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On Riemannian Manifolds of Nonnegative Ricci Curvature Containing Compact Minimal Hypersurfaces

Ryosuke Ichida

§ 0. Introduction

In this paper we study geometric properties of Riemannian manifolds which contain compact minimal hypersurfaces. Our main result, Theorem 4.1, of this paper is stated as follows.

Let N be an n-dimensional ($n \geqslant 2$) connected, complete, real analytic Riemannian manifold without boundary. Let (M_1, f_1) and (M_2, f_2) be compact, connected, real analytic, minimal hypersurfaces immersed in N by real analytic immersions f_1 and f_2. Suppose that N is of nonnegative Ricci curvature and that $f_1(M_1) \cap f_2(M_2) = \phi$. Then N is isometric to one of four types of real analytic Riemannian manifolds described in Section 4.

In case N is a complete, connected, locally symmetric space of nonnegative sectional curvature, such classification was already done by Nakagawa and Shiohama ([6]).

Theorem 1.1 in Section 1 plays important roles in this paper which was proved by the author ([4]). In Section 2 we give an application of Theorem 1.1. Making use of this theorem, we obtain Lemma 2.1 which is a basic lemma of this paper. In Section 3 we study geometric properties of compact, connected Riemannian manifolds with boundary which contain a compact minimal hypersurface. Results of this section will be used to prove Theorems 4.1 and 4.2. Theorem 3.1 was also proved by Kasue independently ([5]).

As applications of Theorem 4.2 we obtain Theorems 4.3 and 4.4. For connected, complete Riemannian manifolds of positive Ricci curvature, Frankel proved the assertions of Theorems 4.3 and 4.4 ([3]). But, in general, Frankel's result does not hold for Riemannian manifolds of nonnegative Ricci curvature. We can easily give counterexamples. Therefore, in our theorems, we need the assumption that Riemannian manifolds are homogeneous.

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Throughout this paper we always assume that manifolds and apparatus on them are of class C^∞, unless otherwise stated.

§ 1. Minimum principle

Let N be an n-dimensional ($n \geqslant 2$) connected Riemannian manifold and let $\langle\ ,\ \rangle$ be the Riemannian metric of N. Let $e_1, \cdots, e_n$ be an orthonormal basis of the tangent vector space T_pN at a point p of N and let X be a unit tangent vector at p. The quantity $\mathrm{Ric}_N(X) = \sum_{i=1}^n \langle R(e_i, X)X, e_i\rangle$ is called the Ricci curvature of N with respect to X direction where R denotes the Riemannian curvature tensor of N. We say that N is of nonnegative (resp. positive) Ricci curvature if $\mathrm{Ric}_N(X) \geqslant 0$ (resp. $\mathrm{Ric}_N(X) > 0$) for every unit tangent vector X at every point of N. Let N be as above and let f: $M \to N$ be an isometric immersion of an $(n-1)$-dimensional Riemannian manifold M into N. We say that (M, f) is a minimal hypersurface in N if the trace of the second fundamental form of M for f is zero everywhere. It is called that (M, f) is a totally geodesic hypersurface in N if the second fundamental form of M for f vanishes identically.

Let D be an open metric ball in the n-dimensional ($n \geq 1$) Euclidean space R^n. Let $(x_1, \cdots, x_n)$ be the canonical coordinate system in R^n. For an $r > 0$, let us consider a Riemannian manifold $N = (D \times (-r, r), ds^2)$ whose line element is given by $ds^2 = \sum_{i,j=1}^n g_{ij}(x, t)dx_i dx_j + dt^2$. Let ∇ be the Riemannian connection of N. For a t, $|t| < r$, we denote the mean curvature (with respect to $\partial/\partial t$) of the level hypersurface $S_t = \{(x, t); x \in D\}$ in N by H_t. In case $n = 1$, by the mean curvature we mean the geodesic curvature.

Lemma 1.1. *Under the above situation, suppose $Ric_N(\partial/\partial t) \geqslant 0$. Then $H_t \leqslant H_{t'}$ holds for any $t < t'$. If $H_t = H_{t'}$ for $t < t'$, then for each s, $t \leqslant s \leqslant t'$, S_s is totally geodesic.*

For the proof, see [4].

Now for a $u \in C^2(D)$, $|u| < r$, we consider a hypersurface $S = \{(x, u(x)); x \in D\}$ in N. We put $X_i = \partial/\partial x_i + u_i\, \partial/\partial t$ and $\tilde{g}_{ij} = g_{ij} + u_i u_j$ where $u_i = \partial u/\partial x_i$, $1 \leqslant i, j \leqslant n$. We can give a unit normal vector field $\xi = \sum_{i=1}^n \xi^i\, \partial/\partial x_i + \xi^{n+1}\, \partial/\partial t$ on S as follows

$$\xi^i = -u^i/(1 + \|\nabla u\|^2)^{1/2}\ (1 \leqslant i \leqslant n) \quad \text{and} \quad \xi^{n+1} = 1/(1 + \|\nabla u\|^2)^{1/2}$$

where $\|\nabla u\|^2 = \sum_{i,j=1}^n g^{ij}(x, u(x))u_i u_j$, $u^i = \sum_{j=1}^n g^{ij}(x, u(x))u_j$ and here g^{ij} is the (i, j)-component of the inverse matrix of (g_{ij}). Let Λ be the mean curvature of S with respect to ξ. Λ is given by $\Lambda = 1/n \sum_{i,j=1}^n \tilde{g}^{ij} \langle \nabla_{X_i} X_j, \xi\rangle$

where $\tilde{g}^{ij}=g^{ij}(x, u(x))-u^i u^j/(1+\|\nabla u\|^2)$. Rewriting it we get

