Advanced Studies in Pure Mathematics 3, 1984
Geometry of Geodesics and Related Topics
pp. 125–181

Comparison and Finiteness Theorems in Riemannian Geometry

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This is a survey article on the above subject. A differentiable manifold admits variety of riemannian structures but we don't know in general what is the most adapted metric to the given differentiable structure. On the other hand, in riemannian geometry we have many important riemannian invariants, e.g., curvatures, volume, diameter, eigenvalues of Laplacians etc., and we know what is the most standard riemannian mani folds (model spaces) in terms of riemannian invariants, e.g., spaces of constant curvature, symmetric spaces, Einstein spaces etc.

We ask here the following problem: if riemannian manifolds are similar to the model spaces with respect to the riemannian invariants, are they also topologically similar?

This is in fact a kind of perturbation problem, but perturbation in terms of riemannian invariants and manifolds may vary during the perturbations. A typical example is the Hadamard-Cartan theorem which states that a complete simply connected riemannian manifold of non-positive curvature is diffeomorphic to the euclidean space. This follows from the fact that geodesic behavior from a point of the manifolds is similar to that of euclidean space. Namely the exponential map gives a diffeomorphism (see e.g. [B-C], [C-E], [G-K-M], [N-K], [K 6], [B 5]). Also many results from the theory of surfaces of fixed signed Gaussian curvature and the theory of space forms of constant curvature motivated such a question.

In 1951 H. E. Rauch proposed the above problem for sphere case and showed that if for sectional curvature K of a compact simply connected riemannian manifold $\min K/\max K$ is sufficiently close to 1, then the manifold is homeomorphic to the sphere. This was further developed by Berger, Klingenberg, Toponogov, Tsukamoto, Cheeger, Gromoll, Shiohama, Karcher, Ruh and other people and their works gave much influence on riemannian geometry. In Chapter 2 we treat the above problem.

On the other hand we may ask more generally: classify all the topological types of riemannian manifolds some of whose riemannian invariants satisfy some conditions. For instance classify manifolds of positive (or

Received January 4, 1983.

more generally fixed signed) curvature. Usually such classification problems are very difficult and we may ask whether there are only finitely many topological types of such riemannian manifolds. This was firstly attacked by J. Cheeger and A. Weinstein around 1967. We will be concerned this problem in Chapter 3.

In Chapter 1 we collected some fundamental tools for the above problems. The riemannian invariants with which we are mainly concerned here are sectional curvature, Ricci curvature, diameter and volume. Of course there are many other important invariants, e.g., eigenvalues of Laplacians and we may also consider the above problems in these cases (see e.g., [Cro], [L-T], [L-Z], [Pi]). Also tools and methods which are treated here are mainly concerned with geodesics. We could not here treat methods from Partial Differential Equations although they are playing important roles (see [Ya]). Since there are survey articles on non-compact manifolds and manifolds of negative curvature in this proceeding we don't touch upon these manifolds here.

Now Gromov's recent works with many brilliant ideas from various branches of mathematics are giving decisive influence on the above problem (in fact on many problems beyond above). Since they are still expanding we could only touch some of them here (see papers by Gromov [G 1–8] and [Bu-K], [B 8,9]).

Also the references given here are far from completeness.

In this article I owe very much to the papers by Buser-Karcher, Cheeger, Gromov, Heintze-Karcher, ImHoff-Ruh, Weinstein and other people to whom I would like to express my sincere thanks.

I also would like to thank M. Berger, W. Klingenberg, K. Shiohama and T. Yoshida for advices and suggestions.

Chapter 1. Preliminary Comparison Results

§ 1. Riemannian invariants

In this section we introduce some fundamental riemannian invariants. From the existence of a riemannian structure g on a smooth manifold M we can introduce the following concepts and notions:

1°. Firstly we have the Levi-Civita connection ∇_X adapted to the given metric. For a given curve c: $[0, l]\to M$ and a vector field Y along c we denote by $\nabla_{\partial/\partial t}Y(:=\nabla Y)$ the covariant differential of Y in direction of tangents to c. We also denote by P_c: $T_{c(0)}M\to T_{c(l)}M$ parallel translation along c. Recall that ∇_X is not a tensor field on M and we lift it to the tangent bundle τ_M: $TM\to M$ so that we can define the bundle map K: TTM

$\to TM$ in the following way: for $\xi \in T_v TM$, $v \in TM$ choose a curve $t \to v_t \in TM$ tangent to ξ at $t=0$, which may be considered as a vector field along a curve $x_t := \tau_M v_t$. We define

$$K(\xi) := \nabla_{\partial/\partial t|t=0} v_t \qquad \text{(this is in fact well defined).} \tag{1.1}$$

Restricting K to the vertical subspace $(T_v TM)^v := d_v \tau_M^{-1}(0) = T_v T_{\tau_M v} TM$, we have

$$K_{|d_v \tau_M^{-1}(0)} = \text{the canonical identification } \iota_v \colon T_v T_{\tau_M v} M \cong T_{\tau_M v} M.$$

Then $d\tau_M(v) \colon K^{-1}(0) \cong T_{\tau_M v} M$ is an isomorphism and we have a splitting

$$T_v TM (= K^{-1}(0) \oplus d_v \tau_M^{-1}(0)) \cong T_{\tau_M v} M \oplus T_{\tau_M v} M. \tag{1.2}$$

$(T_v TM)^h := K^{-1}(0)$ will be called the horizontal subspace. Especially horizontal vector field $S_v := (v, 0)$ on TM is called the geodesic spray.

Now from ∇ we have the curvature tensor

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.3}$$

which is the most fundamental local invariant of (M, g) with its successive covariant derivatives. Geometrically following sectional curvature introduced by Riemann generalizing Gauss curvature in the surface theory is important. Let $G_2(TM)$ be the Grassmann bundle $\{\sigma \subset T_m M;\ 2\text{-planes}, m \in M\}$. Then the sectional curvature K_σ of σ is defined as

$$K_\sigma := g(R(x, y)y, x)/|x \wedge y|^2 \qquad (:= K(x, y)), \tag{1.4}$$

where $|\cdot|$ denotes the riemannian norm, $\{x, y\}$ is a basis of σ and K_σ is independent of the choice of $\{x, y\}$. Then $K \colon G_2(TM) \to \boldsymbol{R}$ is a smooth function which determines R (see e.g. [C-E] p.16).

In the case when K is a constant δ we have for the curvature tensor $R_\delta(x, y)z = \delta\{g(y, z)x - g(x, z)y\}$. These riemannian manifolds of constant curvature cover classical euclidean and non-euclidean geometries and the problem how K_σ controls metrical and manifold structure has been one of the central problems in riemannian geometry.

For example assume that $\delta \leq K_\sigma \leq \Delta$ and put $R^\circ := R - R_{(\Delta+\delta)/2}$. Then recalling that $x \to R^\circ(x, y)y$ is a symmetric linear map we have

$$|R^\circ(x, y)y| \leq (\Delta - \delta)/2 \cdot |x||y|^2. \tag{1.5}$$

Next putting $\|R^\circ\| := \max\{|R^\circ(x, y)z|;\ |x|, |y|, |z| = 1\}$ we have also

$$\|R^\circ\| \leq 2/3(\Delta - \delta). \tag{1.6}$$

Now Ricci curvature is defined as

$$
(1.7) \quad \begin{aligned} & r(x, y) := \text{Trace}\,(z \to R(x, z)y), \quad \text{and for } x \in U(M, g) \\ & r(x) := r(x, x) = \sum_{i=2}^{d} K(x, x_i), \end{aligned}
$$

where $x = x_1, \cdots, x_d$ are orthonormal with $\dim M = d$. This notion was introduced by Ricci and became very important after it played the fundamental role in Einstein's equation for gravitational field. Recently it turns out that Ricci curvature tells much more information about (M, g) than expected by Gromov, Yau and other people.

2°. Secondly from the given riemannian structure we can consider the length L_c and energy E_c of a piecewise smooth (or more general H^1—) curve $c: [0, l] \to M$ as $L_c := \int_0^l |\dot{c}(t)|\,dt$, $E_c := 1/2 \int_0^l |\dot{c}(t)|^2 dt$ respectively. Thus we can define the distance $d(p, q)$ of two points $p, q \in M$ as the infimum of the length of curves joining p and q. With this distance M has the structure of a metric space (M, d) whose topology coincides with the manifold topology. (M, d) is complete if and only if every metric ball $B_r(p) := \{q \in M;\ d(p, q) < r\}$ is relatively compact by Hopf-Rinow theorem. In the following we consider only complete riemannian manifolds. In this case M is compact if and only if its diameter $d_M := \text{Sup}_{p,q \in M}\, d(p, q)$ is finite. The space $\Omega_{p,q}M := \{c: [0, l] \to M,\ H^1\text{-curves with } c(0) = p,\ c(l) = q\}$ has a structure of complete Hilbert manifold such that tangent space $T_c\Omega_{p,q}M = \{H^1\text{-vector fields along } c\}$. Then E is a differentiable function on $\Omega_{p,q}M$ and Morse theory may be developed for $(\Omega_{p,q}M, E)$ ([K 5], [K 6]). From g we have also canonical Lebesgue measure $dv = \sqrt{\det(g_{ij})}\,dx^1 \cdots dx^d$ and we may consider the volume v_M. Riemannian invariants d_M and v_M play important roles in the following.

3°. Riemannian metric defines a one form α on TM by $\alpha_v(\xi) := g(v, d\tau_M \xi)$, $\xi \in T_v TM$. Then $d\alpha$ defines a symplectic form on TM (i.e., a closed 2-form with $\overbrace{d\alpha \wedge \cdots \wedge d\alpha}^{d} \neq 0$ everywhere). This corresponds to the canonical symplectic structure on T^*M by an identification $b: TM \cong T^*M$ $(b(x)y = g(x, y))$ via the metric, which is nothing but the Legendre transformation with respect to the energy function $E: TM \to R$, $E(v) = 1/2\, g(v, v)$. Thus we may consider the Hamiltonian vector field H_E corresponding to the Hamiltonian E. In our case H_E coincides with the geodesic spray defined in 1°. In fact we have $i_{S_v} d\alpha = -dE$ which follows from the fact that $L_{S_v}\alpha = dE$.

4°. Now the notion of geodesic may be introduced in connection with 1°, 2°, 3° respectively as follows: $c: [0, l] \to M$, $c(0) = p$, $c(l) = q$ is a geodesic if and only if

(with respect to 1°) an auto-parallel curve i.e.,

$$\nabla_{\partial/\partial t}\dot{c}(t)=0, \tag{1.8}$$

which is the τ_M-image of an integral curve of the spray S_v.

(with respect to 2°) a critical point of the energy integral E on $\Omega_{p,q}M$, namely, the Euler-Lagrange equation of the fuctional E is nothing but (1.8).

(with respect to 3°) τ_M-image of a solution curve of the Hamiltonian system H_E.

In the above geodesics are parametrized proportionally to arc length. Every geodesic is determined by the initial point p and the initial direction $v \in T_pM$, which will be denoted by $c_v(t)$. Especially geodesics parametrized by arc length will be called normal. For a complete riemannian manifold every geodesic c_v may be defined for all real numbers and any points p, q may be joined by a minimal (i.e., distance realizing) geodesic. We denote by Min (p, q) the set of all minimal and normal geodesics joining p to q. It is also important to consider the flow ϕ_t of $S_v(=H_E)$ on TM, which is called the geodesic flow. Clearly we have $\phi_t(v)=\dot{c}_v(t)$ and $\tau_M\phi_t(v)=c_v(t)$.

Once the notion of geodesics is introduced we have the normal coordinates system. Namely for $p \in M$ we define the exponential map Exp_p: $T_pM \to M$ at p as $\mathrm{Exp}_p\, v := c_v(1)$, which is a diffeomorphism on $B_r(o_p) := \{v \in T_pM, |v|<r\}$ for some $r>0$. Now the normal coordinates system (x^i) at p is determined as $\mathrm{Exp}_p \sum x^i(q)e_i=q$, when an orthonormal basis $\{e_i\}$ of T_pM is given. Then we have the following expansion of the metric tensor $g=(g_{ij})$ around p with respect to the normal coordinate

$$g_{ij}(tx)=\delta_{ij}+1/3 \sum R_{kihj}(p)x^k x^h t^2+O(t^3) \tag{1.9}$$

(for the further expansion see e.g. [Sa 1]). Normal coordinates system gives most adapted local chart to the riemannian structure.

Remark. From (1.9) we have the following interpretation of the curvatures. Let $\sigma \in G_2(TM)$ be a plane section at p and c_r a circle in σ of radius r centered at the origin. Then we have

$$K_\sigma=3/\pi \lim_{r\to 0} (2\pi r-L_{\mathrm{Exp}_p c_r})/r^3. \tag{1.10}$$

Next we have for a unit vector $x \in U_pM$

$$r(x)=3 \lim_{r\to 0} (1-\det g_{ij}(\mathrm{Exp}_p\, tx))/t^2. \tag{1.11}$$

5°. To see the behavior of geodesic, which satisfies second order non-linear equation, it is useful to consider the infinitesimal deformation of geodesics, which satisfies the linearized equation. Namely let $\alpha_s(-\varepsilon \leq s \leq \varepsilon)$ be a family of geodesics with $\alpha_0 = c_v$. Then the vector field Y along c_v defined as $Y(t) := \partial/\partial s_{|s=0}\alpha_s(t)$ satisfies the second order linear differential equation

$$\nabla\nabla Y + R(Y, \dot{c}_v)\dot{c}_v = 0. \tag{1.12}$$

Conversely a vector field Y along a geodesic satisfying (1.12) may be obtained from such a geodesic variation and will be called a Jacobi field. Note that Y is uniquely determined by $Y(0)$ and $\nabla Y(0)$. With respect to 2°, c_v: $[0, l] \to M$ is a critical point of E on $\Omega_{p,q}M(p = c_v(0), q = c_v(l))$. We can consider the Hessian $D^2E(c_v)$, which is a symmetric bilinear form on $T_{c_v}\Omega_{p,q}M$ given by

$$D^2E(c_v)(X, Y) = \int_0^l \{g(\nabla X, \nabla Y) - g(R(X, c_v)c_v, Y)\}dt. \tag{1.13}$$

Then the null space of $D^2E(c_v)$ is nothing but the space of Jacobi fields along c_v vanishing at end points. From geodesic flow view point, we consider the differential $d\phi_t$: $T_vTM \to T_{\phi_{tv}}TM$ of the geodesic flow. Then in terms of the splitting of (1.2) we have

$$d\phi_t(A, B) = (Y(t), \nabla Y(t)), \tag{1.14}$$

where $Y(t)$ is a Jacobi field with $Y(0) = A$ and $\nabla Y(0) = B$. Namely Jacobi fields are characterized as geodesic flow invariant fields. Finally the relationship with the exponential map is given as follows: for $v, w \in T_pM$ we have the linear field $t \to (0, tw) = \iota_{tv}w \in T_{tv}T_pM$. Then the Jacobi field Y along c_v with $Y(0) = 0$ and $\nabla Y(0) = w$ is characterized by

$$Y(t) = d\,\mathrm{Exp}_p\,(tv)(0, tw). \tag{1.15}$$

Roughly speaking curvature controls the behavior of Jacobi fields, which are the infinitesimal deformation of geodesics, and also the behavior of geodesics. Then behavior of geodesics gives information on normal coordinate systems, namely on the structure of manifolds.

6°. Here we remark that we may control the parallel translation by curvature. Let c_0, c_1 be curves with initial point p and c_s: $[0, 1] \to M$ $(0 \leq s \leq 1)$ a homotopy from c_0 to c_1 with $c_s(0) = p$. We put $\gamma(s) := c_s(1)$. Let a be the parallel translation along $c_0 \cup \gamma \cup c_1^{-1}$ which may be considered as an element of $SO(d)$. Then we have

$$\|a\| \left(:= \operatorname*{Max}_{|U|=1} |a(U) - U|\right) \leq \|R\| \cdot \text{Area of the surface generated by } c_s. \tag{1.16}$$

In fact for $U \in U_pM$, let X_s be the parallel vector field along $C_s \cup \gamma_{|[s,1]}$ with $X_s(0)=U$. We put

$$\alpha(s,t)=\alpha_s(t)=\begin{cases} c_s(t) & \text{for } 0\leq t\leq 1 \\ \gamma((1-s)t+2s-1) & \text{for } 1\leq t\leq 2 \end{cases}.$$

Then we get

$$\begin{aligned}|a(U)-U| &= |X_0(2)-X_1(2)| \leq \int_0^1 |\nabla_{\partial/\partial s}X_s(2)|\,ds \leq \int_0^2 dt \int_0^1 |\nabla_{\partial/\partial t}\nabla_{\partial/\partial s}X_s(t)|\,ds \\ &= \int_0^2 dt \int_0^1 |R(\partial\alpha/\partial t, \partial\alpha/\partial s)X_s(t)|\,ds \leq \|R\| \int_{[0,1]\times[0,2]} |\partial\alpha/\partial t \wedge \partial\alpha/\partial s|\,dsdt \\ &= \|R\| \int_{[0,1]\times[0,1]} |\partial\alpha/\partial t \wedge \partial\alpha/\partial s|\,dsdt.\end{aligned}$$

Remark (1.17). The same result also does hold in case of a metric connection of riemannian vector bundle.

