q)}. \end{aligned}
$$

Set

$$
\hat{v}^{(\infty)}(s;x,y;\lambda)=1+\sum_{j=1}^{\infty} z_{s,s-j}\lambda^{-j},
$$

$$
\hat{v}^{(0)}(s;x,y;\lambda)=\sum_{j=0}^{\infty} \hat{v}^{(0)}_{s,s+j}\lambda^{j}.
$$

Then we get the following proposition.

Proposition 4.7. *We have*

$$
(4.1.32)\qquad \hat{v}^{(\infty)}(s;x,y;\lambda)=\frac{\tau'(s;x-\varepsilon(\lambda^{-1}),y)}{\tau'(s;x,y)},
$$

$$
(4.1.33)\qquad \hat{v}^{(0)}(s;x,y;\lambda)=\frac{\tau'(s+1;x,y-\varepsilon(\lambda))}{\tau'(s;x,y)},
$$

which means, by the remarks in Section 1.3 ((1.3.32)–(1.3.35)) *together with* (4.1.4), *that*

$$
(4.1.34)\qquad \tau(s;x,y)=\tau'(s;x,y)\exp\left(-\sum_{n=1}^{\infty} nx_n y_n\right)
$$

is a τ function for the TL hierarchy. Furthermore it is expressed as

$$
(4.1.35)\qquad \begin{aligned} \tau'(s;x,y)=&\sum_{l=0}^{N}\sum_{i_1<\cdots<i_l} c_{i_1\cdots i_l}a_{i_1}(s)\cdots a_{i_l}(s) \\ &\times\exp\left(\sum_{\mu=1}^{l}\eta(p_{i_\mu})-\eta(q_{i_\mu})\right) \end{aligned}
$$

where

$$
a_i(s)=a_i(p_i/q_i)^s\frac{q_i}{p_i-q_i}
$$

$$
c_{i_1\cdots i_l}=\prod_{1\leqq\mu<\nu\leqq l} c_{i_\mu i_\nu},\quad c_{ij}=\frac{(p_i-p_j)(q_i-q_j)}{(p_i-q_j)(q_i-p_j)}.
$$

Proof. From the definition of $\hat{v}^{(\infty)}(s;x,y;\lambda)$ and (4.1.26), (4.1.28), it follows that

$$
\hat{v}^{(\infty)}(s;x,y;\lambda)=\tau'(s)^{-1}\Big\{\tau'(s)
$$

$$-\sum_{i=1}^{N}\det\begin{pmatrix} 1+a_1\,{}^te^*(q_1;s)e(p_1;s) & \cdots\cdots & a_N\,{}^te^*(q_1;s)e(p_N;s)\\ \vdots & & \vdots\\ a_1(p_1/q_i)^s e^{\eta(p_1)-\eta(q_i)} & \cdots\cdots\cdots & a_N(p_N/q_i)^s e^{\eta(p_N)-\eta(q_i)}\\ \vdots & & \vdots \\ a_1\,{}^te^*(q_N;s)e(p_1;s) & \cdots\cdots\cdots & 1+a_N\,{}^te^*(q_N;s)e(p_N;s)\end{pmatrix}$$

$$\times\frac{q_i/\lambda}{1-q_i/\lambda}\Big\}.$$

On the other hand, applying (4.1.30) to (4.1.28), one finds

$$\begin{aligned}\tau'(s;x-\varepsilon(\lambda^{-1}),y)=\det\Big(&\delta_{ij}+a_j\,{}^te^*(q_i;s)e(p_j;s)\\ &-a_j\frac{q_i/\lambda}{1-q_i/\lambda}(p_j/q_i)^s e^{\eta(p_j)-\eta(q_i)}\Big)_{1\leq i,j\leq N}.\end{aligned}\tag{4.1.36}$$

Comparing these identities leads us to (4.1.32).

Next we wish to prove (4.1.33). For the purpose, we prepare the following notation: Let M be a matrix. By $M^{(k)}$, we mean a matrix obtained from M by setting the (k,j), (j,k) $(j\neq k)$ entries to be 0, and leaving the other entries.

By the way, it is easy to see

$$\begin{aligned}&\hat{v}^{(0)}(s;x,y;\lambda)\\ &=1+\sum_{k=1}^{N}a_k(p_k/q_k)^s e^{\eta(p_k)-\eta(q_k)}\hat{v}^{(\infty)}(s;x,y;p_k)\frac{1}{1-\lambda/q_k}.\end{aligned}\tag{4.1.37}$$

Using the notation prepared above, one finds $\hat{v}^{(\infty)}(s;x,y;p_k)$ to be given by

$$\begin{aligned}&\hat{v}^{(\infty)}(s;x,y;p_k)\\ &=\tau'(s)^{-1}\det\left(\delta_{ij}+a_j\frac{1-p_j/p_k}{1-q_i/p_k}\,\frac{q_i/p_j}{1-q_i/p_j}(p_j/q_i)^s\,e^{\eta(p_j)-\eta(q_i)}\right)^{(k)}.\end{aligned}\tag{4.1.38}$$

On the other hand, applying (4.1.31) one sees that

$$\tau'(s+1;x,y-\varepsilon(\lambda))=\tau'(s)+\sum_{k=1}^{N}\det$$

$$\times\begin{array}{cc} \begin{array}{c} \qquad\qquad\qquad\quad k\\ \qquad\qquad\qquad\quad\downarrow \end{array} & \\ \begin{pmatrix} 1+a_1\,{}^te^*(q_1s)e(p_1s) & \cdots & \dfrac{q_1/p_k}{1-q_1/p_k}q_1^{-s}e^{\eta(q_1)} & \cdots & a_N\,{}^te^*(q_Ns)e(p_Ns)\\ \vdots & & \vdots & & \vdots\\ a_1p_1^s e^{\eta(p_1)} & \cdots\cdots & 1 & \cdots\cdots & a_Np_N^s e^{\eta(p_N)}\\ \vdots & & \vdots & & \vdots\\ a_1\,{}^te^*(q_Ns)e(p_1s) & \cdots & \dfrac{q_N/p_k}{1-q_N/p_k}q_N^{-s}e^{\eta(q_N)} & \cdots & 1+a_N\,{}^te^*(q_Ns)e(p_Ns) \end{pmatrix} & \begin{array}{c} \\ \\ \leftarrow k\\ \\ \\ \end{array} \end{array}$$

(4.1.39)
$$\times a_k \frac{1}{1-\lambda/q_k}(p_k/q_k)^s e^{\eta(p_k)-\eta(q_k)}.$$

In the above determinants, let us perform the following fundamental operations: Multiply the k-th line by $((q_i/p_k)/(1-q_i/p_k))q_i^{-s}e^{-\eta(q_i)}$ and subtract it from the i-th $(i \neq k)$ line so that the (k, i) entries becomes 0. After these operations, further perform fundamental operations to let the (i, k) entries $(i \neq k)$ be 0. Substitute (4.1.38) into (4.1.37), and compare (4.1.37) with (4.1.39). Noting that $(1-p/\lambda)/(1-q/\lambda)\cdot(q/p)/(1-q/p)=(q/p)/(1-q/p)-(q/\lambda)/(1-q/\lambda)$, we conclude (4.1.33).