$$\begin{aligned}(1.1)\qquad &\sum_{i,j=1}^{n}\{(1+\|\nabla u\|^2)g^{ij}(x, u(x))-u^i u^j\}u_{ij}\\ &=n\Lambda(x)(1+\|\nabla u\|^2)^{3/2}-nH(x, u(x))(1+\|\nabla u\|^2)\\ &+\frac{1}{2}\sum_{i,j=1}^{n}(\partial g_{ij}/\partial t)(x, u(x))u_i u_j\\ &+\sum_{i,j,k=1}^{n}\{(1+\|\nabla u\|^2)g_{ij}(x, u(x))-u^i u^j\}\Gamma_{ij}^{k}(x, u(x))u_k\end{aligned}$$

where

$$u_{ij}=\partial^2 u/\partial x_i\partial x_j,\ nH(x, u(x))=-\frac{1}{2}\sum_{i,j=1}^{n}g^{ij}(x, u(x))(\partial g_{ij}/\partial t)(x, u(x))$$

and Γ_{ij}^{k} denotes the Christoffel's symbol.

In (1.1), if we regard Λ as a given real-valued continuous function on D, then (1.1) is a nonlinear differential equation of second order on D. Then the following theorem holds.

Theorem 1.1 ([4]). *Suppose* $\mathrm{Ric}_N(\partial/\partial t)\geqslant 0$ *on* $D\times[0, r)$. *Let* Λ *be a given real-valued continuous function on* D *such that* $\Lambda\leqslant H_0$ *on* D. *Then any solution* u *of the equation* (1.1) *such that* $0\leqslant u<r$ *can not take the minimum value in* D *unless* u *is constant.*

§ 2. An application of the minimum principle

Let N be an n-dimensional ($n\geqslant 2$) connected, complete Riemannian manifold without boundary. We denote by d the distance function on N. For a subset S of N we set $S(r)=\{p\in N;\ d(p, S)=r\}$, $r\geqslant 0$. A geodesic $\sigma\colon[0, 1]\to N$ is called a minimal geodesic if its length is equal to the distance between its end points. Let (S_1, ι_1) and (S_2, ι_2) be connected hypersurfaces embedded in N such that $S_1\cap S_2=\phi$ where $\iota_k\colon S_k\to N$ be the inclusion map. We suppose that there exist points $p_1\in S_1$ and $p_2\in S_2$ such that $d(p_1, p_2)=d(S_1, S_2)=r>0$. Let $\sigma\colon[0, r]\to N$ be a minimal geodesic from p_1 to p_2. By minimality, the velocity vector $\dot{\sigma}(0)$ (resp. $\dot{\sigma}(r)$) is perpendicular to S_1 (resp. S_2), respectively. Let ξ_1 (resp. ξ_2) be the unit normal vector field on an open neighborhood W_1 (resp. W_2) of p_1 (resp. p_2) in S_1 (resp. S_2) such that $\xi_1=\dot{\sigma}(0)$ at p_1 (resp. $\xi_2=-\dot{\sigma}(r)$ at p_2), respectively. We denote by H_k the mean curvature of S_k with respect to ξ_k on W_k, $k=1, 2$.

Lemma 2.1. *Under the situation stated above, suppose that* N *is of nonnegative Ricci curvature and that* $H_k\geqslant 0$ *on* W_k, $k=1, 2$. *Then there* *exist open neighborhoods V_k, $V_k \subset W_k$, of p_k in S_k, $k=1, 2$, with the following properties:*

(1) *V_1 and V_2 are totally geodesic.*

(2) *The map Φ: $V_1 \times [0, r] \to N$ defined by $\Phi(p, t) = \exp_p t\xi_1(p)$ is an isometric imbedding of the Riemannian product manifold $V_1 \times [0, r]$ into N where $\exp_p$ denotes the exponential map at p.*

(3) $\Phi(V_1 \times \{r\}) = V_2$ *and* $\Phi(V_1 \times (0, r)) \cap S_k = \phi$, $k=1, 2$.

Proof. Taking W_2 sufficiently small if necessary, we may assume that for an $\varepsilon > 0$ $(2\varepsilon < r)$ exp: $\perp^+_{2\varepsilon}(W_2) \to N$ is an embedding and for each t, $0 \leqslant t \leqslant 2\varepsilon$, $d(\exp_q t\xi_2(q), S_2) = t$ for $q \in W_2$, where $\perp^+_{2\varepsilon}(W_2) = \{t\xi_2(q); q \in W_2, 0 \leqslant t \leqslant 2\varepsilon\}$. We put $W_2(s) = \{\exp_q s\xi_2(q); q \in W_2\}$ and $\tau_q(s) = \exp_q s\xi_2(q)$, $0 \leqslant s \leqslant \varepsilon$, $q \in W_2$. We note that $\tau_{p_2}(\varepsilon) = \sigma(r-\varepsilon)$ and that by Gauss Lemma the velocity vector $\dot{\tau}_q(s)$ is perpendicular to $W_2(s)$, $0 \leqslant s \leqslant \varepsilon$, $q \in W_2$. For each s, $0 \leqslant s \leqslant \varepsilon$, let $H_{2,s}$ be the mean curvature of $W_2(s)$ with respect to $\dot{\tau}_q(s)$, $q \in W_2$. We put $\Lambda = -H_{2,\varepsilon}$. Then by Lemma 1.1 $\Lambda \leqslant 0$ on $W_2(\varepsilon)$. Now, $\sigma(r-\varepsilon)$ is not a focal point of S_1 along σ because σ is a minimal geodesic from S_1 to S_2. Therefore, by the implicit function theorem, there exist an open neighborhood V_1 of p_1 in S_1, which is diffeomorphic to an open metric ball in R^{n-1}, and a function u of class C^∞ on V_1 such that $u \geqslant r-\varepsilon$ on V_1, $u(p_1) = u-\varepsilon$ and an open neighborhood W of $\tau_{p_2}(\varepsilon)$ in $W_2(\varepsilon)$ can be expressed by $W = \{\exp_p u(p)\xi_1(p); p \in V_1\}$. We note that Λ is the mean curvature of W with respect to $-\dot{\tau}_q(\varepsilon)$. Then we can apply Theorem 1.1 to the present case. Hence, by Theorem 1.1, $u = r-\varepsilon$ on V_1. Then by Lemma 1.1 $\Lambda = 0$ on W. Furthermore, by Lemma 1.1 $V_1(t) = \{\exp_p t\xi_1(p); p \in V_1\}$ is a totally geodesic hypersurface, $0 \leqslant t \leqslant r-\varepsilon$. We note $W = V_1(r-\varepsilon)$. Let V_2 be an open neighborhood of p_2 in S_2 such that $W = \{\exp_q \varepsilon\xi_2(q); q \in V_2\}$. By Lemma 1.1 $V_2(s) = \{\exp_q s\xi_2(q); q \in V_2\}$ is totally geodesic, $0 \leqslant s \leqslant \varepsilon$. Then the map Φ: $V_1 \times [0, r] \to N$ defined by $\Phi(p, t) = \exp_p t\xi_1(p)$ is an isometric imbedding and $\Phi(V_1 \times \{r\}) = V_2$, $\Phi(V_1 \times (0, r)) \cap S_k = \phi$, $k=1, 2$.

