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Convexity in Riemannian Manifolds without Focal Points

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§ 0. Introduction

Throughout this paper let M' be a complete Riemannian manifold and let a geodesic $\alpha: (-\infty, \infty) \to M'$ be parametrized by its arc-length. M' is said to have no *focal points* if every geodesic $\alpha: (-\infty, \infty) \to M'$ has no focal points as a 1-dimensional submanifold in M'. In this paper we shall deal with complete Riemannian manifolds N without focal points from the point of view of geometry of geodesics. In particular, we shall investigate relations between the existence of totally convex sets in such a manifold N (or convex functions on N) and the topological and metric structure of N.

However our starting point of the study is different from usual ones. In a paper of O'Sullivan [20] we find a nice exposition of having no focal points. Namely, he has stated that (1) M' has nonpositive sectional curvature if and only if $\langle Y, Y\rangle'' \geqq 0$ for every Jacobi field along every geodesic α, (2) M' has no focal points if and only if $\langle Y, Y\rangle' > 0$ for $t > 0$ where Y is any nontrivial Jacobi field along any geodesic vanishing at $t=0$, (3) M' has no conjugate points if and only if $\langle Y, Y\rangle > 0$ for $t > 0$ where Y is any non-trivial Jacobi field along any geodesic α vanishing at $t=0$. Therefore if M' has nonpositive sectional curvature, then it has no focal points, and if M' has no focal points, then it has no conjugate points. These three classes of Riemannian manifolds are actually distinct as was shown in Gulliver [16]. From this fact it is a natural question to ask whether there exists a condition which define a new class of Riemannian manifolds relative to the classification of these three classes. The condition in (1) means that $\|Y\|^2$ is a convex function for $t \in \boldsymbol{R}$. Near peaklessness of a function which is explained by Busemann-Phadke [5] is weaker than convexity. A continuous function f on $\boldsymbol{R}$ is by definition *nearly peakless* if $f(t_2) \leqq \mathrm{Max}\,\{f(t_1), f(t_3)\}$ for any $t_1 < t_2 < t_3$. A differentiable nearly peakless function f on $\boldsymbol{R}$ has a property: $f''(t) \geqq 0$ for those t such that $f'(t)=0$.

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Hence M' has nonpositive sectional curvature even if $\|Y\|^2$ is nearly peakless for every Jacobi field Y along every geodesic in M'. In this meaning no class of Riemannian manifolds exist between (1) and (2). Since the condition in (2) is stronger than near peaklessness of $\|Y\|^2$, it is natural to have a question if the condition in (2) is equivalent to the near peaklessness of $\|Y\|^2$. In other words, is the condition, $\langle Y, Y\rangle' > 0$, in (2) equivalent to the condition, $\langle Y, Y\rangle' \geqq 0$? Here the latter implies the near peaklessness of $\|Y\|^2$ and hence of $\|Y\|$ where Y is as in (2). Remark 2.5 will say that these two conditions are equivalent. From these reasons it is significant that a Riemannian manifold in which near peaklessness of $\|Y\|$ holds is said to have no focal points. It should be noted that if Y is a nontrivial perpendicular Jacobi field along a geodesic α in M' vanishing at $t=0$ and if $\langle Y, Y\rangle'(t_0)=0$ for some $t_0>0$, then $\alpha(0)$ is a focal point of the geodesic tangent to $Y(t_0)$ along the geodesic $\alpha_0\colon (-\infty, \infty)\to M'$ given by $\alpha_0(t) := \alpha(t_0 - t)$.

Hereafter let N be a complete Riemannian manifold which satisfies the condition; $\|Y\|$ *is nearly peakless where* Y *is any Jacobi field along any geodesic in* N *vanishing at* $t=0$. Equivalently $\|Y\|$ is monotone nondecreasing for $t\geqq 0$. If N is simply connected, it is always denoted by M instead of N. M necessarily has no conjugate points, so that M is diffeomorphic to $\boldsymbol{R}^n$, $n=\dim M$, and every geodesic $\alpha\colon(-\infty,\infty)\to M$ is a straight line, i.e., any sub arc of it is a distance minimizing geodesic. N is actually a quotient manifold M/D, where D is the group of isometries of M corresponding to the fundamental group of N.

In Section 1 we shall give the new proofs of the divergence property and the flat strip theorem on M which are crucial in our geometry. Goto [12] and O'Sullivan [21]-Eschenburg [11] have shown these properties on manifolds M' without focal points by using the idea of the stable Jacobi tensor. Although this idea is very interesting and useful, it may be unnecessary by our simplification in this paper. Indeed without the idea of the stable Jacobi tensor we can prove

Theorem 1.7 (Divergence property). *Let* α *and* β *be distinct geodesics in* M *with* $\alpha(0)=\beta(0)$ *and let* $a\geqq 0$. *Then* $d(\alpha(at), \beta(t))$ *goes to infinity as* $t\to\infty$.

Theorem 1.13 (Flat strip theorem). *Let geodesics* α *and* β *in* M *be biasymptotic. Then there is a unique totally geodesic flat strip in* M *such that* $\alpha(\boldsymbol{R})\cup\beta(\boldsymbol{R})$ *is its boundary.*

Also we can prove the angle vanishing property which have played an important role in the study of Riemannian manifolds with nonpositive curvature à la Eberlein-O'Neill [8]. Let p and q be any distinct points in M and let α be the geodesic such that $\alpha(0)=p$ and $\alpha(d(p, q))=q$. Then we denote $\dot{\alpha}(0)$ by $V(p, q)$. $\sphericalangle_p(q, r)$ is by definition the angle between $V(p, q)$ and $V(p, r)$.

Proposition 1.8 (Angle vanishing property). *Let p_n, q_n and r_n be sequences of points in M such that $p_n \to p$, $q_n \to q$ and $d(p, r_n) \to \infty$ as $n \to \infty$. Then $\sphericalangle_{r_n}(p_n, q_n) \to 0$ as $n \to \infty$.*

These results will be deduced by the direct method à la Busemann from the following basic propositions.

Proposition 1.1. *Let α and β be geodesics in M with $\alpha(0)=\beta(0)$ and $\alpha \neq \beta$. If $F(t) := d(\alpha(at), \beta(t))$ for any $a \geqq 0$, then the function F is monotone nondecreasing for $t \geqq 0$.*

Due to the notion of the angular measure which makes sense in Riemannian geometry, Proposition 1.1 is equivalent to

Proposition 1.4. *For each point $p \in M$ the distance function $F(\cdot) := d(p, \cdot)$ is convex on M.*

From Proposition 1.4 it follows that the Busemann function of every ray in M is convex on M. And convexity of Busemann functions ensures the divergence property and the angle vanishing property in M. Let α: $(-\infty, \infty) \to M$ be a geodesic and β_t: $(-\infty, \infty) \to M$ be a geodesic for each $t \in \boldsymbol{R}$ such that $\beta_t(0) \to p \in M$ as $t \to \infty$ and it passes $\alpha(t)$ in its positive direction. Then β_t converges to a geodesic β as $t \to \infty$ (see [3]). We call β an *asymptote* to α through p. From Proposition 1.1 the asymptote relation is characterized as follows. β is asymptotic to α in M if and only if $F(t) := d(\alpha(t), \beta(t))$ is monotone nonincreasing for $t \in \boldsymbol{R}$ (Proposition 1.9). Hence, if β is biasymptotic to α, then F is constant for $t \in \boldsymbol{R}$. This property yields the flat strip theorem.

In Section 2 we shall investigate the structure of N having totally convex sets. A subset Q of M' is said to be *totally convex* if $p, q \in Q$ implies that all geodesic curves from p to q are entirely contained in Q. If C is a closed set in M', then each point $p \in M'$ has a nearest point $q \in C$ from p to C, which is called a *foot* of p on C. We say that a geodesic α: $[0, \infty) \to M'$ is a *perpendicular* to C in M' if $\alpha(0) \in \dot{C}$, *the boundary of C in* M', and $d(\alpha(t), C)=t$ for all $t \geqq 0$. If α and β are distinct perpendiculars to C, then $\alpha(t) \neq \beta(t')$ for all $t, t' > 0$, $\alpha(0)=\beta(0)$ admitted. The fundamental and useful result is that if there is a nonempty closed totally convex set Q in N, then N is the union of Q and the point set carrying all perpendiculars to Q (Theorem 3.3). Busemann-Phadke [5] have pointed out that this property is very important in the proofs of some results of Bishop-O'Neill [2], which are, in fact, generalized in G-spaces with convex capsules. The author learned the implication of consequence from them. As a direct consequence, we have that if there exists a nonempty closed totally convex set Q in N such that $\partial Q=\phi$, *the boundary of* Q *as a submanifold in* N, then the exponential map of the normal bundle of Q onto N is a diffeomorphism (Corollary 2.4). Remark 2.5 is a particular case, i.e., the exponential map of the normal bundle of an arbitrary geodesic in M onto M is a diffeomorphism, since the point set carrying the geodesic is a nonempty closed totally convex set in M. The author would like to emphasize as a main theorem in Section 2

Theorem 2.11. *Suppose that there exist at most countably many nonempty closed totally convex sets,* $Q_1, Q_2, \cdots$, *in* N *such that* $Q_i \cap Q_j = \phi$ *for* $i \neq j$. *If* $Y := N - \bigcup_{i=1}^{\infty} Q_i$ *is nonempty and bounded, then the following hold.*

(1) *Exactly two of* Q_i*'s are noncompact and other* Q_i*'s are compact. We assume that* Q_1 *and* Q_2 *are noncompact.*

(2) *If* $X := \partial Q_1$ *(or* ∂Q_2*), then* X *is a compact totally geodesic submanifold in* N *with* $\partial X = \phi$.