7°. As mentioned before simply connected riemannian manifolds $M^d(\delta)$ of constant curvature δ are the simplest riemannian manifolds. Take $p \in M^d(\delta)$, $v \in U_pM$ and an orthonormal basis $\{e_1, \cdots, e_d\}$ of $T_pM^d(\delta)$. We put

$$(1.18) \qquad s_\delta(t):=\begin{cases} \sin\sqrt{\delta}t/\sqrt{\delta} & \text{if } \delta>0 \\ t & \text{if } \delta=0, \\ \sinh\sqrt{|\delta|}\,t/\sqrt{|\delta|} & \text{if } \delta<0 \end{cases} \qquad c_\delta(t):=s'_\delta(t).$$

Then Jacobi field $Y(t)$ along c_v, $Y \perp \dot{c}_v$ with $Y(0)=\sum a_ie_i$, $\nabla Y(0)=\sum b_ie_i$ takes the form

$$Y(t)=\sum (a_ic_\delta(t)+b_is_\delta(t))E_i(t),$$

where $E_i(t)$ is the parallel translation of e_i along c_v.

Next typical examples of riemannian manifolds are symmetric spaces, on which sectional curvature K_{σ_t} is constant if σ_t is parallel along a curve c_t. In this case behavior of geodesics is explicitly known (see [Hel], [Sa 4]). Especially for rank one symmetric spaces, which are various projective spaces with their canonical riemannian structures, all geodesics are simple closed geodesics of the same length (so-called C_L-manifolds [Be]).

Also invariant metrics on homogeneous spaces give nice examples in riemannian geometry ([BB 1–3], [B 3], [Su 2–4], [Wa 1–3], [Z 1–2]). Here we only mention Berger's spheres ([Cha 1], [S 5], [W 4]), which are one parameter normal homogeneous metrics of positive curvature on odd dimensional spheres, and Wallach's examples, which are (not normal) homogeneous metrics of positive curvature on $SU(3)/T(p, q)$, where $T(p, q)$ are circles defined by

$$\left\{\begin{pmatrix} \exp(2\pi p\theta\sqrt{-1}) & 0 & 0 \\ 0 & \exp(2\pi q\theta\sqrt{-1}) & 0 \\ 0 & 0 & \exp(-2\pi(p+q)\theta\sqrt{-1}) \end{pmatrix}; \theta \in \boldsymbol{R}\right\}$$

with relatively prime $p, q \in \boldsymbol{Z}$. It is known that $SU(3)/T(p, q)$ are simply connected and $H^4(SU(3)/T(p, q)\colon \boldsymbol{Z}) \simeq \boldsymbol{Z}_r$, $r := |p^2+pq+r^2|$. Huang computed explicitly $\min K_\sigma / \max K_\sigma$ for some homogeneous metric on $SU(3)/T(p, q)$ (see [Wal–A], [Hu], [Es]).

For more general homogeneous manifolds of positive curvature see [BB 1–3], [Wal 1–3]). The geodesic behavior on homogeneous manifolds is not known completely ([Z 1–2]).

More generally Cheeger constructed metrics of non-negative curvature using group actions ([C 4], [Gr-M], [Po]).

§ 2. Jacobi fields comparison theorems

Recall that Jacobi fields satisfy the second order linear differential equation. Extending classical Sturm-Liouville comparison theorem to riemannian case, Rauch ([R 1]) obtained comparison theorems on Jacobi fields in terms of curvature of manifolds. Here we give generalized version by Warner, Heintze-Karcher etc. ([H-K], [War 2]).

We consider Jacobi fields satisfying the boundary condition.

1°. Let $N^e \hookrightarrow M^d$ be an immersed submanifold of dimension e with the induced riemannian structure, $\nu\colon TN^\perp \to M$ the normal bundle. For a normal vector $v \in T_pN$ we define the second fundamental form S_v as

$$S_v(u, w) := g(\nabla_u V, w), \qquad u, w \in T_pM, \tag{2.1}$$

where V is a local section of TN around p with $V_p = v$. S_v is a symmetric bilinear form on T_pN and we denote the corresponding linear transformation by the same letter S_v. Now a Jacobi field Y along a geodesic c_v will be called an N-Jacobi field if Y satisfies

$$Y(0) \in T_pN, \qquad \nabla Y(0) - S_v Y(0) \in T_pN^\perp \tag{2.2}$$

(namely in terms of the splitting (1.2), initial condition $(Y(0), \nabla Y(0))$ belongs to a Lagrangian subspace $\mathscr{L} := \{(A, B) \in T_vTM;\, A \in T_pN,\, B - S_vA \in T_pN^\perp\}$). N-Jacobi fields may be characterized as the variation vector fields of geodesics with initial direction in $TN^\perp$. It is also useful to consider the splitting of $TTN^\perp$ with respect to the normal connection $\nabla^\perp$. For a section $Z: N \to TN^\perp$ and $x \in T_pN$ we set $\nabla_x^\perp Z := (\nabla_x Z)^\perp$ (orthogonal projection to $T_pN^\perp$). We can define as before the bundle map $K_N: TTN^\perp \to TN^\perp$ by the condition $K_N(dZ \cdot x) = \nabla_x^\perp Z$. Then we see that for $v \in T_pN^\perp$,

$$K_{N|d\nu^{-1}(0)}: d\nu^{-1}(0) = T_vT_pN^\perp \cong T_pN^\perp \text{ is the canonical identification,}$$
$$d\nu_{|K_N^{-1}(0)}: K_N^{-1}(0) \cong T_pN \text{ is an isomorphism}$$

and we have a splitting

$$T_vTN^\perp (= K_N^{-1}(0) \oplus d\nu^{-1}(0)) \cong T_pN \oplus T_pN^\perp \qquad (p = \nu(v)). \tag{2.3}$$

We denote this splitting by $(A, B)_N$, $A \in T_pN$, $B \in T_pN^\perp$. Especially we can define the riemannian structure on $TN^\perp$ so that $K_{N|d\nu^{-1}(0)}$, $d\nu_{|K_N^{-1}(0)}$ are linear isometries and $d\nu^{-1}(0) \perp K_N^{-1}(0)$.

Now we consider a linear field $U(t) := (A, tB)_N \in T_{tv}TN^\perp$ along $t \to tv$. Then an N-Jacobi field $Y(t)$ with $Y(0) = A$, $\nabla Y(0) = S_vA + B$ is given by

$$Y(t) = d\,\mathrm{Exp}_\nu\, U(t), \tag{2.4}$$

where $\mathrm{Exp}_\nu := \mathrm{Exp}_{|TN}$. Thus we have $d\tau_M d\phi_t(A, B + S_vA) = d\,\mathrm{Exp}_\nu\,(A, tB)_N$.

Next for a geodesic c_v, $v \in TN^\perp$, a point $c_v(t)$ $(t > 0)$ is called a focal point of N along c_ν if there exists a non-zero N-Jacobi field Y with $Y(t) = 0$. For $v \in TN^\perp$ we define the *focal distance* of N in direction v as $\min\{t > 0;\ c_v(t)$ is a focal point of $N\}$. In case when N reduces to a point this will be called the *conjugate distance*.

We consider the following situation: Let c_v, $v \in U_pN^\perp$ be a perpendicular normal geodesic, $k(t) := \min\{K_\sigma;\ \sigma \ni \dot{c}_v(t)\}$, $K(t) := \max\{K_\sigma;\ \sigma \ni \dot{c}_v(t)\}$, $\lambda_1, \cdots, \lambda_e$ eigenvalues of S_v (principal curvatures). We also consider another immersed manifold $\bar{N}^e \hookrightarrow \bar{M}^d$, $c_{\bar{v}}$, $\bar{v} \in U_{\bar{p}}\bar{N}$, $\bar{k}(t)$, $\bar{K}(t)$, $\bar{\lambda}_1, \cdots, \bar{\lambda}_e$ will be defined similarly. Let $t_0 > 0$ be smaller than the focal distance of N in v. We shall assume that

$$
\begin{aligned}
&(*)_0: \quad d = \dim M \geqq \bar{d} = \dim \bar{M}. \qquad \dim N = \dim \bar{N} := e\\
(2.5)\quad &(*)_1: \quad k(t) \geq \bar{K}(t), \qquad 0 \leq t \leq t_0\\
&(*)_2: \quad \mathrm{Max}\,\lambda_i \leq \mathrm{Min}\,\bar{\lambda}_i, \text{ or } (*)_2': \lambda_i \leq \bar{\lambda}_i \ (i = 1, \cdots, e) \text{ for some fixed order of principal curvatures.}
\end{aligned}
$$

Let $Y_1, \cdots, Y_r$ (resp. $\bar{Y}_1, \cdots, \bar{Y}_r$), $1 \leq r \leq d-1, \bar{d}-1$, be linearly independent N(resp. $\bar{N}$)-Jacobi fields given by $Y_i(t) = d\,\mathrm{Exp}_\nu\, U_i(t)$ (resp. $\bar{Y}_i(t) = d\,\mathrm{Exp}_{\bar{\nu}}\, \bar{U}_i(t)$), which are perpendicular to c_v (resp. $c_{\bar{v}}$). Putting

$$(2.6)\quad \begin{aligned} f(t) &:= \log(|Y_1(t)\wedge\cdots\wedge Y_r(t)|/|U_1(t)\wedge\cdots\wedge U_r(t)|),\\ \bar f(t) &:= \log(|\bar Y_1(t)\wedge\cdots\wedge \bar Y_r(t)|/|\bar U_1(t)\wedge\cdots\wedge \bar U_r(t)|),\end{aligned}$$

we want to compare $f(t)$ and $\bar f(t)$. Note that $\lim_{t\to 0} f(t)=\lim_{t\to 0}\bar f(t)=0$. For that purpose we estimate

$$(2.7)\quad g(t):=\{\log|\bar Y_1(t)\wedge\cdots\wedge\bar Y_r(t)|-\log|Y_1(t)\wedge\cdots\wedge Y_r(t)|\}'.$$

Fix $t_1\leq t_o$, $\bar t_o$ ($<$focal distance of $\bar N$ in $\bar v$). We give a condition guaranteeing $g(t_1)\geq 0$.

Lemma (2.8). *Assume* $(*)_0$, $(*)_1$, $(*)_2$ *or* $(*)_2'$ *and that there is a linear isometric injection* $\iota_{t_1}\colon T_{\bar p}\bar M\to T_pM$ *such that*

(i) $\iota_{t_1}\bar v=v$, $\iota_{t_1}T_{\bar p}\bar N=T_pN$,

(ii) $\iota_{t_1}\bar V_{t_1}=V_{t_1}$ *where* $V_{t_1}:=P_{c_v}^{-1}\{Y_i(t_1)\}_R$ *etc.*

We assume furthermore

(iii) ι_{t_1} *maps eigenvectors of* $\bar\lambda_i$ *to that of* λ_i,

when we assume $(*)_2'$. *Then we have* $g(t_1)\geq 0$.

Proof. Main idea is to use the index form. Namely on $\chi_{c_v}:=\{X(t)$; H^1-vector fields along $c_{v|[0,t_1]}$ with $X(0)\in T_pN$ and $X(t)\perp\dot c_v(t)\}$, we define

$$(2.9)\quad I_N(X,X):=\int_0^{t_1}\left\{g(\nabla X,\nabla X)-K(\dot c_v(t),X(t))|X(t)|^2\right\}dt + S_v(X(0),X(0)).$$

Then one of the fundamental properties of I_N is as follows: for $t_1\leq t_o$, we have $I_N(X,X)\geq I_N(Y,Y)$, where Y is the uniquely determined N-Jacobi field with $Y(t_1)=X(t_1)$ and the equality holds if and only if $Y=X$ (see e.g. [B-C]). For the proof we may firstly assume that $\bar Y_i(t_1)$ are orthonormal by taking a linear combination of $\bar Y_1(t_1),\cdots,\bar Y_r(t_1)$. Since $\bar Y_i$ are N-Jacobi fields, we have

$$\begin{aligned}(\log|\bar Y_1(t)\wedge\cdots\wedge\bar Y_r(t)|)'_{t=t_1} &= \sum(\log|\bar Y_i(t_1)|)'=\sum g(\nabla\bar Y_i(t_1),\bar Y_i(t_1))\\ &=\sum I_N(\bar Y_i,\bar Y_i).\end{aligned}$$

Let $W_i(t):=P_{c_v}\circ\iota_{t_1}\circ P_{c\bar v}^{-1}(\bar Y_i(t))$ be an element of χ_{c_v} (by (i)) with $|W_i(t)|=|\bar Y_i(t)|$, $|\nabla W_i(t)|=|\nabla\bar Y_i(t)|$. From the assumption (ii), taking appropriate linear combination of $Y_i(t_1)$'s, we may assume that $Y_i(t_1)=W_i(t_1)$. Then note that $\{Y_i(t_1)\}$ are orthonormal and $W_i(0)=\iota_{t_1}\bar Y_i(0)$. Then we have from (iii)

$$\begin{aligned}(\log|Y_1(t)\wedge\cdots\wedge Y_r(t)|)'_{t=t_1} &=\sum I_N(Y_i,Y_i)\leq\sum I_N(W_i,W_i)\\ (2.10)\qquad &=\sum\left\{\int_0^{t_1}(g(\nabla W_i,\nabla W_i)-K(\dot c_v,W_i)|W_i|^2)dt+S_v(W_i(0),W_i(0))\right\}\end{aligned}$$

$$\leq \sum \left\{ \int_0^{t_1} (g(\bar{\nabla}\bar{Y}_i, \bar{\nabla}\bar{Y}_i) - K(\dot{\bar{c}}_{\bar{v}}, \bar{Y}_i)|\bar{Y}_i|^2)dt + S_{\bar{v}}(\bar{Y}_i(0), \bar{Y}_i(0)) \right\}$$
$$= \sum I_{\bar{N}}(\bar{Y}_i, \bar{Y}_i) = (\log |\bar{Y}_1(t_1) \wedge \cdots \wedge \bar{Y}_r(t_1)|)'_{t=t_1}. \qquad \text{q.e.d.}$$

Now the problem is that (i), (ii), and (iii) are not consistent in general. We consider the following cases:

(I) $(*)_0$: $d \geq \bar{d}$, $e=0$, $1 \leq r \leq \bar{d}-1$, $(*)_1$: $k(t) \geq \bar{K}(t)$ $(0 \leq t \leq t_o)$

(II) $(*)_0$: $d=\bar{d}$, $e=d-1$, $1 \leq r \leq d-1$, $(*)_1$: $k(t) \geq \bar{K}(t)$ $(0 \leq t \leq t_o)$
$(*)_2$: $\max \lambda_i \leq \min \bar{\lambda}_i$.

(III) $(*)_0$: $d=\bar{d}$, $r=d-1$. $(*)_1$: $k(t) \geq \bar{K}(t)$ $(0 \leq t \leq t_o)$.
$(*)_2$: $\lambda_i \leq \bar{\lambda}_i$ for some fixed order of principal curvatures.

(IV) $(*)_0$: $d=\bar{d}$ and $\bar{M}$ is a space form of constant curvature δ, $r=d-1$.
$(*)_1$: $r(\dot{c}_v(t)) \geq (d-1)\delta$ $(0 \leq t \leq t_o)$.
$(*)_2$: $e=0$, or $e=d-1$ and $\bar{N}$ is totally umbilical at p (i.e., $S_{\bar{v}} = \lambda\,\mathrm{id}$) and $\mathrm{tr}\, S_v \leq e\lambda$.

Then in these cases we may easily check that assumptions of (2.8) are satisfied for $t_1 \leq t_0, \bar{t}_0$. In the last case (IV) we make the assumption on Ricci curvature only but the last inequality in (2.10) also holds in the same way. To see that $\sum S_v(W_i(0), W_i(0)) \leq \sum S_{\bar{v}}(\bar{Y}_i(0), \bar{Y}_i(0))$ note that every $\bar{N}$-Jacobi field $\bar{Y}(t)$ in $\bar{M}$ takes the form $\bar{Y}(t) = (c_\delta(t) + \lambda s_\delta(t))\bar{E}(t)$ with a parallel vector field $\bar{E}(t)$.

