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On the Manifolds of Periodic Geodesics

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Let S^n be the n-dimensional sphere with a Riemannian metric g. If all geodesics are periodic with the same period l, we say that the Riemannian manifold (S^n, g) is a C_l-manifold, or g is a C_l-metric on S^n. (For detail see Besse [2]). Let g_0 be the canonical metric of S^n. Then (S^n, g_0) is a $C_{2\pi}$-manifold. There are some examples of C_l-metric on S^n (Zoll [10], Weinstein [2], Guillemin [5]) other than the canonical metric. These examples are all obtained from deformations of g_0 in the space of C_l-metrics.

Let $T_1(S^n, g)=T_1(S^n)$ denote the tangent sphere bundle of radius 1 of a C_l-manifold (S^n, g). Then the geodesic flow induces a free S^1-action on T_1S^n. Since the geodesic flow vector field is a contact vector field on T_1S^n, the quotient space T_1S^n/S^1 is a $(2n\text{-}2)$-dimensional symplectic manifold. We call T_1S^n/S^1 the manifold of geodesics and denote by $\mathrm{Geod}(S^n, g)$. The manifold $\mathrm{Geod}(S_n, g_0)$ is symplectically diffeomorphic to the Kähler manifold Q^{n-1}, called hyperquadric and defined by the equation

$$Z_0^2+Z_1^2+\cdots+Z_n^2=0$$

in CP^n. Since every known example of C_l-manifold (S^n, g) is a deformation of (S^n, g_0), the manifold of geodesics $\mathrm{Geod}(S^n, g)$ for such manifold is symplectically diffeomorphic to Q^{n-1}.

A result of Weinstein [7] says that, if $\mathrm{Geod}(S^n, g_1)$ and $\mathrm{Geod}(S^n, g_2)$ are symplectically diffeomorphic, then the eigenvalues of the Laplacian on two C_l-manifolds (S^n, g_1) and (S^n, g_2) are asymptotically similar.

Our problem is as follows. For any C_l-metric g on S^n, is $\mathrm{Geod}(S^n, g)$ diffeomorphic to Q^{n-1}? In this paper we study the tangent bundle of $\mathrm{Geod}(S^n, g)$ and its characteristic classes.

Since $Sp(n-1, \boldsymbol{R})$ is homotopy equivalent to $U(n-1)$, the symplectic manifold $\mathrm{Geod}(S^n, g)$ has the unique almost complex structure up to homotopy.

Let γ denote the complex line bundle associated to the S^1-principal

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bundle $T_1S^n \to \text{Geod}(S^n, g)$. Recall the structure of the tangent bundle τ_0 of $\text{Geod}(S^n, g_0) = Q^{n-1}$. We have

$$\tau_0 \otimes \gamma \oplus \gamma \oplus \gamma^* = \varepsilon^{n+1}$$

where γ^* is the dual of γ and ε^{n+1} is the trivial $(n+1)$-dimensional complex vector bundle over Q^{n-1}. Let $g \in H^2(\text{Geod}(S^n, g); Z)$ be the class represented by the symplectic form. For the Chern class C_i of $\tau_0 \otimes \gamma$, we have

$$C_i(\tau_0 \otimes \gamma) = \begin{cases} 0 & i\text{: odd} \\ g^i & i\text{: even.} \end{cases}$$

The characterictic class of $\text{Geod}(S^n, g_0)$ is determined by this equation. The bundle $\gamma \oplus \gamma^*$ is isomorphic to $\gamma_R \otimes_R C$, where γ_R denote the underlying real vector bundle of γ.

By a stable class of a bundle η, we mean the Whitney sum of η with a trivial bundle of sufficient dimension. We write η^{st} for the stable class of η.

Our main result is as follows.

Theorem. *Let (S^n, g) be a C_l-manifold and let τ be the tangent bundle of* Geod (S^n, g). *Then there exists a real vector bundle ξ over* Geod (S^n, g) *such that*

$$(\tau \otimes \gamma)^{st} = \xi \underset{R}{\otimes} C.$$

Remark that there are many examples of free S^1-action on T_1S^n, not coming from a geodesic vector field, such that the tangent bundle of the orbit space does not satisfy the relation in Theorem.

The cohomology ring $H^*(\text{Geod}(S^n, g); Z)$ is known to be isomorphic to $H^*(Q^{n-1}; Z)$ which has no torsion (Yang [9]).

Corollary 1. *Every odd dimensional Chern class of the bundle $\tau \otimes \gamma$ vanishes.*

If $n \leq 4$, then the non-zero Chern class of $\tau \otimes \gamma$ is only $C_2(\tau \otimes \gamma) \in H^4(\text{Geod}(S^n, g); Z)$. The $(n-1)$-dimensional Chern class $C_{n-1}(\tau)$ of τ is representable by $C_2(\tau \otimes \gamma)$, which must be equal to the Euler class.

By identifying the isomorphic cohomology rings $H^*(\text{Geod}(S^n, g); Z)$ and $H^*(Q^{n-1}; Z)$, we have the following.

Corollary 2. *If $n \leq 4$, then the Chern classes of the manifold* Geod (S^n, g) *are equal to that of* Geod (S^n, g_0).

S. Sasao has shown the author the following result.

Proposition. *Let M be a simply connected 6-dimensional closed manifold with cohomology ring $H^*(M; Z)$ isomorphic to $H^*(Q^3; Z)$. If the second Stiefel class $w_2(M) \neq 0$, then M is homotopy equivalent to Q^3.*

By using the Browder-Novikov's surgery technique, we obtain

Corollary 3. *For any C_l-metric g on S^4, the manifold* Geod(S^4, g) *is diffeomorphic to Q^3.*

For the proof of Theorem, we need an inverse of Thom isomorphism in K-theory, a global Jacobi equation written in terms of the horizontal lift of connections and a topological study of Sturm-Liouville equations by means of Morse theory.

We outline our argument. Detailed proof will appear elsewhere.

I. Topological Preliminaries

1. Projectable bundles

Let X be a smooth manifold and let $\pi: L \to X$ be the projection of an S^1-principal bundle.