The expansion formula (4.1.35) is easily verified by applying Lemma 4.2 (2) to (4.1.27). Q.E.D.

The τ function (4.1.35) coincides with the N-soliton τ function discussed in [19]. We denote the τ function (4.1.34) by

(4.1.40)
$$\tau\left(s;\, \begin{matrix} a_1 & \cdots & a_N \\ p_1 q_1 & \cdots & p_N q_N \end{matrix};\, x, y\right).$$

Next we consider a N-soliton solutions of the $(TL)_l$ hierarchy, the one-dimensional TL hierarchy, and so on.

The $(TL)_l$ hierarchy: In (4.1.35), we set [21, 25]

(4.1.41)
$$q_j = \omega p_j \quad (\omega^l = 1,\, \omega \neq 1,\, 1 \leqq j \leqq N).$$

It is evident that the resulting τ functions $\tau'(s; x, y)$ satisfy

$$\partial_{x_j}\tau'(s) = \partial_{y_j}\tau'(s) = 0 \quad \text{for } j \equiv 0 \bmod l,\ \tau'(s+l) = \tau'(s).$$

Hence they belong to the l-periodic hierarchy. We remark that if we set $q_j = \omega p_j$ in (4.1.24), then the infinite series ${}^t e^*(q_j; s)e(p_j; s)$ diverges. Namely the RH decomposition cannot be directly solved under this constraint.

The one-dimensional TL hierarchy. In the RH decomposition (4.1.24), we impose the following constraint compatible with the assumption on p_j, q_j (see (4.1.24));

(4.1.42)
$$p_j q_j = 1 \quad \text{for } 1 \leqq j \leqq N.$$

After a little computation, we find $[X_{pp^{-1}}, \Lambda + \Lambda^{-1}] = 0$. Thus Theorem 4.1 (3) assures that the resulting wave matrices fall into the one-dimensional sector. The τ function (4.1.35) takes the form

$$\tau'(s; t) = \sum_{l=0}^{N} \sum_{i_1 < \cdots < i_l} \tilde{c}_{i_1 \cdots i_l} \tilde{a}_{i_1}(s) \cdots \tilde{a}_{i_l}(s) \exp\left(\sum_{\mu=1}^{l} \tilde{\eta}(p_{i_\mu})\right)$$

where $t=(t_1, t_2, \cdots)=(\frac{1}{2}(x_1-y_1), \frac{1}{2}(x_2-y_2), \cdots)$, and

$$\tilde{\eta}(p)=2\sum_{n=1}^{\infty}(p^n-p^{-n})t_n, \quad \tilde{a}_i(s)=a_i\frac{p_i^{2s}}{p_i^2-1},$$

$$\tilde{c}_{i_1\cdots i_l}=\prod_{1\leq\mu<\nu\leq l}\tilde{c}_{i_\mu i_\nu}, \quad \tilde{c}_{ij}=\frac{(p_i^2-p_j^2)^2}{(p_ip_j-1)^2}.$$

The *BTL*, *CTL* hierarchies: Define

$$\tau_B(s;x,y)=\tau\left(s;\begin{matrix}a_1 & -a_1 & \cdots & a_N & -a_N\\ p_1,-q_1 & q_1,-p_1 & \cdots & p_N,-q_N & q_N,-p_N\end{matrix};x,y\right),$$
(4.1.45)
$$\tau_C(s;x,y)=\tau\left(s;\begin{matrix}-q_1^{-1}a_1 & -p_1^{-1}a_1 & \cdots & -q_N^{-1}a_N & -p_N^{-1}a_N\\ p_1,-q_1 & q_1,-p_1 & \cdots & p_N,-q_N & q_N,-p_N\end{matrix};x,y\right).$$

Then we have

Proposition 4.8. *The τ functions $\tau_B(s;x,y)$, $\tau_C(s;x,y)$ have the symmetries,*

(4.1.46) $$\tau_B(-s;x,y)=\tau_B(s+1;\iota(x),\iota(y)),$$

(4.1.47) $$\tau_C(-s;x,y)=\tau_C(s;\iota(x),\iota(y)).$$

Proof. Note that there are more general symmetries,

(4.1.48)
$$\tau\left(s;\begin{matrix}a_1 & \cdots & a_N\\ p_1q_1 & \cdots & p_Nq_N\end{matrix};x,y\right)=\tau\left(s;\begin{matrix}a_1 & \cdots & a_N\\ -p_1-q_1 & \cdots & -p_N-q_N\end{matrix};-\iota(x),-\iota(y)\right)$$

(4.1.49) $$\tau\left(s;\begin{matrix}(p_1/q_1)a_1 & \cdots & (p_N/q_N)a_N\\ p_1q_1 & \cdots & p_Nq_N\end{matrix};x,y\right)=\tau\left(s+1;\begin{matrix}a_1 & \cdots & a_N\\ p_1q_1 & \cdots & p_Nq_N\end{matrix};x,y\right)$$

(4.1.50)
$$\tau\left(s;\begin{matrix}a_1 & \cdots & a_N\\ p_1q_1 & \cdots & p_Nq_N\end{matrix};x,y\right)=\tau\left(-s+1;\begin{matrix}-a_1 & \cdots & -a_N\\ q_1p_1 & \cdots & q_Np_N\end{matrix};-x,-y\right).$$

It is an easy task to verify these symmetries. Applying these to our case, we see that

$$\tau_B(s+1;\iota(x),\iota(y))$$
$$=\tau\left(s+1;\begin{matrix}a_1 & -a_1 & \cdots\\ p_1,-q_1 & q_1,-p_1 & \cdots\end{matrix};\iota(x),\iota(y))\right)$$

$$=\tau\left(s+1;\begin{matrix}a_1 & -a_1 & \cdots \\ -p_1, q_1 & -q_1, p_1 & \cdots\end{matrix}; -x, -y\right) \quad \text{(by (4.1.48))}$$

$$=\tau\left(-s;\begin{matrix}-a_1 & a_1 & \cdots \\ q_1, -p_1 & p_1, -q_1 & \cdots\end{matrix}; x, y\right) \quad \text{(by (4.1.50))}$$

$$=\tau_B(-s; x, y).$$

Likewise we can show the symmetry of the C-type (4.1.47). Q.E.D.

The τ function (4.1.45) are N-soliton τ functions of the *BTL*, *CTL* hierarchies [23, 26]. These τ functions may be thought of to come from the following *RH* decomposition (however, it is impossible to achieve it in a rigorous sense):

We set

$$\begin{aligned}
X_{B,pq} &= \sum_{m,n\in Z}\{(-)^n E_{m,-n}-(-)^m E_{n,-m}\}p^m q^{-n} \\
&= X_{p,-q}-X_{q,-p} \in \mathfrak{o}((\infty)), \\
X_{C,pq} &= \sum_{m,n\in Z}\{(-)^n E_{m,-n-1}\}-(-)^{m+1}E_{n,-m-1}\}p^m q^{-n} \\
&= -q^{-1}X_{p,-q}-p^{-1}X_{q,-p} \in \mathfrak{sp}((\infty)).
\end{aligned}$$

We apply the *RH* decomposition to the matrices

$$\begin{aligned}
H_B(x, y) &= \exp(\xi(x, \Lambda)+\xi(y, \Lambda^{-1}))\left(I+\sum_{j=1}^N a_j X_{B,p_j q_j}\right) \\
&\quad \times \exp(\xi(-x, \Lambda)+\xi(-y, \Lambda^{-1})), \\
H_C(x, y) &= \exp(\xi(x, \Lambda)+\xi(y, \Lambda^{-1}))\left(I+\sum_{j=1}^N a_j X_{C,p_j q_j}\right) \\
&\quad \times \exp(\xi(-x, \Lambda)+\xi(-y, \Lambda^{-1})).
\end{aligned}$$

Then Proposition 4.7 suggests that the resulting τ functions should be given by (4.1.45).