Proposition 2.1. *Let N be an n-dimensional $(n \geqslant 2)$ connected, complete Riemannian manifold of nonnegative Ricci curvature without boundary. Let (M_1, f_1) and (M_2, f_2) be compact, connected, minimal hypersurfaces immersed in N by immersions f_1 and f_2, respectively. Suppose that $S_1 = f_1(M_1)$ and $S_2 = f_2(M_2)$ do not intersect. Then the following holds.*

(1) *$S_1 \subset S_2(r)$, $S_2 \subset S_1(r)$ where $r = d(S_1, S_2)$.*

(2) *(S_k, ι_k) is a compact, connected, totally geodesic hypersurface imbedded in N where ι_k: $S_k \to N$ is the inclusion map, $k=1, 2$.*

(3) *S_1 is locally isometric to S_2.*

Proof. By the assumption we can choose points $p_1 \in S_1$ and $p_2 \in S_2$ such that $d(p_1, p_2)=d(S_1, S_2)=r$. Then $C=\{q \in M_2;\ d(S_1, f_2(q))=r\}$ is a nonempty closed subset of M_2. We shall show C is open in M_2. This implies $C=M_2$ because M_2 is connected. Hence, $S_2 \subset S_1(r)$. Now let q_2 be an arbitrary point of C and take points $p_1 \in S_1$ and $q_1 \in M_1$ such that $d(p_1, f_2(q_2)) = d(S_1, f_2(q_2)) = r$, $f_1(q_1) = p_1$. Let U_k be a connected open neighborhood of q_k in M_k such that $f_{k|U_k}: U_k \to N$ is an embedding, $k=1, 2$. We put $W_k=f_k(U_k)$, $k=1, 2$, and $p_2=f_2(q_2)$. Let $\sigma: [0, r] \to N$ be a minimal geodesic with unit speed from p_1 to p_2. By minimality of σ, $\dot\sigma(0)$ (resp. $\dot\sigma(r)$) is perpendicular to W_1 (resp. W_2) and $\sigma((0, r)) \cap S_k=\phi$, $k=1, 2$. Since W_1 and W_2 are minimal hypersurfaces in N, by Lemma 2.1 there are open neighborhoods V_1 of p_1 in W_1 and V_2 of p_2 in W_2 with properties (1) to (3) in Lemma 2.1. Thus there exists an open neighborhood of q_2 in M_2 which is contained in C. Hence C is open in M_2. By the same argument, we have $S_1 \subset S_2(r)$. Using Lemma 2.1, we see that (M_1, f_1) and (M_2, f_2) are totally geodesic hypersurfaces in N. In the following, we shall show that (S_1, ι_1) and (S_2, ι_2) are embedded hypersurfaces in N. Let p be a point of S_1. We choose a $q \in M_1$ such that $f_1(q)=p$. By Lemma 2.1 there exists a connected open neighborhood U of q in M_1 with the following properties: (1) $f_{1|U}: U \to N$ is an embedding, (2) $\Phi_p: V_p \times [0, r] \to N$, $\Phi_p(p', t)=\exp_{p'} t\xi_p(p')$, is an isometric imbedding where $V_p= f_1(U)$ and ξ_p is a unit normal vector field on V_p, (3) $\Phi_p(V_p \times (0, r)) \cap S_k=\phi$, $k=1, 2$, and $\Phi_p(V_p \times \{r\}) \subset S_2$. Then there exists an $\varepsilon>0$ such that $B_\varepsilon(p) \cap S_1 \subset V_p$ where $B_\varepsilon(p)=\{p' \in N;\ d(p, p')<\varepsilon\}$. In fact, suppose for contradiction that there is a sequence of points p_j of $S_1 \backslash V_p$, $j=1, 2, \cdots$, which converges to p in N. Then $p_j \notin \Phi_p(V_p \times [0, r])$ for each p_j. Let $\{q_j\}$, $j=1, 2, \cdots$, be a sequence of points in M_1 such that $f_1(q_j)=p_j$. By compactness of M_1, we can choose a subsequence of $\{q_j\}$ which converges to a $q' \in M_1$ so that $q' \neq q$ and $f_1(q')=p$. Then there is a connected open neighborhood U' of q' in M_1 such that $q \notin U'$ and $f_{1|U'}: U' \to N$ is an embedding. We put $V'_p=f_1(U')$. By $V'_p \subset S_2(r)$, $V'_p \cap \Phi_p(V_p \times (0, r])=\phi$. Hence, V_p and V'_p have the same tangent vector space at p. Since V_p and V'_p are totally geodesic, taking U' sufficiently small if necessary, $V'_p \subset V_p$ holds. Then, for a sufficiently large number j, $p_j \in V'_p \subset V_p$. This is a contradiction because $p_j \notin S_1 \backslash V_p$. Thus we can choose an $\varepsilon>0$ so that $B_\varepsilon(p) \cap S_1 \subset V_p$. Therefore (S_1, ι_1) is a compact, connected, totally geodesic hypersurface imbedded in N. By the same argument as above, so is (S_2, ι_2). From the above argument, it is clear that S_1 is locally isometric to S_2.