Definition. A vector bundle $p: E \to L$ over L is projectable onto X, if there exists a vector bundle $\hat{P}: \hat{E} \to X$ over X such that $\pi^*\hat{E} = E$. The map π induces the bundle map $\pi_!: E \to \hat{E}$, which we call the projection. The bundle $\hat{E}$ is called the projected bundle.

Let x be a point in X. For any $a, b \in \pi^{-1}(x) = S^1$, we have a linear isomorphism

$$\Phi_{ab}: p^{-1}(a) \longrightarrow p^{-1}(b)$$

of vector spaces defined by $\Phi_{ab}(u) = v$, where $\pi_!(u) = \pi_!(v)$. Then we have, for $a, b, c \in \pi^{-1}(x)$,

$$(1) \qquad \Phi_{bc}\Phi_{ab} = \Phi_{ac}.$$

Let $\pi^*L = \{(a, b) \in L \times L,\ \pi(a) = \pi(b)\}$ be the induced S^1-bundle over L from L. We have two projections $\pi_1, \pi_2: \pi^*L \to L$ defined by $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$. Let π_i^*E $(i = 1, 2)$ be the induced vector bundle. The map $\Phi: \pi^*L \to \mathrm{Iso}(\pi_1^*E, \pi_2^*E)$ defined by $\Phi(a, b) = \Phi_{ab}$ is a continuous cross section of the bundle $\mathrm{Iso}(\pi_1^*E, \pi_2^*E)$ over π^*L.

We call Φ the projecting isomorphism associated with the projectable bundle E. Given a cross section Φ satisfying (1) and a vector bundle E

over L, we can regard E as a projectable bundle with Φ as the projecting isomorphism.

Assume that the vector bundle $p: E \to L$ is isomorphic to the trivial bundle. Then we may regard Φ_{ab} as an element of the general linear group GL. The bundle $\mathrm{Iso}(\pi_1^* E, \pi_2^* E)$ is isomorphic to the trivial bundle $GL \times \pi^* L$ over $\pi^* L$. The space $\pi^* L$ is naturally homeomorphic to $S^1 \times L$. Let G be a subgroup of GL such that Φ_{ab} is contained in G for any $(a, b) \in \pi^* L$. Then Φ is a map from $S^1 \times L$ to G. Define a map $\tilde{\Phi}: L \to L \times \Omega G$ by $\tilde{\Phi}(a) = (a, \Phi(\cdot, a))$. We have the action of S^1 on L. On the loop space ΩG, the group S^1 acts by $(t\omega)(s) = \omega(t+s) \cdot \omega(t)^{-1}$, where $t, s \in S^1$, $\omega \in \Omega G$, and $\cdot$ denote the composition of G. Thus we have the product action of S^1 on $L \times \Omega G$. Using (1), we can easily see that $\tilde{\Phi}$ is S^1-equivariant. The factor space $L \times_{S^1} \Omega G$ is the total space of a fiber bundle over $X = L/S^1$. Thus $\tilde{\Phi}$ is a cross section of the bundle $L \times_{S^1} \Omega G$, i.e. $\tilde{\Phi} \in \Gamma(X, L \times_{S^1} \Omega G)$.

2. Homotopy theorem

Let Λ denote the complex line bundle over X associated to L and let X^{Λ} denote the Thom space of Λ. Let $\mathrm{Vect}(X^{\Lambda})$ denote the set of isomorphism classes of vector bundles over X. Let $\Pi\Gamma(X, L \times_{S^1} \Omega G)$ denote the set of homotopy classes of $\Gamma(X, L \times_{S^1} \Omega G)$. By taking the homotopy classes of $\tilde{\Phi}$, we have the set map

$$\Phi: \mathrm{Vect}(X^{\Lambda}) \longrightarrow \Pi\Gamma(X, L \underset{S^1}{\times} \Omega G).$$

Now suppose that we are given an element f in $\Gamma(X, L \times_{S^1} \Omega G)$. Let E be a trivial bundle over L. Then we naturally obtain a projecting isomorphism Ψ_f of E which satisfy (1). We write $\Psi_f(E)$ for the projected bundle over X.

Proposition. *If f_1, f_2 in $\Gamma(X, L \times_{S^1} \Omega G)$ are homotopic, then $\Psi_{f_1}(E)$ and $\Psi_{f_2}(E)$ are isomorphic vector bundles over X.*

3. Stable case

Suppose that the dimension of the fiber of the trivial bundle E is sufficiently large. Let $G = GL(\boldsymbol{R}, \infty)$, $GL(\boldsymbol{C}, \infty)$ or $GL(\boldsymbol{H}, \infty)$. According to Bott [4], the space ΩG is homotopy equivalent to the space of minimal geodesics, on which S^1 acts trivially.

Proposition. *The fiber bundle $L \times_{S^1} \Omega G$ is homotopy equivalent to the trivial bundle $X \times \Omega G$.*

If $G=GL(\boldsymbol{C}, \infty)$, then $\Gamma\Pi(X, L\times_{S^1}\Omega G)=K(X)$. The map Φ in Section 2 may be regarded as a map

$$\Phi: \tilde{K}(X^{\Lambda})\longrightarrow K(X).$$

Let $\tau: K(X)\to\tilde{K}(X^{\Lambda})$ be the Thom isomorphism defined by $\tau(x)=\lambda_{\Lambda}x$, where $\lambda_{\Lambda}\in\tilde{K}(X^{\Lambda})$ is defined by the exterior algebra of Λ ([1]).

Proposition. *The map Φ is the inverse of τ.*

Corollary. *For $f\in\Gamma(X, L\times_{S^1}\Omega GL(\boldsymbol{C}, \infty))$,*

$$\{\psi_f(E)\}=(1-[\Lambda])\{f\}\in\tilde{K}(X).$$

If $G=GL(\boldsymbol{R}, \infty)$, then the projected bundle is isomorphic to the complexification of a real vector bundle. Let $i: [X, \Omega GL(\boldsymbol{R}, \infty)]\to [X, \Omega GL(\boldsymbol{C}, \infty)]$ be the natural homomorphism.