(3) *If* $W := N - (Q_1 \cup Q_2)$, *then the closure* $\overline{W}$ *of* W *is isometric to a Riemannian product* $X \times [0, L]$, *where* $L := d(Q_1, Q_2)$.

(4) *For each* $i \geqq 3$ *there exist numbers,* a_i *and* a_i', *such that* $0 < a_i \leqq a_i' < L$ *and* Q_i *is isometric to* $X \times [a_i, a_i'] \subset X \times [0, L]$.

In Section 3 we shall give some conditions that N splits isometrically as $N_1 \times \boldsymbol{R}$. In fact, the purpose is to prove

Theorem 3.12. *Let* γ *be a ray in* N. *If the Busemann function* f_γ *is convex on* N *and if the diameter function* δ *of the levels of* f_γ *is bounded, then* N *is isometric to a Riemannian product* $N_1 \times \boldsymbol{R}$.

Proposition 3.13. *Suppose that* N *has two ends. If there exist convex functions* f_1 *and* f_2 *on* N *such that they have no minimum and* $f_1^{-1}((-\infty, t_0]) \cap f_2^{-1}((-\infty, s_0]) = \phi$ *for some* $t_0 > \inf f_1$ *and some* $s_0 > \inf f_2$, *then* N *is isometric to a Riemannian product* $N_1 \times \boldsymbol{R}$.

Proposition 3.13 will be shown as an application of Theorem 2.11. However, for the proof of Theorem 3.12 we must establish the notion of points at infinity as in Eberlein-O'Neill [8] and need to prepare many lemmas which will be obtained in the same way as in [8]. The asymptote relation on the set of all geodesics in M is an equivalence relation (Corollary 1.10). We denote by $M(\infty)$ the set of all asymptote classes of geodesics in M and put $\bar{M}:=M\cup M(\infty)$. Then we can give a topology in $\bar{M}$ such that $\bar{M}$ is homeomorphic to the closed unit ball in $\boldsymbol{R}^n$, $n=\dim M$. This topology is called the *cone topology* of $\bar{M}$. All isometries of M leaves each asymptote relation invariant, so that we should naturally consider isometries of M as mappings of $\bar{M}$ into $\bar{M}$. The extension of an isometry of M is a homeomorphism of $\bar{M}$ onto itself with the cone topology. We shall mainly investigate how the group D of isometries on M acts on $M(\infty)$ under the assumption of Theorem 3.12 if $N=M/D$.

§ 1. Fundamental properties

In this section we obtain the fundamental properties which are used later. We again emphasize that we do not use the idea of the stable Jacobi tensor at all. We have already promised to denote by α a geodesic such that its domain is $\boldsymbol{R}$ and it is parametrized by its arc-length. When we need to change the parameter of α, we use the notation, for example, $\alpha(at)$, whose speed is a. The first observation is

Proposition 1.1. *Let α and β be geodesics in M with $p:=\alpha(0)=\beta(0)$ and $\alpha\neq\beta$. If $F(t):=d(\alpha(at),\beta(t))$ for any $a\geqq 0$, then the function F is monotone nondecreasing for $t\geqq 0$.*

Proof. Let $0\leqq t_1<t_2$ and let $\gamma\colon[0,L]\to M$ be the geodesic joining $\alpha(at_2)=\gamma(L)$ and $\beta(t_2)=\gamma(0)$. Define a curve $Z\colon[0,L]\to T_pM$ in such a way that $\gamma(s)=\exp_p Z(s)$ for each $s\in[0,L]$. We can construct the geodesic variation $r\colon[0,1]\times[0,L]\to M$ so that $r(u,s)=\exp_p uZ(s)$ for every $(u,s)\in[0,1]\times[0,L]$. The vector field $Y:=r_*(\partial/\partial s)$ is a non-trivial Jacobi field vanishing at $u=0$ along each geodesic passing through p and $\gamma(s)$, so that for each $s\in[0,L]$ the norm $\|Y\|$ is nondecreasing for $u\in[0,1]$. The curve $r(t_1/t_2,s)\colon[0,L]\to M$ goes from $\beta(t_1)$ to $\alpha(at_1)$. Hence we have

$$\begin{aligned}F(t_1)&=d(\alpha(at_1),\beta(t_1))\\&\leqq\int_0^L\|Y(t_1/t_2,s)\|\,ds\leqq\int_0^L\|Y(1,s)\|\,ds=\int_0^L\|\dot{\gamma}(s)\|\,ds\\&=L=d(\alpha(at_2),\beta(t_2))=F(t_2).\end{aligned}$$

This completes the proof.

Using the notion of angles we can change this property to the distance function property from each point in M. The author does not know whether Proposition 1.1 implies Proposition 1.2 in more general spaces, for example, in Finsler G-spaces.

Before stating Proposition 1.2, we give two definitions. A function F on M' is said to be *convex* if, along each geodesic α in M', $F\circ\alpha$ is a onevariable convex function for the parameter of α. If M' is noncompact and if $\alpha\colon [0, \infty)\to M'$ is a ray, then we can define a function on M' by $f_\alpha(\cdot):=\lim_{t\to\infty}\{d(\cdot, \alpha(t))-t\}$, which is called the *Busemann function* of α.

According to Busemann-Phadke [4], the following three propositions are equivalent in a straight G-space in which all geodesics are by definition distance minimizing. So we have only to prove Proposition 1.2.

Proposition 1.2. *For each point $p \in M$ the square distance function $F(\cdot):=d(p, \cdot)^2$ is convex on M.*

Proposition 1.3. *Let α be a geodesic in M. Then f_α is convex on M.*

Proposition 1.4. *For each point $p \in M$ the distance function $F(\cdot):= d(p, \cdot)$ is convex on M.*

A continuous function F on M' is said to be *peakless* if for every geodesic α in M' the function F satisfies that $F\circ\alpha(t_2)\leqq \mathrm{Max}\,\{F\circ\alpha(t_1), F\circ\alpha(t_3)\}$ for $t_1<t_2<t_3$ and whenever the equality holds, $F\circ\alpha(t_1)=F\circ\alpha(t_2)=F\circ\alpha(t_3)$. The condition on the equality implies that $F\circ\alpha(t)=c$ on a proper interval only if $\min F\circ\alpha((-\infty, \infty))=: c$. If the equality cannot hold for any $t_1<t_2<t_3$, then F is said to be *strictly peakless*. All convex functions are peakless. According to [3], we have from Proposition 1.4

Corollary 1.5. *For each point $p \in M$ the distance function $d(p, \cdot)$ is strictly peakless on M.*

We return to the proof of Proposition 1.2.

Proof of Proposition 1.2. If a geodesic α passes through $p=\alpha(0)$, then it is trivial that $F\circ\alpha$ is convex, because $F\circ\alpha(t)=t^2$ for every $t \in \boldsymbol{R}$. We assume that a geodesic α does not pass through p. Let $G:=F\circ\alpha$ and $H:=\sqrt{G}$. We will prove that $G'(t_2)\geqq G'(t_1)$ for $t_2>t_1$. Let γ_1 and γ_2 be geodesics such that $\gamma_1(0)=\gamma_2(0)=p$, $\gamma_1(L_1)=\alpha(t_1)$ and $\gamma_2(L_2)=\alpha(t_2)$, where $L_1:=d(p, \alpha(t_1))=H(t_1)$ and $L_2:=d(p, \alpha(t_2))=H(t_2)$. Then, by Proposition 1.1, if $I(s):=d(\gamma_2(L_2 s/L_1), \gamma_1(s))$ for $s\geqq 0$, I is monotone nondecreasing for $s\geqq 0$. Let θ_i be the angles between $\gamma_i(L_i)$ and $\alpha(t_i)$ for $i=1, 2$. Then $I'(L_1)=L_2\cos\theta_2/L_1-\cos\theta_1\geqq 0$. Since $H'(t_i)=\cos\theta_i$ at $t=t_i$ for $i=1, 2$, $H(t_2)H'(t_2)\geqq H(t_1)H'(t_1)$. Hence $G'(t_2)\geqq G'(t_1)$. This completes the proof.

The following lemma will be used in the proofs of the divergence property (Theorem 1.7) and the angle vanishing property (Proposition 1.8).

Lemma 1.6. *Let α and β be geodesics with $\alpha(0)=\beta(0)$ in M and let $a>0$. Then $d(\alpha(at), \beta(t))\geqq t(1-a\langle\dot\alpha(0), \dot\beta(0)\rangle)$ for all $t>0$.*

For the proof we need some properties of Busemann functions. If

f_α is the Busemann function of a ray α on M', then $|f_\alpha(p)-f_\alpha(q)|\leqq d(p, q)$ for any points p and q in M'. And if M' is simply connected and has no conjugate points, then f_α is at least (C^1-)differentiable and the gradient vector at each point $p \in M'$ is the negative velocity vector of the asymptote γ to α through $p=\gamma(0)$ at $t=0$, and hence $f_\alpha(\gamma(t))=f_\alpha(p)-t$ for all $t \in \boldsymbol{R}$ (see [11] and [18]).