At the end of this section, we give some remarks.

Remark 1. Though we have not considered here, it is possible to generalize the *RH* decomposition to the multi-components. In the l-reduced *KP* or *TL* hierarchy, the *RH* decomposition reduces to the ordinary Riemann-Hilbert problem. These topics will be investigated in detail in a future paper.

Remark 2. Taking into account the remark after Lemma 4.2, the τ functions in Propositions 4.4, 4.7 take the form

$$\tau(s)=\det(A_s^* \exp \xi(x, \Lambda) A \exp \xi(-x, \Lambda) A_s)$$

where A is the matrix that appeared in (4.1.11) or (4.1.24). The rectangular matrices A_s^*, A_s are defined by

$$A_s^*=(\delta_{mn})_{\substack{m<s,\\ n\in Z}} \quad A_s=(\delta_{mn})_{\substack{m\in Z.\\ n<s}}$$

In fact, it is known [33, 34, 22] that the τ functions of the *KP* hierarchy are expressed in the above form (see also the Appendix 1 in this paper).

Remark 3. Let $X(p, q)$ be the vertex operator [22]

$$X(p, q)=e^{\xi(x,p)-\xi(x,q)}\, e^{-\xi(\tilde{\partial},p^{-1})+\xi(\tilde{\partial},q^{-1})}.$$

By a simple calculation we see

$$\prod_{l=1}^{N} e^{b_l X(p_l,q_l)} \cdot \exp\left(-\sum_{n=1}^{\infty} n x_n y_n\right)=\tau\left(0; \begin{matrix} a_1 & \cdots & a_N \\ p_1 q_1 & \cdots & p_N q_N \end{matrix}; x, y\right)$$

where $b_l=((q_l/p_l)/(1-q_l/p_l))a_l$. Expanding $X(p, q)$ into a formal Laurent series in p, q,

$$\text{(4.1.51)} \qquad \frac{q/p}{1-q/p} X(p, q)=\sum_{i,j\in Z} Z_{ij} p^i q^{-j},$$

then we see that the coefficients Z_{ij} satisfy the same commutation relations that the matrix units E_{ij} do [22]. Hence X_{pq} (4.1.25) can be identified with (4.1.51).

4.2. Special solutions of the Wronskian type

In this section we shall show a direct method for the construction of special solutions of the Wronskian type, which is a modification of the construction in [33] of rational solutions to the *KP* equation (see Appendix 1) and in a special case coincides with Date's method [6] for the soliton solutions.

In the following we shall mainly consider the one component case.

Consider the following functions

$$\text{(4.2.1)} \qquad p_i(x, y)=\sum_{j\in Z} p_{i+j}(x) p_j(y) \quad (i \in Z).$$

$p_i(x)$ and $p_i(y)$ are polynomials, while $p_i(x, y)$ is an infinite series of x and y with the generating function

$$\text{(4.2.2)} \qquad \sum_{i\in Z} p_i(x, y)\lambda^i=\exp[\xi(x, \lambda)+\xi(y, \lambda^{-1})].$$

As the data for the solution we give constant vectors $f_j=(f_{i,j})_{i\in Z}$, $j=1,\cdots,N$, of infinite size, and set

$$f_j(s;x,y)=\sum_{i\in Z}p_{i-s}(x,y)f_{i,j}. \tag{4.2.3}$$

Furthermore we assume the following condition

$$\det[f_j(s+i-1;x,y)]_{i,j=1,\dots,N}\neq 0,\quad s\in Z. \tag{4.2.4}$$

Then we can define the functions $w_1(s;x,y),\cdots,w_N(s;x,y)$ such that

$$f_j(s+N;x,y)+\sum_{i=0}^{N-1}w_{N-i}(s;x,y)f_j(s+i;x,y)=0,\quad j=1,\cdots,N. \tag{4.2.5}$$

Using Cramer's formula we have

$$w_{N-k}=-\det\begin{pmatrix} f_j(s+i-1;x,y)\begin{pmatrix} i=1,\cdots,k\\ j=1,\cdots,N\end{pmatrix} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ f_j(s+N;x,y)\quad (j=1,\cdots,N) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ f_j(s+i-1;x,y)\begin{pmatrix} i=k+2,\cdots,N\\ j=1,\cdots,N\end{pmatrix}\end{pmatrix} \Big/ \det[f_j(s+i-1;x,y)]_{i,j=1,\dots,N} \tag{4.2.6}$$

for $k=1,\cdots,N$. In particular

$$w_N=-\det[f_j(s+i;x,y)]_{i,j=1,\dots,N}/\det[f_j(s+i-1;x,y)]_{i,j=1,\dots,N}\neq 0 \tag{4.2.7}$$

for any $s\in Z$.

Now we set

$$W_N(x,y)=\sum_{j=0}^{N}\operatorname{diag}[w_j(s;x,y)]\Lambda^{N-j},\quad w_0(s;x,y)=1, \tag{4.2.8}$$

$$\begin{cases} W^{(\infty)}(x,y)=W_N(x,y)\Lambda^{-N}\exp[\xi(x,\Lambda)+\xi(y,\Lambda^{-1})],\\ W^{(0)}(x,y)=W_N(x,y)\exp[\xi(x,\Lambda)+\xi(y,\Lambda^{-1})]. \end{cases} \tag{4.2.9}$$

Then we have

Theorem 4.8. *$W^{(\infty)}$ and $W^{(0)}$ solve the linear problem* (1.2.8) *for certain suitable matrices B_n and C_n, so that they solve the Toda lattice hierarchy. The corresponding τ function $\tau'(s;x,y)$ is given by*

$$\tau'(s;x,y)=\det[f_j(s+i-1;x,y)]_{i,j=1,\dots,N}. \tag{4.2.10}$$

It is remarkable that the τ function is obtained in the Wronskian form (cf. Lemma 4.11). Therefore we call the solution obtained above a special solution of the Wronskian type.

Example 4.9. Suppose that $f_{i,j}$ takes the form

$$f_{i,j}=\sum_{l=1}^{M} k_l^i a_{l,j}, \tag{4.2.11}$$

where k_l and $a_{l,j}$ $(l=1, \cdots, M, j=1, \cdots, N)$ are constants. Then

$$f_j(s; x, y)=\sum_{l=1}^{M} k_l^s a_{l,j} \exp [\xi(x, k_l)+\xi(y, k_l^{-1})], \tag{4.2.12}$$

and we obtain a soliton-type solution.