Corollary. *For any $\gamma\in K(X)$ represented by a line bundle over X, and for any $\varepsilon\in[X, \Omega GL(\boldsymbol{R}, \infty)]$, there exists a real vector bundle β on X, such that*

$$(1-\gamma)i(\varepsilon)=\beta_C\in\tilde{K}(X),$$

where β_C is the complexification $\beta\otimes_{\boldsymbol{R}}\boldsymbol{C}$.

II. The Manifold of Geodesics

4. Global Jacobi differential equation

Let ∇ be a connection on the tangent bundle TM of a smooth manifold M, and let ∇^H be the horizontal lift of ∇ (Yano-Ishihara [8]). Then ∇^H is a connection on the tangent bundle TTM of TM. We decompose the tangent space T_vTM, $v\in TM$, as the sum of the horizontal part and the vertical part

$$T_vTM=H_vTM+V_vTM.$$

We write $X=\begin{pmatrix}x_1\\x_2\end{pmatrix}$ for $X=x_1^H+x_2^V\in T_vTM$, where $x_1, x_2\in T_{\pi(v)}M$, x_1^H and x_2^V are horizontal and vertical lifts, π is the projection $TM\to M$. For a vector field X on M, by definition, we have

$$\nabla^H_{X^V}\begin{pmatrix}Y_1\\Y_2\end{pmatrix}=0,\qquad \nabla^H_{X^H}\begin{pmatrix}Y_1\\Y_2\end{pmatrix}=\begin{pmatrix}\nabla_X Y_1\\\nabla_X Y_2\end{pmatrix}.$$

Let g be a Riemannian metric on M. For $v \in T_xM$, we define a linear symmetric transformation R_v of T_xM by

$$R_v w = R(w, v)v,$$

where $w \in T_xM$ and R is the curvature tensor of (M, g). To each $v \in T_xM$, we define a linear endomorphism P_v of T_vTM by

$$P_v\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -R_v & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This defines a smooth cross section P of the bundle Hom(TTM, TTM) over TM. Let Z be the geodesic flow vector field. Then we have $Z(v) = \begin{pmatrix} v \\ 0 \end{pmatrix}$ for $v \in TM$. We define a linear differential equation on TM by

$$\nabla_Z^H Y = PY, \tag{2}$$

where $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ is a time dependent smooth cross section of the bundle TTM over TM. We call (2) the global Jacobi differential equation. It is a second order differential equation.

5. The tangent bundle of the manifold of geodesics

Let (M, g) be a C_l-manifold. Then the geodesic flow is periodic and defines a free S^1-action on the unit tangent sphere bundle T_1M. The manifold of geodesics Geod(M, g) is the quotient space T_1M/S^1. We give the canonical metric on TM. Let $\tilde{F}$ be the subbundle of TTM consisting of vectors orthogonal to X^H and x^V at $x \in TM$. Let F be the restriction of $\tilde{F}$ on T_1M. Then the global Jacobi equation (2) can be restricted on F and it defines a projecting isomorphism on F. Since the solution of the Jacobi equation is the integral curve of the complete lift ([8]) of the geodesic flow vector field, we obtain

Proposition. *The projected bundle $\hat{F}$ defined by the global Jacobi differential equation is isomorphic to the tangent bundle TGeod(M, g) of Geod(M, g).*

III. Differential Equation and Morse Theory

6. Sturm-Liouville equation and a symmetric space

We want to study geometrically families of curves defined by vector-valued Sturm-Liouville equations

$$y'' - q_x(t)y = 0, \tag{3}$$

where y'' means d^2y/dt^2 and $q_x(t)$ is a symmetric matrix-valued continuous function on $t \in \boldsymbol{R}$ parameterized by a point x in a smooth manifold X. Put $z = \begin{pmatrix} y \\ y' \end{pmatrix}$ and we obtain

$$z' - \begin{pmatrix} 0 & I \\ q_x & 0 \end{pmatrix} z = 0. \tag{4}$$

Since $\begin{pmatrix} 0 & I \\ q_x & 0 \end{pmatrix}$ is contained in the Lie algebra $\mathfrak{sp}(m, \boldsymbol{R})$, the fundamental solution $W_x(t)$ with $W_x(0) = I$ is contained in the Lie group $Sp(m, \boldsymbol{R})$.

We embed $GL^+(m, \boldsymbol{R})$ in $Sp(m, \boldsymbol{R})$ by regarding $x \in GL^+(m, \boldsymbol{R})$ as $\begin{pmatrix} x & 0 \\ 0 & {}^t x^{-1} \end{pmatrix}$ in $Sp(m, \boldsymbol{R})$. Let N be the subgroup of $Sp(m, \boldsymbol{R})$ defined by

$$N = \left\{ \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}; {}^tC = C \right\}.$$

Then $N \cap GL^+(m, \boldsymbol{R}) = I$ and $xNx^{-1} = N$ for all $x \in GL^+(m, \boldsymbol{R})$. Thus $N \cdot GL^+(m, \boldsymbol{R}) = GL^+(m, \boldsymbol{R}) \cdot N$, which we denote by H. Let Q be the space of right cosets $H \backslash Sp(m, \boldsymbol{R})$ and let

$$\mu \colon Sp(m, \boldsymbol{R}) \longrightarrow Q = N \cdot GL^+(m, \boldsymbol{R}) \backslash Sp(m, \boldsymbol{R})$$

be the natural projection. Since $U(m) \cap H = SO(m)$, we have $SO(m) \backslash U(m) = H \backslash Sp(m, \boldsymbol{R}) = Q$. We also denote by μ the projection $U(m) \to Q$. Let $\mathfrak{h}$ be the Lie algebra of H and let $\mathfrak{M}$ be the subspace of $\mathfrak{u}(m) \subset \mathfrak{sp}(M, \boldsymbol{R})$ defined by

$$\mathfrak{M} = \left\{ \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}; {}^tB = B \right\}.$$

Then we have

$$\mathfrak{sp}(m, \boldsymbol{R}) = \mathfrak{M} + \mathfrak{h}, \qquad \mathfrak{u}(m) = \mathfrak{M} + \mathfrak{so}(m),$$
$$[\mathfrak{M}, \mathfrak{so}(m)] \subset \mathfrak{M}, \qquad [\mathfrak{M}, \mathfrak{M}] \subset \mathfrak{so}(m).$$

The space Q has a $U(m)$-invariant metric such that Q is a Riemannian symmetric space.