Remark (2.11). In each case $U_i(t)$ (and $\bar{U}_i(t)$) are given as follows:

(I) $U_i(t) = (0, tB_i)$, $B_i = \nabla Y_i(0)$.

(II) $U_i(t) = (A_i, 0)_N$, $A_i = Y_i(0)$.

(III) $U_i(t) = (A_i, tB_i)_N$. Note that we may take $U_i(t) = (A_i, 0)$ $(i = 1, \cdots, e)$, where A_i are eigenvectors of S_v, and $U_j(t) = (0, tB_j)_N$ $(j = e+1, \cdots, d-1)$, $B_j \in T_pN^\perp$.

(IV) $U_i(t) = (0, tB_i)$ or $U_i(t) = (A_i, 0)_N$.

Now under the assumptions of one of (I)$\sim$(IV) and $t_1 \leq t_o, \bar{t}_o$, we have $g(t_1) \geq 0$. From (2.11) we see that $\log|U_1(t) \wedge \cdots \wedge U_r(t)| - \log|\bar{U}_1(t) \wedge \cdots \wedge \bar{U}_r(t)| = \text{constant}$, and we get $f(t) \geq \bar{f}(t)$ for $t \leq t_o, \bar{t}_o$. But this implies that t_o is smaller than the focal distance of $\bar{N}$ in $\bar{v}$. Thus we have the following ([H-K]):

Theorem (2.12) (Heintze-Karcher). *Assume one of* (I)$\sim$(IV). *Then we get*

(i) $t \to |\bar{Y}_1(t) \wedge \cdots \wedge \bar{Y}_r(t)| / |Y_1(t) \wedge \cdots \wedge Y_r(t)|$ *is monotone increasing for* $0 \leq t \leq t_o$.

(ii) $|\bar{Y}_1(t) \wedge \cdots \wedge \bar{Y}_r(t)| / |\bar{U}_1(t) \wedge \cdots \wedge \bar{U}_r(t)| \geq$
$|Y_1(t) \wedge \cdots \wedge Y_r(t)| / |U_1(t) \wedge \cdots \wedge U_r(t)|$.

(iii) *Focal distance of* N *in* $v \leq$ *focal distance of* $\bar{N}$ *in* $\bar{v}$.

2°. From the above we have many important consequences. First we give original Rauch comparison theorems ([R 1], [R 3], [C-E]).

Theorem (2.13) (R.C.T.-I). *Assume that* $\dim M \geq \dim \bar{M}$, $k(t) \geq \bar{K}(t)$ *for* $t \leq t_o$ ($<$*conjugate distance in direction* v). *Let* $Y(t)$, $\bar{Y}(t)$ *be Jacobi fields along* c_v, $c_{\bar{v}}$ *resp. such that* $Y(0)$, $\bar{Y}(0)$ *are tangent to* c_v, $c_{\bar{v}}$ *resp. Assume furthermore that* $g(v, Y(0)) = g(v, Y(0))$, $g(v, \nabla Y(0)) = g(\bar{v}, \nabla \bar{Y}(0))$ *and* $|\nabla Y(0)| = |\nabla \bar{Y}(0)|$. *Then we have* $|Y(t)| \leq |\bar{Y}(t)|$ *for* $0 \leq t \leq t_o$.

Proof. We decompose $Y(t) = Y^{\top}(t) + Y^{\perp}(t)$, where $Y^{\top}(t)$ is the orthogonal projection of $Y(t)$ to $\dot{c}_v(t)$. Clearly we have $g(Y(t), \dot{c}_v(t)) = g(Y(0), v) + g(\nabla Y(0), v)t$ and $|Y^{\top}(t)| = |\bar{Y}^{\top}(t)|$. From (2.12-(I)) $|Y^{\perp}(t)| \leq |\bar{Y}^{\perp}(t)|$ holds, because of $Y^{\perp}(0) = \bar{Y}^{\perp}(0) = 0$, $|\nabla Y^{\perp}(0)| = |\nabla \bar{Y}^{\perp}(0)|$. q.e.d.

Next integrating the above we get

Theorem (2.14) (R.C.T.-II). *Suppose that* $\dim M \geq \dim \bar{M}$ *and*

(i) $K_\sigma \geq K_{\bar{\sigma}}$ *for all* $\sigma \in G_2(TM)$, $\bar{\sigma} \in G_2(T\bar{M})$,

(ii) $\mathrm{Exp}_{\bar{p}|B_r(o_{\bar{p}})}$ *is an embedding and* $\mathrm{Exp}_{p|B_r(o_p)}$ *is regular.*

Let I: $T_{\bar{p}}\bar{M} \to T_p M$ *be a linear isometric injection. Then for any curve* $\bar{c}$: $[0, 1] \to \mathrm{Exp}_{\bar{p}}(B_r(o_{\bar{p}}))$ *we have* $L_c \leq L_{\bar{c}}$, $c = \mathrm{Exp}_p \circ I \circ \mathrm{Exp}_{\bar{p}}^{-1}(\bar{c})$.

Proof. Put $\alpha(t, s) := \mathrm{Exp}_p\, (tI(\mathrm{Exp}_{\bar{p}}^{-1} \bar{c}(s)/|\mathrm{Exp}_{\bar{p}}^{-1} \bar{c}(s)|))$, $0 \leq t \leq |\mathrm{Exp}_{\bar{p}}^{-1} \bar{c}(s)|$. Then $L_c = \int_0^1 |\partial\alpha/\partial s(1, s)| ds$ and $t \to (\partial\alpha/\partial s)(t, s)$ is a Jacobi field Y_s along a geodesic $t \to \alpha(t, s)$ with $Y_s(0) = 0$. Similarly define $\bar{\alpha}(t, s) := \mathrm{Exp}_{\bar{p}}\, (t\, \mathrm{Exp}_{\bar{p}}^{-1}\bar{c}(s)/|\mathrm{Exp}_{\bar{p}}^{-1} \bar{c}(s)|)$ and $\bar{Y}_s$. Noting that $|\nabla Y_s(0)| = |\nabla \bar{Y}_s(0)|$ we get our result from (2.13). q.e.d.

Similarly we have Berger's comparison theorems ([B 4], [C-E]).

Theorem (2.15) (B.C.T.-I). *Assume that* $\dim M = \dim \bar{M}$. *For* $v \in U_p M$ *let* $N := \mathrm{Exp}_p \{x \in B_r(o_p) \subset T_p M, g(x, v) = 0\}$ *be a hypersurface with a normal vector* v *and* $S_v = 0$. *For* $\bar{v} \in U_{\bar{p}}\bar{M}$ *define* $\bar{N}$ *similarly. Suppose that for* N *(resp.* $\bar{N}$*)-Jacobi field* Y *(resp.* $\bar{Y}$*)*

(i) $k(t) \geq \bar{K}(t)$ *for* $0 \leq t \leq t_0$ ($<$*focal distance of* N *in direction* v).

(ii) $\nabla Y(0)$, $\nabla \bar{Y}(0)$ *are tangent to* c_v, $c_{\bar{v}}$ *resp.*

(iii) $g(v, Y(0)) = g(\bar{v}, \bar{Y}(0))$, $g(v, \nabla \bar{Y}(0)) = g(\bar{v}, \nabla \bar{Y}(0))$, $|Y(0)| = |\bar{Y}(0)|$.

Then we have $|Y(t)| \leq |\bar{Y}(t)|$.

Theorem (2.16) (B.C.T.-II). *Let* $c_v(c_{\bar{v}})$: $[0, l] \to M(\bar{M})$ *be a geodesic and* $E(\bar{E})$ *parallel vector field along* $c_v(c_{\bar{v}})$. *Put* $e(t) := \mathrm{Exp}\, (f(t)E(t))$, $\bar{e}(t) := \mathrm{Exp}\, (f(t)\bar{E}(t))$, *where* f: $[0, l] \to \boldsymbol{R}$ *is a smooth function such that* $f(t) <$ *focal distance of* $\mathrm{Exp}\, \{w \in T_{c_v(t)}M;\ w \perp E(t)\}$ *in direction* $E(t)$. *Suppose that* $K_\sigma \geq K_{\bar{\sigma}}$ *for all* $\sigma \in G_2(TM)$, $\bar{\sigma} \in G_2(T\bar{M})$. *Then we have* $L_e \leq L_{\bar{e}}$.

Remark (2.17). Let M^d be a riemannian manifold with $K_\sigma \leq \Delta$ and $M^d(\Delta)$ space form of constant curvature Δ. Suppose that $\mathrm{Exp}_p\colon B_r(o_p) \to B_r(p)$ is a diffeomorphism ($r \leq \pi/\sqrt{\Delta}$). Take $\bar{p} \in M^d(\Delta)$ and a linear isometry $I\colon T_pM \to T_{\bar{p}}\bar{M}$. For $q, r \in B_r(o_p)$ take a minimal geodesic $\gamma \in \mathrm{Min}(q, r)$. Assume that $\gamma \subset B_r(p)$. Then we have $d(q, r) \geq d(\bar{q}, \bar{r})$ with $\bar{q} := \mathrm{Exp}_p I(\mathrm{Exp}_p^{-1} q)$ etc. from (2.14). If the equality holds $\bar{\gamma} := \mathrm{Exp}_p I(\mathrm{Exp}_p^{-1} \gamma)$ is a minimal geodesic and we have a totally geodesic triangle $S := \mathrm{Exp}_p I(\mathrm{Exp}_p^{-1} \bar{S})$ of constant curvature Δ, where $\bar{S} = (\bar{p}, \bar{q}, \bar{r})$ is a geodesic triangle in $M^d(\Delta)$.

Next we consider the case when M or $\bar{M}$ is of constant curvature.

Theorem (2.18) ([Ka], [Bu-K]). *Let M be a riemannian manifold, $Y(t)$ a Jacobi field along a normal geodesic c_v with $Y(t) \perp \dot{c}_v(t)$.*

(i) *Suppose that $K_\sigma \leq \Delta$ for all $\sigma \in G_2(TM)$. Then as far as $y_\Delta(t) := |Y(0)| c_\Delta(t) + |Y|'(0) s_\Delta(t)$ is positive we get*

$$g(Y, \nabla Y) y_\Delta \geq g(Y, Y) y'_\Delta \quad \textit{and} \quad |Y(t)| \geq y_\Delta(t).$$

(ii) *Suppose that $K_\sigma \geq \delta$ and that $\nabla Y(0)$ and $Y(0)$ are linearly dependent. Let $t_o (> 0)$ be smaller than the focal distance in direction v of hypersurface N with normal v such that $S_v = (|Y|'(0)/|Y|(0))\,\mathrm{id}$ if $Y(0) \neq 0$ (conjugate distance in v when $Y(0) = 0$). Then we have $|Y(t)| \leq y_\delta(t)$ $(0 \leq t \leq t_o)$ and that $t \to y_\delta(t)/|Y(t)|$ is monotone increasing $(0 \leq t \leq t_o)$.*

Proof. If $Y(0) = 0$ both cases follow from (2.12–I). We assume $Y(0) \neq 0$.

(i) Take a hypersurface N with a normal v and with respect to which $Y(t)$ is an N-Jacobi field. In $\bar{M} := M^d(\Delta)$ take a point $\bar{p}$, $\bar{v} \in U_{\bar{p}}M$ and a hypersurface $\bar{N}$ with a normal $\bar{v}$ such that $S_{\bar{v}} = \lambda\,\mathrm{id}$, $\lambda = |Y|'(0)/|Y|(0)$. It suffices to show $(\log |Y(t_1)|)' \geq (\log y_\Delta(t_1))'$ for $t_1 \leq t_o$. This follows from the arguments in the proof of (2.11) changing the role of M and $\bar{M}$. Note that in our case $S_{\bar{v}}(\bar{W}(0), \bar{W}(0)) = \lambda |\bar{W}(0)|^2 = \lambda |Y(0)|^2 = |Y|'(0) |Y(0)| \geq g(\nabla Y(0), Y(0)) = S_v(Y(0), Y(0))$.

(ii) Put $\nabla Y(0) = \lambda Y(0)$ and take a hypersurface N with a normal vector v such that $S_v = \lambda\,\mathrm{id}$. Then Y is an N-Jacobi field. Considering the same situation in $\bar{M} := M^d(\delta)$ we have our result from (2.12-II). q.e.d.

Corollary (2.19). *For a riemannian manifold with the curvature restriction $\delta \leq K_\sigma \leq \Delta$, let c_v be a normal geodesic, $Y(t)$ a Jacobi field along c_v with $Y(0) = 0$, $Y(t) \perp \dot{c}_v(t)$. Then we have*

$$s_\delta(s)/s_\delta(t) \leq |Y(s)|/|Y(t)| \leq s_\Delta(s)/s_\Delta(t) \qquad \textit{for } o \leq s \leq t \leq \pi/\sqrt{\Delta}.$$

Corollary (2.20) (R.C.T.-III). *Suppose again that $\delta \leq K_\sigma \leq \Delta$. Then* *for* $u \in T_pM$ *with* $|u|<\pi/\sqrt{\Delta}$ *and for any* $v \in T_pM$ *we have*

$$s_\Delta(|u|)/|u| \leq |d\,\mathrm{Exp}_p(u)\cdot v|/|v| \leq s_\delta(|u|)/|u|, \qquad u \perp v.$$

Remark. For a map $f: X \to Y$ between metric spaces define $\mathrm{dil}\, f := \sup\{d(f(x_1), f(x_2))/d(x_1, x_2);\ x_1, x_2 \in X(x_1 \neq x_2)\}$, $\mathrm{dil}_x f := \lim_{\varepsilon\to 0} \mathrm{dil}\, f_{|B_\varepsilon(x)}$. Then the above means that

$$\mathrm{dil}_u\, \mathrm{Exp}_p \leq \max\{s_\delta(|u|)/|u|, 1\},\ \mathrm{dil}_{\mathrm{Exp}_p u}\, \mathrm{Exp}_p^{-1} \leq \max\{|u|/s_\Delta(|u|), 1\}.$$

3°. Now we apply (2.12) to the volume comparison. Let $\{u_1, \cdots, u_{d-1}, u_d := v\}$ be an orthonormal basis of T_pM and $Y_i(t) = d\,\mathrm{Exp}_p(tv)(0, tu_i)$ $(i=1, \cdots, d-1)$ Jacobi fields along c_v. Then $\Theta_p^M(v, t) := |Y_1(t) \wedge \cdots \wedge Y_{d-1}(t)|/t^{d-1}$ $(t>0)$ is independent of the choice of u_i and equals $|\det d\,\mathrm{Exp}_p(tv)|$. Then we get

Theorem (2.21) (Bishop [B-C]).

(i) *Suppose that* $r(\dot c_v(t)) \geq (d-1)\delta$ *for* $t \leq t_o$ ($<$*conjugate distance in* v). *Then* $t \to \Theta_p^M(v, t)\,(t/s_\delta(t))^{d-1}$ *is monotone decreasing and we get* $\Theta_p^M(v, t) \leq (s_\delta(t)/t)^{d-1}$. *Especially conjugate distance is smaller than or equal to* $\pi/\sqrt{\delta}$. *Thus for a complete* M *with* $r(v) \geq (d-1)\delta(>0)$, M *is compact and* $d_M \leq \pi/\sqrt{\delta}$.

(ii) *Assume that* $K(t) \leq \Delta$ *for* $t \leq \pi/\sqrt{\Delta}$. *Then* $t \to \Theta_p^M(v, t)(t/s_\Delta(t))^{d-1}$ *is monotone increasing and we get* $\Theta_p^M(v, t) \geq (s_\Delta(t)/t)^{d-1}$.

Proof. First note that $(s_\delta(t)/t)^{d-1} = \Theta_p^{M^d(\delta)}(v, t)$ for any p and $v \in U_pM$. Then (i) follows from $g(t) \geq 0$ for (2.12-ii). (ii) follows similarly from (2.12-i). q.e.d.

Next we generalize the above to submanifold case.

Theorem (2.22) ([H-K]). (i) *Let* M, $N \hookrightarrow M$, $v \in U_pN^\perp$ *be as in Theorem* (2.12). *Suppose that* $k(t) \geq \delta$ *for* $t \leq t_o$ ($<$*focal distance of* N *in* v). *Then we get*

$$|\det d\,\mathrm{Exp}_\nu(tv)|t^{d-e-1} \leq \prod_{i=1}^{e} (c_\delta(t) + \lambda_i s_\delta(t)) s_\delta(t)^{d-e-1}$$
$$\leq (c_\delta(t) + \eta s_\delta(t))^e \cdot s_\delta(t)^{d-e-1},$$

where $\eta := (\sum \lambda_i)/e$ *is the mean curvature in direction* v.