Lemma 2.2. *Let N be as in Proposition* 2.1. *Let (S_1, ι_1) and (S_2, ι_2) be compact, connected, totally geodesic hypersurfaces imbedded in N which* *are disjoint. Then there exists an isometric imbedding* $f\colon M\times[0, \varepsilon]\to N$ $(\varepsilon>0)$ *of a Riemannian product manifold* $M\times[0, \varepsilon]$ *into* N *where* M *is an* $(n-1)$*-dimensional compact, connected Riemannian manifold without boundary.*

Proof. By Proposition 2.1, $S_1\subset S_2(r)$ and $S_2\subset S_1(r)$ where $r=d(S_1, S_2)>0$. Since S_1 and S_2 are compact hypersurfaces, for each $p\in S_1$ (resp. S_2) there are at most two minimal geodesics from p to S_2 (resp. S_1). For each k, $k=1, 2$, let $G_{k,1}$ be a subset of S_k such that for any $p\in G_{k,1}$ there is a unique minimal geodesic from p to S_{k+1}, where $S_3=S_1$, and let $G_{k,2}=S_k\backslash G_{k,1}$. Using Lemma 2.1, we can show that either $G_{k,1}=S_k$ or $G_{k,2}=S_k$, $k=1, 2$. Therefore the following four cases are possible: (1) $G_{1,1}=S_1$, $G_{2,1}=S_2$, (2) $G_{1,1}=S_1$, $G_{2,2}=S_2$, (3) $G_{1,2}=S_1$, $G_{2,1}=S_2$ and (4) $G_{1,2}=S_1$, $G_{2,2}=S_2$. For each case we shall show that the assertion of the lemma holds.

Case (1). For each $p\in S_1$ let $\sigma_p\colon[0, r]\to N$ be a unique minimal geodesic with unit speed from p to S_2. It follows from Lemma 2.1 that the map $\Phi\colon S_1\times[0, r]\to N$ defined by $\Phi(p, t)=\sigma_p(t)$ is an isometric imbedding.

Case (2). We put $D=\{p\in N;\ d(p, S_2)<r\}$. Then D is a connected open subset of N with boundary S_1. Let $\xi(p)$ be the unit normal vector to S_1 at p which directs to D. By Lemma 2.1 the map $\Phi_\varepsilon\colon S_1\times[0, \varepsilon]\to N$ defined by $\Phi_\varepsilon(p, t)=\exp_p t\xi(p)$ is an isometric imbedding, $0<\varepsilon<r$.

In the case (3), by the same argument as in the case (2) the assertion holds.

Case (4). We put $L=\{p\in N;\ d(p, S_1)=d(p, S_2)\}$. By virtue of Lemma 2.1 and Proposition 2.1, L is a compact, totally geodesic hypersurface imbedded in N which has at most two connected components. In case L is connected, we see that the pair (L, S_2) satisfies the condition of the case (2). Therefore the assertion holds. We now suppose that L has two connected components. Let L_1 be a connected component of L. Then we see that the pair (S_1, L_1) satisfies the condition of the case (1). Hence the assertion holds.

§ 3. Riemannian manifolds with boundary

Let $\bar{N}=N\cup\partial N$ be an n-dimensional $(n\geqslant 2)$ connected, complete Riemannian manifold with compact boundary ∂N of class C^∞. We denote by d the distance function on $\bar{N}$. Let p be a point of N. For each $q\in\bar{B}_r(p)\backslash\{p\}$, $r=d(p, \partial N)$, there is a minimal geodesic $\sigma\colon[0, r']\to\bar{B}_r(p)$, $r'=d(p, q)$, with unit speed from p to q where $\bar{B}_r(p)$ is the closed metric ball of radius r centered at p.

Making use of Lemma 2.1, we can prove the following theorem.

Theorem 3.1 ([4]). *Let $\bar{N}=N\cup\partial N$ be an n-dimensional $(n\geqslant 2)$ connected, complete Riemannian manifold with compact boundary ∂N of class C^∞. Suppose that N is of nonnegative Ricci curvature and that the mean curvature of ∂N with respect to the inner normal direction is nonnegative. Then ∂N has at most two connected components. If ∂N has exactly two connected components, then $\bar{N}$ is isometric to a Riemannian product manifold $M\times[0, r]$, $r>0$, where M is an $(n-1)$-dimensional compact, connected Riemannian manifold without boundary.*

In the following let $\bar{N}=N\cup\partial N$ be an n-dimensional $(n\geqslant 2)$ compact, connected Riemannian manifold with connected boundary ∂N of class C^∞. We will denote by $\xi(p)$ the inner unit normal vector to ∂N at p. We set $L(t)=\{p\in N;\ d(p, \partial N)=t\}$, $t\geqslant 0$. For a subset S of N we put $S(t)=\{p\in\bar{N};\ d(p, S)=t\}$, $t\geqslant 0$.