Proof of Lemma 1.6. From the properties above and convexity of f_β, we have

$$\begin{aligned} d(\alpha(at), \beta(t)) &\geqq |f_\beta(\alpha(at))-f_\beta(\beta(t))| \geqq t+f_\beta(\alpha(at)) \\ &\geqq t+at(f_\beta \circ \alpha)'(0)=t(1-a\langle\dot\beta(0), \dot\alpha(0)\rangle), \end{aligned}$$

which is our goal.

Theorem 1.7 (Divergence property). *Let α and β be distinct geodesics in M with $\alpha(0)=\beta(0)$ and let $a\geqq 0$. Then $d(\alpha(at), \beta(t))$ goes to infinity as $t\to\infty$.*

Proof. If $a\neq 1$, then $d(\alpha(at), \beta(t))\geqq|d(\alpha(at), \beta(0))-d(\alpha(0), \beta(t))|=|at-t|=|a-1||t|$, and therefore $d(\alpha(at), \beta(t))\to\infty$ *as* $t\to\infty$. If $a=1$, then $1-\langle\dot\alpha(0), \dot\beta(0)\rangle>0$, because $\alpha\neq\beta$. Thus, from Lemma 1.6, we have that $d(\alpha(t), \beta(t))\to\infty$ as $t\to\infty$. This completes the proof.

Proposition 1.8 (Angle vanishing property). *Let p_n, q_n and r_n be sequences of points of M such that $p_n\to p$, $q_n\to q$ and $d(p, r_n)\to\infty$ as $n\to\infty$. Then $\measuredangle_{r_n}(p_n, q_n)\to 0$ as $n\to\infty$.*

Proof. Let α_n and β_n be geodesics in M such that $\alpha_n(0)=r_n=\beta_n(0)$, $\alpha_n(d(r_n, p_n))=p_n$ and $\beta_n(d(r_n, q_n))=q_n$ for each n. If $t_n:=d(r_n, q_n)$ and $a_n:=d(r_n, p_n)/d(r_n, q_n)$, then $d(p_n, q_n)=d(\alpha_n(a_nt_n), \beta_n(t_n))\geqq t_n(1-a_n\langle\dot\alpha_n(0), \dot\beta_n(0)\rangle)=t_n(1-a_n \cos\measuredangle_{r_n}(p_n, q_n))$. Since $t_n\to\infty$ and $a_n\to 1$ as $n\to\infty$, $\cos\measuredangle_{r_n}(p_n, q_n)\to 1$, and hence $\measuredangle_{r_n}(p_n, q_n)\to 0$ as $n\to\infty$, which is our goal.

The following characterization for asymptotes, which is stated in O'Sullivan [21] and Goto [12], is very important. Goto's proof is somewhat complicated at least to the author.

Proposition 1.9. *Let α and β be geodesics in M and let $F(t):=d(\alpha(t), \beta(t))$ for all $t \in \boldsymbol{R}$. Then the following are equivalent*:

(1) *β is an asymptote to α.*

(2) *F is monotone nonincreasing in $t \in \boldsymbol{R}$.*

(3) *F is bounded above in $t\geqq 0$.*

Proof. (1)$\Rightarrow$(2). Let β_n be the geodesic in M such that $\beta_n(0)=\beta(0)$

and $\beta_n(d(\beta(0), \alpha(t_n)))=\alpha(t_n)$ and let $F_n(t):=d(\beta_n(a_n t), \alpha(t))$ for all $t \in \boldsymbol{R}$, where $a_n:=d(\beta(0), \alpha(t_n))/t_n$. For each n it follows from Proposition 1.1 that F_n is monotone nonincreasing for $t \leqq t_n$. Since $a_n \to 1$ as $n \to \infty$, $F_n(t) \to F(t):=d(\beta(t), \alpha(t))$ for all $t \in \boldsymbol{R}$ as $n \to \infty$. Thus F is monotone nonincreasing for $t \in \boldsymbol{R}$.

(2)$\Rightarrow$(3). There is nothing to prove.

(3)$\Rightarrow$(1). Suppose that β is not asymptotic to α and that γ is the asymptote to α through $\beta(0)=\gamma(0)$. By the argument above there exists a $C>0$ such that $d(\gamma(t), \alpha(t))<C$ for all $t \geqq 0$. Also, by Theorem 1.7, $d(\beta(t), \gamma(t)) \to \infty$ as $t \to \infty$. On the other hand, from the triangle inequality, $d(\beta(t), \alpha(t)) \geqq d(\beta(t), \gamma(t))-d(\gamma(t), \alpha(t))$. Hence $d(\beta(t), \alpha(t)) \to \infty$ as $t \to \infty$, which completes the proof.

As applications of the above characterization we have very important and useful implements in Section 3.

Corollary 1.10. *The asymptote relation on the set of all geodesics in M is symmetric and transitive.*

This follows from (3) in Proposition 1.9.

Proposition 1.11. *If α and β are asymptotic in M, then $f_\alpha - f_\beta$ is constant on M.*

Because $\operatorname{grad} f_\alpha = \operatorname{grad} f_\beta$ on M by Corollary 1.10 and by the remark following Lemma 1.6.

Proposition 1.12. *Let p_n, q_n, r_n and s_n be sequences of points in M such that $p_n \to p$, $q_n \to q$, $d(p, r_n) \to \infty$ and $d(q, s_n) \to \infty$ as $n \to \infty$ and $\sup\{d(r_n, s_n); n \in \boldsymbol{N}\}=: C<\infty$. Let α_n and β_n be geodesics in M such that $\alpha_n(0)=p_n$, $\beta_n(0)=q_n$, $\alpha_n(d(p_n, r_n))=r_n$ and $\beta_n(d(q_n, s_n))=s_n$. If α_n converges to a geodesic α, then β_n converges to the asymptote β to α with $\beta(0)=q$.*

Proof. We have only to prove that if a subsequence β_k of β_n converges to a geodesic γ, then γ is always asymptotic to α, because of the uniqueness of the asymptote through given point in M. Let ω_k be the geodesic for each k such that $\omega_k(0)=p_k$ and $\omega_k(d(p_k, s_k))=s_k$. Then, by Proposition1 .1, $d(\alpha_k(t), \omega_k(a_k t)) \leqq d(r_k, s_k)$ for $0 \leqq t \leqq d(p_k, r_k)$, where $a_k:=d(p_k, s_k)/d(p_k, r_k)$. Also $d(\omega_k(a_k t), \beta_k(b_k t)) \leqq d(p_k, q_k)$ for $0 \leqq t \leqq d(p_k, r_k)$, where $b_k:=d(q_k, s_k)/d(p_k, r_k)$. Since $|d(p_k, r_k)-d(q_k, s_k)| \leqq d(r_k, s_k)+d(p_k, q_k)$ and $|d(p_k, r_k)-d(p_k, s_k)| \leqq d(r_k, s_k)$, $a_k \to 1$ and $b_k \to 1$ as $k \to \infty$. Therefore $d(\alpha(t), \gamma(t))=\lim d(\alpha_k(t), \beta_k(t))=\lim d(\alpha_k(t), \beta_k(b_k t)) \leqq \lim\{d(\alpha_k(t), \omega_k(a_k t))+d(\omega_k(a_k t), \beta_k(b_k t))\} \leqq C+d(p, q)$ for all $t \geqq 0$. This completes the proof in conjunction with Proposition 1.9.

The last in this section is the flat strip theorem which was proved by O'Sullivan [21] and Eschenburg [11]. Our proof is elementary and simple rather than theirs. Let α and β be asymptotic in M. If the reversed geodesic β_- of β is asymptotic to the reversed geodesic α_- of α, then α and β are by definition *biasymptotic*. If α and β are biasymptotic and if $F(t):=d(\alpha(t), \beta(t))$ for $t \in \boldsymbol{R}$, then F is constant for $t \in \boldsymbol{R}$.

Theorem 1.13 (Flat strip theorem). *Let α and β be biasymptotic. Then there is a unique totally geodesic flat strip in M such that $\alpha(\boldsymbol{R}) \cup \beta(\boldsymbol{R})$ is its boundary.*

Proof. We first prove that if $p \in \alpha(\boldsymbol{R})$ and $q \in \beta(\boldsymbol{R})$, then there exists a flat strip such that its boundary consists of $\alpha(\boldsymbol{R})$ and $\beta(\boldsymbol{R})$ and it contains the segment $T(p, q)$. More precisely, if we assume that $p=\alpha(0)$ and $q=\beta(0)$ by changing their representations, if necessary, then the surface $S:=\bigcup_{-\infty<t<\infty} T(\alpha(t), \beta(t))$ is flat or equivalently is isometric to a strip in $\boldsymbol{R}^2$. To do this it is sufficient to prove that for each $r \in S$ if $r \in T(\alpha(s), \beta(s))$ and if γ is the asymptote to α (or equivalently β) with $\gamma(s)=r$, then γ is biasymptotic to α (or β) and $\gamma(t) \in T(\alpha(t), \beta(t))$ for all $t \in \boldsymbol{R}$. Because, F is constant if $F(t):=d(\alpha(t), \gamma(t))$ for all $t \in \boldsymbol{R}$. And further, because, if δ: $[0,L] \rightarrow M$ and ε: $[0, L] \rightarrow M$ are representations of $T(\alpha(t_0), \beta(t_0))$ and $T(\alpha(t_1), \beta(t_1))$ for each $t_0, t_1 \in \boldsymbol{R}$ and if $G(u):=d(\delta(u), \varepsilon(u))$ for $0 \leqq u \leqq L$, then G is constant $|t_0-t_1|$ in $0 \leqq u \leqq L$. The claim is proved as follows.