(ii) *Let* $N \hookrightarrow M$ *be an immersed hypersurface,* $v \in U_pN^\perp$ *and suppose that* $r(\dot c_v(t)) \geq (d-1)\delta$ *for* $t \leq t_o$. *Then we have*

$$|\det d\,\mathrm{Exp}_\nu(tv)| \leq (c_\delta(t) + \eta s_\delta(t))^{d-1}.$$

Proof. In a space form $\bar M = M^d(\delta)$, $\bar p \in \bar M$, $\bar v \in U_{\bar p}\bar M$, take a locally immersed submanifold $\bar{N}$ with a normal $\bar{v}$ such that $S_{\bar{v}}$ has the same eigenvalues as S_v. Put $Y_i(t)=d\,\mathrm{Exp}_\nu\, U_i(t)$, $\bar{Y}_i(t)=d\,\mathrm{Exp}_{\bar{\nu}}\,\bar{U}_i(t)$. From (2.11-III) we may take $\bar{Y}_i(t)=(c_\delta(t)+\lambda_i s_\delta(t))\bar{E}_i(t)$ $(1\leq i\leq e)$ and $\bar{Y}_j(t)=s_\delta(t)\bar{E}_j(t)$ $(e+1\leq j\leq d-1)$, where $\{\bar{E}_i(0), \bar{E}_j(0)\}$ are orthonormal. Then from (2.12-ii) we get

$$\begin{aligned}|\det d\,\mathrm{Exp}_\nu(tv)| &= |Y_1(t)\wedge\cdots\wedge Y_{d-1}(t)|/|U_1(t)\wedge\cdots\wedge U_{d-1}(t)|\\ &\leq |\bar{Y}_1(t)\wedge\cdots\wedge\bar{Y}_{d-1}(t)|/|\bar{U}_1(t)\wedge\cdots\wedge\bar{U}_{d-1}(t)|\\ &\leq \prod_{i=1}^{e}(c_\delta(t)+\lambda_i s_\delta(t))s_\delta(t)^{d-e-1}/t^{d-e-1}.\end{aligned}$$

(ii) follows similarly from (2.12-iii). q.e.d.

4° (**Toponogov's comparison theorem**). In surface theory Gauss-Bonnet theorem plays very important roles. In higher dimensional case following Toponogov comparison theorem plays a similar role. Let $(\gamma_1, \gamma_2, \gamma_3)$ be a geodesic triangle, which consists of normal geodesics γ_i with $L_{\gamma_i}+L_{\gamma_{i+1}}\leq L_{\gamma_{i+2}}$. Put $\alpha_i=\measuredangle(-\dot{\gamma}_{i+1}(L_{\gamma_{i+1}}), \dot{\gamma}_{i+2}(0))$. $(i+3\equiv i)$

Theorem (2.23) (T.C.T-I). *Suppose that $K_\sigma\geq\delta$ for all $\sigma\in G_2(TM)$. For a geodesic triangle $(\gamma_1, \gamma_2, \gamma_3)$, where γ_1, γ_3 are minimal and $L_{\gamma_2}\leq\pi/\sqrt{\delta}$ (no condition if $\delta\leq 0$), there exists in $M^2(\delta)$ a geodesic triangle $(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)$ such that $L_{\bar{\gamma}_i}=L_{\gamma_i}$ and $\bar{\alpha}_1\leq\alpha_1$, $\bar{\alpha}_3\leq\alpha_3$.*

Theorem (2.24) (T.C.T.-II). *Suppose that $K_\sigma\geq\delta$ for all $\sigma\in G_2(TM)$. Let (γ_1, γ_2) be normal geodesics emanating from p such that γ_1 is minimal and $L_{\gamma_2}\leq\pi/\sqrt{\delta}$. Put $\alpha=\measuredangle(\dot{\gamma}_1(0), \dot{\gamma}_2(0))$. Then for a pair of geodesics $(\bar{\gamma}_1, \bar{\gamma}_2)$ in $M^2(\delta)$ emanating from $\bar{p}$ such that $L_{\bar{\gamma}_i}=L_{\gamma_i}$, $\measuredangle(\dot{\bar{\gamma}}_1(0), \dot{\bar{\gamma}}_2(0))=\alpha$, we have*

$$d(\gamma_1(L_{\gamma_1}), \gamma_2(L_{\gamma_2}))\leq d(\bar{\gamma}_1(L_{\bar{\gamma}_1}), \bar{\gamma}_2(L_{\bar{\gamma}_2})).$$

These are global version of R.C.T. and B.C.T., and proof reduces to R.C.T., B.C.T. by dividing geodesic triangle into small or thin geodesic triangles and requires many steps (see [To], [B 4], [C-E], [K 6]). We need the case when the equality holds in T.C.T.-II.

Remark (2.25). Under the situation of T.C.T-II assume that $0<\alpha<\pi$ and $d(\gamma_1(L_{\gamma_1}), \gamma_2(L_{\gamma_2}))=d(\bar{\gamma}_1(L_{\bar{\gamma}_1}), \bar{\gamma}_2(L_{\bar{\gamma}_2}))$. Let $\bar{\gamma}_3$ be the unique minimal geodesic from $\bar{\gamma}_1(L_{\bar{\gamma}_1})$ to $\bar{\gamma}_2(L_{\bar{\gamma}_2})$ and D be the domain of $T_{\bar{p}}M^2(\delta)$ obtained by lifting $(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)$ via $\mathrm{Exp}_{\bar{p}}^{-1}$. We choose a linear isometry I: $T_{\bar{p}}M^2(\delta)\to T_pM$ with $I(\dot{\bar{\gamma}}_i(0))=\dot{\gamma}_i(0)$ $(i=1, 2)$. Then $\mathrm{Exp}_p\, I(D)$ is an embedded surface of constant curvature δ with totally geodesic interior.

§ 3. Injectivity radius estimate

In this section we want to estimate the size of domains uniformly from below, over which normal coordinates are valid. For a complete riemannian manifold M we define the injectivity radius at $p \in M$ as

$$i_p(M) := \operatorname{Sup}\{r>0;\ \operatorname{Exp}_{p|B_r(o_p)} \text{ is a diffeomorphism}\}.$$

Then $p \to i_p(M)$ is a continuous function on M. The injectivity radius i_M is defined as $\min\{i_p(M);\ p \in M\}$. Now we assume that M is compact. Then i_M is positive and known to be characterized by each of the following:

(I) $\operatorname{Sup}\{r>0;\ \operatorname{Exp}_{p|B_r(o_p)}$ is injective for every $p \in M\}$,

(II) $\operatorname{Sup}\{t>0;\ d(c_v(0), c_v(t))=t$ for all $v \in UM\}$

(III) minimum of half the length of the shortest (simple) closed geodesic and the shortest conjugate distance.

For these fundamental facts see e.g. ([B-C], [C-E], [G-K-M], [K 6], [N]). Thus to estimate i_M we need to estimate conjugate distance and the length of closed geodesics. There is a standard way to estimate the first one in terms of curvature (§2). Namely conjugate distance $\geq \pi/\sqrt{\Delta}$ if $K_\sigma \leq \Delta$. On the other hand estimate of the second one is more difficult and Cheeger observed that there exists a positive constant $c_d(\rho, V, \delta)$, where ρ, V are positive, with the following property: Closed geodesics c in compact riemannian manifold of dimension d with $K_\sigma \geq \delta$, $d_M \leq \rho$, $v_M \geq V$ have length $L_c \geq c_d(\rho, V, \delta)$. Cheeger proved this fact by showing that the existence of short closed geodesic implies small volome by T.C.T. ([C-1.2]). Here we give a proof due to Heintze-Karcher by more direct volume estimate using (2.22). (see also [Ma]).

Theorem (3.1) ([C2], [H-K]). *In a compact riemannian manifold M with $K_\sigma \geq \delta$, every closed geodesic c has length L_c*

$$L_c \geq 2\pi(v_M/\omega_d)s_\delta(\min(d_M, \pi/2\sqrt{\delta}))^{1-d}, \qquad \omega_d := v_{S^d(1)}.$$

Especially, if $\delta \leq K \leq \Delta$, then we have

$$i_M \geq \min\{\pi/\sqrt{\Delta},\ \pi(v_M/\omega_d)s_\delta(\min(d_M, \pi/2\sqrt{\delta}))^{1-d}\}.$$

$$(\pi/2\sqrt{\delta} = +\infty \text{ if } \delta \leq 0).$$

Proof. Let c be a closed geodesic of M, which is a totally geodesic immersed submanifold of M. Then focal distance of c in any direction $v \in Tc^\perp$ is not greater than $\pi/2\sqrt{\delta}$ by (2.12-III). Thus the maximal domain over which Exp_ν is a diffeomorphism is contained in $D := \{v \in Tc^\perp;\ |v| < l := \min\{d_M, \pi/2\sqrt{\delta}\}\}$. Then we have by (2.22)

$$\begin{aligned} v_M &\leq \int_D |\det d\,\mathrm{Exp}_\nu(v)|\,dv_D \\ &\leq \int_c dv_c \int_0^l dt \int_{\{x\in T_{c(s)}c^\perp;\,|x|=t\}} c_\delta(t)s_\delta(t)^{d-2}/t^{d-2}dv_{S^{d-2}t^{-2}} \\ &= \int_c dv_c \int_0^l \omega_{d-2}c_\delta(t)s_\delta(t)^{d-2}dt = L_c s_\delta(l)^{d-1}\omega_{d-2}/(d-1) \\ &= \omega_d L_c s_\delta(l)^{d-1}/2\pi. \end{aligned}$$

q.e.d.

Remark. If δ is positive then we have $L_c \geq 2\pi/\sqrt{\delta}\cdot v_M/v_{S^d(\delta)}$.

Problem. Is it possible to have a similar estimate if we only assume Ricci curvature restriction?

In Chapter 2 we need more precise estimate assuming only (strong) curvature restriction. Firstly Klingenberg obtained

Theorem (3.2) ([K 1]). *Let M be a compact simply connected even dimensional riemannian manifold with $0<K_\sigma\leq\Delta$, where Δ is a positive constant. Then we have $i_M\geq\pi/\sqrt{\Delta}$.*

For the proof assume that $i_M<\pi/\sqrt{\Delta}$. Then there exists a simple closed geodesic c with $L_c=2i_M<2\pi/\sqrt{\Delta}$. Even dimensionality implies that there exists a parallel periodic vector field $X(t)$ ($X(t)\perp\dot{c}(t)$) along c. For the second variation we get

$$D^2E(c)(X,X)=\int_0^{L_c}\{g(\nabla X,\nabla X)-g(R(X,\dot{c})\dot{c},X)\}dt<0,$$

which means that closed curves c_s defined by $c_s(t):=\mathrm{Exp}_{c(t)}\,sX(t)$ have length smaller than $2i_M$ for $s>0$. Then c_s may be lifted to smooth closed curves $\tilde{c}_s$ ($\subset T_{c_s(0)}M$) with $\tilde{c}_s(0)=o_{c_s(0)}$, $\mathrm{Exp}_{c_s(0)}\tilde{c}_s=c_s$. Since $\Phi\colon TM\to M\times M$ defined by $\Phi(v)=(\tau_M v, \mathrm{Exp}_{\tau_M v}\,v)$ is regular on $\{v\in TM;\,|v|<\pi/\sqrt{\Delta}\}$, $\tilde{c}_s$ ($s\to 0$) converge to a smooth closed curve $\tilde{c}$ in $T_{c(0)}M$, which covers c, a contradiction.

Then it was conjectured that the same fact holds also for odd dimensional case. But Berger showed that on Berger's spheres with $\delta\leq K_\sigma\leq 1$, $\delta<1/9$, there are closed geodesics of length less than 2π.

When $\min K/\max K$ is rather large we have

Theorem (3.3) ([C-G], [K-S]). *Let M be a compact simply connected riemannian manifold with $(0<)\delta\leq K_\delta\leq\Delta$, $4\delta\geq\Delta$. Then we have $i_M\geq\pi/\sqrt{\Delta}$.*

We only comment about the proof given in [K-S]. We need global considerations. Let $\Lambda M:=\{c\colon S^1\to M;\ H^1\text{-closed curves on } M\}$ be the space of closed curves which has a structure of complete separable Hilbert manifold. The energy integral E on ΛM is a differentiable function whose critical points are closed geodesics and point curves. Let ϕ_t be a flow generated by $-\operatorname{grad} E$. Now for the proof we may assume that $\Delta=1$. Suppose that there exists a closed geodesic c_1 of length $<2\pi$ (i.e. $E_{c_1}<2\pi^2$). We take the space of homotopies from a fixed point curve c_0 to c_1: $\mathscr{H}:=\{H: [0, 1]\to\Lambda M$; continuous curve with $H_0=c_0$ and $H_1=c_1\}$. Then $\mathscr{H}$ is non empty because M is simply connected and $\mathscr{H}(I):=\{H([0, 1]); H\in\mathscr{H}\}$ is a ϕ-family, namely, $\mathscr{H}(I)$ is ϕ_t-invariant, because c_0, c_1 are fixed under ϕ_t. We define the critical value κ of $\mathscr{H}(I)$ as $\kappa:=\operatorname{Inf}_{H\in\mathscr{H}} \operatorname{Max}_{s\in[0,1]} E(H_s)$. Then the essential part of the proof is to show that $\kappa=2\pi^2$. For this we need lifting argument as above to see $\kappa\geq 2\pi^2$ and the following modified Lyusternik-Schnirellman lemma: Let K' be the set of critical points of E with E-value κ and of index less than or equal to 1. Then for every open neighborhood W' of K' there exists an $H\in\mathscr{H}$ such that $H([0, 1])\subset\Lambda^{\kappa-}\cup W'$, where $\Lambda^{\kappa-}:=\{c\in\Lambda M; E(c)<\kappa\}$. We need the assumption $K_\sigma\geq 1/4$ to see that every closed geodesic of length greater than 2π has index ≥ 2 and this implies that $\kappa\leq 2\pi^2$.

Now once we have $\kappa=2\pi^2$ we have a closed geodesic c of index 1 and length 2π and a sequence of closed curves γ_n of length $<2\pi$, which converges to c in ΛM. Then we can see that $c(1/2)$ is a conjugate point to $c(0)$. Comparing the situation with the case of sphere of constant curvature 1, we have a parallel periodic vector field X ($\perp\dot{c}$) along c. At this point we assume that $\dim M$ (≥ 3) is odd. Then by the same argument as in (3.2) we have the second parallel periodic vector field Y ($\perp\dot{c}$) along c. As before $D^2E(c)(X, X)$, $D^2E(c)(Y, Y)<0$ and this means that index of c, which is the number of negative eigenvalue of $D^2E(c)$, is greater than or equal to 2, a contradiction.

Remark (3.4). P. Hartmann ([Har]) showed that the condition "$\Delta\geq K_\sigma$ and $r(v)\geq(d+2)\Delta/4$" implies that every geodesic of length greater than $2\pi/\sqrt{\Delta}$ has index $d-1$. Thus the same conclusion holds under the weaker curvature condition " ".

Recalling the Berger's spheres we may ask for the compact simply connected riemannian manifolds M whether there exists $\varepsilon(\delta)>0$ such that we have $i_M\geq\varepsilon(\delta)$ whenever $\delta\leq K_\sigma\leq 1$. But this doesn't hold in general. In fact Wallach's examples give a family of compact simply connected homogeneous spaces M of seven dimension whose elements satisfy $\delta\leq K_\sigma\leq 1$ for some positive constant δ, but such that $\inf i_M=0$ ([Hu], [Es]).

On the other hand in 3-dimensional case we have

Theorem (3.5) ([Bu-T], [S 6]). *Let M be a compact simply connected* *riemannian manifold of dimension* 3. *Assume that* $K_\sigma \leq 1$, *and* $r(v) \geq R$ *for all* $v \in UM$, *where* R *is a positive constant. Then for any* $b>1$ *we have*

$$i_M \geq \min\{2b(b-1)\pi^2/(b^2\pi^2+(b-1)^2),\ \pi[1+(e^{b/R}-1)^4/\pi^2]^{-1/2}\}.$$

In this case we consider in stead of homotopy the Plateau problem for a short (simple) closed geodesic c and reduce the estimate of L_c to the estimate of the first eigenvalue of Laplacian of the area minimizing surface bounding c.