Furthermore if $M=2N$ and

$$a_{l,j}=\begin{cases}\delta_{l,j} & (1\leqq l\leqq N),\\ \delta_{l,j+N}c_j & (N+1\leqq l\leqq 2N),\end{cases} \qquad k_l=\begin{cases}q_l & (1\leqq l\leqq N),\\ p_{l-N} & (N+1\leqq l\leqq 2N),\end{cases}$$

then we recover the classical soliton solution of the Gram determinant type (up to simple exponential factors)

$$\tau'(s; x, y)=\prod_{i=1}^{N} e^{\eta(q_i)}q_i^s\cdot \prod_{i>j} (q_i-q_j)$$
$$\times\det\left[\delta_{ij}+c_j e^{\eta(p_j)-\eta(q_j)}\left(\frac{p_j}{q_j}\right)^s \frac{(p_j-q_j)\prod_{l(\neq j)}(p_j-q_l)}{(p_j-q_i)\prod_{l(\neq i)}(q_i-q_l)}\right]_{i,j=1,\cdots,N}.$$

Applying the expansion formula for $\det(1+X)$, remarked in the previous section, to the last determinant, we get

$$\begin{aligned}\tau'(s; x, y)&=\prod_{i=1}^{N} e^{\eta(q_i)}q_i^s\cdot \prod_{i>j} (q_i-q_j)\\ &\times\sum_{l=0}^{N}\sum_{i_1<\cdots<i_l} c_{i_1}\cdots c_{i_l}a_{i_1}(s)\cdots a_{i_l}(s)\exp\sum_{\mu=1}^{l}(\eta(p_{i_\mu})-\eta(q_{i_\mu})).\end{aligned} \tag{4.2.13}$$

Here the notations are the same as in (4.1.35) and

$$a_i=\frac{c_i\prod_{l(\neq i)}(p_i-q_l)}{\prod_{l(\neq i)}(q_i-q_l)}.$$

Thus we get the soliton solution (4.1.35) up to the trivial multiplier $\prod_{i=1}^{N} e^{\eta(q_i)}q_i^s\cdot\prod_{i>j}(q_i-q_j)$ which can be absorbed in the trivial arbitrariness of wave matrices indicated in Theorem 1.2.

Remark 4.10. In the expression of the solution there appeared infinite series of the form $\sum_{n\in Z} c_n p_n(x, y)$, where $c_n (n \in z)$ are constants. Using the integral representation

$$p_j(x, y) = \frac{1}{2\pi\sqrt{-1}} \oint_{|\lambda|=r} \lambda^{-j-1} \exp[\xi(x, \lambda) + \xi(y, \lambda^{-1})] d\lambda,$$

we can estimate $|p_j(x, y)|$, where the integration contour is chosen to be in the convergence domain of the Laurent series $\xi(x, \lambda) + \xi(y, \lambda^{-1})$. In this way we can easily prove, under the condition $\lim.\sup_{|n|\to\infty} |c_n|^{1/n} < \infty$, that the series $\sum_{n\in Z} c_n p_n(x, y)$ converges absolutely in the domain

$$\begin{cases} \lim\limits_{n\to\infty}.\sup. |x_n|^{1/n} \cdot \lim\limits_{n\to\infty}.\sup. |c_n|^{1/n} < 1, \\ \lim\limits_{n\to\infty}.\sup. |y_n|^{1/n} \cdot \lim\limits_{n\to-\infty}.\sup. |c_n|^{-1/n} < 1, \\ \lim\limits_{n\to\infty}.\sup. |x_n|^{1/n} \cdot \lim\limits_{n\to\infty}.\sup. |y_n|^{1/n} < 1. \end{cases}$$

Now we proceed to the proof of Theorem 4.8.
We prepare two lemmas.

Lemma 4.11. *We have the following formulas.*

$$\partial_{x_j} p_i(x, y) = p_{i-j}(x, y), \qquad \partial_{y_j} p_i(x, y) = p_{i+j}(x, y),$$
$$\partial_{x_j} f_k(s; x, y) = f_k(s+j; x, y), \quad \partial_{y_j} f_k(s; x, y) = f_k(s-j; x, y).$$

This is an immediate consequence of (4.2.2) *and* (4.2.3).

Lemma 4.12. *For any matrix* $U = \sum_{j\geq 0} \operatorname{diag}[u_j(s)] \Lambda^j$ *there exist two matrices* Q *and* R *uniquely such that*

$$\begin{cases} U = QW_N + R, \\ Q = \sum\limits_{j\geqq 0} \operatorname{diag}[q_j(s)]\Lambda^j, \quad R = \sum\limits_{j=0}^{N-1} \operatorname{diag}[r_j(s)]\Lambda^j. \end{cases} \tag{4.2.14}$$

Similarly, for any matrix $U' = \sum_{j\leq 0} \operatorname{diag}[u'_j(s)]\Lambda^j$ *there exist two matrices* Q' *and* R' *uniquely such that*

$$\begin{cases} U' = Q'W_N\Lambda^{-N} + R', \\ Q' = \sum\limits_{j\leqq 0} \operatorname{diag}[q'_j(s)]\Lambda^j, \quad R' = \sum\limits_{j=1-N}^{0} \operatorname{diag}[r'_j(s)]\Lambda^j. \end{cases} \tag{4.2.14$'$}$$

Proof. Equating the coefficient matrices of Λ^j in the equalities

$$U = QW_N + R, \quad U' = Q'W_N\Lambda^{-N} + R',$$

we get a series of linear equations for q_j, r_j, q'_j, r'_j. Since $w_0 = 1$ and w_N is invertible (cf. (2.2.7)) we can solve them recursively and uniquely. This proves Lemma 4.12. Q.E.D.

Let us prove, by use of these lemmas, that there exist an upper triangular matrix B_n and a lower triangular one C_n of infinite size such that the following equations are satisfied for $n = 1, 2, \cdots$.

(4.2.15) $$\partial_{x_n} W_N + W_N \Lambda^n = B_n W_N,$$

(4.2.16) $$\partial_{y_n} W_N + W_N \Lambda^{-n} = C_n W_N.$$

Rewrite (4.2.5) in the form

(4.2.17) $$W_N f_j(x, y) = 0 \quad (j = 1, \cdots, N),$$

where we set $f_j(x, y) = (f_j(i; x, y))_{i \in Z}$. Differentiating (4.2.17) with respect to x_n and using Lemma 4.11, we have

$$(\partial_{x_n} W_N + W_N \Lambda^n) f_j(x, y) = 0 \quad (j = 1, \cdots, N).$$

On the other hand the former half of Lemma 4.12 implies that there exist certain matrices B_n and R_n of the form

$$B_n = \sum_{j \geq 0} \operatorname{diag}[b_{n,j}(s; x, y)]\Lambda^j, \quad R_n = \sum_{j=0}^{N-1} \operatorname{diag}[r_{n,j}(s; x, y)]\Lambda^j$$

such that

$$\partial_{x_n} W_N + W_N \Lambda^n = B_n W_N + R_n.$$

Hence

$$R_n f_j(x, y) = 0 \quad (j = 1, \cdots, N),$$

or equivalently,

$$(r_0, \cdots, r_{N-1})(f_j(s+i; x, y))_{i,j=1,\dots,N} = 0.$$

In view of (4.2.4) we conclude $R_n = 0$, and hence (4.2.15).