Let r_n be the point in $T(\alpha(n), \beta(n))$ such that $d(r_n, \alpha(n))=d(r, \alpha(s))$ for each $n \in Z_-$ and let γ_n be the asymptote to α with $\gamma_n(n)=r_n$. Then, by Proposition 1.9, $d(\gamma_n(s), \alpha(s)) \leqq d(\gamma_n(n), \alpha(n))$ and $d(\gamma_n(s), \beta(s)) \leqq d(\gamma_n(n), \beta(n))$ for every $n \leqq s$. Hence $d(\alpha(n), \beta(n))=d(\alpha(s), \beta(s)) \leqq d(\gamma_n(s), \alpha(s))+d(\gamma_n(s), \beta(s)) \leqq d(\gamma_n(n), \alpha(n))+d(\gamma_n(n), \beta(n))=d(\alpha(n), \beta(n))$, and thus $d(\gamma_n(s), \alpha(s))=d(\gamma_n(n), \alpha(n))=d(r, \alpha(s))$ and $d(\gamma_n(s), \beta(s))=d(\gamma_n(n), \beta(n))=d(r, \beta(s))$. This implies that $\gamma_n(s)=r$, because of the uniqueness of the existence of the segment $T(\alpha(s), \beta(s))$. By the uniqueness of the existence of the asymptote through given point in M, we conclude that $\gamma_n(t)=\gamma(t)$ for all $t \geqq n$, and hence $\gamma_n \rightarrow \gamma$ as $n \rightarrow -\infty$. It follows from the construction of γ_n that γ is biasymptotic to α and $\gamma(t) \in T(\alpha(t), \beta(t))$ for all $t \in \boldsymbol{R}$. This completes the first step of the proof.

We change a representation of β, if necessary, to satisfy that $f_\alpha(\beta(0))=0$ and $f_\alpha(\beta(t))=-t$ for all $t \in \boldsymbol{R}$. Let $S_0:=\bigcup_{-\infty<t<\infty} T(\alpha(t), \beta(t))$. We prove that S_0 is totally geodesic. Suppose that S_0 is not totally geodesic. Then there exist p and q in S_0 such that the segment $T(p, q)$ is not contained in S_0. Suppose that $p \in T(\alpha(t_0), \beta(t_0))$ and $q \in T(\alpha(t_1), \beta(t_1))$ for some $t_0 \neq t_1$. Let μ and τ be asymptotes to α with $\mu(t_0)=p$ and $\tau(t_1)=q$ respectively. Since μ and τ are biasymptotic, we determine the flat surfaces S_0' and S_1 as above such that $S_0' \ni p, \tau(t_0)$ and $S_1 \ni p, q$. Obviously

$S'_0 \subset S_0$. If $c: [0, L] \to S_1$ is the geodesic in S_1, where $L := d(p, q) \sin \measuredangle (V(p, q), \dot{\mu}(t_0))$, such that $c(0) = p$ and $\measuredangle(\dot{c}(0), \dot{\mu}(t_0)) = \pi/2$, then $f_\alpha \circ c$ is constant on $[0, L]$, and hence $c(L) = \tau(t_0)$ and $\measuredangle(\dot{c}(L), \dot{\tau}(t_0)) = \pi/2$. Hence we find from the Pythagoras theorem in S'_0 and S_1 that $d(p, q)^2 - L^2 = |t_0 - t_1|^2 = K^2 - d(p, \tau(t_0))^2$, where K is the distance between p and q in S'_0. This is impossible, because $d(p, q) < K$ and $L > d(p, \tau(t_0))$. Thus S_0 is totally geodesic.

§ 2. Totally convex sets

In this section we see to what extent the existence of totally convex sets influences the structure of N. A closed totally convex set Q in M' has the structure of a topological submanifold such that the interior of Q as a manifold is a smooth totally geodesic submanifold in M' (see [6] and [7]). If Q is a totally convex set in N, then the closure of Q is also totally convex in N, because of the uniqueness of the existence of the segment joining given two points in M. Each point $p \in M$ has a unique foot q on Q, since the distance function from p is strictly peakless. And the point q is the foot of each point $r \in T(p, q)$.

Lemma 2.1. *Let Q $(\neq M)$ be a closed totally convex set in M. Then for every point $r \in \dot{Q}$ there exists a perpendicular to Q which emanates from r, and M is covered by Q and the point set carrying all perpendiculars to Q. Furthermore, if $p \in M - Q$, then there exists a unique perpendicular to Q through p.*

Proof. Let $p \in M - Q$ and let q be the foot of p on Q. Let $\alpha: [0, \infty) \to M$ be the geodesic in M such that $\alpha(0) = q$ and $\alpha(d(p, q)) = p$. We want to prove that if $q \neq q' \in Q$, then $d(\alpha(t), q') > d(\alpha(t), q)$ for all $t > 0$. Let β be the geodesic in M such that $\beta(0) = q$ and $\beta(d(q, q')) = q'$. Then $\measuredangle(\dot{\beta}(0), \dot{\alpha}(0)) \geqq \pi/2$. In fact, otherwise there exists a point $q'' \in \beta([0, d(q, q')]) \subset Q$ such that $d(p, q) > d(p, q'')$, a contradiction. It follows from the convexity and the strict peaklessness of the distance function F from $\alpha(t)$ for each $t > 0$ that $F \circ \beta$ assumes a minimum only at a point $s_0 \in \boldsymbol{R}$ such that $(F \circ \beta)'(s_0) = 0$. Hence $d(\alpha(t), q') > d(\alpha(t), q)$ by the convexity and the strict peaklessness of $F \circ \beta$ again. This completes the first step.

Let $r \in \dot{Q}$ and $p_n \in M - Q$ such that $p_n \to r$ as $n \to \infty$. Let α_n be perpendiculars to Q through p_n for each n. If α is a limit of converging subsequence α_k of α_n, then α is a perpendicular to Q emanating from r. This completes the proof.

Lemma 2.2. *Let $\overline{M}'$ be the universal covering space of M' and π its* *projection. Suppose that Q is a closed set in M'. If $F(\cdot):=d(\cdot, Q)$ and $\bar{F}(\cdot):=d(\cdot, \pi^{-1}Q)$, then $\bar{F}=F\pi$.*

Proof. Let $\bar{p} \in \bar{M}'$ and $p:=\pi(\bar{p}) \in M'$. There exists a $q \in Q$ such that $d(p, q)=d(p, Q)$. Let $\bar{q}$ be the point over q such that the segment $T(\bar{p}, \bar{q})$ lies over a segment $T(p, q)$. Then $\bar{F}(\bar{p})\geqq F(p)=d(p, q)=d(\bar{p}, \bar{q})$ $\geqq d(\bar{p}, \pi^{-1}Q)=\bar{F}(\bar{p})$, since π is distance nonincreasing and $q \in \pi^{-1}Q$. This completes the proof.

Theorem 2.3. *Let Q $(\neq N)$ be a nonempty closed totally convex set in N. Then for every point $r \in \dot{Q}$ there exists a perpendicular to Q which emanates from r, and N is covered by Q and the point set carrying all perpendiculars to Q. Furthermore, if $p \in N-Q$, then there is a unique perpendicular to Q through p. In particular, N is noncompact.*

Proof. Let M be the universal covering space of N and π the projection. $\pi^{-1}Q$ is a closed totally convex set. If α is a perpendicular to $\pi^{-1}Q$, then $d(\pi\alpha(t), Q)=d(\alpha(t), \pi^{-1}Q)=t$ for all $t\geqq 0$. Hence $\pi\alpha$ is a perpendicular to Q. This completes the proof.

Busemann-Phadke [5] have indicated that this property is important in the proofs of some results in [2], which are generalized in G-spaces with convex capsules. The author learned the implication of consequence from [5].

Corollary 2.4. *If there exists a nonempty closed totally convex set Q in N such that $\partial Q=\phi$, then the exponential map of the normal bundle of Q onto N is a diffeomorphism.*

Remark 2.5. By Corollary 2.4, if α is a geodesic in M, then α has no focal points in N as a 1-dimensional submanifold. Hence the condition that for every nontrivial Jacobi field Y along every geodesic vanishing at $t=0$ the norm of Y is nondecreasing for $t>0$ implies that $\langle Y, Y\rangle'$ >0 for $t>0$ (see [10] and [20]).

The following is a generalization of a note in [21].

Lemma 2.6. *Let Q $(\neq M)$ be a nonempty closed totally convex set in M and let γ be a geodesic in M. If $F(t):=d(\gamma(t), Q)$, then either F is monotone nonincreasing for $t \in \boldsymbol{R}$ or $F(t)$ goes to infinity as $t\to\infty$.*

Proof. Suppose that there exists a $C>0$ such that $F(t_j)<C$ for some sequence $t_j \in \boldsymbol{R}$ with $t_j\to\infty$. Let $s<s'$. We define a point $q(t) \in Q$ for each $t \in \boldsymbol{R}$ as follows. If $\gamma(t) \in Q$, then $q(t):=\gamma(t)$. Otherwise $q(t)$ is the foot of $\gamma(t)$ on Q. Since $\{F(t_j); t_j\}$ is bounded above, we have; $d(q(s),$

$q(t_j)) \to \infty$ as $t_j \to \infty$, since $d(q(s), q(t_j)) \geqq |s-t_j| - F(t_j) - F(s)$ for all t_j. For each $t \in \boldsymbol{R}$ let α_t be a geodesic such that $\alpha_t(s) = q(s)$ and $\alpha_t(d(q(s), q(t)) + s) = q(t)$. Then, by Proposition 1.12, α_{t_j} converges to the asymptote α to γ with $\alpha(s) = q(s)$. Thus $F(s) = d(\gamma(s), Q) = d(\gamma(s), \alpha(s)) \geqq d(\gamma(s'), \alpha(s')) \geqq d(\gamma(s'), Q) = F(s')$, since the total convexity of Q implies that $\alpha([s, \infty)) \subset Q$. This completes the proof.