7. Sturm-Liouville curve

We identify $\mathfrak{sp}(m, \boldsymbol{R})$ with the right invariant vector field on $Sp(m, \boldsymbol{R})$. Define a subspace $\mathrm{stl}\,(m)$ of $\mathfrak{sp}(m, \boldsymbol{R})$ by

$$\mathrm{stl}\,(m)=\left\{\begin{pmatrix}0 & I\\ C & 0\end{pmatrix};\ {}^tC=C\right\}.$$

We say that a smooth curve c: [0, 1]$\to Sp(m, \boldsymbol{R})$ is a Sturm-Liouville curve (abbrev. SL-curve) if dc/dt is contained in the set stl(m) for all $t \in [0, 1]$. Let $\mathfrak{M}^+$ be the subspace of $\mathfrak{M}$ defined by

$$\mathfrak{M}^+=\left\{\begin{pmatrix}0 & B\\ -B & 0\end{pmatrix};\ {}^tB=B>0\right\}.$$

For $x \in Sp(m, \boldsymbol{R})$, we write $\mathfrak{M}_x^+$ for the subspace of $T_xSp(m, \boldsymbol{R})$ corresponding to $\mathfrak{M}^+$. If $\mu(x_1)=\mu(x_2)$ for $x_1, x_2 \in Sp(m, \boldsymbol{R})$, then

$$\mu_*(\mathfrak{M}_{x_1}^+)=\mu_*(\mathfrak{M}_{x_2}^+).$$

For $q \in Q$, we define a subspace $\mu_*(\mathfrak{M}^+)$ of T_qQ by $\mu_*(\mathfrak{M}_x^+)$ for some $x \in Sp(m, \boldsymbol{R})$, $\mu(x)=q$.

A smooth curve c; [0, 1]$\to Q$ is called a positive curve (abbrev. (+)-curve) if dc/dt is contained in $\mu_*(\mathfrak{M}^+)$ for all $t \in [0, 1]$. The image $\hat{c}=\mu c$: [0, 1]$\to Q$ of a SL-curve c: [0, 1]$\to Sp(m, \boldsymbol{R})$ is a (+)-curve. Let us define a space $\Omega^+(Q)$ by the set of all piecewise smooth (+)-curves c: [0, 1] $\to Q$ with $c(0)=c(1)=p_0$, where $p_0=\{H\} \in Q$. An element in $\Omega^+(Q)$ is called a (+)-loop. We give a topology on $\Omega^+(Q)$ as the subspace of the loop space $\Omega(Q)$. The energy function E is given by $E(\omega)=\int_0^1\left\|\frac{d\omega}{dt}\right\|^2 dt$ for $\omega \in \Omega^+(Q)$. A (+)-loop ω in $\Omega^+(Q)$ is a critical point for the function E if and only if ω is a geodesic (+)-loop. Remark that a geodesic of Q emanating from p_0 is given by

$$\gamma_X(t)=p_0 \exp tX$$

for $X \in \mathfrak{M}$.

Remark that we can define the space of (+)-curves in $\Omega(U(m))$ to be the inverse image of $\Omega^+(Q)$ by the natural projection $\Omega(U(m))\to\Omega(Q)$. The following arguments are also valid. But for our purpose, the definition is sufficient.

8. Morse theory on $\Omega^+(Q)$

We study the weak homotopy type of the space $\Omega^+(Q)$ by using the Morse theory, where $Q=SO(m)\backslash U(m)$. On $\Omega=\Omega(Q)$, we have the energy function E defined by

$$E(\omega)=\int_0^1\left\|\frac{d\omega}{dt}\right\|^2 dt,$$

for $\omega \in \Omega(Q)$. Given $c>0$, let $\Omega_c=\Omega_c(Q)$ denote the closed subset $E^{-1}([0, c])\subset\Omega(Q)$ and let Int Ω_c denote the open subset $E^{-1}([0, c])$. Put $\Omega_c^+=\Omega_c\cap\Omega^+(Q)$ and Int $\Omega_c^+=\text{Int }\Omega_c\cap\Omega^+(Q)$. Regard S^1 as $[0, 1]/\{0\}\sim\{1\}$. We can choose a subdivision $0=t_0<t_1<\cdots<t_k=1$ of $[0, 1]$ so that, for any $\omega \in \Omega_c$, the geodesic connecting $\omega(t_{i-1})$ and $\omega(t_i)$ is uniquely and differentiably determined by the two end points, for each $i=1, 2, \cdots, k$ (cf. [6, § 16]). For any $j \in \boldsymbol{N}$, let $0=t_0<t_1<\cdots<t_{2^{j-1}k}=1$ be the vertexes of the $(j-1)$-th barycentric subdivision of the polyhedron $[0, 1]$ with the vertexes $t_0, t_1, \cdots, t_k$. We define ${}^j\Omega_c$ to be the subspace of Ω_c consisting of loops $\omega \in \Omega_c$ such that $\omega|_{[t_{i-1}, t_i]}$ is the geodesic for each $i=1, 2, \cdots, 2^{j-1}k$. Put ${}^j\Omega_c^+={}^j\Omega_c\cap\Omega^+$, and Int $\Omega_c^+=\text{Int }{}^j\Omega_c\cap\Omega^+$. We have the natural inclusions

$$
{}^1\Omega_c^+\subset{}^2\Omega_c^+\subset\cdots\subset{}^j\Omega_c^+\subset\cdots.
$$

We study the homotopy type of ${}^j\Omega_c^+$ and show that the inclusions are homotopy equivalences. Note that Int ${}^j\Omega_c$ has the structure of a smooth finite dimensional manifold, and Int ${}^j\Omega_c^+$ is an open submanifold of Int ${}^j\Omega_c$. Choose a Riemannian metric on Int ${}^j\Omega_c$, and consider the gradient vector field $\text{grad}(-E)=-\text{grad}\, E$. Let ϕ_s be the associated local one-parameter group of transformations. The main result of this section is the following

Proposition. *For any* $\omega \in$ Int ${}^j\Omega_c^+$, *the maximal integral curve* $\phi_s(\omega)$ *for* $s\geq 0$ *of* $-\text{grad}\, E$ *in* Int ${}^j\Omega_c$ *is contained in* Int ${}^j\Omega_c^+$.