Similarly, from the equalities

$$(\partial_{y_n}(W_N \Lambda^{-N}) + (W_N \Lambda^{-N})\Lambda^{-n})\Lambda^N f_j(x, y) = 0 \quad (j = 1, \cdots, N),$$

we can show (4.2.16), using the latter half of Lemma 4.12.

(4.2.15) and (4.2.16) implies that $W^{(\infty)}$ and $W^{(0)}$ defined by (4.2.9) solve the linear equations (1.2.8). Hence B_n and C_n solve the Toda lattice hierarchy (cf. (ii) of Theorem3.3).

For the proof of (4.2.10) it suffices to prove the following.

$$\tag{4.2.18} \begin{cases} 1+\sum_{i=0}^{N-1} w_{N-i}(s; x, y)\lambda^{i-N} = \dfrac{\det [f_j(s+i-1; x-\varepsilon(\lambda^{-1}), y)]_{i,j=1,\cdots,N}}{\det [f_j(s+i-1; x, y)]_{i,j=1,\cdots,N}}, \\ \lambda^N+\sum_{i=0}^{N-1} w_{N-i}(s; x, y)\lambda^{i} = \dfrac{\det [f_j(s+i; x, y-\varepsilon(\lambda))]_{i,j=1,\cdots,N}}{\det [f_j(s+i-1; x, y)]_{i,j=1,\cdots,N}}. \end{cases}$$

If we notice the formula

$$\tag{4.2.19} \begin{cases} f_j(s; x-\varepsilon(\lambda^{-1}), y)=f_j(s; x, y)-\lambda^{-1}f_j(s+1; x, y), \\ f_j(s; x, y-\varepsilon(\lambda))=f_j(s; x, y)-\lambda f_j(s-1; x, y), \end{cases}$$

we can show (4.2.18) by a simple calculation of linear algebra, comparing (4.2.6) with the right hand side of (4.2.18). (4.2.19) is an immediate consequence of the formulas

$$\begin{cases} p_j(x-\varepsilon(\lambda^{-1}), y)=p_j(x, y)-\lambda^{-1}p_{j-1}(x, y), \\ p_j(x, y-\varepsilon(\lambda))=p_j(x, y)-\lambda p_{j+1}(x, y), \end{cases}$$

which are derived from (4.2.2) and the formula

$$\exp \xi(-\varepsilon(\lambda), \lambda')=1-\lambda\lambda'.$$

Thus we have proved Theorem 4.8.

Next, let us consider a condition for the l-periodicity, i.e. a condition under which we have

$$\tag{4.2.20} [W^{(\infty)}, \Lambda^l]=0, \quad [W^{(0)}, \Lambda^l]=0.$$

Theorem 4.13. *Suppose that for the* $\boldsymbol{Z}\times N$ *matrix* $\boldsymbol{f}=(f_{i,j})_{i\in \boldsymbol{Z},\ j=1,\cdots,N}$ *there exists a constant* $N\times N$ *matrix* C *such that*

$$\tag{4.2.21} \Lambda^l \boldsymbol{f}=\boldsymbol{f}C.$$

Then (4.2.20) *holds. Moreover we have*

$$\tag{4.2.22} \begin{cases} [W_N, \Lambda^l]=0, \\ \partial_{x_{ln}}W_N=0, \quad \partial_{y_{ln}}W_N=0, \quad n=1, 2, \cdots. \end{cases}$$

Proof. Set $\boldsymbol{f}(x, y)=(f_j(i; x, y))_{i\in \boldsymbol{Z},\ j=1,\cdots,N}$. (2.2.21) implies

$$\Lambda^l f(x, y) = f(x, y) C,$$

and in view of (4.2.17) it leads to

$$W_N \Lambda^l f(x, y) = 0.$$

Then we can show, as we derived (4.2.15), that there exists a matrix $Q = \sum_{j=0}^{l} \mathrm{diag}\,[q_j(s; x, y)] \Lambda^j$ such that

$$W_N \Lambda^l = Q W_N. \tag{4.2.23}$$

Hence we have two expressions for Q in terms of $L = \hat{W}^{(\infty)} \Lambda \hat{W}^{(\infty)-1}$ and $M = \hat{W}^{(0)} \Lambda^{-1} \hat{W}^{(0)-1}$,

$$Q = \hat{W}^{(\infty)} \Lambda^l \hat{W}^{(\infty)-1} = L^l, \quad Q = \hat{W}^{(0)} \Lambda^{-l} \hat{W}^{(0)-1} = M^l,$$

which immediately imply the following.

$$L^l = M^{-l} = \Lambda^l, \quad Q = \Lambda^l. \tag{4.2.24}$$

From (4.2.23), (4.2.24), (4.2.15) and (4.2.16) we have (4.2.21) and (4.2.22). This proves Theorem 4.13.

At the end of this section we shall briefly comment on the multi-component case. Also in this case special solutions of the Wronskian type are constructed in the same way as we have just discussed. We shall show only the results:

In the r component case $f_{i,j}$ and w_i are replaced by matrices of size $r \times r$, and we set

$$f_j(s; x, y) = \sum_{\alpha=1}^{r} \sum_{i \in Z} p_{i-s}(x^{(\alpha)}, y^{(\alpha)}) E_\alpha f_{i,j}. \tag{4.2.25}$$

w_i $(i = 1, \cdots, N)$ are defined by (4.2.5) under the condition (4.2.4).

Since Lemma 4.12 is also valid in the multi-component case under the condition that w_N is invertible, we can derive

$$\begin{cases} \partial_{x_n^{(\alpha)}} W_N + W_N \boldsymbol{\Lambda}^n \boldsymbol{E}_\alpha = B_n^{(\alpha)} W_N, \\ \partial_{y_n^{(\alpha)}} W_N + W_N \boldsymbol{\Lambda}^{-n} \boldsymbol{E}_\alpha = C_n^{(\alpha)} W_N, \end{cases} \tag{4.2.26}$$

for the matrix $W_N = \sum_{j=0}^{N} \mathrm{diag}\,[w_j(s; x, y)]\ \boldsymbol{\Lambda}^{N-j}$ with $w_0 = 1_r$. Hence $W^{(\infty)}$ and $W^{(\infty)}$ defined by

$$\begin{cases} W^{(\infty)} = W_N \boldsymbol{\Lambda}^{-N} \exp\left(\sum_{\alpha=1}^{r} \xi(x^{(\alpha)}, \boldsymbol{\Lambda}) \boldsymbol{E}_\alpha + \sum_{\alpha=1}^{r} \xi(y^{(\alpha)}, \boldsymbol{\Lambda}^{-1}) \boldsymbol{E}_\alpha\right), \\ W^{(0)} = W_N \exp\left(\sum_{\alpha=1}^{r} \xi(x^{(\alpha)}, \boldsymbol{\Lambda}) \boldsymbol{E}_\alpha + \sum_{\alpha=1}^{r} \xi(y^{(\alpha)}, \boldsymbol{\Lambda}^{-1}) \boldsymbol{E}_\alpha\right), \end{cases}$$

solve the linearized equation of the r component theory.