In conjunction with Lemma 2.2 and Lemma 2.6 we have

Proposition 2.7. *Let Q $(\neq N)$ be a nonempty closed totally convex set in N and let γ be a geodesic in N. If $F(t) := d(\gamma(t), Q)$ for all $t \in \boldsymbol{R}$, then either F is monotone nonincreasing in $t \in \boldsymbol{R}$ or $F(t)$ goes to infinity as $t \to \infty$.*

Proposition 2.8. *Let Q $(\neq N)$ be a nonempty closed totally convex set in M and let γ be a geodesic such that $\gamma(\boldsymbol{R}) \not\subset Q$ and $\{d(\gamma(t_j), Q)\}$ is bounded above for a sequence t_j which diverges in both directions of $\boldsymbol{R}$. For each $t \in \boldsymbol{R}$ let $q(t) \in Q$ be the foot of $\gamma(t)$ on Q. If $\alpha(t) := q(t)$ for all $t \in \boldsymbol{R}$, then α is a geodesic in M and biasymptotic to γ.*

Proof. By Lemma 2.6, $d(\gamma(t), Q)$ is constant in $t \in \boldsymbol{R}$, say $L > 0$. For each $n \in \boldsymbol{Z}$ let α_n be the asymptote to γ with $\alpha_n(n) = q(n)$. As in the proof above, $\alpha_n([n, \infty)) \subset Q$. Since $L \leqq d(\gamma(t), \alpha_n(t)) \leqq d(\gamma(n), \alpha_n(n)) = L$ for all $t \geqq n$, $d(\gamma(t), \alpha_n(t)) = L$ for all $t \geqq n$. From the uniqueness of the existence of the foot of $\gamma(t)$ for all $t \in \boldsymbol{R}$ it follows that $\alpha_n(t) = q(t) = \alpha(t)$ for all $t \geqq n$. This completes the proof as $n \to -\infty$.

As an application of Proposition 2.8, we have

Theorem 2.9. *Let Q_1 and Q_2 be nonempty compact totally convex set in N such that $Q_1 \neq Q_2$, $\partial Q_1 = \phi$ and $\partial Q_2 = \phi$. For each $q \in Q_2$ if $f(q) \in Q_1$ is the foot of $q \in Q_2$ on Q_1 and if $Q := \bigcup_{q \in Q_2} T(q, f(q))$, then Q is isometric to a Riemannian product $Q_2 \times [0, L]$, where $L := d(Q_1, Q_2)$, and Q_1 is isometric to Q_2.*

Proof. It follows that $d(q, Q_1) = L$ for every $q \in Q_2$. Otherwise there exists a geodesic α in N such that $\alpha(\boldsymbol{R}) \subset Q_2$ and $F(t) = d(\alpha(t), Q_1)$ is not constant in $t \in \boldsymbol{R}$. If $F(t_0) < F(t_1)$ for $t_0 < t_1$, then $F(t) \to \infty$ as $t \to \infty$ by Proposition 2.7, contradicting that $\alpha(\boldsymbol{R}) \subset Q_2$. Since $Q_1 \neq Q_2$, $Q_1 \cap Q_2 = \phi$. Next we want to construct an isometric map I of Q onto $Q_2 \times [0, L]$. The map I is defined as follows. Let $p \in Q$ and let q be the point in Q_2 such that $p \in T(q, f(q))$. If $t := d(p, q)$, then we define a map I by sending p to $(q, t) \in Q_2 \times [0, L]$. The map I is bijective. It remains to prove that $d(p, p') = d(I(p), I(p'))$ for any $p, p' \in Q$, i.e., if $I(p) = (q, t)$ and $I(p') = (q', t')$, then $d(p, p') = \sqrt{d(q, q')^2 + |t - t'|^2}$. Let M be the universal covering space of N and let π the projection. Fix a point $\bar{p} \in M$ with $\pi\bar{p} = p$.

Let $T(\bar{p}, \bar{p}')$ in M lie over a segment $T(p, p')$ in N and let $T(\bar{q}, \overline{f(q)})$, $T(\bar{q}', \overline{f(q')})$ lie over $T(q, f(q))$, $T(q', f(q'))$ which contain $\bar{p}$ and $\bar{p}'$ respectively. Using the flat strip which is determined by the geodesics through $\bar{q}$, $\bar{q}'$ or through $\overline{f(q)}$, $\overline{f(q')}$ respectively, we have that $d(p, p') = \sqrt{d(q, q')^2 + |t - t'|^2}$. The remainder is to prove that $d(\bar{q}, \bar{q}') = d(q, q')$. Suppose that there exist $\bar{\bar{q}}$ and $\bar{\bar{q}}'$ in $\pi^{-1}Q_2$ such that $d(q, q') = d(\bar{\bar{q}}, \bar{\bar{q}}') < d(\bar{q}, \bar{q}')$. Then we can conclude from Proposition 2.8 and Lemma 2.2 that there exists a curve from p to p' in N whose length is less than $d(p, p')$, a contradiction. Thus $d(\bar{q}, \bar{q}') = d(q, q')$. This ensures that I is an isometric map of Q onto $Q_2 \times [0, L]$.

If f is surjective, then Q_1 is isometric to Q_2. Suppose that there exists a point $p \in Q_1 - f(Q_2)$. By the definition of f, $d(p, Q_2) > L$. Let α be a geodesic in N such that $\alpha(0) = p$ and $\alpha(\boldsymbol{R}) \cap f(Q_2) \neq \phi$. If $F(t) := d(\alpha(t), Q_2)$ for each $t \in \boldsymbol{R}$, then F is not constant, contradicting Proposition 2.7 because $\alpha(\boldsymbol{R}) \subset Q_1$.

In order to continue the investigation we need the notion of ends. An *end* ε is by definition an assignment to each compact set K in M' a component $\varepsilon(K)$ of $M' - K$ in such a way that $\varepsilon(K_1) \supset \varepsilon(K_2)$ if $K_1 \subset K_2$. Greene-Shiohama [14] have studied the end structure of manifolds which admit locally nonconstant convex functions. Such a manifold has at most two ends. We can here obtain an analogous result.

Theorem 2.10. *Suppose that there exists a nonempty closed totally convex set Q in N which has no interior point in N. Then N has at most two ends. Furthermore, N has exactly two ends only if Q is a compact hypersurface in N such that $\partial Q = \phi$ and it separates N. In particular, in that case, N is diffeomorphic to a product $Q \times \boldsymbol{R}$, where Q is compact.*

Proof. Define a function F on N by $F(p) := d(p, Q)$ for each point $p \in N$. We first prove that if Q does not separate N, then N has one end. Secondly we will treat the other case. Then, if Q is compact, N has just two ends, and otherwise one end.

Suppose that Q does not separate N. For a $t > 0$ let $p, q \in F^{-1}(t)$. There exists a curve $c\colon [0, L] \to N - Q$ such that $c(0) = p$ and $c(L) = q$. If $\alpha_s\colon [0, \infty) \to N$ is the perpendicular to Q through $c(s)$ for each $s \in [0, L]$, then a map $c'\colon [0, L] \to F^{-1}(t)$ with $c'(s) := \alpha_s(t)$ is continuous, since the perpendicular to Q through $p \in N - Q$ continuously depends on p. This implies the connectedness of $F^{-1}(t)$ for all $t > 0$. Therefore for any compact set K the number of unbounded components of $N - K$ is one, so that N has one end.

Next we assume that Q separates N. It is well-known that Q is a hypersurface in N possibly with $\partial Q \neq \phi$. If $\partial Q = \phi$, then, by Corollary

2.4, the exponential map of the normal bundle of Q onto N is a diffeomorphism. Hence N is diffeomorphic to a product $Q\times \boldsymbol{R}$, since $F^{-1}(t)$ is not connected for $t>0$. If Q is compact, then N has just two ends. Otherwise N has one end. It remains to prove what we suppose above. Suppose that there exists a boundary point $p\in\partial Q$ as a manifold. Let q, r be points which are contained in distinct components of $N-Q$ and let α_0, β_0 be perpendiculars to Q through q, r respectively. Assume that α_1 and β_1 are perpendiculars to Q emanating from p such that $\dot\alpha_1(0)$ and $\dot\beta_1(0)$ are the resulting vectors of parallel translations of $\dot\alpha_0(0)$ and $\dot\beta_0(0)$ along segments $T(\alpha_0(0), p)$ and $T(\beta_0(0), p)$ respectively. Also we can use these parallel translations to obtain a curve from $\alpha_0(t)$ to $\alpha_1(t)$ (and from $\beta_0(t)$ to $\beta_1(t)$) in $N-Q$. Since p is a boundary point of Q any neighborhood of p in N can never be disconnected by Q. Hence we obtain a curve from $\alpha_1(\varepsilon)$ to $\beta_1(\varepsilon)$ in $N-Q$, where $\varepsilon>0$ is sufficiently small. Then there exists a curve from $\alpha_1(t)$ to $\beta_1(t)$ obtained by perpendiculars to Q through the curve as above. Consequently there exists a curve from q to r in $N-Q$, contradicting the choice of q and r. This completes the proof.