For the proof, we need the following. For a real symmetric $(m\times m)$-matrix A, we associate a real number $\sigma(A)$ defined by

$$
\sigma(A)=\min_{\|a\|=1}\langle Aa, a\rangle,
$$

where $a \in \boldsymbol{R}^m$, and $\langle\ \ \rangle$ is the usual inner product in $\boldsymbol{R}^m$.

Obviously we have

Lemma. *For any two real symmetric* $(m\times m)$*-matrices* A *and* B, *we have*

$$
\sigma(A+B)\geq\sigma(A)+\sigma(B).
$$

Let p be a point in Q. We can express p by $p=SO(m)g$ for some g in $U(m)$. We identify T_pQ with $\mathfrak{M}=T_{p_0}Q$ by the right translation $(R_g)_*$. Thus a vector $X \in T_pQ$ is identified with $A \in \mathfrak{M}$, where $X=(R_g)_*A$. The space $\mathfrak{M}$ is naturally identified with the set of real symmetric $(m\times m)$-matrices. We define a real number $\sigma(X)$ by

$$\sigma(X)=\sigma(A).$$

Lemma. *The value $\sigma(X)$ is independent of the choice of g in $U(m)$.*

Thus σ is a well-defined function on $T_p(Q)$. Let h be an element of $U(m)$. For any $p \in Q$ and $X \in T_pQ$, we have

$$\sigma((R_h)_*X)=\sigma(X).$$

Now let ω be an element in Int ${}^j\Omega_c$. Then ω is a broken geodesic with $E(\omega)<c$. Put $V_t=d\omega/dt$, and

$\varDelta_t V=V_{t+}-V_{t-}=$discontinuity in the velocity vector at t, where $0<t<1$.

Then $\varDelta_t V=0$ except for $t=t_1, t_2, \cdots, t_{2^j-1k-1}$. We define a real number $\sigma(\omega)$ by

$$\sigma(\omega)=\min_{i=0,1,\cdots,2^j-1k-1}\sigma(V_{t_i+}).$$

Remark that $\sigma(V_{t_i+})=\sigma(V_{t_{i+1}-})=\sigma(V_t)$ for $t_i<t<t_{i+1}$. Obviously σ is a continuous function on the manifold Int ${}^j\Omega_c$. More precisely, we have the following

Lemma. *For any smooth curve ψ: $\boldsymbol{R}\to$Int ${}^j\Omega_c$, the function $\psi^*\sigma$ is a piecewise smooth function on $\boldsymbol{R}$.*

Remark that $\omega \in$ Int ${}^j\Omega_c$ is contained in Int ${}^j\Omega_c^+$ if and only if $\sigma(\omega)>0$.

Let ω be an element in Int ${}^j\Omega_c^+$, and let ϕ: $(-\varepsilon, \varepsilon)\to$Int ${}^j\Omega_c^+$ be the integral curve of the vector field $-$grad E with $\phi(0)=\omega$.

Lemma. *Suppose that $\phi^*\sigma$ is smooth at* 0. *Then we have*

$$(-\text{grad } E)_\omega(\sigma)\geq 0.$$

The proof is given as follows. By the first variational formula, we have

$$(-\text{grad } E)_\omega=\sum_i \varDelta_{t_i}V.$$

We show that, for each i, $(\varDelta_{t_i}V)(\sigma)\geq 0$. The integral curve $\omega(s)(t)=\omega^s(t)$ of $\varDelta_{t_i}V$ with $\omega^0(t)=\omega(t)$ in Int ${}^j\Omega_c^+$ for $-\varepsilon<s<\varepsilon$, ε small, is given as follows. Let $\eta(t)$, $0\leq t$, be the geodesic with $\eta(0)=\omega(t_i)$ such that $d\eta/dt(0)=\varDelta_{t_i}V$. We define the loop ω^s by

$$\omega^s(t)=\begin{cases}\omega(t) & \text{for } 0\leq t\leq t_{i-1} \text{ and } t_{i+1}\leq t\leq 1,\\ \text{geodesic connecting } \omega(t_{i-1}) \text{ and } \eta(s) & \text{for } t_{i-1}\leq t\leq t_i,\\ \text{geodesic connecting } \eta(s) \text{ and } \omega(t_{i+1}) & \text{for } t_i\leq t\leq t_{i+1}.\end{cases}$$

Put $V_t^s = d\omega^s/dt$. By the definition of $\sigma(\omega)$, it is sufficient to show that $\lim_{s\to 0} 1/s(\sigma(V_{ti-}^s) - \sigma(V_{ti-})) \geq 0$ and $\lim_{s\to 0} 1/s(\sigma(V_{ti+}^s) - \sigma(V_{ti+})) \geq 0$. For fixed s, let $\alpha^s(t)$ $(0 \leq t \leq s)$ denote the geodesic such that $\alpha^s(0) = \omega(t_i - s)$ and $\alpha^s(s) = \eta(s)$. Let $\exp\colon \mathfrak{u}(m) \to U(m)$ be the exponential mapping of the group $U(m)$. Fix an identification of $T_{\omega(t_i-s)}Q$ with $\mathfrak{M}$ and identify $T_{\omega(t_i)}Q$ with $T_{\omega(t_i-s)}Q$ by the action of $\exp(V_{ti-})$. Then the velocity vector $d\alpha^s/dt(0)$ is given by $\mu_*\{\exp^{-1}(\exp(s\Delta_{t_i}V)\exp(sV_{t_i-s}))\}$, where $\mu_*\colon \mathfrak{u}(m) \to \mathfrak{M}$ is the projection. Since $V_{t_i-s} = V_{t_i-}$ by the identification, we have $\Delta_{t_i}V + V_{t_i-s} = V_{t_i+}$. Consequently we have