Similar argument as in the proof of Theorem 4.13 leads to a condition for the reduction to the system of the Zakharov-Mikhailov type: If there exists a constant matrix C of size $Nr \times Nr$ such that

$$\Lambda f = fC \quad \text{for } f = (f_{i,j})_{\substack{i \in Z, \\ j=1,\cdots,N}}, \tag{4.2.27}$$

then we have

$$[W^{(\infty)}, \Lambda] = [W^{(0)}, \Lambda] = [W_N, \Lambda] = 0. \tag{4.2.28}$$

Appendix. A Brief Summary of the *KP* Theory.

In this appendix, for the reader's convenience, we shall briefly summarize the recent results [12], [20–25], [33], [34] in the study of the *KP* hierarchy.

1.1. Microdifferential operators.

Let $\mathcal{O}$ be a differential algebra with a derivation ∂. A microdifferential (or pseudodifferential) operator with coefficients in $\mathcal{O}$ is, by definition, a formal sum $\sum_{j \in Z} a_j \partial^j$ with $a_j \in \mathcal{O}$ and $a_j = 0$ for any sufficiently large j (the integer $m = \max\{j; a_j \neq 0\}$ is called the order of $\sum_{j \in Z} a_j \partial^j$), and the sum and the product of two microdifferential operators are defined by the following.

$$\begin{cases} \sum_j a_j \partial^j + \sum_j b_j \partial^j = \sum_j (a_j + b_j) \partial^j, \\ \sum_j a_j \partial^j \cdot \sum_j b_j \partial^j = \sum_j c_j \partial^j \quad \text{where} \\ c_j = \sum_{\substack{k, l \in Z, \alpha \geqq 0 \\ k+l-\alpha = j}} \binom{j}{\alpha} a_k \cdot \partial^\alpha b_l. \end{cases} \tag{A. 1}$$

We denote by $\mathcal{E}$ (resp. $\mathcal{D}$, $\mathcal{E}^{(-1)}$) the totality of microdifferential operators (resp. differential operators, microdifferential operators of order <0). Then $\mathcal{D}$ is a subalgebra of $\mathcal{E}$, and there is a direct sum decomposition

$$\begin{cases} \mathcal{E} = \mathcal{D} \oplus \mathcal{E}^{(-1)}, \\ \sum_{j \in Z} a_j \partial^j = \sum_{j \geqq 0} a_j \partial^j + \sum_{j < 0} a_j \partial^j. \end{cases} \tag{A. 2}$$

We denote by $(\quad)_\pm$ the projections to $\mathcal{D}$ and $\mathcal{E}^{(-1)}$;

(A. 3) $$(\sum_{j\in Z} a_j\partial^j)_+=\sum_{j\geq 0} a_j\partial^j,\quad (\sum_{j\in Z} a_j\partial^j)_-=\sum_{j<0} a_j\partial^j.$$

The formal adjoint P^* of a microdifferential operator P is defined by

(A. 4) $$(\sum_j a_j\partial^j)^*=\sum_j (-\partial)^j a_j,$$

which induces an anti-isomorphism of $\mathscr{E}$.

1.2. One component theory

In this case $\mathcal{O}$ is a suitable differential algebra consisting of functions in the independent variables $x=(x_1, x_2, \cdots)$ with the derivation

(A. 5) $$\partial=\partial_{x_1}.$$

As the dependent variable we introduce a microdifferential operator L of the form

(A. 6) $$L=\partial+u_{-1}\partial^{-1}+u_{-2}\partial^{-2}+\cdots,\quad u_j=u_j(x)\in\mathcal{O}.$$

We set

(A. 7) $$B_n=(L^n)_+,\quad n=1, 2, \cdots.$$

Then the one component hierarchy is defined by the system of the Lax-type equations

(A. 8) $$\partial L/\partial_{x_n}=[B_n, L],\quad n=1, 2, \cdots,$$

where ∂/∂_{x_n} denotes the differentiation of the coefficients of L with respect to x_n.

(A. 8) is equivalent to the system of the Zakharov-Shabat type

(A. 9) $$\partial B_m/\partial_{x_n}-\partial B_n/\partial_{x_m}+[B_m, B_n]=0,\quad m, n=1, 2, \cdots.$$

The equation $\partial B_2/\partial_{x_3}-\partial B_3/\partial_{x_2}+[B_2, B_3]=0$ is nothing but the *KP* (Kadomtsev-Petviashvili) equation

(A. 10) $$3u_{yy}+(-4u_t+u_{xx}+6uu_x)_x=0,$$

where $u=u_{-1}$ and $(x, y, t)=(x_1, x_2, x_3)$. Thus (A. 8) and (A. 9) give a hierarchy for the *KP* equation.

The linearization is achieved by the system

(A. 11) $$Lw=\lambda w,$$

(A. 12) $$\partial_{x_n} w = B_n w, \quad n=1, 2, \cdots,$$

where $w=w(x;\lambda)$ is a formal Laurent series of λ of the form

(A. 13) $$\begin{cases} w(x;\lambda)=\left(\sum_{j=0}^{\infty}\hat{w}_j(x)\lambda^{-j}\right)\exp\xi(x,\lambda), \\ w_j(x)\in\mathcal{O}, \quad w_0(x)=1, \quad \xi(x,\lambda)=\sum_{n=1}^{\infty}x_n\lambda^n, \end{cases}$$

or equivalently, given by

(A. 14) $$\begin{cases} w(x;\lambda)=\hat{W}(x;\partial)\exp\xi(x,\lambda), \\ \hat{W}(x;\partial)=\sum_{j=0}^{\infty}\hat{w}_j(x)\partial^{-j}\in\mathcal{E}. \end{cases}$$

Remark. Here we used the convention that the action of microdifferential operators on $\exp\xi(x;\lambda)$, or on a series of the form

$$\sum_j b_j\lambda^j \exp\xi(x,\lambda)\ (\sum_j b_j\partial^j\in\mathcal{E}),$$

is defined by the formulas

(A. 15) $$\begin{cases} (\sum_j a_j\partial^j)\exp\xi(x,\lambda)=\sum_j a_j\lambda^j\exp\xi(x,\lambda), \\ (\sum_j a_j\partial^j)(\sum_j b_j\lambda^j\exp\xi(x,\lambda))=\sum_j c_j\lambda^j\exp\xi(x,\lambda), \end{cases}$$

where c_j is the element defined in (A. 1). Thus $\exp\xi(x,\lambda)$ generates a free $\mathcal{E}$-module of rank one.

We notice that in terms of $\hat{W}$, (A. 11) and (A. 12) are rewritten in the form

(A. 16) $$L=\hat{W}\partial\hat{W}^{-1},$$

(A. 17) $$\partial\hat{W}/\partial_{x_n}=B_n\hat{W}-\hat{W}\partial^n, \quad n=1, 2, \cdots.$$

The equivalence of three systems (A. 8), (A. 9) and (A. 11)+(A. 12) are established in the same way as we did in the case of the Toda lattice. We call a solution to (A. 11)+(A. 12) a wave function of the *KP* hierarchy.