Problem. For compact simply connected riemannian manifolds, what is $\inf\{\delta;\ i_M \geq \pi$ for all M with $\delta \leq K_\sigma \leq 1\}$? (we only know that this is not greater than 1/4 and not smaller than 1/9) and what is $\inf\{\delta$; there exists $\varepsilon(\delta)>0$ such that $i_M \geq \varepsilon(\delta)$ for all M with $\delta \leq K_\sigma \leq 1\}$?

Remark. We consider the space $\mathfrak{M}$ of smooth riemannian structures on a compact manifold M with C^2-topology. Then the function $g \rightarrow i_M(g)$, the injectivity radius with respect to g, is continuous on $\mathfrak{M}$ ([Eh]). But we don't know whether there exists $\delta(\varepsilon)$ such that $i_M \geq \pi - \varepsilon$ for any simply connected compact riemannian manifolds with $1 \geq K \geq 1/4 - \delta(\varepsilon)$.

With respect to the volume estimate we can ask whether there exists a point $p \in M$ such that $i_p(M)$ may be estimated from below. For instance

Theorem (3.6) (Heintze-Gromov [Bu-K], [G 2]). *Let* M^d *be a compact riemannian d-manifold with* $-1 \leq K_\sigma < 0$. *Then there exists a point* $p \in M$ *such that* $i_p(M) \geq 4^{-(d+3)}$.

§ 4. Cut locus and distance function

1°. Next we define the notion of the cut locus. Let M be a compact riemannian manifold. For $v \in U_pM$, $p \in M$ the cut point of p along c_v is defined as the last point on c_v to which geodesic arc of c_v is minimal. Namely setting $t(v) := \mathrm{Sup}\{t>0;\ d(c_v(t), p)=t\}(<\infty)$, $\mathrm{Exp}_p\, t(v)v$ is the cut point of p along c_v. We also call $t(v)v \in T_pM$ the tangent cut point. The set of (tangent) cut points of p along all normal geodesics emanating from p is called the (tangent) cut locus of p and denoted by C_p ($\tilde{C}_p$). It is not difficult to see that $v \rightarrow t(v)$ is a continuous function on UM and $\tilde{C}_p$ is homeomorphic to S^{d-1}. Then a d-cell $\mathscr{I}_p := \{tv \in T_pM;\ 0 \leq t < t(v),\ v \in U_pM\}$ is a maximal domain over which Exp_p is a diffeomorphism, and its boundary $\tilde{C}_p$ is mapped onto C_p via Exp_p. Thus M is obtained from C_p by attaching a d-cell and cut locus contains the essence of the topology of M. The structure of cut locus is interesting in connection with the singularity of the exponential mapping. See e.g., [Bu 1–3], [Gl-S], [I],

[Ko], [My 2], [N-S 1,2], [Su 1], [Wa], [Wa 1,3], [W 2,3]. But still we don't know much about the structure of cut locus, e.g., we can ask

Problem. What can we say about the structure of cut loci of compact simply connected homogeneous manifolds? Do they have the intersection with the conjugate loci? (for symmetric spaces see [Cr], [Nai], [Sa 2,3], [Ta]).

2°. Next we return to the volume comparison theorem. Integrating the volume element comparison theorem (2.21) we get

Theorem (4.1) (Bishop-Gromov). *Let M be a complete riemannian manifold such that $r(v)\geq(d-1)\delta$ for all $v\in UM$. Then we have for $0<r\leq R$, $v_{B_R(p)}/v_{B_r(p)}\leq b_\delta^d(R)/b_\delta^d(r)$, where $b_\delta^d(r)$ denotes the volume of r-ball in $M^d(\delta)$ which is independent of the choice of the center.*

Proof. Put

$$\bar{\Theta}_p^M(v,t):=\begin{cases}\Theta_p^M(v,t) & \text{for } t<t(v)(\leq\pi/\sqrt{\delta}) \text{ and}\\ 0 & \text{for } t\geq t(v)\end{cases}$$

$$\bar{w}(t):=\begin{cases}(s_\delta(t)/t)^{d-1} & \text{for } t<\pi/\sqrt{\delta}\\ 0 & \text{for } t\geq\pi/\sqrt{\delta},\end{cases}$$

where we set $\pi/\sqrt{\delta}=+\infty$, if $\delta\leq 0$. We may assume that $r<\pi/\sqrt{\delta}$, otherwise both sides of our inequality equal 1. Then from (2.21) we get for $0\leq s\leq r$, $r\leq t\leq R$

$$\bar{\Theta}_p^M(v,t)\bar{w}(s)\leq\bar{\Theta}_p^M(v,s)\bar{w}(t), \quad \text{and by integration}$$

$$\int_r^R \bar{\Theta}_p^M(v,t)t^{d-1}dt\Big/\int_r^R \bar{w}(t)t^{d-1}dt\leq\int_0^r \bar{\Theta}_p^M(v,s)s^{d-1}ds\Big/\int_0^r \bar{w}(s)s^{d-1}ds.$$

Now from the above we have

$$\begin{aligned}\frac{v_{B_R(p)}-v_{B_r(p)}}{b_\delta(R)-b_\delta(r)}&=\frac{\int_{S^{d-1}(1)}dv\int_r^R\bar{\Theta}_p^M(v,t)t^{d-1}dt}{\omega_{d-1}\cdot\int_r^R w(t)t^{d-1}dt}\\ &=1/\omega_{d-1}\cdot\int_{S^{d-1}}dv\int_r^R\bar{\Theta}_p^M(v,t)t^{d-1}dt\Big/\int_r^R\bar{w}(t)t^{d-1}dt\\ &\leq 1/\omega_{d-1}\cdot\int_{S^{d-1}}dv\int_0^r\bar{\Theta}_p^M(v,s)s^{d-1}ds\Big/\int_0^r\bar{w}(s)s^{d-1}ds\\ &=v_{B_r(p)}/b_\delta^d(r),\end{aligned}$$

from which we have easily our result. q.e.d.

Corollary (4.2) (Bishop). *Under the hypothesis of the theorem we get*

(i) $v_{B_r(p)} \geq v_M \cdot \int_0^t s_\delta^{d-1}(s)\,ds \Big/ \int_0^{d_M} s_\delta^{d-1}(s)ds$, *if M is compact.*

(ii) $v_{B_r(p)} \leq b_\delta^d(r)$.

(iii) *Suppose that δ is positive. Then we have $v_M \leq v_{S^d(\delta)}$ and the equality holds if and only if M is isometric to $S^d(\delta)$.*

Remark (4.3). Let M be a complete riemannian manifold with $K_\sigma \leq \Delta$. Then we have from (2.21) that $v_{B_r(p)} \geq b_\Delta(r)$, if $r < i_M$.

3°. Now we consider convexity. A subset $S \subset M$ is called strongly convex if S has the following property: for any $x, y \in S$ we have the unique minimal geodesic $\gamma \in \mathrm{Min}(x, y)$ and $\gamma([0, L_\gamma]) \subset S$. Suppose that $K_\sigma \leq \Delta$ for all $\sigma \in G_2(TM)$. Then it is known that for $p \in M$, every open ball $B_r(p)$ is strongly convex if $r < 1/2 \min\{i_p(M), \pi/\sqrt{\Delta}\}$. In particular there exists a positive continuous function $p \to r(p)$ such that $B_r(p)$ is strongly convex if $r < r(p)$ (J.H.C. Whitehead). Next assume that $r < \{i_p(M), \pi/2\sqrt{\Delta}\}$. We consider the distance function d_p: $B_r(p) \to \boldsymbol{R}^+$, defined by $d_p(m) := d(p, m)$. Then d_p is smooth except for p and we get

Proposition (4.4). (i) $\mathrm{grad}\, d_p(m) = \dot{c}_v(d_p(m))$, *where c_v is the unique normal minimal geodesic from p to m.*

(ii) *If $x \perp \mathrm{grad}\, d_p$, then $\nabla_x \mathrm{grad}\, d_p = \nabla Y(d_p(m))$, where Y is the Jacobi field along c_v with $Y(0)=0$, $Y(d_p(m))=x$.*

(iii) $c_\Delta/s_\Delta(r)|x|^2 \leq \mathrm{Hess}\, d_p(x, x) \leq |x|^2\{1+\Delta/2 \cdot d_p(m)^2\}/d_p(m)$ *for* $x \perp \mathrm{grad}\, d_p(m)$. $\mathrm{grad}\, d_p(m)$ *belongs to the null space of* $\mathrm{Hess}\, d_p$.

Proof. Fox $x \in T_m M$, $m \in B_r(p)$ put $\alpha(s, t) := \mathrm{Exp}_p\, t(v + s/l \cdot w)$, $0 \leq t \leq l := d_p(m)$, where $w \in T_m M$ with $d\,\mathrm{Exp}_p\, \iota_{lv} w = x$. Then we have $x \cdot d_p$ $(= g(x, \mathrm{grad}\, d_p)) = d/ds_{|s=0} \int_0^l |\partial\alpha/\partial t|dt = g(\partial\alpha/\partial s(0, l),\ \partial\alpha/\partial t(0, l))$, namely $\mathrm{grad}\, d_p = \partial\alpha/\partial t(0, l) = \dot{c}_v(l)$. Next note that $\nabla_x \mathrm{grad}\, d_p = \nabla_{\partial/\partial s|_{s=0}} \partial\alpha/\partial t(s, l)/|\partial\alpha/\partial t(s, l)| = \nabla_{\partial/\partial t|_{t=l}} (\partial\alpha/\partial s)(0, l) = \nabla Y(l)$, if $x \perp \mathrm{grad}\, d_p$. Hence $\mathrm{Hess}\, d_p(x, x) = g(\nabla_x \mathrm{grad}\, d_p, x) = g(Y(l), \nabla Y(l))$ if $x \perp \mathrm{grad}\, d_p$. First applying (2.18) we get $\mathrm{Hess}\, d_p(x, x) \geq c_\Delta/s_\Delta(l)|x|^2 \geq c_\Delta/s_\Delta(r)|x|^2$. On the other hand we get $|Y(l) - l\nabla Y(l)| \leq |Y(l)|\Delta t^2/2$. In fact for $0 \leq s \leq 1$ and for any unit parallel vector field P, we get

$$|g(Y(s) - s\nabla Y(s), P(s))|' \leq |g(s\nabla\nabla Y(s), P(s))| = |sg(R(\dot{c}_v, Y(s))\dot{c}_v, P(s))|$$
$$\leq \Delta|Y(s)|s \leq |Y(l)|s_\Delta(s)/s_\Delta(l) \cdot \Delta s \leq |Y(l)|\Delta s \quad \textit{by } (2.19).$$

By integration we have

$$|g(Y(s) - s\nabla Y(s), P(s))| \leq |Y(l)|\Delta s^2/2, \quad \text{and consequently}$$

$$-|x|^2+l\,\mathrm{Hess}\,d_p(x,x)\leq|g(Y(l)-l\nabla Y(l),\ Y(l))|\leq|x|^2\Delta l^2/2.$$

The last assertion is clear from $\nabla_{\mathrm{grad}\,d_p}\,\mathrm{grad}\,d_p=0$. q.e.d.

In the same way we get

Proposition (4.5). *We put $f:=d_p^2/2$ which is smooth on $B_r(p)$.*
(i) $\mathrm{grad} f(m)=-\mathrm{Exp}_m^{-1}p$
(ii) $r\cdot c_\Delta/s_\Delta(r)|x|^2\leq\mathrm{Hess} f(x,x)\leq(1+\Delta r^2/2)|x|^2$.

4°. Now we consider the global behavior of the distance function d_p from a fixed point $p\in M$ for a compact riemannian manifold. d_p is smooth except $M\backslash C_p\cup\{p\}$ by the same reason as above. More precisely

Lemma (4.6). *Let $C^1(p):=\{q\in C_p$; there exist at least two minimal geodesics joining p to $q\}$. Then $C^1(p)$ is dense in C_p and d_p is not differentiable at any point in $C^1(p)$.*

The proof is not so difficult and a nice excercise (see [Bi], [Wol]). Nevertheless Gromov defined the notion of critical points of d_p:

Definition (4.7). p is by definition a critical point of d_p, at which d_p takes the unique minimum. Next for $q\neq p$ it is called a critical point of d_p if for any $v\in U_qM$ there exists a minimal geodesic $\gamma\in\mathrm{Min}(p,q)$ such that $g(v,\dot\gamma(d_p(q))\geq0$.

From the definition if $q(\neq p)$ is critical then $q\in C^1(p)$.

Lemma (4.8) (Berger). *Let q satisfy $d(q,p)=\mathrm{Max}_{x\in M}\,d(p,x)$. Then q is a critical point of d_p.*

Proof. Take a curve $c(s):=\mathrm{Exp}_p(-sv)$ and $\gamma_s\in\mathrm{Min}(p,c(s))$. Put $\alpha_s:=\sphericalangle(\dot c(s),\dot\gamma_s(d(p,c(s)))$. We may assume that $K_\delta\geq-\delta$ $(\delta>0)$. By T.C.T. and $d(p,q)\geq d(p,c(s))$ we get

$$\cosh\sqrt{\delta}\,d(p,c(s))\leq\cosh\sqrt{\delta}\,d(p,q)\leq\cosh\sqrt{\delta}\,s\cosh\sqrt{\delta}\,d(p,c(s))$$
$$-\cos\alpha_s\sinh\sqrt{\delta}\,s\sinh\sqrt{\delta}\,d(p,c(s)),\ \text{from which follows}$$
$$\cos\alpha_s\cosh\sqrt{\delta}\,s/2\,\sinh\sqrt{\delta}\,d(p,c(s))\leq\cosh\sqrt{\delta}\,d(p,c(s))\sinh\sqrt{\delta}\,s/2.$$

Letting $s\to0$, we may assume $\dot\gamma_s(0)$ converge to $w\in U_qM$. Then $\gamma(t):=\mathrm{Exp}_p\,tw$ is a desired minimal geodesic. q.e.d.

On the other hand if $m\in M$ is not a critical point, there exists a $t(m)\in U_mM$ such that $g(t(m),\dot\gamma(d(p,m))<0$ (or equivalently>0) for every $\gamma\in\mathrm{Min}(p,m)$. Moreover we easily see that we may choose $\pi\geq\alpha(m)>\pi/2$ such that $\sphericalangle(t(m),\dot\gamma(d(p,m)))>\alpha(m)$ for every $\gamma\in\mathrm{Min}(p,m)$.

Lemma (4.9). *Let m be not critical for d_p. Then there exists a neighborhood U of m and a smooth vector field $t(n)$, $n \in U$ such that $\measuredangle(t(n), \dot{\gamma}(d(p, n))) > \alpha(m)$ for all $\gamma \in$* Min (p, n).

In fact, take a convex open ball $B_r(m)$ and define $t(n)$, $n \in B_r(m)$ as the parallel translation of $t(m)$ along the unique $\gamma_{m,n} \in \text{Min}(m, n)$, from which we have a smooth vector field t on $B_r(m)$. Then it is not difficult to see that there exists $0 < r_o < r$ such that the assertion of the lemma holds for the above t and $U = B_{r_0}(m)$.

Now by T.C.T. we may see that d_p is strictly monotone decreasing along trajectories of $t(n)$. In fact we may show using T.C.T.

Lemma (4.10). *Let ϕ_t be the flow generated by t and $V = B_{r_1}(m)$, $0 < r_1 < r_o$. Then for $l_o > 0$, there exist $\delta > 0$ and $\varepsilon(t, l_o)$, which is continuous and positive for $t > 0$ with the following property: $d_p(n) - d_p(\phi_t n) \geq \varepsilon(t, l_o)$ for $0 \leq t \leq \delta$ and $n \in \bar{V}$ with $d(p, n) \geq l_0$.*

Now we give Gromov's isotopy lemma.

Lemma (4.11). *Let $B_{r_2}(p) \subset B_{r_1}(p)$ be concentric metric balls centered at p. Suppose that $A := \overline{B_{r_1}(p) \setminus B_{r_2}(p)}$ contains no critical points of d_p. Then for any open neighborhood U of $B_{r_1}(p)$ there exists an isotopy of M sending $B_{r_1}(p)$ into $B_{r_2}(p)$ and fixing outside U.*

Proof. Take a finite open cover $\{U_i\}$ of the compact set A such that $\{U_i, t_i\}$ are pairs given in (4.9) with $\bar{U}_i \subset U$. Let $\{\varphi_i\}$ be the partition of unity subordinated to $\{U_i\}$ and we define a vector field t on a neighborhood of A as $t(y) := \sum \varphi_i(y) t_i(y)$. Then $t_{|A}$ satisfies also the same property as t_i and we may extend t to a smooth vector field on M by setting 0 outside U. Then this vector field provides a desired isotopy. In fact it is easy to see that there exists an $R > 0$ such that $\varphi_R(B_{r_1}(p)) \subset B_{r_2}(p)$, where φ_t denotes the flow generated by t. q.e.d.