Concerning the existence of totally convex sets in N we have

Theorem 2.11. *Suppose that there exist at most countably many nonempty closed totally convex sets, $Q_1, Q_2, \cdots$, in N such that $Q_i\cap Q_j=\phi$ for $i\neq j$. If $Y:=N-\bigcup_{i=1}^{\infty} Q_i$ is nonempty and bounded, then the following hold.*

(1) *Exactly two of Q_i's are noncompact and other Q_i's are compact. We assume that Q_1 and Q_2 are noncompact.*

(2) *If $X:=\partial Q_1$ (or ∂Q_2), then X is a compact totally geodesic hypersurface in N with $\partial X=\phi$.*

(3) *If $W:=N-(Q_1\cup Q_2)$, then the closure $\overline{W}$ of W is isometric to a Riemannian product $X\times[0, L]$, where $L:=d(Q_1, Q_2)$.*

(4) *For each $i\geqq 3$ there exist numbers, a_i and a_i', such that $0<a_i\leqq a_i'<L$ and Q_i is isometric to $X\times[a_i, a_i']\subset X\times[0, L]$.*

It should be noted that the following property is often used in the proof: Any closed interval A in $\boldsymbol{R}$ can never be covered by countably many, mutually disjoint, proper closed intervals of A, a single point admitted.

Proof. Since Q_i is closed for every $i\geqq 1$ and since $Q_i\cap Q_j=\phi$ for $i\neq j$, there exists a point of Y near by any point p such that $p\in\dot{Q}_i$ for some i. Thus we note that there exists a compact set K in N such that K contains Y and $\bigcup_i \dot{Q}_i$.

Since the existence of nonempty closed totally convex set in N implies that N is noncompact, it follows from the existence of a ray in N and the remark above that one of Q_i's contains the ray, and hence is noncompact, say Q_1. If we use a perpendicular to Q_1 instead of the ray which is just now used, then we can find another noncompact set Q_i, $i \neq 1$, say Q_2. The interiors of Q_1 and Q_2 in N are nonempty. Otherwise Y is unbounded, a contradiction. If $X := \partial Q_1 = \dot{Q}_1$ and $X' := \partial Q_2 = \dot{Q}_2$, then X and X' are closed hypersurfaces in N, i.e., compact, $\partial X = \phi$, $\partial X' = \phi$.

We prove that X (and X') is connected. Suppose that it is false. Let X_1 be the component of X which contains the starting point p of a perpendicular α to Q_1 a sub-ray of which is contained in Q_2 and let X_2 be a different component of X. If $q \in X_2$, then a perpendicular β to Q_1 from q does not intersect Q_2. In fact, if there exist t_0, $t_1 > 0$ such that $\alpha(t_0) \in Q_2$ and $\beta(t_1) \in Q_2$, then we can obtain a curve from $\alpha(t_0)$ to $\beta(t_1)$ in Q_2, because Q_2 is totally convex in N. Since the starting point of the perpendicular to Q_1 through $r \in N - Q_1$ continuously depends on r, there exists a curve from p to q in X, a contradiction. There is a unique Q_{i_0} which contains a sub-ray of β. Let γ be a ray from q in Q_1 and let $c\colon \boldsymbol{R} \to N$ be a curve such that $c(\boldsymbol{R})$ is the union of $\gamma([0, \infty))$ and $\beta([0, \infty))$. Then $c(\boldsymbol{R}) \subset N - Q_2$. Hence it will be asserted from the continuity of perpendiculars to Q_2 that there is a curve $w\colon [0, 1] \to N - Y$ such that $w(0) \in \gamma([0, \infty)) \cap Q_1$, $w(1) \in \beta([0, \infty)) \cap Q_{i_0}$ and $w([0, 1])$ consists of finitely many segments. This contradicts the remark preceding the proof, and hence X is connected. In fact, the curve w is constructed as follows. Let $L' > 0$ be a number greater than the diameter of K. Let $w'\colon [0, 1] \to N - Q_2$ be a sub-arc of c such that $w'(0) = \gamma(L') \in Q_1$ and $w'(1) = \beta(L') \in Q_{i_0}$. For each $s \in [0, 1]$ if $\alpha_s\colon [0, \infty) \to N$ is the perpendicular to Q_2 through $w'(s)$ and if $w''(s) := \alpha_s(L')$, then $w''\colon [0, 1] \to N - Y$ is continuous. Hence we can obtain a desired curve w as an approximation of w''.

Next we prove that Q_i is compact for $i \geqq 3$. Suppose that Q_3, for example, is not compact. Let p be a point in Q_3 such that $d(p, Q_1) > L'$, where L' is as above, and let β' be the perpendicular to Q_1 through p. We can connect $\beta'(L')$ and $\alpha(L')$ by a curve in $N - Y$ as follows, where α is as above. To accomplish this, if $F(r) := d(r, Q_1)$ for each $r \in N$, then we prove that $F^{-1}(t)$ is connected for every $t > 0$. Since X is compact, $F^{-1}(t)$ is compact for each $t > 0$, and therefore the number of components of $F^{-1}(t)$ is finite. For $r \in N - Q_1$ let $f(r)$ denote the starting point of the perpendicular to Q_1 through r, i.e., the foot of r on Q_1. Since f is continuous on $N - Q_1$, for each $t > 0$ the image under f of each components of $F^{-1}(t)$ is closed in X. Hence there exists a point $p' \in X$ such that there exist perpendiculars, α_1 and β_1, to Q_1 from p' which pass different components, V_1 and V_2, of $F^{-1}(t)$, since X is connected. There exists a neighborhood B of p' such that $B - Q_1$ is connected, because of the total convexity of Q_1 with interior points in N. Hence for a sufficiently small $\varepsilon>0$ we can connect $\alpha_1(\varepsilon)$ and $\beta_1(\varepsilon)$ by a curve in $B-Q_1$. Using this curve we can obtain a curve in $F^{-1}(t)$ from $\alpha_1(t) \in V_1$ to $\beta_1(t) \in V_2$, a contradiction. Thus $F^{-1}(t)$ is connected for all $t>0$, so that there exists a curve from $\alpha(L')$ to $\beta'(L')$ in $F^{-1}(L')$, contradicting that $Q_2 \cap Q_3=\phi$. We have just finished the proof of (1).

We prove (2) and (3). Under the notation above $F^{-1}(t)$ is a closed topological submanifold in N for each $t>0$. If $F^{-1}(t)$ is totally convex for all $0<t<L$, then X is a totally convex set in N, because $F^{-1}(t)\to X$ as $t\to +0$. By the same reasoning the boundary X' of Q_2 in N is totally convex. Since X and X' have no boundarys as manifolds, it follows from Theorem 2.9 that the closure $\overline{W}$ of W is isometric to a Riemannian product $X\times[0, L]$. It remains to prove that $F^{-1}(t)$ is totally convex for each $0<t<L$. Suppose that it is false. Then there exist points p and q in $F^{-1}(t)$ such that there is a geodesic curve c from p to q which is not contained in $F^{-1}(t)$. If α is the extension of c, then $F\circ\alpha$ is not monotone nonincreasing in both directions of $\boldsymbol{R}$, and hence, by Proposition 2.7, $F\circ\alpha(t)\to\infty$ as $t\to \pm\infty$. Thus there exist $t_0\neq t_1 \in \boldsymbol{R}$ such that $\alpha(t_0) \in Q_2$ and $\alpha(t_1) \in Q_2$, contradicting the total convexity of Q_2.

For the proof of (4) it is sufficient to notice that if a totally convex set $C\subset X\times[0, L]$ contains a point $(q, t) \in X\times[0, L]$, then $C\supset X\times\{t\}$, since it follows from Theorem 2.3 that a compact manifold without focal points can never contain any proper totally convex set.

§3. Splitting theorems

In this section we give some conditions that N splits isometrically as $N_1\times\boldsymbol{R}$. We first need to extend the notion of the asymptotic closure in the sense of Eberlein-O'Neill [8] to manifolds without focal points.

It follows from Corollary 1.10 that the asymptote relation in the set of all geodesics in M is an equivalence relation. We denote by $\alpha(\infty)$ the class containing a geodesic α and by $M(\infty)$ the set of all asymptote classes. Let $\overline{M}:=M\cup M(\infty)$. We define a topology of $\overline{M}$ as follows. For a point $p \in M$ let $B(p)\subset T_pM$ be the closed unit ball and $S(p)$ its boundary sphere. Define a map F_p: $B(p)\to\overline{M}$ as follows; $F_p(v):=\exp_p v/(1-\|v\|)$ for $v \in B(p)-S(p)$, $F_p(v):=\gamma_v(\infty)$ for $v \in S(p)$, which γ_v is the geodesic with $\gamma_v(0)=v$. Obviously F_p is bijective, and also $F_p|(B(p)-S(p))$ is diffeomorphic onto M. Thus F_p yields a topology k on $\overline{M}$ such that $U\subset\overline{M}$ is open if and only if $F_p^{-1}(U)$ is open in $B(p)$. We have to prove that the definition is independent of the choice of $p \in M$. This is ensured by the following. The idea of the proof have already appeared in [9].