$$\frac{d\alpha^s}{dt}(0) = \mu_*\left\{sV_{t_i+} + \frac{s^2}{2}[\Delta_{t_i}V, V_{t_i-}] + o(s^3)\right\},$$

where $o(s^3)$ denote a $\mathfrak{a}(m)$-valued function of order s^3. Since $\sigma(V_{t_i+}) > 0$, we have $((d\alpha^s/dt)(0)) > 0$ for small $s > 0$. Let $\mathrm{Exp} = \mathrm{Exp}_{\eta(s)}\colon T_{\eta(s)}Q \to Q$ be the exponential mapping of the symmetric space Q. In the vector space $T_{\eta(s)}Q$, we have equalities

$$\mathrm{Exp}^{-1}(\omega(t_i - s)) = -\frac{d\alpha^s}{dt}(s), \qquad \mathrm{Exp}^{-1}(\omega(t_{i-1})) = -V_{ti-}^s.$$

Since

$$\{\mathrm{Exp}^{-1}(\omega(t_i - s) - \mathrm{Exp}^{-1}(\omega(t_{i-1}))\} + \{-\mathrm{Exp}^{-1}(\omega(t_i - s))\} = -\mathrm{Exp}^{-1}(\omega(t_{i-1})),$$

we have

$$\sigma(\mathrm{Exp}^{-1}(\omega(t_i - s)) - \mathrm{Exp}^{-1}(\omega(t_{i-1}))) + \sigma\left(\frac{d\alpha^s}{dt}(s)\right) \leq \sigma(V_{ti-}^s).$$

Note that

$$\lim_{s\to 0}\{\mathrm{Exp}^{-1}(\omega(t_i - s)) - \mathrm{Exp}^{-1}(\omega(t_{i-1}))\} = V_{ti-} \in T_{\omega(t_i)}Q$$

and

$$\sigma\left(\frac{d\alpha^s}{dt}(s)\right) = \sigma\left(\frac{d\alpha^s}{dt}(0)\right).$$

Consequently we have

$$\lim_{s\to 0+}\{\sigma(V_{ti-}^s) - \sigma(V_{ti-})\} \geq 0,$$

and

$$\lim_{s\to 0}\frac{1}{s}\{\sigma(V_{ti-}^s) - \sigma(V_{ti-})\} \geq 0$$

if it exists. The proof that

$$\lim_{s\to 0}\frac{1}{s}\{\sigma(V^s_{ti+})-\sigma(V_{ti+})\}\geq 0$$

is quite similar.

Let W_i $(i=1, 2, \cdots, p)$ be the collection of critical manifolds in Int ${}^j\Omega_c^+$ (Bott [4]). Remark that these collections are equal for every $j=1, 2, \cdots$. Let ξ_i be the negative bundle of W_i ([4]).

Corollary. *For any $j\geq 1$, the space ${}^j\Omega_c$ is homotopy equivalent to the CW-complex*

$$K=\xi_1\cup\xi_2\cup\cdots\cup\xi_p.$$

Corollary. *The inclusions*

$${}^1\Omega_c^+\subset{}^2\Omega_c^+\subset\cdots\subset{}^j\Omega_c^+\subset\cdots$$

are homotopy equivalences.

Put ${}^{pf}\Omega_c^+=\varinjlim_j {}^j\Omega_c^+$. Then ${}^{pf}\Omega_c^+$ is homotopy equivalent to ${}^j\Omega_c^+$ for any $j\geq 1$.

The following is easy to see.

Proposition. *For any compact topological space X, the inclusion $i\colon {}^{pf}\Omega_c^+\hookrightarrow\Omega_c^+$ induces an isomorphism*

$$i_*\colon [X, {}^{pf}\Omega_c^+]\cong[X, \Omega_c^+].$$

9. Degree and index of (+)-loops

For $x\in Q=SO(m)\backslash U(m)$, we have the determinant $\det(x)\in S^1=\{z\in \boldsymbol{C}; \|z\|=1\}$. For a map $\omega\colon S^1\to Q$, the degree $d(\omega)$ is defined to be the winding number of the composition $\det\cdot\omega\colon S^1\to S^1$. Two elements ω_1 and ω_2 in $\Omega(Q)$ are contained in the same connected component if and only if $d(\omega_1)=d(\omega_2)$. For $k\in \boldsymbol{Z}$, put ${}_k\Omega(Q)=\{\omega\in\Omega(Q); d(\omega)=k\}$ and ${}_k\Omega^+(Q)=\Omega^+(Q)\cap{}_k\Omega(Q)$.

The following follows from results of Section 8.

Proposition. *Every $(+)$-loop ω is homotopic in $\Omega^+(Q)$ to a geodesic loop.*

Each geodesic issuing from $p_0=\{SO(m)\}$ is written as $p_0\exp tA$ for some $A\in\mathfrak{M}$. Diagonalize A, and we obtain

Proposition. *For $Q=SO(m)\backslash U(m)$, ${}_k\Omega^+(Q)$ is non-vacuous if and only if $k\geq m$.*

The critical manifold in ${}_m\Omega^+(Q)$ consists of one point γ_0 defined by $p_0 \exp \frac{t}{2\pi}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

Proposition. *The space ${}_m\Omega^+(SO(m)\backslash U(m))$ is contractible (up to weak homotopy type).*

Now to each $(+)$-loop $\omega \in \Omega^+(Q)$, we will associate a non-negative integer $i(\omega)$ called the index of ω. Let $\Pi\colon \mathfrak{sp}(m, \boldsymbol{R})=\mathfrak{M}+\mathfrak{h}\to\mathfrak{M}$ be the projection. A $(+)$-loop $\omega \in \Omega^+(Q)$ is the image by μ of a piecewise smooth curve $\zeta\colon [0, 1]\to Sp(m, \boldsymbol{R})$ satisfying the relations

$$\zeta(0)=I, \quad \zeta(1)\subset H,$$
$$\Pi\left(\frac{d\zeta}{dt}(t)\zeta^{-1}(t)\right)\subset\mathfrak{M}^+ \qquad \text{for all } t \in [0, 1].$$