Definition. Let $\bar{N}=N\cup\partial N$ be an n-dimensional $(n\geqslant 2)$ compact, connected Riemannian manifold with connected boundary ∂N of class C^∞. We will call that $\bar{N}$ is of Möbius type if ∂N admits an isometric involution ψ of fixed point free and $\bar{N}$ is isometric to a quotient manifold $\partial N\times[0, \delta]/\sim$ where $\delta=\max\{d(p, \partial N);\ p\in N\}$ and $\sim$ is an equivalence relation in the Riemannian product manifold $\partial N\times[0, \delta]$ defined by $(p, t)\sim(p, t)$ for $p\in\partial N$, $0\leqslant t\leqslant\delta$, and $(p, \delta)\sim(\psi(p), \delta)$ for $p\in\partial N$.

Theorem 3.2. *Let $\bar{N}=N\cup\partial N$ be an n-dimensional $(n\geqslant 2)$ compact, connected Riemannian manifold with connected boundary ∂N of class C^∞. Let (M, f) be a compact, connected, minimal hypersurface immersed in N by an immersion f. Suppose that N is of nonnegative Ricci curvature and that the mean curvature of ∂N with respect to the inner normal direction is nonnegative. Then we have following.*

(1) *$f(M)=L(r)$, $r=d(f(M), \partial N)$, and $L(t)$ is a compact, connected, totally geodesic hypersurface imbedded in $\bar{N}$, $0\leqslant t\leqslant r$.*

(2) *If $r<\delta=\max\{d(p, \partial N);\ p\in N\}$, then $\Phi_r\colon\partial N\times[0, r]\to\bar{N}$, $\Phi_r(p, t)=\exp_p t\xi(p)$, is an isometric imbedding of the Riemannian product manifold $\partial N\times[0, r]$ into $\bar{N}$ and $\Phi_r(\partial N\times\{t\})=L(t)$, $0\leqslant t\leqslant r$.*

(3) *If $r=\delta$, then $\bar{N}$ is of Möbius type.*

Proof. The assumption $\mathrm{Ric}_N\geqslant 0$ implies $\mathrm{Ric}_{\bar{N}}(\xi)\geqslant 0$ for the inner unit normal vector ξ at any point of ∂N. We put $S=f(M)$. Let p and q be points of S and ∂N such that $d(p, q)=d(S, \partial N)=r$, respectively. Then there exists a minimal geodesic $\sigma\colon[0, r]\to\bar{N}$ with unit speed from p to q. By minimality of σ, $\sigma([0, r))\subset N$ and $\dot\sigma(0)$ (resp. $\dot\sigma(r)$) is perpendicular to S (resp. ∂N). Then we can apply Lemma 2.1 to the present case. Therefore there exists an connected open neighborhood V of q in ∂N

such that $\Phi: V\times[0, r]\to\bar{N}$, $\Phi(p', t)=\exp_{p'} t\xi(p')$, is an isometric imbedding and $\Phi_r(V\times\{r\})\subset S$. Using a similar argument as in the proof of Proposition 2.1, $S\subset L(r)$ and $\partial N\subset S(r)$. Then we see that $S=L(r)$ and $L(t)$ is a compact, connected, totally geodesic hypersurface imbedded in $\bar{N}$, $0\leqslant t\leqslant r$. Since S is a hypersurface in N, for each $q\in S$ there are at most two minimal geodesics from q to ∂N. Let G_1 be a subset of S such that for each $q\in G_1$ there exists a unique minimal geodesic from q to ∂N. We put $G_2=S\backslash G_1$. Using Lemma 2.1, we see that either $G_1=S$ or $G_2=S$. We now assume $r<\delta$. Then we shall show $G_1=S$. Suppose, for contradiction, $G_2=S$. Let q be a point of $L(\delta)$ and let $\tau: [0, \delta]\to\bar{N}$ be a minimal geodesic with unit speed such that $\tau(0)\in\partial N$ and $\tau(\delta)=q$. By minimality of τ, $\dot{\tau}(0)$ is the inner unit normal vector to ∂N at $\tau(0)$ and $\dot{\tau}(r)$ is perpendicular to $S=L(r)$ ar $\tau(r)$. Since $G_2=S$, there is a minimal geodesic $\gamma: [0, r]\to\bar{N}$ with unit speed from $\tau(r)$ to ∂N such that $\dot{\gamma}(0)\neq -\dot{\tau}(r)$. By minimality of γ, $\dot{\gamma}(0)=\dot{\tau}(r)$. This implies $\delta=d(q, \partial N)=2r-\delta$, which is a contradiction. Thus $G_1=S$. Then $\Phi_r: \partial N\times[0, r]\to\bar{N}$, $\Phi_r(p, t)=\exp_p t\xi(p)$, is an isometric imbedding and $\Phi_r(\partial N\times\{t\})=L(t)$, $0\leqslant t\leqslant r$.

Next we assume $r=\delta$. Then $G_2=S$. For if $G_1=S$, then $\Phi_\delta: \partial N\times[0, \delta]\to\bar{N}$ is an isometric imbedding onto $\bar{N}$. Therefore ∂N has just two connected components. This contradicts the connectedness of ∂N. $G_2=S$ implies that $\Phi_\delta: \partial N\times[0, \delta]\to\bar{N}$ is an isometric immersion onto $\bar{N}$ and $\Phi_\delta: \partial N\times[0, \delta)\to\bar{N}$ is an isometric imbedding. For each $q\in L(\delta)$ $\Phi_\delta^{-1}(q)=\{(p_1, \delta), (p_2, \delta)\}$ where $p_2=\exp_{p_1} 2\delta\xi(p_1)$. We now define a map $\psi: \partial N\to\partial N$ by $\psi(p)=\exp_p 2\delta\xi(p)$. Then ψ is an isometric involution of ∂N which is fixed point free. From the above argument we see that $\bar{N}$ is of Möbius type.