Lemma 3.1. *For any $p, q \in M$, $F_q^{-1} \circ F_p \colon B(p) \to B(q)$ is a homeomorphism.*

Proof. $F_q^{-1} \circ F_p$ is bijective, so that we have only to prove that $F_q^{-1} \circ F_p$ is continuous at any $v \in S(p)$, because $F_q^{-1} \circ F_p | B(p) - S(p)$ is diffeomorphic to $B(q) - S(q)$. Suppose that $B(p) \ni v_n \to v \in S(p)$. For each n there exists a unique $w_n \in B(p)$ such that $F_q(w_n) = F_p(v_n)$. Since $B(q)$ is compact, a subsequence w_k of w_n converges to a point $w \in S(p)$. It follows from Proposition 1.9 and Proposition 1.12 that γ_v and γ_w are asymptotic, and hence the sequence w_n converges to w. This completes the proof.

For each $v \in S(p)$ and $\pi > \varepsilon > 0$ the set $C(v, \varepsilon) := \{b \in \bar{M} \quad \{p\}$ $\sphericalangle_p(v, \dot{\gamma}_{pb}(0)) < \varepsilon$, where γ_{pb} is the geodesic through p and b and with $\gamma_{pb}(0) = p\}$, is called the *cone of vertex p, axis v and angle ε.* There exists a unique topology k' on $\bar{M}$ which is generated by canonical topology on M and the set of all cones. Obviously $k \supset k'$. The following lemma implies that $k = k'$.

Lemma 3.2. *For any $v \in S(p)$, any $0 < \varepsilon < \pi$ and any $L > 0$ there exists a cone $C(v', \varepsilon')$ such that $C(v', \varepsilon') \ni \gamma_v(\infty)$ and $C(v', \varepsilon') \subset C(v, \varepsilon) - \{q \in M;$ $d(p, q) \leqq L\} =: C$.*

Proof. Let $q := \gamma_v(L+1)$. Since $F_q^{-1}(C)$ is open in $B(q)$ and since $F_q^{-1}(\gamma_v([L+1, \infty])$ is a radius of $B(q)$, it follows that if $v' := \dot{\gamma}_v(L+1)$, then there exists an ε', $0 < \varepsilon' < \pi$, such that the set $\{u \in B(p); \sphericalangle(u, v') < \varepsilon'\}$ is contained in $F_q^{-1}(C)$. This completes the proof.

From this fact it is meaningful to call k the *cone topology* on $\bar{M}$. Hereafter we consider only $\bar{M}$ which is attached the cone topology. If μ: $TM \to M$ is the projection, then $\mu \times \exp \colon TM \to M \times M$ is a diffeomorphism. Using this map we define a distance δ on SM as follows. $\delta(v, w) := d(\mu v, \mu w) + d(\exp v, \exp w)$ for any v. $w \in SM$.

Lemma 3.3 (see [8]). *The map $\psi \colon SM \times [-\infty, \infty] \to \bar{M}$ given by $\psi(v, t) := \gamma_v(t)$ is continuous.*

Proof. It will be sufficient to prove that $\gamma_{v_n}(t_n) \to \gamma_v(\infty)$ as $v_n \to v$ and $t_n \to \infty$. Let $p := \mu v$ and $p_n := \mu v_n$. There exists a unique $w_n \in S(p)$ such that $\gamma_{w_n}(s_n) = \gamma_{v_n}(t_n)$, where $s_n := d(p, \gamma_{v_n}(t_n))$. Obviously $s_n \to \infty$ as $n \to \infty$. By Proposition 1.1, $d(\exp (t_n/s_n)v_n, \exp w_n) \leqq d(p, p_n)$. Hence

$$\begin{aligned}\delta(v_n, w_n) &= d(p_n, p) + d(\exp v_n, \exp w_n) \\ &\leqq d(p, p_n) + d(\exp v_n, \exp (t_n/s_n)v_n) \\ &\quad + d(\exp (t_n/s_n)v_n, \exp w_n) \to 0\end{aligned}$$

as $n \to \infty$, since $p_n \to p$ and $t_n/s_n \to 1$, and thus $w_n \to v$ as $n \to \infty$. Since $\gamma_{v_n}(t_n)$

$=\gamma_{w_n}(s_n)=F_p((s_n/1+s_n)w_n)$ and $(s_n/1+s_n)w_n \to v$, $\gamma_{v_n}(t_n) \to \gamma_v(\infty)$. This completes the proof.

The following will be used in the proof of Theorem 3.12.

Lemma 3.4 (see [8]). *Let $E:=\{(p, b) \in M \times \overline{M}; p \neq b\}$. The map V: $E \to SM$ given by $V(p, b):=\dot{\gamma}_{pb}(0)$ is continuous.*

For we have just proved that $\lim V(p_n, b_n)=\lim V(p, b_n)=\dot{\gamma}_{pb}(0)$ if $(p_n, b_n) \to (p, b) \in M \times \overline{M}$.

Lemma 3.5 (see [8]). *Let $E':=\{(p, a, b) \in M \times \overline{M} \times \overline{M}; p \neq a, p \neq b\}$. The function $\sphericalangle: E' \to \boldsymbol{R}$ given by $\cos \sphericalangle_p(a, b)=\langle V(p, b), V(p, b)\rangle$ $(0 \leqq \sphericalangle_p(a, b) \leqq \pi)$ is continuous.*

This is because V and $\langle \cdot, \cdot \rangle$ is continuous.

Let ϕ be an isometry on M. Then ϕ preserves the asymptote relation on the set of all geodesics in M. Hence we can naturally extend ϕ to a map of $\overline{M}$ into itself, which is denoted by the same notation ϕ. Obviously ϕ: $\overline{M} \to \overline{M}$ is bijective. If C is a cone in M, then ϕC is a cone in M. Thus we have

Lemma 3.6. *Let ϕ be an isometry on M. Then ϕ: $\overline{M} \to \overline{M}$ is a homeomorphism. In particular, ϕ always has a fixed point in $\overline{M}$.*

If $x \in M(\infty)$ is a fixed point of an isometry ϕ of M, then $\phi^{-1}\alpha \in x$ for all $\alpha \in x$. It follows from Proposition 1.11 that $f_\alpha \circ \phi - f_\alpha = f_{\phi^{-1}\alpha} - f_\alpha$ is constant on M. By the same reasoning, $f_\alpha \circ \phi - f_\beta \circ \phi = f_\alpha - f_\beta$ on M for any $\alpha, \beta \in x$. Thus $f_\alpha \circ \phi - f_\alpha$ is independent of the choice of $\alpha \in x$, so that we can define a function T_x: $I_x(M) \to \boldsymbol{R}$ by $T_x(\phi)=f_\alpha \circ \phi - f_\alpha$, where $I_x(M)$ is the set of all isometries of M which have a fixed point x.

Lemma 3.7 (see [8]). *T_x is a homomorphism into the additive group of real numbers.*

A geodesic γ in N is by definition *almost minimizing* if there is a number $c>0$ such that $d(\gamma(0), \gamma(t)) \geqq t-c$ for all $t \geqq 0$. We say that $x \in M(\infty)$ is (*almost*) *D-minimizing* if $\pi\alpha$ is (almost) minimizing in $N:=M/D$ for every $\alpha \in x$. Obviously it follows that if $\pi\alpha$ is almost minimizing in N, then $\alpha(\infty)$ is almost D-minimizing. However we do not know whether the property hold for minimizing geodesics.

Lemma 3.8 (see [8]). *Let D be a properly discontinuous group of isometries of M, and let $z \in M(\infty)$ be a common fixed point of D. Then the following are equivalent.*

(1) *z is almost D-minimizing.*
(2) $T_z(\phi)=0$ *for every* $\phi \in D$, *where* T_z *is as in Lemma* 3.7.
(3) *z is D-minimizing.*

Lemma 3.9. *Let D, z be as in Lemma* 3.8 *and let* $\pi\colon M \to M/D$ *is the covering projection. Then* $f_{\pi\alpha} \circ \pi = f_\alpha$, *where* $\alpha \in z$ *and* f_α *and* $f_{\pi\alpha}$ *is the Busemann functions of* α *and* $\pi\alpha$ *respectively. In particular,* $f_{\pi\alpha}$ *is convex and* (C^1-)*differentiable.*

Proof. From (2) of Lemma 3.8 there exists a function f on $N := M/D$ such that $f \circ \pi = f_\alpha$, where $\alpha \in z$. Then f is convex and C^1-differentiable, since so is f_α on M. We assert that $f = f_{\pi\alpha}$. Let $p \in N$ and let $\bar{p}$ lie over p. Let $T(\bar{p}, \bar{q}_t)$ lie over $T(p, \pi\alpha(t))$ for each t. Then $\bar{q}_t \in f_\alpha^{-1}(-t)$. If β is an asymptote to α in M such that $\beta(\boldsymbol{R}) \cap f_\alpha^{-1}(0) = \beta(0)$ and $\bar{p} = \beta(-s)$ for some s, then $s + t \leqq d(\bar{p}, \bar{q}_t) = d(p, \pi\alpha(t)) \leqq d(\bar{p}, \alpha(t))$ for every $t \in \boldsymbol{R}$, and therefore $f_\alpha(\bar{p}) = s \leqq \lim\{d(\bar{p}, \bar{q}_t) - t\} = \lim\{d(p, \pi\alpha(t)) - t\} \leqq \lim\{d(\bar{p}, \alpha(t)) - t\} = f_\alpha(\bar{p})$. Thus $f_{\pi\alpha} = f_\alpha \circ \pi = f$. This completes the proof.