We say that ζ is a lifting of ω. Express the $\mathfrak{sp}(m, \boldsymbol{R})$-valued function $d\zeta/dt\cdot\zeta^{-1}$ on $[0, 1]$ as

$$\frac{d\zeta}{dt}=\begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}\zeta.$$

Then $\Pi(d\zeta/dt\cdot\zeta^{-1})\subset\mathfrak{M}^+$ if and only if $B>0$. The curve

$$\zeta(t)=\begin{pmatrix} X(t) & Y(t) \\ Z(t) & W(t) \end{pmatrix}$$

is the fundamental matrix of the differential equation

$$\text{(5)} \qquad \begin{pmatrix} U \\ V \end{pmatrix}'=\begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix}.$$

We say that $0<t\leq 1$ is conjugate to 0 if there exists a non-zero solution $\begin{pmatrix} U \\ V \end{pmatrix}$ of (5) such that

$$\text{(6)} \qquad U(0)=U(t)=0.$$

Thus t with $0<t\leq 1$ is conjugate to 0 if and only if $\det Y(t)=0$. The multiplicity of the conjugate point t is defined to be the dimension of the solutions $\begin{pmatrix} U \\ V \end{pmatrix}$ of (5) which satisfy (6). We define the k-th conjugate point by counting multiplicities. The index of the curve $\zeta\colon [0, 1]\to Sp(m, \boldsymbol{R})$ is defined to be the number of conjugate points in $(0, 1)$ counted with their multiplicities.

By a (+)-variation of the curve $\zeta(t)=\zeta_0(t)$, we mean a continuous mapping

$$\zeta\colon (-\varepsilon, \varepsilon)\times[0, 1]\longrightarrow Sp(m, \boldsymbol{R})$$

for some $\varepsilon>0$, such that

*) ζ_s is piecewise smooth,

**) $\Pi((d\zeta_s/dt)(t)) \in \mathfrak{M}^+$ for all $s \in (-\varepsilon, \varepsilon)$ and $t \in [0, 1]$, where we put $\zeta_s(t)=\zeta(s, t)$.

By the classical method in the calculus of variations, we obtain

Proposition. *For a (+)-variation, the k-th conjugate points vary continuously for all $k \in \boldsymbol{N}$.*

Suppose that ζ_s is a (+)-variation of ζ, such that $\zeta_s(0)=e$, and $\zeta_s(1) \in H$ for all $s \in (-\varepsilon, \varepsilon)$. Then the point 1 is conjugate to 0 with multiplicity equal to m. Since m is the maximum of possible multiplicities, conjugate points do not cross the point 1.

Proposition. *Suppose that a (+)-variation ζ_s satisfies the relation $\zeta_s(0)=e$, $\zeta_s(1) \in H$ for all $s \in (-\varepsilon, \varepsilon)$. Then, for any $s \in (-\varepsilon, \varepsilon)$, the index of ζ_s is equal to the index of ζ_0.*

For an element $\omega \in \Omega^+(Q)$, we define the index $i(\omega)$ to be the index of a lifting ζ. Obviously we have

Lemma. *If ω_0 and ω_1 lie in the same arcwise connected component of $\Omega^+(Q)$, then $i(\omega_0)=i(\omega_1)$.*

The sum of conjugate points in (0, 1] of a geodesic loop $\gamma \in \Omega(Q)$ is equal to 2d(γ).

Lemma. *For a geodesic loop $\gamma \in \Omega^+(Q)$, we have*

$$i(\gamma)=2d(\gamma)-m.$$

Combining lemmas, we obtain

Proposition. *For a (+)-loop $\omega \in \Omega^+(Q)$,*

$$i(\omega)=2d(\omega)-m.$$

IV. C_l-metrics on S^n

10. A stable bundle

Let g be a C_l-metric on S^n. We may assume that $l=1$. We embed

S^n in $\boldsymbol{R}^{n+1}$ as the unit sphere. We give a Riemannian metric h on $\boldsymbol{R}^{n+1}$ such that h is equal to the product $g\times g_0$ on a tubular neighborhood of S^n in $\boldsymbol{R}^{n+1}$, where g_0 is the standard metric on $(-1, 1)$. The metric h naturally induces the Riemannian metric h on $T\boldsymbol{R}^{n+1}$.

Let $L=T_1S^n$ be the unit tangent sphere bundle of S^n. Define a $(2n+2)$-dimensional vector bundle E over L by $E=TT\boldsymbol{R}^{n+1}|L$. Of course E is isomorphic to the trivial bundle. The geodesic flow defines a free S^1-action on L such that the base space X is equal to Geod(S^n, g). We give a projecting isomorphism Φ on E as follows. Let B be the $(2n-2)$-dimensional subbundle of E consisting of tangent vectors of L orthogonal to the geodesic flow vector. Thus we have the orthogonal decomposition

$$E=A\oplus B,$$

where A is the real 4-dimensional vector bundle isomorphic to the trivial bundle. Consider the following differential equation on E,

$$\nabla^H_{\dot{\gamma}}\begin{pmatrix}Y_1\\Y_2\end{pmatrix}=\tilde{P}\begin{pmatrix}Y_1\\Y_2\end{pmatrix}, \tag{7}$$

where $\dot{\gamma}$ is the geodesic flow vector, and $\tilde{P}$ is the linear endomorphism of E defined by,

$$\tilde{P}_v=\begin{cases}\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix} & \text{on } A\\ P_v=\begin{pmatrix}0 & I\\ -R_v & 0\end{pmatrix} & \text{on } B,\end{cases}$$

where $v\in L\subset T\boldsymbol{R}^{n+1}$ and R is the curvature tensor of $(\boldsymbol{R}^{n+1}, h)$. We write $\tilde{P}_v=\begin{pmatrix}0 & I\\ P^* & 0\end{pmatrix}$ on E. Then all the solutions of (7) are periodic and it defines a projecting isomorphism Φ on E. Remark that Φ can be restricted to subbundles A and B. Let $\hat{A}$ and $\hat{B}$ be the projected bundle. Then the projected bundle $\hat{E}$ is isomorphic to the Whitney sum $\hat{A}\oplus\hat{B}$. The natural complex structure of the bundle $TT\boldsymbol{R}^{n+1}$ induces the complex structure on $\hat{E}$ such that $\hat{A}$ and $\hat{B}$ are complex subbundles. By the result of Section 5, $\hat{B}$ is isomorphic to the tangent bundle TX.