Theorem 3.3. *Let $\bar{N}=N\cup\partial N$ be an n-dimensional ($n\geqslant 2$) compact, connected, real analytic Riemannian manifold with connected, real analytic boundary ∂N. Let (M, f) be a compact, connected, minimal hypersurface of class C^∞ immersed in N by an immersion f. Suppose that N is of nonnegative Ricci curvature and that the mean curvature of ∂N with respect to the inner normal direction is nonnegative. Then we have the following.*

(1) *$f(M)=L(r)$, $r=d(f(M), \partial N)$, and $L(t)$ is a compact, connected, totally geodesic, real analytic hypersurface imbedded in $\bar{N}$, $0\leqslant t\leqslant\delta=\max\{d(p, \partial N); p\in N\}$.*

(2) *$\bar{N}$ is of Möbius type.*

Proof. We shall show that $L(\delta)$ is a compact, connected, totally geodesic hypersurface of class C^ω imbedded in N. First we assume $r=d(f(M), \partial N)<\delta$. Since ∂N is real analytic, by Theorem 3.2 $\Phi_r: \partial N\times$

$[0, r] \to \bar{N}$, $\Phi_r(p, t) = \exp_p t\xi(p)$, is an isometric imbedding of class C^ω and $\Phi_r(\partial N \times \{t\}) = L(t)$, $0 \leqslant t \leqslant r$. Let t_0 be the supremum of those numbers t for which $\Phi_t \colon \partial N \times [0, t] \to \bar{N}$, $\Phi_t(p, t') = \exp_p t'\xi(p)$, is an isometric imbedding of class C^ω. Then $L(t_0)$ is a compact, connected, totally geodesic hypersurface of class C^ω imbedded in N. Suppose $t_0 < \delta$. Since $L(t_0)$ is compact and real analytic, for a sufficiently small $\varepsilon > 0$, $t_0 + \varepsilon < \delta$, $L(t)$ is a compact, connected, hypersurface of class C^ω imbedded in N, $t_0 \leqslant t \leqslant t_0 + \varepsilon$. By analyticity, $L(t)$ is totally geodesic, $t_0 \leqslant t \leqslant t_0 + \varepsilon$. Hence $\Phi_{t_0+\varepsilon} \colon \partial N \times [0, t_0 + \varepsilon] \to \bar{N}$ is an isometric imbedding of class C^ω. This is a contradiction. Thus $t_0 = \delta$. Hence $L(\delta)$ is a compact, connected, totally geodesic hypersurface of class C^ω imbedded in N. In case $r = \delta$, it follows from the previous theorem that $L(\delta)$ is a compact, connected, totally geodesic hypersurface of class C^ω imbedded in N. The assertions of theorem follow from the previous theorem.

§ 4. Main theorems

In order to state our main theorems we need to describe four types of Riemannian manifolds which are model spaces in our consideration.

In the following, manifolds and apparatus on them are of class C^∞ or C^ω unless otherwise stated.

Let N be an n-dimensional ($n \geqslant 2$) compact, connected Riemannian manifold without boundary. We will call that N is of type I if there exists a Riemannian submersion $\Psi \colon N \to S^1(r)$ such that for each $z \in S^1(r)$ $\Psi^{-1}(z)$ is connected and totally geodesic where $S^1(r)$ denotes a circle of radius r in the Euclidean plane.

Let M be an $(n-1)$-dimensional ($n \geqslant 2$) compact, connected Riemannian manifold without boundary which admits isometric involutions ψ_1 and ψ_2 of fixed point free. In a Riemannian product manifold $M \times [0, r]$, $r > 0$, we define an equivalence relation $\sim$ as follows: $(p, 0) \sim (\psi_1(p), 0)$, $(p, t) \sim (p, t)$, $0 \leqslant t \leqslant r$, and $(p, r) \sim (\psi_2(p), r)$, where $p \in M$. We will say that the quotient manifold $N = M \times [0, r]/\sim$ is of type II.

Let M be an $(n-1)$-dimensional ($n \geqslant 2$) compact, connected Riemannian manifold without boundary which admits an isometric involution ψ of fixed point free. We define an equivalence relation $\sim$ in the Riemannian product manifold $M \times [0, \infty)$ as follows: $(p, 0) \sim (\psi(p), 0)$ and (p, t) (p, t), $t \geqslant 0$, where $p \in M$. The quotient manifold $N = M \times [0, \infty)/\sim$ is called a manifold of type III.

Finally, we will call that a Riemannian product manifold $M \times R$ is of type IV where M is an $(n-1)$-dimensional ($n \geqslant 2$) compact, connected Riemannian manifold without boundary and R is a real line.