The wave function $w(x;\lambda)$ is characterized by the following bilinear equation

(A. 18) $$\oint w(x;\lambda)w^*(x';\lambda)d\lambda=0 \quad \text{for any } x \text{ and } x',$$

where the integration contour is a small circle around $\lambda=\infty$, while

(A. 19) $$w^*(x;\lambda)=(\hat{W}(x,\partial)^*)^{-1}\exp\xi(-x,\lambda),$$

and $\hat{W}^*$ is the formal adjoint operator of $\hat{W}$. (A. 18) is a generating functional expression of infinitely many equations with the indeterminate $x-x'$.

The τ function $\tau(x)$ is consistently introduced by the formula

(A. 20) $$w(x;\lambda)=\frac{\tau(x-\varepsilon(\lambda^{-1}))\exp\xi(x,\lambda)}{\tau(x)},\quad \varepsilon(\lambda^{-1})=\left(\lambda^{-1},\frac{\lambda^{-2}}{2},\frac{\lambda^{-3}}{3},\cdots\right).$$

Then the original hierarchy for the dependent variable L is transformed into the bilinear equation for the τ function of the form

(A. 21) $$\sum_{j=0}^{\infty}p_j(-2u)p_{j+1}(\tilde{D}_x)e^{\langle u,D_x\rangle}\tau\cdot\tau=0,\quad \tilde{D}_x=\left(D_{x_1},\frac{D_{x_2}}{2},\frac{D_{x_3}}{3},\cdots\right),$$

which is a generating functional expression, with the indeterminate $u=(u_1,u_2,\cdots)$, of infinitely many bilinear equations of the Hirota type. The first one is

(A. 22) $$(D_{x_1}^4+3D_{x_2}^2-4D_{x_1}D_{x_3})\tau\cdot\tau=0,$$

which is equivalent to (A. 10) with $u=\partial^2(\log\tau)/\partial x_1^2$.

Remark. The wave functions of the *BKP* and *CKP* hierarchies [23–25] are characterized by the following bilinear equations

(A. 23) $$\oint w(x,\lambda)w(x',-\lambda)\lambda^n d\lambda=\delta_{n0}\quad\text{for any } x, x',$$

($n=0$ for *BKP*, $n=1$ for *CKP*), where the evolution is restricted to the odd sector $\{x_2=x_4=\cdots=0\}$.

Sato [34] discovered a remarkable fact that the structure of the τ functions is completely discribed in terms of the (infinite-dimensional) Grassmann manifold as follows:

(A. 24) $$\begin{aligned}\tau(x)&=\det({}^t f_0\exp(x_1\Lambda+x_2\Lambda^2+\cdots)f)\\&=\sum_{Y:\text{Young diagram}}\chi_Y(x)f_Y,\end{aligned}$$

where f and f_0 are constant matrices of size $Z\times N^c$, $f=(f_{ij})_{\substack{i\in Z\\ j\in N^c}}$, $f_0=(\delta_{ij})_{\substack{i\in Z\\ j\in N^c}}$, $N^c=\{-1,-2,\cdots\}$. f_Y is the Plücker coordinate of the "frame" f corresponding to the Young diagram Y. $\chi_Y(x)$ is the character polynomial (the Schur function) which we encountered in Section 4.1.

We omit the precise definitions of these concepts (cf. [34]).

The rational solutions, i.e. the solutions with polynomial τ functions, are constructed and parametrized as follows [33]: As the data we give a constant matrix $f=(f_{ij})_{\substack{i=-m,1-m,\cdots,n-1\\ j=-m,1-m,\cdots,-1}}$ of size $(m+n)\times m$ (m and n are positive integers), and set

$$f(x)=(f_{ij}(x))_{\substack{i=-m,1-m,\cdots,n-1\\ j=-m,1-m,\cdots,-1}}=\exp(x_1\Lambda+x_2\Lambda^2+\cdots)f, \tag{A. 25}$$

where $\Lambda=(\delta_{i-j+1})_{i,j=-m,1-m,\cdots,n-1}$. Notice that we have the Wronskian structure

$$f_{i,j}(x)=\partial^{i+m}f_{-m,j}(x),\quad i=-m,1-m,\cdots,n-1. \tag{A. 26}$$

We assume the condition

$$\operatorname{rank} f=m. \tag{A. 27}$$

Then $\det(f_{i,j}(x))_{i,j=-m,1-m,\cdots,-1}\not\equiv 0$. Hence the functions $w_1(x),\cdots,w_m(x)$ are uniquely determined by

$$(\partial^m+w_1\partial^{m-1}+\cdots+w_m)f_{-m,j}(x)=0,\ j=-m,\cdots,-1. \tag{A. 28}$$

Furthermore in the same way as we discussed in Section 4.2, using a division theorem for differential operators instead of that for matrices of infinite size, we can conclude that the microdifferential operator $\hat{W}=1+w_1\partial^{-1}+\cdots+w_m\partial^{-m}$ solves (A. 7). Hence the L defined by (A. 16) solves the hierarchy. The corresponding τ function is given by

$$\begin{aligned}\tau(x)&=\det({}^t f_0\exp(x_1\Lambda+x_2\Lambda^2+\cdots)\xi)\\ &=\sum_{-m\leq l_{-m}<\cdots<l_{-1}<n}\chi_{l_{-m}\cdots l_{-1}}(x)f_{l_{-m}\cdots l_{-1}},\end{aligned} \tag{A. 29}$$

where

$$f_0=(\delta_{ij})_{\substack{i=-m,\cdots,n-1,\\ j=-m,\cdots,-1}}\quad \chi_{l_{-m}\cdots l_{-1}}(x)=\det(p_{l_i-j}(x))_{i,j=-m,\cdots,-1}$$

and $f_{l_{-m}\cdots l_{-1}}=\det(f_{l_i,j})_{i,j=-m,\cdots,-1}$.

The transformation $f\mapsto fC(C\in GL(m))$ changes τ into $\tau\det C$. Thus the polynomial τ functions are parametrized, up to constant multipliers, by the equivalence classes of "frames" f (i.e. $(m+n)\times n$-matrices with (A. 27)) with respect to the equivalence relation $f\sim fC(C\in GL(m))$, namely by the Grassmann manifold $GM(m,n)$.

We note here that the method stated above is also valid in the case

$n=\infty$, $m<\infty$. Then we obtain the special solutions of the Wronskian type to the *KP* hierarchy (cf. § 4.2).

The formula (A. 24) is established in a suitable limit procedure as $m, n\to\infty$.

An alternative expression of the τ functions is given in terms of the vacuum expectation values of Clifford operators [20, 22].