Let γ denote the complex line bundle associated to the S^1-principal bundle $T_1S^n\to X$, and γ^* be the dual bundle.

Proposition. *The complex bundle A is isomorphic to the bundle $2\gamma^*=\gamma^*\oplus\gamma^*$.*

11. Proof of Theorem

Let $Z_1, Z_2, \cdots, Z_{n+1}$ be a global orthonormal frame of $(\boldsymbol{R}^{n+1}, h)$. We have the horizontal and the vertical lifts

$$Z_1^H, Z_2^H, \cdots, Z_{n+1}^H,$$
$$Z_1^V, Z_2^V, \cdots, Z_{n+1}^V,$$

on the manifold $T\boldsymbol{R}^{n+1}$. Define an $(n+1)\times(n+1)$-matrix valued function $A=(a_{ij})$ on $L=T_1S^n$ by

$$\nabla_{\dot{\gamma}} Z_j = \sum_{k=1}^{n+1} a_{ik} Z_k,$$

where $\dot{\gamma}$ is the geodesic flow vector field. Remark that $a_{ij}=-a_{ji}$. For unknown vectors Y_1 and Y_2, write

$$Y_1^i = \sum_j f_{ij} Z_j,$$
$$Y_2^i = \sum_j g_{ij} Z_j.$$

The vector $Y=\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ is represented by $(2n+2, n+1)$-matrix $\begin{pmatrix} F \\ G \end{pmatrix}$, where $F=(f_{ij})$ and $G=(g_{ij})$. Consider the following differential equation,

$$(Y, Z)' = (Y, Z)\begin{pmatrix} -A & 0 \\ 0 & -A \end{pmatrix} + \begin{pmatrix} 0 & I \\ P^* & 0 \end{pmatrix}(Y, Z), \tag{8}$$

where Y and Z are $(2n+2, n+1)$-matrices and $'$ means d/dt. The solutions Y and Z of (8) give the solutions of (7). Let U and V be the fundamental matrix of the differential equations

$$U' = U\begin{pmatrix} -A & 0 \\ 0 & -A \end{pmatrix}, \qquad V' = \begin{pmatrix} 0 & I \\ P^* & 0 \end{pmatrix} V.$$

Then the matrix equation W of (8) is given by

$$W = VU.$$

Since $A \in \mathfrak{o}(n+1, \boldsymbol{R})$, U is contained in $SO(n+1, \boldsymbol{R})$. Since $\begin{pmatrix} 0 & I \\ P^* & 0 \end{pmatrix} \in \mathfrak{sp}(n+1, \boldsymbol{R})$, $V \in Sp(n+1, \boldsymbol{R})$. Thus W is contained in $Sp(n+1, \boldsymbol{R})$. By the projection μ: $Sp(n+1, \boldsymbol{R}) \to Q = H\backslash Sp(n+1, \boldsymbol{R}) = SO(n+1)\backslash U(n+1)$, U is mapped trivially. Since $W(0)=W(1)=e$, we have $\mu V(0)=\mu V(1)=p_0$. Thus μV can be regarded as a map : $X \to \Omega Q$. For $W^{-1}=U^{-1}V^{-1}$, we have $\mu(W^{-1})=\mu(V^{-1})$. Since dim $X=2n-2$, the set of homotopy classes

$[X, \Omega Sp(n+1, \boldsymbol{R})]$ and $[X, \Omega Q]$ are abelian groups and μ induces a homomorphism. As elements in $[x, \Omega Q]$,

$$\{\mu v\} = -\{\mu(V^{-1})\} = -\{\mu(W^{-1})\} = \{\mu W\}.$$

Note that μV belongs to $\Omega^+(Q)$.

Using the fact that the index of geodesics of a C_l-metric of S^n is equal to $n-1$ (Bott [3]), we obtain

Lemma. *The degree of the* $(+)$*-loop* μV *is equal to* $n+1$.

From Section 8, it follows that the homotopy class of μV is trivial. The trivial homotopy class gives the bundle $(n+1)\gamma^*$.

The proof of the main theorem is given as follows. From Section 10, we see that the equation (8) gives the projecting isomorphism Φ such that

$$\hat{E} = TX \oplus 2\gamma^*.$$

Let $i: [X, \Omega GL(\boldsymbol{R}, \infty)] \to K(X) = [X, \Omega GL(\boldsymbol{C}, \infty)]$ be the natural homomorphism. The difference between $\Phi(\hat{E})$ and the trivial class lies in the image of $[X, \Omega GL(\boldsymbol{R}, \infty)]$. By Corollary of Section 3, we obtain

$$\hat{E} - (n+1)\gamma^* = (1-\gamma)i(\varepsilon),$$

for some $\varepsilon \in [X, \Omega GL(\boldsymbol{R}, \infty)]$. Thus

$$TX + 2\gamma^* = (n+1)\gamma^* + (1-\gamma)i(\varepsilon).$$

Write τ for TX. Tensoring γ to both sides, we obtain

$$\tau \otimes \gamma = (n-1) + (1-\gamma)\gamma i(\varepsilon).$$

Since $(1-\gamma)\gamma = (1-\gamma^2) - (1-\gamma)$, by Corollary in Section 3, we have real vector bundles ξ_1 and ξ_2 such that

$$(1-\gamma)\gamma i(\varepsilon) = \xi_1 \underset{\boldsymbol{R}}{\otimes} \boldsymbol{C} - \xi_2 \underset{\boldsymbol{R}}{\otimes} \boldsymbol{C}.$$

Thus we have

$$(\tau \otimes \gamma)^{st} = \xi \underset{\boldsymbol{R}}{\otimes} \boldsymbol{C},$$

for some real vector bundle ξ over X, which is the concluson of Theorem.

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