Let ϕ be an isometry of M'. Define a function g_ϕ on M' by $g_\phi(p) := d(p, \phi p)$ for each point $p \in M'$. g_ϕ is called the *displacement function* of ϕ. The displacement function divides the isometry of M' into three types. ϕ is said to be *elliptic* if g_ϕ has minimum zero, *axial* if g_ϕ has positive minimum and *parabolic* if g_ϕ has no minimum. In M an isometry ϕ is axial if and only if there exists a geodesic α in M such that, for some $a \neq 0$, $\phi\alpha(t) = \alpha(t+a)$ for all $t \in \boldsymbol{R}$. Such a geodesic α is called an *axis* of ϕ. By Proposition 1.12, all axes of ϕ are biasymptotic to each other. Also $a = \min g_\phi$.

Let D be a properly discontinuous group of isometries of M without fixed points. We say that D is *axial* if every $1 \neq \phi \in D$ is *axial* and $\phi, \psi \in D - \{1\}$ have biasymptotic axes. We say that D is *parabolic* if there exists a $z \in M(\infty)$ that is the unique fixed point of every $1 \neq \phi \in D$. Also we say that N is *axial* (or *parabolic*) if the fundamental group D is *axial* (or *Parabolic*). It is *axial* and *parabolic* manifolds that have simplest relationships between the fundamental group and its limit set (see [8]). Although we can extend results of [8] concerning axial and parabolic manifolds, it is not interesting to do so, because the material change of methods in [8] may be unnecessary. But another important and simplest cases exist which are related to the splitting theorems of manifolds and treated in [17]. Without material change of methods of [17] we have the following. But we improve it in Proposition 3.13.

Proposition 3.10. *Let D be a properly discontinuous group of isometries of M without fixed points. If there exist distinct points x and y in* $M(\infty)$ *such that* (1) *they are common fixed points of D,* (2) *they are almost*

D-minimizing and (3) $\pi(x)$ *and* $\pi(y)$ *are in different ends of* $N:=M/D$, *then* N *is isometric to a Riemannian product* $N_1 \times \boldsymbol{R}$.

In [14] there arises a problem concerning the growth of the diameter of levels of a convex function. We treat a special case in Theorem 3.12.

Lemma 3.11. *Let* γ *be a ray in* N *and let* $\bar{\gamma}$ *be a lift of* γ *to the universal covering space* M *of* N. *Then* f_γ *is convex on* N *if and only if* $\bar{\gamma}(\infty)$ *is an almost D-minimizing fixed point of* D, *the fundamental group of* N.

Proof. The sufficient condition is shown in the same way as the proof of Theorem 3.9.

Suppose that f_γ is convex on N. Let $1 \neq \phi \in D$. For each $t \geqq 0$ let α_t: $[0, L_t] \rightarrow N$ be the geodesic loop at $\gamma(t)$ such that α_t corresponds to ϕ. For each $t \geqq 0$ if $\beta(u):=\gamma(-u+t)$ for each $u \geqq 0$, where the domain of γ is extended to $\boldsymbol{R}$, then β is a perpendicular to $f_\gamma^{-1}((-\infty, -t])$. Hence $\sphericalangle(\dot{\gamma}(t), \dot{\alpha}_t(0)) \leqq \pi/2$ and $\sphericalangle(\dot{\gamma}(t), -\dot{\alpha}_t(L_t)) \leqq \pi/2$ for every $t \geqq 0$. This means that L_t is monotone nonincreasing in $t \geqq 0$, and equivalently $\bar{\gamma}(\infty)$ is a fixed point of ϕ. Since γ is a ray, $\bar{\gamma}(\infty)$ is almost D-minimizing. These completes the proof.

Theorem 3.12. *Let* γ *be a ray in* N. *If* f_γ *is convex on* N *and if the diameter function* δ *of the levels of* f_γ *is bounded, then* N *is isometric to a Riemannian product* $N_1 \times \boldsymbol{R}$.

Proof. Let M be the universal covering space of N and $\bar{\gamma}$ be a lift of γ to M. Let D be the fundamental group of N. By Lemma 3.11, $\bar{\gamma}(\infty)$ $=:z$ is an almost D-minimizing fixed point of D. For each point $p \in M$ let $W(p):=\gamma_{pz}(-\infty) \in M(\infty)$. If ψ and V are maps as in Lemma 3.3 and 3.4, then $W(p)=\psi(V(p, z), -\infty)$ for all $p \in M$. Hence the map W: $M \rightarrow M(\infty)$ is continuous, and thus $W(M)$ is connected in $M(\infty)$. Once we establish that $W(M) \subset D\bar{\gamma}(-\infty)$, since $D\bar{\gamma}(-\infty)$ is a countable set in $M(\infty)$, $W(M)=\bar{\gamma}(-\infty)$, and then M is isometrically a Riemannian product $M_1 \times \boldsymbol{R}$. The product decomposition $M_1 \times \boldsymbol{R}$ is hereditable to N through the projection π. This is our goal. It remains to prove that $W(M)=D\bar{\gamma}(-\infty)$. Let $p \in M$ and let $\bar{\beta}$ be a geodesic through p in M such that $f_{\bar{\gamma}}(\bar{\beta}(0))=0$. Assume that $L>0$ is an upper bound of δ. For each $t<0$ there exists a $\phi_t \in D$ such that $d(\bar{\beta}(t), \phi_t\bar{\gamma}(t))<L$. Since $\phi_t\bar{\gamma}$ and $\bar{\beta}$ are asymptotic, $d(\bar{\beta}(0), \phi_t\bar{\gamma}(0))<L$. By the proper discontinuity of D, there exists a sequence t_n and a $\phi \in D$ such that $t_n \rightarrow -\infty$ and $\phi_{t_n}=\phi$. Then $d(\bar{\beta}(t), \phi\bar{\gamma}(t))<L$ for all $t \in \boldsymbol{R}$, since for each n it holds that $d(\bar{\beta}(t), \phi\bar{\gamma}(t))$ $<L$ for all $t \geqq t_n$. Hence $\bar{\beta}$ and $\phi\bar{\gamma}$ is biasymptotic, and therefore $\phi\bar{\gamma}(-\infty)$ $=\bar{\beta}(-\infty)$. This completes the proof.

Concerning Theorem 3.12, it should be noted that if N has non-positive sectional curvature, then "f_γ is convex" in the assumption can be replaced by "there exists a convex function f without minimum". In fact, using (4) of 3.4 Proposition in [2], it is possible to prove the analogous result to Lemma 3.11: For each point $p \in N$ let γ be a ray such that if $q_s \in N$ is the foot of p on $f^{-1}((-\infty, s])$ for each s, then $\gamma([0, \infty))$ is the limit of a subsequence of $T(p, q_s)$ as $s \to \inf f$. Then $\bar{\gamma}(\infty)$ is an almost D-minimizing fixed point of D, where $\bar{\gamma}$ is a lift of γ to M. Also we can obtain the analogous result to Theorem 3.12 if we replace $f_{\bar{\gamma}}$ with the lift $\bar{f}$ of f to M in the proof.

The following is an application of Theorem 2.11 and gives another version of Proposition 3.10.

Proposition 3.13. *Suppose that N has two ends. If there exist convex functions f_1 and f_2 on N such that they have no minimum and $f_1^{-1}((-\infty, t_0]) \cap f_2^{-1}((-\infty, s_0]) = \phi$ for some $t_0 > \inf f_1$ and some $s_0 > \inf f_2$, then N is isomertic to a Riemannian product $N_1 \times \boldsymbol{R}$.*

Proof. From Theorem 2.11 we need only to prove that $W := N - f_1^{-1}((-\infty, t_1]) - f_2^{-1}((-\infty, s_1])$ is bounded for any t_1, $t_0 > t_1 > \inf f_1$ and s_1, $s_0 > s_1 > \inf f_2$, since $f_1^{-1}((-\infty, t_1]) \cap f_2^{-1}((-\infty, s_1]) = \phi$. Let γ_1 and γ_2 be rays in N such that $f_1(\gamma_1(u)) \to \inf f_1$ as $u \to \infty$ and $f_2(\gamma_2(u)) \to \inf f_2$ as $u \to \infty$. Suppose that W is unbounded. Then there exist p_0 and p_1 in W such that $f_1(p_0) < f_1(p_1)$ and $f_2(p_0) < f_2(p_1)$. The reasoning is as follows. By [14] all levels of f_1 and f_2 are compact and N is topologically a product $N_1 \times \boldsymbol{R}$, where N_1 is a level of f_1. Hence, if p'_n is a sequence of points in W which goes to infinity, then $f_1(p'_n), f_2(p'_n) \to \infty$ as $n \to \infty$. Thus there exist points in W which satisfy the condition. Let γ be a geodesic in N such that $\gamma(0) = p_0$ and $\gamma(L) = p_1$ for some $L > 0$. Let K be a compact set in N containing $f_1^{-1}(t_1)$ and $f_2^{-1}(s_1)$. Since all $\gamma(u)$, $\gamma_1(u)$ and $\gamma_2(u)$ go to infinity as $u \to \infty$, there exists an $L' > L > 0$ such that each of $\gamma([L', \infty))$, $\gamma_1([L', \infty))$ and $\gamma_2([L', \infty))$ is contained in an unbounded component of $N - K$. Hence $\gamma([L', \infty))$ and $\gamma_1([L', \infty))$ are, for example, contained in the same component of $N - K$, since N has two ends. There exists a curve $c: [0, 1] \to N - K$ such that $c(0) = \gamma(L')$ and $c(1) = \gamma_1(L')$. However this is impossible, because $f_1(\gamma_1(L')) < t_1$ and $f_1(\gamma(L')) > t_1$, which implies the existence of a $v \in [0, 1]$ such that $f_1(c(v)) = t_1$. This completes the proof.

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