1.3. Multi-component theory

In the r component theory we introduce the independent variables $x=(x^{(1)},\cdots,x^{(r)})$, $x^{(\alpha)}=(x_1^{(\alpha)},x_2^{(\alpha)},\cdots)$ $(\alpha=1,\cdots,r)$, and $\mathcal{O}$ is a suitable differential algebra consisting of matrix-valued functions of x of size $r\times r$ with the derivation

$$\partial=\sum_{\alpha=1}^{r}\partial_{x_1^{(\alpha)}} \tag{A. 30}$$

As the dependent variables we consider microdifferential operators L and $U_\alpha(\alpha=1,\cdots,r)$ of the form (cf. § 3.1)

$$\begin{cases} L=\sum\limits_{j=-\infty}^{1}u_j\partial^j \quad \text{with } u_j\in\mathcal{O},\ u_1=1_r,\ u_0=0,\\ U_\alpha=\sum\limits_{j=-\infty}^{0}u_{j,\alpha}\partial^j \quad \text{with } u_{j,\alpha}\in\mathcal{O},\ u_{0,j}=E_\alpha,\end{cases} \tag{A. 31}$$

(our notations are slightly different from those used in [34]), and assume the following algebraic conditions

$$\begin{cases} [L,U_\alpha]=0, \quad [U_\alpha,U_\beta]=0,\\ \sum\limits_{\alpha=1}^{r}U_\alpha=1_r, \quad U_\alpha U_\beta=\delta_{\alpha\beta}U_\beta, \quad \alpha,\beta=1,\cdots,r.\end{cases} \tag{A. 32}$$

We set

$$B_n^{(\alpha)}=(L^nU_\alpha)_+, \quad \alpha=1,\cdots,r,\ n=1,2,\cdots. \tag{A. 33}$$

Then the r component hierarchy is defined by the system of the Lax type

$$\partial L/\partial_{x_n^{(\alpha)}}=[B_n^{(\alpha)},L], \quad \partial U_\beta/\partial_{x_n^{(\alpha)}}=[B_n^{(\alpha)},U_\beta], \quad \alpha,\beta=1,\cdots,r,\ n=1,2,\cdots, \tag{A. 34}$$

which is equivalent to the system of the Zakharov-Shabat type

$$\partial B_m^{(\alpha)}/\partial_{x_n^{(\beta)}}-\partial B_n^{(\beta)}/\partial_{x_m^{(\alpha)}}+[B_m^{(\alpha)},B_n^{(\beta)}]=0, \quad \alpha,\beta=1,\cdots,r, \quad m,n=1,2,\cdots. \tag{A. 35}$$

The linearization is achieved by

$$LW=\lambda W, \quad U_\alpha W=WE_\alpha, \quad \alpha=1, \cdots, r, \tag{A. 36}$$

$$\partial_{x_n^{(\alpha)}} W=B_n^{(\alpha)}W, \quad \alpha=1, \cdots, r, \ n=1, 2, \cdots, \tag{A. 37}$$

where $W=W(x;\lambda)$ is a matrix-valued formal Laurent series of λ of the form

$$W(x;\lambda)=\sum_{j=0}^{\infty} \hat{w}_j(x)\lambda^{-j}\cdot\exp\left(\sum_{\alpha=1}^{r}\xi(x^{(\alpha)}, \lambda)E_\alpha\right), \quad w_j \in \mathcal{O}, \ w_0=1_r. \tag{A. 38}$$

Using the microdifferential operator

$$\hat{W}(x;\partial)=\sum_{j=0}^{\infty} w_j(x)\partial^{-j}, \tag{A. 39}$$

we can rewrite (A. 36) and (A. 37) into

$$L=\hat{W}\partial\hat{W}^{-1}, \quad U_\alpha=\hat{W}E_\alpha\hat{W}^{-1}, \tag{A. 40}$$

$$\partial\hat{W}/\partial_{x_n^{(\alpha)}}=B_n^{(\alpha)}\hat{W}-\hat{W}E_\alpha\partial^n. \tag{A. 41}$$

In the r component case we need several τ functions $\tau(x)$ and $\tau_{\alpha\beta}(x)$ $(\alpha\neq\beta)$ which are consistently introduced by

$$W(x;\lambda)_{\alpha\beta}=\begin{cases}\dfrac{\tau(x-\varepsilon_\alpha(\lambda^{-1}))\exp\xi(x^{(\alpha)}, \lambda)}{\tau(x)} & (\alpha=\beta), \\[2ex] \dfrac{\tau_{\alpha\beta}(x-\varepsilon_\beta(\lambda^{-1}))\exp\xi(x^{(\beta)}, \lambda)}{\tau(x)} & (\alpha\neq\beta),\end{cases} \tag{A. 42}$$

where $\varepsilon_\beta(\lambda^{-1})=(0, \cdots, 0, \overset{(\beta)}{\varepsilon(\lambda^{-1})}, 0, \cdots, 0)$, and the subindex (α, β) indicates the (α, β) component of a matrix of size $r\times r$.

The τ functions have a parametrization like (A. 24) in terms of the (infinite-dimensional) Grassmann manifolds. Also in terms of the vacuum expectation values $\tau_l(x)$ $(l=(l_1, \cdots, l_r)\in \boldsymbol{Z}^r$ with $\sum_{\alpha=1}^r l_\alpha=0)$ introduced in [22] they are parametrized as follows.

$$\begin{cases}\tau(x)=(\text{a signature factor})\cdot\tau_{0\cdots0}(x) \\ \tau_{\alpha\beta}(x)=(\text{a signature factor})\cdot\tau_{0\cdots\underset{(\alpha)}{1}\cdots\underset{(\beta)}{-1}\cdots0}(x)\end{cases} \tag{A. 43}$$

The wave functions $W_l(x;\lambda)$ and $W_l^*(x;\lambda)$ are introduced by the formula

(A. 44)

$$\begin{cases} W_l(x;\lambda)_{\alpha\beta}=\dfrac{\sigma_{\alpha\beta}(l)\tau_{l_1\cdots l_\alpha+1\cdots l_\beta-1\cdots l_r}(x-\varepsilon_\beta(\lambda^{-1}))\lambda^{l_\beta+\delta_{\alpha\beta}-1}\exp\xi(x^{(\beta)},\lambda)}{\tau_l(x)}, \\ W_l^*(x;\lambda)_{\alpha\beta}=\dfrac{\sigma_{\alpha\beta}(l)\tau_{l_1\cdots l_\alpha-1\cdots l_\beta+1\cdots l_r}(x+\varepsilon_\beta(\lambda^{-1}))\lambda^{-l_\beta+\delta_{\alpha\beta}-1}\exp\xi(-x^{(\beta)},\lambda)}{\tau_l(x)}, \end{cases}$$

and satisfy the bilinear equation

(A. 45) $$\oint W_l(x;\lambda)^t W_{l'}^*(x';\lambda)d\lambda=0 \quad \text{for any } l,\ l',\ x \text{ and } x',$$

where $\sigma_{\alpha\beta}(l)=(-1)^{l_{\alpha+1}+\cdots l_\beta}(\alpha<\beta)$, 1 $(\alpha=\beta)$, $(-1)^{l_{\beta+1}+\cdots+l_\alpha}(\alpha>\beta)$, and $(l_1\cdots l_\alpha\pm1\cdots l_\beta\mp1\cdots l_r)$ is replaced by $(l_1\cdots l_r)$ when $\alpha=\beta$. (Here our normalization of wave functions is slightly different from the original one used in [22].)

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