Corollary (4.12). *Let M be a compact riemannian manifold. If d_p has only two critical points, then M is homeomorphic to the sphere.*

In fact from (4.8) we see that there exists a unique point $q \in M$ with $d(p, q) = \text{Max}\, d(p, m)$. Then from (4.11) M may be covered by two differentiably embedded disks. Then M is homeomorphic to the sphere (with respect to this I would like to thank T. Yoshida for showing me a simple proof, see also [*Ru*]).

§ 5. Center of mass techniques ([Ka], [Bu-K], [Gro-K])

For a locally finite open cover $\{U_\alpha\}$ of M suppose that we have a family of smooth maps $g_\alpha: U_\alpha \to N$ into a fixed manifold N. If N is a linear space then by a partition of unity $\{\phi_\alpha\}$ which is subordinate to $\{U_\alpha\}$ we can glue $\{g_\alpha\}$ to a smooth map $g := \sum \phi_\alpha g_\alpha: M \to N$. But in non-linear case this breaks down. Nevertheless if N is a riemannian manifold and each $g_\alpha(U_\alpha)$ is contained in a convex ball we can take the average of $\{g_\alpha(p)\}$ by considering the "center" of $\{g_\alpha(p)\}$. More precisely we consider firstly the following situation: Let A be a normalized measure space with total volume 1 (e.g., finite set of points, compact riemannian manifolds etc.) and $B_r(n)$, $n \in N$, a strongly convex neighborhood in N. Then for a measurable map $f: A \to B_r(n)$ we want to define the center of mass $C_f \in B_r(n)$ of f. We define as in the euclidean case,

$$P_f(p) := 1/2 \int_A d^2(p, f(a))da, \qquad p \in B_r(n). \tag{5.1}$$

Then we have from (4.5)

Lemma (5.2). *Suppose that* $K_\sigma \le \Delta$ *on* $B = B_r(n)$ $(r < \pi/4\sqrt{\Delta})$. *Then* P_f *is smooth on* B *and the following hold.*

(i) $\operatorname{grad} P_f(p) = -\int_A \operatorname{Exp}_p^{-1} f(a)da$.

(ii) $(1+2\Delta r^2)|x|^2 \ge \operatorname{Hess} P_f(x,x) \ge 2r\, c_\Delta(2r)/s_\Delta(2r) \cdot |x|$.

Then from (i) $-\operatorname{grad} P_f$ points inward at the boundary of B and (ii) means that P_f is convex on B. Thus P_f admits the unique minimum point C_f in B, which is called the center for mass of f. Note that C_f is characterized by

$$\operatorname{grad} P_f(C_f) = 0. \tag{5.3}$$

C_f has the following natural property: let $\varphi: A \to A$ be a measure preserving transformation and $\Phi: N \to N$ an isometry. Then we get

$$C_{\Phi \circ f \circ \varphi} = \Phi(C_f). \tag{5.4}$$

Remark (5.5). Consider finite points $\{n_i\} \subset B$ with weights $\{\phi_i\}$ ($\phi_i \ge 0$, $\sum \phi_i = 1$). We define $P_{\{n_i, \phi_i\}}(p) := 1/2 \sum \phi_i(p) d^2(p, n_i)$. In this case the center of $\{n_i, \phi_i\}$ will be denoted by $C_{\{n_i, \phi_i\}}$.

The notion of center of mass has many applications ([C 2], [Gro-K], [Bu-K], [Gro-K-R 1], [IH-R], [MO-R 1], [Ru 3], [Y 1] etc.). We just mention the average of differentiable maps.

Let M be a complete riemannian manifold and $\{m_i\}_{i \in Z^+}$ a discrete $r/3$-dense subset such that $d(m_i, m_j) > r/3$ ($r <$ convexity radius) and $\bigcup_i B_{r/3}(m_i) = M$. Let $F_i: B_r(m_i) \to N (i \in Z^+)$ be smooth maps into a riemannian manifold N such that for any $m \in M$ $\{F_i(m); d(m, m_i) \le r\}$ is contained in a strongly convex neighborhood B_m of N. Now to glue together F_i's to a smooth map $F: M \to N$ we define weights $\{\phi_i, i \in \boldsymbol{Z}^+\}$ as follows: take a C^∞-function $\psi: \boldsymbol{R}^+ \to [0, 1]$ with $\psi[0, 2]=1$, $\psi[3, \infty]=0$, $\psi'(t) \leq 0$ and define $\phi_i(m):=\psi(3d(m, m_i)/r)/\sum_j \psi(3d(p, m_j)/r)$. Now for $p \in M$ we set $F(m):=C_{\{F_i(m), \phi_i(m)\}}$. Also we define v: $D(\subset M \times N) \to TN$ as $v(m, n):=-\sum \phi_i(m) \operatorname{Exp}_n^{-1} F_i(m) \in T_n N$ $(=\operatorname{grad} P_{\{m_i, \phi_i\}}(n)$, with $P_{\{m_i, \phi_i\}}(q):=1/2 \sum \phi_i(m) d^2(q, F_i(m)))$, where D is a sufficiently small neighborhood of graph F. Then we have by definition $v(m, F(m))=0$. We want to show that F is smooth. For that purpose set $D_1 v(m, n)$: $T_m M \to T_n N$ (resp. $D_2 v(m, n)$: $T_n N \to T_n N$) by

$$D_1 v(m, n)(\dot{m}(0)):=d/dt_{t=0} v(m(t), n)$$

$$(\text{resp. } D_2 v(m, n)(\dot{n}(0)):=\nabla_{\partial/\partial t|_{t=0}} v(m, n(t))).$$

Theorem (5.6). *F is smooth and we get*

(i) *$D_2 v(m, F(m))$ is invertible*

(ii) *$D_1 v(m, F(m))+D_2 v(m, F(m))\, dF(m)=0$.*

Proof. From (4.5) we have $g(\nabla_x v(m, F(m)), x)=\operatorname{Hess} P_{\{m_i, \phi_i\}}(x, x) \geq 2s\, c_\Delta/s_\Delta(2s) \cdot |x|^2$ (s: radius of B_m), from which (i) is clear. Next we consider the horizontal and vertical components of $d/dt_{t=0} v(p, n(t)) \in T_{v(m,n)} TN$: $(d/dt_{t=0} v(m, n(t)))^h=d/dt_{t=0}(\tau_N v(m, n(t)))=\dot{n}(0)$,

$$(d/dt_{t=0} v(m, n(t)))^v=\nabla_{\partial/\partial t|_{t=0}} v(m, n(t))=D_2 v(m, n)\dot{n}(0).$$

Thus if $v(m, n)=0$, the horizontal components span the tangent space to the zero section and $\{d/dt_{t=0} v(m, n(t))\}$ is transversal to the zero section by (i). Now our assertion follows from the implicit function theorem.

Remark. $dF(m)$ has maximal rank if and only if $D_1 v(m, F(m))$ has maximal rank.

Next we give another application. Let M be a compact riemannian manifold, and τ_E: $E \to M$ a riemannian vector bundle with a metric connection.

Let τ_P: $P \to M$ be the principal bundle of orthonormal frames associated to τ_E. Then P carries a riemannian structure so that τ_P: $P \to M$ is a riemannian submersion with totally geodesic fibers.

Now let u: $M \to P$ be a continuous cross section. We want to approximate u by smooth cross sections. For any $m \in M$ we have a strongly convex open ball $B_r(m)$ ($r<$ convexity radius). Firstly we define v_m: $B_r(m) \to P_m:=\tau_P^{-1}(m)$ as follows: for $n \in B_r(m)$ we define $v_m(n)$ as the parallel translation of $u(n)$ along the unique minimal geodesic from n to m. Note that taking r sufficiently small $\{v_m(n); n \in B_r(m)\}$ is contained in a convex neighborhood C_m in P_m. We need as before a weight function $\eta: M\times M \to \boldsymbol{R}^+$ with the following properties: put $\eta^n(m):=\eta(m,n)$. Then $\eta^n: M\to \boldsymbol{R}^+$ is a smooth function with supp $\eta^n\subset B_s(n)$ and $\int_M \eta^n(m)\,dn=1$ (see (2.3.5)).

Now for $m\in M$, we define a function $P_{m,\eta}: C_m\to \boldsymbol{R}^+$ as

$$P_{m,\eta}(p):=1/2\int_M d^2(p, v_m(n))\eta^n(m)\,dn.$$

As in (5.2) $P_{m,\eta}$ is a smooth function which has the unique minimum point $u^{(s)}(m)\in P_m$ (the center of mass). Also as in (5.6) $m\in M\to u^{(s)}(m)\in P$ gives a smooth section of P. Letting $s\to 0$, η converge to the Dirac measure and $u^{(s)}$ converge to u in the C^1-topology.

Note that because of (5.4) the above construction may be done equivariantly.

Chapter 2. Comparison Theorems

§1. 1/4-pinched manifolds

Rauch proposed the following approach to global riemannian geometry ([R 1–3]): Recall that if M is a complete simply connected riemannian manifold of positive constant curvature δ then M is isometric to the sphere $S^d(\delta)$. Now if curvature K_σ of M varies in the range $[\delta, \Delta]$, where pinching number δ/Δ is close to 1, does M have similar topological property as sphere? Rauch gave an affirmative answer when $\delta/\Delta\approx 3/4$. Then pinching constant δ/Δ was improved by Berger, Klingenberg, Toponogov and Tsukamoto and their ideas provided many useful tools for riemannian geometry ([B 1–2], [K 1-3], [T 2], [Ts 1], [C-E], [G-K-M], [K 6]).

Theorem (1.1) (sphere theorem). *Let M be a complete simply connected riemannian manifold whose sectional curvature satisfies*

$$(0<)\delta\le K_\delta\le\Delta, \quad \textit{with} \quad \delta/\Delta>1/4.$$

Then M is homeomorphic to the sphere.

Proof. We may assume that $\Delta=1$. M is compact and $d_M\le\pi/\sqrt{\delta}$ by (1.1.21). Proof depends on the following two facts:

(i) Injectivity radius estimate ((1.3.3)), i.e., $i_M\ge\pi$.

(ii) Toponogov comparison theorem ((1.2.24)).

Now take two points $p, q\in M$ with $d(p,q)=d_M\ge\pi$. For any point $m\in M$ we show that either $d(p,m)<\pi$ or $d(q,m)<\pi$ holds. In fact assume that $d(p, m) \geq \pi$. Take a minimal geodesic $\gamma_{p,m} \in \mathrm{Min}(p, m)$. From (1.4.8) there exists a $\gamma_{p,q} \in \mathrm{Min}(p, q)$ with $\measuredangle(\dot{\gamma}_{p,m}(0), \dot{\gamma}_{pq}(0)) \leq \pi/2$. Then T.C.T.-(II) implies that $d(m, q) < \pi$. Then for any normal goedesic c_v emanating from p, there is the uniquely determined $(0<)\ t(v) < \pi$ with $d(p, c_v(t(v))) = d(q, c_v(t(v)))$. $v \in U_pM \to c_v(t(v))$ is continuous and injective by (1.3.3). Namely we have a homeomorphism φ from $U_pM(\cong S^{d-1})$ onto the equator $E := \{m;\ d(p, m) = d(q, m)\}$. Similarly we get a homeomorphism ψ; $U_qM \cong E$. Since we have two disks M^+ (resp. M^-) $:= \{m \in M;\ d(p, m) \leq d(q, m)\}$ (resp. $:= \{m \in M;\ d(p, m) \geq d(p, m)\}$) with $M = M^+ \cup M^-$ and common boundary E, it is not difficult to get a homeomorphism between S^d and M. q.e.d.

Next we give Berger's rigidity theorem ([B 2], [Cha 2], [C-E], [K 6]).

Theorem (1.2). *Let M be a complete riemannian manifold whose sectional curvature K_σ satisfies*

$$(0<)\delta \leq K_\sigma \leq \Delta \quad \textit{with} \quad \delta/\Delta \geq 1/4.$$

Then we have the following:

(i) *If $d_M = \pi/\sqrt{\delta}$, then M is isometric to the sphere $S^d(\delta)$.*

(ii) *If $d_M > \pi/2\sqrt{\delta}$, then M is homeomorphic to the sphere.*

(iii) *If $d_M = \pi/2\sqrt{\delta}$ and simply connected, then M is isometric to one of the simply connected rank one symmetric spaces of compact type* (i.e. *CROSS, sphere or various projective spaces with canonical metric*).

We only give outline of proof (for details see the above papers). We may assume $\Delta = 1$, $\delta = 1/4$. First case will be treated more generally in (2.1). For (ii) we show firstly that M is simply connected in this case. In fact otherwise let $\pi: \tilde{M} \to M$ be the universal covering of M. Take $p, q \in M$ with $d(p, q) = d_M$. For different points $\tilde{p}_1, \tilde{p}_2 \in \pi^{-1}(p)$ take a minimal geodesic $\tilde{\gamma}_{\tilde{p}_1, \tilde{p}_2} \in \mathrm{Min}(\tilde{p}_1, \tilde{p}_2)$. By (1.4.8) we have a minimal geodesic $\gamma_{p,q} \in \mathrm{Min}(p, q)$ with $\alpha := \measuredangle(\dot{\gamma}_{p,q}(0), d\pi \cdot \dot{\tilde{\gamma}}_{p_1, p_2}(0)) \leq \pi/2$, which may be lifted to a minimal geodesic $\tilde{\gamma}_{\tilde{p}_1, \tilde{q}}$ from $\tilde{p}_1$ to $\tilde{q}$. Note that $2\pi \geq d(\tilde{p}_2, \tilde{q}) \geq d(\tilde{p}_1, \tilde{q}) > \pi$. By T.C.T.we get

$$\cos d(\tilde{p}_1, \tilde{q})/2 \geq \cos d(\tilde{p}_2, \tilde{q})/2 \geq \cos d(\tilde{p}_1, \tilde{q})/2 \cdot \cos d(\tilde{p}_1, \tilde{p}_2)/2 + \cos\alpha \sin d(\tilde{p}_1, \tilde{q})/2 \sin d(\tilde{p}_1, \tilde{p}_2)/2,$$

which implies that

$$0 > \cos d(\tilde{p}_1, \tilde{q})/2 \cdot (1 - \cos d(\tilde{p}_1, \tilde{p}_2)/2) \geq \cos\alpha \cdot \sin d(\tilde{p}_1, \tilde{q})/2 \cdot \sin d(\tilde{p}_1, \tilde{p}_2)/2.$$

Since $\cos\alpha \geq 0$ we have a contradiction. Thus we have the injectivity radius estimate $i_M \geq \pi$. If we show that for any $m \in M$ either $d(p, m) < \pi$ or $d(q, m) < \pi$ holds, then we can proceed exactly in the same way as (1.1). This may be proved by T.C.T with (1.2.25) and is rather complicated. We omit this (see (2.2) for more general case). For (iii) we have again $i_M \geq \pi$. Since d_M equals π we see that every normal geodesic is minimal just until the parameter value π or equivalently tangential cut locus $\tilde{C}_p$ is a sphere $S_\pi^{d-1}(o_p)$ of radius π centered at the origin for every $p \in M$. Now main step of proof is to show that the cut locus $C_p = \mathrm{Exp}_p \tilde{C}_p$ is a totally geodesic submanifold for any $p \in M$. This follows from the following considerations: For any $m, n \in C_p$ and normal geodesic γ from m to n of length $L_\gamma < 2\pi$, we can show by T.C.T. with (1.2.25) that γ is contained in C_p, every minimal geodesic from p to an interior point of γ is orthogonal to γ at parameter value π and that (p, m, n) forms a totally geodesic triangle of constant curvature 1/4. In particular we see that all geodesics are closed geodesics of length 2π (so-called $C_{2\pi}$-manifold, see [Be]) and intersect $C_{\gamma(0)}$ perpendicularly at parameter value π. Now fix $p \in M$ and take any normal geodesic c_v emanating from p. Put $q := c_v(\pi)$. Then for any unit vector $w \in T_q C_p$, by considering a totally geodesic triangle (p, q, $\mathrm{Exp}_q\, sw$) of constant curvature 1/4, we have a Jacobi field along c_v which takes the form $Y(t) = \sin t/2 \cdot E(t)$, where $E(t)$ is a parallel vector field along c_v with $E(\pi) = w$. Such Jacobi fields form a vector space $\mathscr{I}_{1/4}$ of dimension k ($:= \dim C_p$). On the other hand the null space of $d\,\mathrm{Exp}_p\,(\pi v)$ gives a subspace $\mathscr{I}_1$ of Jacobi fields along c_v of dimension d-k-1. Comparing with $S^d(1)$ we may show that every element Y of $\mathscr{I}_1$ may be expressed as $Y(t) = \sin t \cdot E(t)$ with parallel E. Note that for $0 < t < \pi$, $\dot{c}_v(t)^\perp (\subset T_{c(t)}M) = \mathscr{I}_{1/4}(t) \oplus \mathscr{I}_1(t)$ and this shows that the geodesic symmetry s_p at p is an isometry because $ds_{p|c_v(t)^\perp}$: $\dot{c}_v(t)^\perp \ni Y(t) \to Y(-t) \in c_v(-t)^\perp$. Thus M is locally symmetric and simply connectivity implies that M is a symmetric space. Since M is of positive curvature M must be of rank one.