Theorem 4.1. *Let N be an n-dimensional $(n \geqslant 2)$ connected, complete, real analytic Riemannian manifold without boundary. Let (M_1, f_1) and (M_2, f_2) be compact, connected, real analytic, minimal hypersurfaces immersed in N by real analytic immersions f_1 and f_2. Suppose that N is of nonnegative Ricci curvature and that $f_1(M_1) \cap f_2(M_2) = \phi$. Then N is isometric to one of four types of real analytic Riemannian manifolds described above.*

Proof. By Proposition 2.1 and Lemma 2.2, there is an isometric imbedding of class C^ω $f: M \times [-2r_1, 2r_1] \rightarrow N$ $(r_1 > 0)$ of a Riemannian product manifold $M \times [-2r_1, 2r_1]$ into N where M is an $(n-1)$-dimensional compact, connected, real analytic Riemannian manifold without boundary. We set $L_1 = f(M \times \{r_1\})$, $L_2 = f(M \times \{-r_1\})$ and $D = N \backslash f(M \times [-r_1, r_1])$. L_1 and L_2 are compact, connected, real analytic, totally geodesic hypersurfaces imbedded in N. D has at most two connected components and $\partial D = L_1 \cup L_2$. We first suppose D is connected. Then, by virtue of Theorem 3.1, $\bar{D} = D \cup \partial D$ is real analytically isometric to a Riemannian product manifold $L_1 \times [0, r_2]$, $r_2 > 0$. We define a map $\Phi: L_1 \times R \rightarrow N$ by $\Phi(p, t) = \exp_p t\xi(p)$ where $\xi(p)$ is the unit normal vector to L_1 at p which directs to D. We see that $\Phi: L_1 \times R \rightarrow N$ is a Riemannian covering map and that $L_1 = \Phi(L_1 \times \{0\}) = \Phi(L_1 \times \{r\})$, $r = 2r_1 + r_2$, and $\Phi(L_1 \times \{t\}) \cap \Phi(L_1 \times \{t'\}) = \phi$, $0 \leqslant t < t' < r$. We put $L(t) = \Phi(L_1 \times \{t\})$, $0 \leqslant t < r$, and $S^1(r/2\pi) = \{z \in C; |z| = r/2\pi\}$, where z is a complex number. We now define a map $\Psi: N \rightarrow S^1(r/2\pi)$ by $\Psi(q) = (r/2\pi) \exp(i2\pi t/r)$, $q \in L(t)$, $i^2 = -1$. Then, by the above argument, $\Psi: N \rightarrow S^1(r/2\pi)$ is a Riemannian submersion and for each $(r/2\pi) \exp(i\theta) \in S^1(r/2\pi)$, $0 \leqslant \theta < 2\pi$, $\Psi^{-1}((r/2\pi) \times \exp(i\theta)) = L(r\theta/2\pi)$. Therefore we see that N is of type I.

Next we consider the case where D has two connected components. Let D_1 and D_2 be connected components of D such that $\partial D_1 = L_1$ and $\partial D_2 = L_2$. We have to consider the following three cases: (1) $\bar{D}_1$ and $\bar{D}_2$ are compact, (2) one of them is compact and the other is noncompact, (3) $\bar{D}_1$ and $\bar{D}_2$ are noncompact. We note that D_1 (resp. D_2) contains a compact, connected, totally geodesic hypersurface $f(M \times \{2r_1\})$ (resp. $f(M \times \{-2r_1\})$ of class C^ω, respectively.

In the case (1), both of $\bar{D}_1$ and $\bar{D}_2$ satisfy the hypotheses of Theorem 3.3. Hence $\bar{D}_1$ and $\bar{D}_2$ are of Möbius type. Therefore N is of type II.

For the case (2), we may suppose that $\bar{D}_1$ is compact and $\bar{D}_2$ is noncompact. By the same reason as above, $\bar{D}_1$ is of Möbius type. On the other hand, since $f: M \times [-2r_1, -r_1] \rightarrow \bar{D}_2$ is an isometric imbedding of class C^ω, using the analyticity of N we can show that $\bar{D}_2$ is real analytically isometric to $L_2 \times [0, \infty)$. Then N is of type III.

In the case (3), by a similar argument as above we see that N is real analytically isometric to $L_1 \times R$. We complete the proof.

In the following, we assume that manifolds and apparatus on them are of class C^∞.

Theorem 4.2. *Let N be an n-dimensional ($n \geqslant 3$) connected, homogeneous Riemannian manifold. Let (M_1, f_1) and (M_2, f_2) be compact, connected, minimal hypersurfaces immersed in N by immersions f_1 and f_2. Suppose that N is of nonnegative Ricci curvature and that $f_1(M_1) \cap f_2(M_2) = \phi$. Then N is of type* I *or type* IV.

Proof. By Proposition 2.1 and Lemma 2.2, there is an isometric imbedding $f: M \times [-2r, 2r] \to N (r > 0)$ where M is an $(n-1)$-dimensional compact, connected, Riemannian manifold without boundary. We set $L_1 = f(M \times \{r\})$ and $L_2 = f(M \times \{-r\})$, and put $D = N \backslash f(M \times [-r, r])$. D has at most two connected components and $\partial D = L_1 \cup L_2$. In case D is connected, by the same argument as in the proof of Theorem 4.1, we see N is of type I. Next we consider the case where D has two connected components. Let D_1 and D_2 be connected components of D such that $\partial D_1 = L_1$ and $\partial D_2 = L_2$. We shall show that $\bar{D}_1$ and $\bar{D}_2$ are noncompact. Suppose for contradiction that $\bar{D}_1$ is compact. We let $\delta = \max\{d(p, L_1); p \in D_1\}$. Let p be a point of D_1 such that $d(p, L_1) = \delta$. Since N is homogeneous and connected, the identity component $I_0(N)$ of the isometry group $I(N)$ of N is transitive on N. We can choose an isometry $F \in I_0(N)$ so that $p \in F(L_1)$. Let $F_t (0 \leqslant t \leqslant 1)$ be a continuous curve in $I_0(N)$ such that F_0 is the identity transformation and $F_1 = F$. By continuity of F_t $(0 \leqslant t \leqslant 1)$, there exists a t_1 $(0 < t_1 < 1)$ such that $F_t(L_1) \cap M_0 = \phi$ for each t, $0 \leqslant t \leqslant t_1$, where $M_0 = f(M \times \{0\})$. By Proposition 2.1, for each t, $0 \leqslant t \leqslant t_1$, there is a positive $r(t)$ such that $F_t(L_1) \subset M_0(r(t)) = \{q \in N; d(q, M_0) = r(t)\}$. Using Proposition 2.1 and the continuity of F_t $(0 \leqslant t \leqslant 1)$, we can show that for each t, $0 \leqslant t \leqslant 1$, there exists an $r(t) \geqslant 0$ such that $F_t(L_1) \subset M_0(r(t))$. Since $p \in F(L_1)$ and $F(L_1)$ is connected, $F(L_1) \subset L_1(\delta) = \{q \in D_1; d(q, L_1) = \delta\}$. Then it follows from Theorem 3.2 that $\bar{D}_1$ is of Möbius type. Hence L_1 is a double covering manifold of $F(L_1)$. This is a contradiction. Therefore $\bar{D}_1$ is noncompact. By the same argument, so is $\bar{D}_2$.