Theorem (1.3) ([Ts 2], [Sug]). *Let M be a complete simply connected riemannian manifold whose sectional curvature satisfies*

$$\delta \leq K \leq \Delta, \qquad \delta/\Delta \geq 1/4.$$

(i) *Suppose that there exists a simple closed geodesic of length $2\pi/\sqrt{\delta}$. Then M is isometric to a sphere of constant curvature δ.*

(ii) *Suppose that there exists a closed geodesic of length $\pi/\sqrt{\delta}$. Then M is isometric to one of CROSS.*

Next we consider what happens when M is not simply connected. In this case we have from (1.2-(ii)) that $d_M \leq \pi/2\sqrt{\delta}$ and we ask which nonsimply connected manifolds carry the riemannian structure with maximal diameter $\pi/2\sqrt{\delta}$.

Theorem (1.4) ([S-S], [Sa 4]). *Let M be a complete nonsimply connected riemannian manifold with* $(0<)\delta\leq K_\sigma\leq\Delta$, $\delta/\Delta\geq 1/4$. *Then* $d_M=\pi/2\sqrt{\delta}$ *if and only if M is one of the following*:

(i) *M is of constant curvature* δ *and its fundamental group* $\pi_1(M)$ *has a fully reducible orthogonal representation (namely the universal covering* $\tilde{M}$ *of M is the sphere* $S^d(\delta)$ *and* $\pi_1(M)$ *may be represented by elements of* $O(d+1)$. *Then this representation should have a nontrivial invariant subspace. Typical examples are real projective space and lens spaces etc. see* [Wo 1]).

(ii) $M=P_{2n-1}(C)/\{\text{id}, \psi\}$, *where* $P_{2n-1}(C)$ *denotes the complex projective space of complex dimension* $2n-1$, *which carries the canonical riemannian structure with* $\delta\leq K_\sigma\leq 4\delta$, *and* ψ *denotes the involution of* $P_{2n+1}(C)$ *which is defined in terms of homogeneous coordinates as*

$$\psi(z_1; z_2; \cdots; z_{2n-1}; z_{2n}):=(\bar{z}_{n+1}; \cdots; \bar{z}_{2n}; -\bar{z}_1; \cdots; -\bar{z}_n).$$

For proof we take $p, q\in M$ with $d_M=d(p, q)$. We consider the antipodal set defined as $A_p:=\{m\in M; d(m, p)=d_M\}$. Then using T.C.T. we see that A_p is a convex totally geodesic submanifold without boundary. We consider the universal covering $\pi\colon \tilde{M}\to M$ and put $\tilde{A}_p:=\pi^{-1}(A_p)$, which is connected and of dimension≥ 1. Thus $\tilde{A}_p$ is again compact totally geodesic submanifold of $\tilde{M}$ and is invariant under deck transformations. Moreover $\tilde{A}_p$ is simply connected if dim $A_p>1$. Then we have $d_{\tilde{A}}\geq\pi/2\sqrt{\delta}$ by the injectivity radius estimate. We may see that $d_{\tilde{A}_p}=\pi/\sqrt{\delta}$ or $\pi/2\sqrt{\delta}$. For the first case we have in fact $d_{\tilde{M}}=\pi/\sqrt{\delta}$ and $\tilde{M}$ is isometric to a sphere and $\tilde{A}_p$ is a great sphere in $S_d(\delta)$ which is invariant under deck transformations. For second case we see that $\tilde{M}$ is one of CROSS and (ii) follows. In the second case note that $d_{\tilde{M}}$ is equal to d_M.

Problem. What can we say about riemannian manifolds with $\delta\leq K_\sigma\leq\Delta$, $\delta/\Delta<1/4$? (see (3.4.5)).

§ 2. Curvature and diameter

Firstly we give Toponogov's maximal diameter theorem.

Theorem (2.1) ([T 1]). *Let M be a complete riemannian manifold with* $K_\sigma\geq\delta(>0)$. *Then M is compact and* $d_M\leq\pi/\sqrt{\delta}$. *If* $d_M=\pi/\sqrt{\delta}$, *then M is isometric to the sphere* $S^d(\delta)$ *of constant curvature* δ.

In fact $d_M\leq\pi/\sqrt{\delta}$ follows from (1.2.21). Suppose that $d_M=\pi\sqrt{\delta}$ and take points $p, q \in M$ with $d(p, q)=d_M$. Now by T.C.T. for any normal geodesic c_v emanating from p with $v \in U_pM$ we have $c_v(d_M)=q$. Then $\mathrm{Exp}_{p|B_{d_M}(o_p)}$ is a diffeomorphism. Moreover we see from (1.2.25) that every Jacobi field Y along c_v with $Y(0)=0$ may be written in the form $Y(t)=s_\delta(t)E(t)$ with parallel E. Then $B_{d_M}(p)$ is isometric to $B_{d_M}(\bar{p})$ in $S^d(\delta)$. Then it is not difficult to see M is isometric to $S^d(\delta)$. See also (4.1) for a generalization.

Next we show the following which generalizes the sphere theorem.

Theorem (2.2) ([Gr-S]). *Let M be a complete riemannian manifold with $K_\sigma \geq \delta (>0)$. Suppose that $d_M > \pi/2\sqrt{\delta}$. Then M is homeomorphic to the sphere.*

Remark. This was firstly treated by Berger who proved that under the assumption M is a homotopy sphere. Then Grove-Shiohama constructed a homeomorphism between M and S^d. Here we shall give a simplified proof.

Proof. Take points p, q with $d(p, q)=d_M$. Lemma (4.8) and T.C.T. imply that for given such p there is uniquely determined q with $d(p, q)=d_M$. Next for $m \in M$ note that either $d(p, m) \leq \pi/2\sqrt{\delta}$ or $d(q, m) \leq \pi/2\sqrt{\delta}$ hold by the same reason as in (1.1). Now for $m \neq p, q$ we show that there exists a unit vector $t(m)$ which satisfies the property

(*) $g(t(m), \dot{\gamma}(d(p, m))) > 0$ for all $\gamma \in \mathrm{Min}\,(p, m)$ (i.e. m is not critical for d_p). In fact take a minimal geodesic $\sigma \in \mathrm{Min}\,(m, q)$. Put $\alpha := \measuredangle(\dot{\sigma}(0), \dot{\gamma}(d(p, m)))$ for any $\gamma \in \mathrm{Min}\,(p, m)$. First assume that $d(p, m) \leq \pi/2\sqrt{\delta}$. Then from T.C.T. we get

$$\cos\sqrt{\delta}\, d_M \geq \cos\sqrt{\delta}\, d(m, p) \cdot \cos\sqrt{\delta}\, d(m, q) \\ + \cos(\pi-\alpha) \sin\sqrt{\delta}\, d(m, p) \sin\sqrt{\delta}\, d(m, q),$$

from which follows

$$0 > \cos\sqrt{\delta}\, d_M \cdot (1-\cos\sqrt{\delta}\, d(p, m)) \\ \geq \cos(\pi-\alpha) \sin\sqrt{\delta}\, d(p, m) \sin\sqrt{\delta}\, d(m, q),$$

namely $\alpha < \pi/2$. In case when $d(q, m) \leq \pi/2\sqrt{\delta}$ the same argument holds. Thus $\dot{\sigma}(0)$ satisfies (*) and we may apply (1.4.12).

Remark. In this case $\sigma \in \mathrm{Min}\,(m, q)$ is unique if $m \notin C_q$. Then in a small neighbourhood U of q we may define a vector field $t(n)$, $n \in U$ as $t(n)=\dot{\sigma}_{n,q}(0)$ where $\sigma_{n,q} \in \mathrm{Min}\,(n, q)$. Then t is transversal to $\partial B_r(q)$ for small r and we can construct a homeomorphism more directly.

Remark (2.3). Grove-Gromoll ([Gr-Gro]) announced the following result without detailed proof: If a complete riemannian manifold M satisfies

$$K_\sigma \geq \delta(>0), \qquad d_M \geq \pi/2\sqrt{\delta},$$

then M is homeomorphic to S^d or isometric to one of a CROSS, (i) or (ii) of (1.4).

Now we should mention about almost flat manifolds. Gromov ([G 1]) gave a completely new approach to the problem among curvature, diameter and the manifold structure. He considered the situation when $d_M^2 \max |K_\sigma|$ is very small and studied the structure of $\pi_1(M)$ of M very deeply by geodesic loops. Since there is a very detailed report on the subject by Buser-Karcher ([B-K]) we only state an improved result by Ruh ([R 3]).

Theorem (2.4) (Gromov-Ruh). *Let M be a compact riemannian manifold of dimension d. Then there exists a positive constant $\varepsilon(d)$ such that if $K_\sigma|d_M^2 < \varepsilon(d)$ hold for all $\sigma \in G_2(TM)$ we have the following: there exists a simply connected nilpotent Lie group N and an extension Γ of a lattice $L \subset N$ by a finite group H so that M is diffeomorphic to $\Gamma \backslash N$.*

This generalizes the Bieberbach's theorem in flat case (compact flat riemannian manifolds are finitely covered by a torus). In this almost flat case we should construct the model space (i.e. nilmanifold) in the way of proof. We may assume that $d_M = 1$. Take $p \in M$. Then from the assumption $\mathrm{Exp}_{p|B\rho(o_p)}$ is non-singular for very large ρ. Put

$$\Gamma_\rho := \{\alpha;\ \text{geodesic loop at } p \text{ with } |t(\alpha)| < \rho,\ |r(\alpha)| < 0.48\},$$

where $|t(\alpha)|$ is the length of α, $r(\alpha)$ denotes the element of $O(d)$ defined by the parallel translation along α and $|r(\alpha)|$ denotes the distance from the identity. A product $\alpha * \beta$ is defined as a geodesic loop given by the end point in T_pM of the lift of $\alpha \cup \beta$ via Exp_p^{-1}. Then Γ_ρ may be considered as the set of elements in $B_\rho(o_p)$ obtained by slightly deforming a lattice in T_pM. Roughly speaking, essential part of proof is to show that for some ρ Γ_ρ carries generators $\{\gamma_1, \cdots, \gamma_d\}$ such that every $\gamma \in \Gamma_\rho$ may be uniquely expressed as $\gamma = \gamma_1^{k_1} * \cdots * \gamma_d^{k_d}$ ($k_i \in Z$) and $[\gamma_i, \gamma_j] \in \langle \gamma_1, \cdots, \gamma_{i-1} \rangle$ for commutators. From Γ_ρ we get a nilpotent torsion free group Γ, which may be embedded in a nilpotent Lie group N as a uniform discrete subgroup.

§ 3. Differentiable pinching problem

Since we have exiotic spheres, many authors have attempted to show that a complete simply connected δ-pinched ($\delta > 1/4$) manifold M is in fact diffeomorphic to the standard sphere. Such differentiable sphere theorem was firstly proved by Calabi, Gromoll ([Gr]) and Shikata ([Sh 1, 2]), where the pinching constant δ_d ($\to 1$ as $d \to \infty$) depended on the dimension of manifold. In fact the number of exiotic spheres increases with dimension. But Sugimoto-Shiohama ([Sug-S]) and then Ruh ([R 1, 2]) succeeded to show that δ can be chosen independently of dimension. The actual value of pinching number was improved successively and the equivariant case also has been treated ([Gro-K-R 1, 2], [IH-R]). Here we give

Theorem (3.1) ([IH-R]). *There exists a decreasing sequence δ_d ($\delta_d \to 0.68$ as $d \to \infty$) with the following property: Let M be a complete simply connected riemannian manifold of dimension d whose sectional curvature satisfies $\delta_d \leq K_\sigma \leq 1$, and μ: $G \times M \to M$ an isometric action of the Lie group G. Then there exists a diffeomorphism F: $M \to S^d$ (standard sphere) and a homeomorphism φ: $G \to O(d+1)$ such that $\varphi(g) \circ F = F \circ \mu_g$ for all $g \in G$.*

As an immediate corollary we get

Corollary (3.2). *If M is a complete d-dimensional riemannian manifold with $\delta_d \leq K_\sigma \leq 1$ (δ_d as above), then M is diffeomorphic to a space of positive constant curvature and isometry group of M is isomorphic to a subgroup of the corresponding space form.*

Here we shall explain main ideas of [IH-R] and only show that exists such a pinching constant.

When M is a hypersurface of positive curvature in $\boldsymbol{R}^{d+1}$, Gauss map gives a diffeomorphism between M and S^d. Ruh took the same approach for general case. We put $E = \tau_M \oplus 1_M$, where 1_M is a trivial line bundle. E carries a fiber metric on which elements of G, with the trivial action on 1_M, act as isometries. Let e be the section defined by $e(m) := (o_m, 1)$. Now we define the connection ∇° on E as

$$\text{(3.3)} \qquad \begin{aligned} &\nabla^\circ_X Y := \nabla_X Y - c g(X, Y) e, \quad \nabla^\circ_X e := cX, \\ &X, Y \in \mathscr{X}(M) \quad \text{and} \quad c := \sqrt{(1+\delta)/2}. \end{aligned}$$

Then ∇° is a G-invariant metric connection whose curvature tensor R° is given by

$$\text{(3.4)} \qquad \begin{aligned} &R^\circ(X, Y)Z = R(X, Y)Z - c^2\{g(Y, Z)X - g(X, Z)Y\}, \\ &R^\circ(X, Y)e = 0. \end{aligned}$$

Then from (1.1.6) we have $\| R^\circ \| \leq 2/3\ (1-\delta)$, which is small from the assumption. Now starting from this ∇° by the iteration method we shall construct a G-invariant flat connection D on E, from which we have a parallel field of orthonormal frames $u_m := (e_1, \cdots, e_{d+1})_m$ for the fiber E_m over $m \in M$, since M is simply connected. Then we can define the map

$$F: M \to S^d \quad \text{as} \quad F(m) := (g(e, e_1)(m), \cdots, g(e, e_{d+1})(m)) \quad (:= g(e, u)).$$

We expect that F is a diffeomorphism as Gauss map is. Since u is parallel we get

$$dF_m(x) = g(D_x e, u) = cg(x, u) + g((D_x - \nabla_x^\circ)e, u),$$

which implies that

$$|dF_m(x)| \geq |x|(c - \|D - \nabla^\circ\|),$$

where the above norm is defined as follows: First norm in $\mathcal{O}(d+1)$ is given by $|A| := \mathrm{Max}_{|x|=1}|Ax|$ and then we define $\|D - \nabla^\circ\| := \mathrm{Max}_{x \in UM}|D_x - \nabla_x^\circ|$, where we consider $(D_x - \nabla_x^\circ)$ at $p \in M$ as an element of $\mathcal{O}(d+1)$. Thus if we have $c > \|D - \nabla^\circ\|$, then F gives a covering map, which is in fact a diffeomorphism since M is simply connected. We also define a homomorphism φ as follows: for $m \in M$ the frame u_m may be identified with an isometry $\boldsymbol{R}^{d+1} \to E_m$. Then we put $\varphi(g) := u^{-1} \circ g \circ u \in O(d+1)$. Since g commutes with parallel translation we may easily see that φ is independent of the choice of $m \in M$ and that φ is a group homomorphism.

Now we return to the construction of the flat connection D by the iteration from ∇^i to ∇^{i+1} starting from ∇°. For the computation we prefer to deal with connection form ω^i and curvature form Ω^i on the principal bundle of orthonormal frames associated to E instead of ∇^i and R^i. We compute with their pull backs by means of a cross section $u = (e_a)$(i.e., $(\omega^i(x))^a_b := g(\nabla_x e_a, e_b)$, $(\Omega^i(x, y))^a_b := 1/2\, g(R(x, y)e_a, e_b)$). Now we need a technical lemma.