We shall show that $\bar{D}_k$ is isometric to $L_k \times [0, \infty)$, $k = 1, 2$. Since L_1 is compact and $\bar{D}_1$ is noncompact, there is a geodesic $\sigma: [0, \infty) \to \bar{D}_1$ with unit speed such that $\sigma(0) \in L_1$, $\dot{\sigma}(0)$ is perpendicular to L_1 and $d(\sigma(t), L_1) = t$, $t > 0$. Let t_0 be the supremum of those numbers t for which $\Phi_t: L_1 \times [0, t] \to \bar{D}_1$, $\Phi_t(p, t') = \exp_p t'\xi(p)$, is an isometric imbedding where $\xi(p)$ is the unit normal vector to L_1 at $p \in L_1$ which directs to D_1. From the definition of D_1, $t_0 \geqslant r$. Suppose t_0 is finite. Since N is homogeneous and L_1 is compact, for a sufficiently large $t_1 > t_0$ there exists an isometry F of N such that $F(\sigma(0)) = \sigma(t_1)$, $F(L_1) \subset D_1$ and $L_1 \cap F(L_1) = \phi$. By (1) of Proposition 2.1, we see $F(L_1) = \{p \in D_1; d(p, L_1) = t_1\}$. Using Lemma 2.1 we see that $\Phi_{t_1}: L_1 \times [0, t_1] \to \bar{D}_1$, $\Phi_{t_1}(p, t) = \exp_p t\xi(p)$, is an isometric imbedding. This is a contradiction. Hence $\bar{D}_1$ is isometric to $L_1 \times [0, \infty)$. By the same argument, $\bar{D}_2$ is isometric to $L_2 \times [0, \infty)$. Therefore N is of type IV.

Corollary 4.1. *Let N be an n-dimensional ($n \geqslant 3$) noncompact, connected, homogeneous Riemannian manifold of nonnegative Ricci curvature. Suppose that N contains a compact, connected, minimal hypersurface (M, f) immersed by an immersion f. Then N is of type* IV.

Proof. Since N is noncompact and homogeneous, there is an isometry F of N such that $f(M) \cap F(f(M)) = \phi$. Then, by the previous theorem, the assertion holds.

As applications of Theorem 4.2 we have the following theorems.

Theorem 4.3. *Let N be an n-dimensional ($n \geqslant 3$) compact, connected, homogeneous Riemannian manifold of nonnegative Ricci curvature whose fundamental group is finite. Let (M_1, f_1) and (M_2, f_2) be compact, connected, minimal hypersurfaces immersed in N by immersions f_1 and f_2. Then $f_1(M_1)$ and $f_2(M_2)$ must intersect.*

Proof. Suppose, for contradiction, $f_1(M_1) \cap f_2(M_2) = \phi$. By Theorem 4.2, N is of type I. On the other hand, the fundamental group of a Riemannian manifold of type I contains an infinite cyclic group. This contradicts the assumption.

Theorem 4.4. *Let N be an n-dimensional ($n \geqslant 3$) compact, connected, homogeneous Riemannian manifold of nonnegative Ricci curvature whose fundamental group is finite. Let (M, ι) be a compact, connected, minimal hypersurface imbedded in N where ι is the inclusion map. Then the natural homomorphism of fundamental groups $\iota_{\#}: \pi_1(M) \to \pi_1(N)$ is surjective.*

Proof. Let $\tilde{N}$ be the universal Riemannian covering manifold of N and let $\pi: \tilde{N} \to N$ be the Riemannian covering map. $\tilde{N}$ is homogeneous. Since N is compact and $\pi_1(N)$ is finite, $\tilde{N}$ is compact. We see that each connected component of $\tilde{M} = \pi^{-1}(M)$ is a compact, minimal hypersurface imbedded in $\tilde{N}$. By Theorem 4.3, $\tilde{M}$ is connected. Let $[\alpha]$ be in $\pi_1(N, p)$, $p \in M$, and let $\alpha: [0, 1] \to N$ be a continuous closed curve which is a representative of $[\alpha]$. Let $\tilde{\alpha}: [0, 1] \to N$ be the lift of α starting from $\tilde{p} \in \tilde{M}$, $\pi(\tilde{p}) = p$. We can joint $\tilde{p}$ and $\tilde{\alpha}(1)$ by a continuous curve $\tilde{\beta}: [0, 1] \to \tilde{M}$

such that $\tilde{\beta}(0)=\tilde{p}$ and $\tilde{\beta}(1)=\tilde{\alpha}(1)$. Then $\beta=\pi\circ\tilde{\beta}$ is homotopic to α fixing the base point p. Hence $\iota_{\#}([\beta])=[\alpha]$. Thus $\iota_{\#}$ is surjective.

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Department of Mathematics
Yokohama City University
22–2 Seto, Kanazawa-ku
Yokohama, 236 Japan