Lemma (3.5). *Suppose that* $1/4 < \delta \leq K_\sigma \leq 1$. *Then for any* $\pi > r > \pi/2\sqrt{\delta}$ *we have a weight function* $\eta: M \times M \to \boldsymbol{R}$ *with the following properties*: $\eta(gm, gn) = \eta(m, n)$, $g \in G$. *Put* $\eta^m(n) := \eta(n, m)$. (i) $\eta^m: M \to R$ *is a smooth function with* supp $\eta^m \subset U_m$ (U_m *is a convex neighborhood around* m). (ii) $\int_M \eta^m(n)dm = 1$. (iii) $\int_M |d\eta^m| dm \leq$ const. $d \sin^{d-1} \sqrt{\delta}\, r$.

Now for any $m \in M$ we define a flat connection $\omega^{i,m}$ on U_m by parallel translating orthonormal frame $u^i(m)$ along the unique shortest geodesic from m. Then we define

$$\omega^{i+1}(x) := \int_M \omega^{i,m}(x)\eta^m(\tau_M x)dm. \tag{3.6}$$

Then ω^{i+1} defines in fact a connection. Now we compute the curvature form Ω^{i+1}. By (ii) of (3.5) and the fact that $\omega^{i,m}$ is flat we have

$$\Omega^{i+1}=\int_M (\omega^{i,m}\wedge d\eta^m)dm-\int_M (\omega^{i,m}-\omega^{i+1})\wedge(\omega^{i,m}-\omega^{i+1})\eta^m dm.$$

Taking the following norm for $\mathcal{O}(d+1)$-valued forms,

$$\|\omega\|:=\max_{x\in UM}|\omega(x)|,\qquad \|\Omega\|:=\max_{x,y\in UM}|\Omega(x,y)|$$

we get by direct computations with Cauchy-Schwarz

$$\|\Omega^{i+1}\|\leq\|\alpha^i\|\int_M|d\eta^m|dm+\|\alpha^i\|^2, \tag{3.7}$$

where

$$\alpha^{i,m}:=\omega^{i,m}-\omega^i \quad\text{and}\quad \|\alpha^i\|:=\max_{m\in M}\{\max\{|\alpha^{i,m}(x)|;\ x\in UM_{|\mathrm{supp}\,\eta^m}\}\}.$$

Now the following is essential for the proof.

Lemma (3.8). *Let r be greater than the radius of the ball on which $\omega^{i,m}$ is defined. Then we have $\|\alpha^i\|\leq 2(1-\cos r)/(\delta\sin r)\cdot\|\Omega^i\|$.*

Proof. For $x\in U_nM$, $n\in U_m$ we estimate $|\alpha^{i,m}(x)|$. Let $n=\mathrm{Exp}_m t_0v$ $(t_o=d(m,n))$ and $d\,\mathrm{Exp}_m(t_0v)w=x$. We put $\alpha(s,t):=\mathrm{Exp}_m t(t_ov+sw/|t_0v+sw|)$, $0\leq t\leq|t_ov+sw|$, and $\gamma_s:=\alpha(s,|t_0v+sw|)$. Then we have a triangle (m,n,γ_s). Let $a(s)(\in O(d+1))$ be the parallel translation w.r.t. ∇^i along the triangle. Then we have for the above section $u^{i,m}=(e_a)$

$$\alpha^{i,m}(x)f=-\nabla^i_x f^a e_a=P_{\gamma m,n}\circ\dot{a}(0)\circ P^{-1}_{\gamma m,n}(f^a e_a),\qquad f\in R^{d+1}.$$

Thus we get

$$|\alpha^{i,m}(x)|=|\dot{a}(0)|\leq d/ds_{|s=0}|a(s)|\qquad (|a(s)|:=\operatorname*{Max}_{|u|=1}\{|a(s)u-u|\}).$$

On the other hand we have $|a(s)|\leq\mathrm{Area}\,(m,n,\gamma_s)\|R^i\|\leq 2\,\mathrm{Area}\,(m,n,\gamma_s)\cdot\|\Omega^i\|$ ((1.16)). Then we get using (1.2.21) and (1.2.20)

$$\mathrm{Area}\,(m,n,\gamma_s)=\iint_{(O_m,t_0v,\,t_0v+sw)}|\det d\,\mathrm{Exp}_m|\,dsdt\leq\int_0^{t_0}dt\int_0^{ts|w|/t_0}(s_\delta(t)/t)\,ds$$
$$\leq s(1-c_\delta(t_0))/(\delta\sin t_0)\leq s(1-\cos r)/(\delta\sin r).\qquad \text{q.e.d.}$$

Then we have

$$\|\Omega^{i+1}\| < \text{const.}\ d \sin^{d-1}\sqrt{\delta}\, r \cdot (1-\cos r)/(\delta \sin r)\|\Omega^i\| + 4\left(\frac{1-\cos r}{\delta \sin r}\|\Omega^i\|\right)^2$$
$$=: (a+b\|\Omega^i\|)\|\Omega^i\|.$$

Since $d \sin^{d-1}\sqrt{\delta}\, r \to 0$ $(d\to\infty)$ if $\sqrt{\delta}\, r \neq \pi/2$, taking $\sqrt{\delta}\, r$ greater than but arbitrary close to $\pi/2$ and choosing $\delta \doteqdot 1$ we have from $\|\Omega^0\| < (1-\delta)/3$ that $a+b\|\Omega^0\|<1$ and consequently it is possible to get $\sum \|\Omega^i\| < c\delta \sin r/2(1-\cos r)$ or equivalently $\sum \|\alpha^i\| < c$. Then from $\|\omega^{i+1}-\omega^i\| \leq \|\alpha^i\|$ we see that ω^i converges to a connection form ω^∞ w.r.t. the C^0-topology. Moreover we have $\|\omega^\infty - \omega^0\| \leq \sum \|\alpha^i\| < c$.

Note that from the construction ∇^i and ∇^∞ is G-invariant. ∇^∞ is only continuous. But the parallel translation w.r.t. ∇^∞ is independent of the path and ∇^∞ is flat in this sense. As final step Im Hof-Ruh approximate ∇^∞ by an invariant smooth connection D which may be chosen arbitrarily close to ∇^∞. This may be done by means of the center of mass technique with above weight η (see Chapter 1, § 5).

Remark. See [Ru 4] for another kind of differentiable pinching problem.

Grove-Shiohama [Gr-S] asked for the differentiable sphere theorem for general case. See also the work of T. Yamaguchi in this proceeding.

Problem. Let M be a complete riemannian manifold with $K_\sigma \geq \delta(>0)$. Is M is diffeomorphic to a sphere if d_M is close to $\pi/\sqrt{\delta}$?

§ 4. Ricci curvature pinching problem

Let M^d be a complete d-dimensional riemannian manifold whose Ricci curvature satisfies

(4.1) $\qquad r(v) \geq (d-1)\delta \qquad$ (δ is a positive constant).

Then M is compact and in fact $d_M \leq \pi/\sqrt{\delta}$ ((1.2.21)). Especially considering the universal covering $\tilde{M}$ of M, which should be again compact, we know that $\pi_1(M)$ is a finite group.

Firstly we give maximal diameter theorem. This was given in ([B 5]) without proof. A proof is given by Cheng ([Che]) using the comparison theorem for the first eigenvalue of Laplacian. Then more direct proof using (1.4.1) has been given by Shiohama ([S 4]) and Itokawa ([It]), which will be presented here.

Theorem (4.2). *Suppose that a complete riemannian manifold satisfy* (4.1). *Then $d_M = \pi/\sqrt{\delta}$ if and only if M is isometric to $S^d(\delta)$.*

Proof. Take points p, q with $d(p, q)=d_M$. Put $R:=\pi/\sqrt{\delta}$, $r=\pi/2\sqrt{\delta}$. Then from $d_M=\pi/\sqrt{\delta}$ we have $B_r(p)\cap B_r(q)=\phi$. On the other hand from (1.4.1) we get

$$v_M/v_{B_r(p)}=v_{B_R(p)}/v_{B_r(p)}\leq b_\delta(R)/b_\delta(r)=2. \tag{4.3}$$

From this $v_{B_r(p)}\geq v_M/2$ and also $v_{B_r(q)}\geq v_M/2$ by the same reason. Then we get $v_M\geq v_{B_r(p)}+v_{B_r(q)}\geq v_M$ which implies that equality holds in (4.3) and that $\overline{B_r(p)}\cup\overline{B_r(q)}=M$. Also it is not difficult to see $\partial B_r(p)=\partial B_r(q)$. Looking at the proof of (1.4.1) when equality does hold we see that cut distance $t(v)=R$ for all $v\in U_pM$ and $\operatorname{Exp}_p Rv=q$ for all $v\in U_pM$. Namely every geodesic c_v emanating from p reaches q at the parameter value R. Next for any $v\in U_pM$ take an orthonormal basis $\{v=e_1, \cdots, e_d\}$ and parallel fields $E_i(t)$ along c_v with $E_i(0)=e_i$ $(i=2, \cdots, d)$. Then by the second variation formula for $Y_i(t)=\sin\delta t\, E_i(t)$ we have

$$\begin{aligned}0&\leq\sum_{i=2}^{d} d^2E(c)(Y_i, Y_i)=\sum\int_0^R \{g(\nabla Y_i, \nabla Y_i)-K(\dot{c}_v, Y_i)|Y_i|^2\}dt\\&=\int_0^R \{(d-1)\delta\cos^2\delta t-r(\dot{c}_v)\sin^2\delta t\}dt\leq 0.\end{aligned}$$

From this we see that $K_\sigma\equiv\delta$ for all $\sigma\ni\dot{c}_v(t)$, $0<t<R$ and Y_i are Jacobi fields. Thus we have $\Theta_p^M(v, t)=(s_\delta(t)/t)^{d-1}$ and consequently $v_M=v_{B_R(p)}=b_\delta(R)=v_{S^d(\delta)}$. Then M is isometric to $S^d(\delta)$ by (1.4.2). q.e.d.

Next Assuming $r(v)\geq(d-1)$ for a complete riemannian d-manifold M we may ask whether there exists $V=V(d)$ (or $\rho=\rho(d)$) such that $v_m>V$ (or $d_M>\rho$) implies that M is topologically similar to the sphere. For this problem Shiohama has obtained

Theorem (4.4) ([S 4], see also the work of Itokawa [It]). *Let M^d be a complete riemannian manifold with $r(v)\geq d-1$ and $K_\sigma\geq-\kappa^2$. Then there exists an $\varepsilon(d, \kappa)>0$ such that if $v_M\geq v_{S^d(1)}-b_1^d(\varepsilon(d, \kappa))$ holds, M is homeomorphic to the sphere. In the above we denote by $b_1^d(\varepsilon(d, \kappa))$ the volume of ε-ball of $S^d(1)$.*

In this case instead of injectivity radius estimate, estimate of radii of contractible metric balls, which is given by the infimum of minimal critical values of distance function, plays important roles. In fact under the assumption of the theorem M may be covered by two contractible balls. See also [It] for further informations.

Remark. As for the diameter pinching problem see the work of Kasue in this proceeding ([Kas]). The following example due to Itokawa is also suggestive.

Example (4.5). Let M be the riemannian product $M=S^j(j+k-1/(j-1))\times S^k(j+k-1/(k-1))$. Then M is an Einstein manifold with $r(v)\equiv j+k-1=\dim M$. By Pythagoras' theorem we have

$$d_M=\sqrt{j+k-2/(j+k-1)}\pi \longrightarrow \pi \qquad (j+k\longrightarrow\infty).$$

On the other hand

$$v_M=(j-1/(j+k-1))^{j/2}(k-1/(j+k-1))^{k/2}\omega_j\omega_k \longrightarrow 0 \quad (j+k\longrightarrow\infty).$$

For three dimensional case we have now complete answer.

Theorem (4.6) (Hamilton [Ha]). *A compact riemannian manifold of dimension* 3 *with positive Ricci curvature admits a metric of positive constant curvature.*

Method of proof heavily depends on P.D.E.

Remark. For noncompact case see Schoen-Yau ([Sc-Y]): A complete open 3-manifold of positive Ricci curvature is diffeomorphic to $\boldsymbol{R}^3$. see also Gromoll-Cheeger ([CG 1]).

§ 5. General comparison theorems

1°. Next we consider comparison problem when the model space is a compact symmetric space. One of the crucial properties of geodesics in a compact simply connected symmetric space is the following. First we define

Definition (5.1) (Cheeger [C 1–2]). For a compact riemannian manifold M and $p\in M$, (M,p) has property CM if

($*$) for any $m,n\in M\backslash C_p$ and $\varepsilon>0$ there exists a curve h_ε with $L_{h_\varepsilon}<d(m,n)+\varepsilon$ whose interior is disjoint from C_p.

($**$) For every geodesic c_v ($v\in U_pM$) cut point of c_v is the first conjugate point to p along c_v.

In fact ($*$) implies ($**$). When (M,p) has property CM for every p, we say that M has property CM.

Example (5.2). Suppose that M is simply connected and for every geodesic c_v emanating from p the first conjugate point $c(t(v))$ to p along c_v has order ≥ 2 (i.e., $\dim\{Y;$ Jacobi field along c_v with $Y(0)=Y(t(v))=0\}\geq 2$). Then (M,p) satisfies CM. This follows from the fact that $(d-1)$-Hausdorff measure of C_p in this case equals 0 ([War 3], [C 1–2]).

Example (5.3). If M is a compact simply connected symmetric space, then M has property CM. (For (**) see [C-E], [Cr], [Sa 2]). For (*) Cheeger proved that $(d-1)$-Hausdorff measure of C_p equals zero using the fact that in this case any conjugate point is induced by a one parameter subgroup in the isotropy subgroup of the isometry group.

Now the problem is as follows: If the metric structure of a compact manifold is similar to that of symmetric spaces, are they also topologically similar? For that purpose we should have a number which measures how close are curvature behaviour of two compact riemannian manifolds M, $\bar{M}$ of same dimension. Let I: $T_pM \to T_{\bar{p}}\bar{M}$ be an isometry and define for $v \in U_pM$ I_v^t: $T_{c_v(t)}M \to T_{c_{\bar{v}}(t)}\bar{M}$ as $I_v^t := P_{c_{\bar{v}}} \circ I \circ P_{c_v}^{-1}$ $(\bar{v} := Iv)$. We define for a positive number d as

$$\rho_d(M, \bar{M}) := \inf_{I:\, T_pM \to T_{\bar{p}}\bar{M},\, p \in M,\, \bar{p} \in \bar{M}} \sup \{\|R - (I_v^t)^{-1}\bar{R}\|;\ 0 \leq t \leq d,\ v \in UM\}$$

and $\rho(M, \bar{M}) := \rho_{2d_c}(M, \bar{M})$, where d_c is the supremum of the conjugate distances in all directions.

We shall give some consequences when $\rho(M, \bar{M})$ is sufficiently small. In the following let I: $T_pM \to T_{\bar{p}}\bar{M}$ be the isometry which minimizes the quantity in the definition of $\rho(M, \bar{M})$. Firstly from the theory of O.D.E.

(5.4) Let $\{x^i\}$, $\{\bar{x}^i\}$ be normal coordinates in T_pM, $T_{\bar{p}}\bar{M}$ based on orthonormal bases $\{e_i\}$ $\{\bar{e}_i := Ie_i\}$ respectively. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho(M, \bar{M}) < \delta$ implies $|\tilde{g}_{ij} - I^*\tilde{\bar{g}}_{ij}| < \varepsilon$, for $\sqrt{\sum x_i^2} < 2d_c$ where $\tilde{g}_{ij}$, $\tilde{\bar{g}}_{ij}$ denote (pseudo)metric tensors induced from g, $\bar{g}$ via the exponential mappings at p, $\bar{p}$ respectively.

Next for $v \in U_pM$ we denote by $t_0(v)$ the conjugate distance in direction v. Considering the second variation $d^2E(c_v)$ we may show

(5.5) Suppose that M satisfies $t_0(v) < +\infty$ for all $v \in UM$. Then for any $\varepsilon > 0$, there exist $\delta > 0$ and $s(v) \in [t_0(v), t_0(v) + \varepsilon]$ for $v \in UM$ such that if $\rho(M, \bar{M}) < \delta$ then $c_{\bar{v}}$ has no conjugate point on $[0, t_0(v) - \varepsilon]$ and c_v and $c_{\bar{v}}$ have the same number of conjugate points on the interval $[0, s(v)]$. In particular we have $d_{\bar{M}} < d_M + \varepsilon$.

(5.6) Suppose that M has property CM and that some real characteristic number of M is non-zero and the corresponding Chern-Weil form B vanishes nowhere. Then there exist $i_0 > 0$ and $\delta > 0$ such that if $\rho(M, \bar{M}) < \delta$ we have $i_{\bar{M}} > i_0$.

In fact let $\int_M \|B\|\, dm > c (>0)$. If $\rho(M, \bar{M}) < \delta$ for sufficiently small

δ then we have $\int_{\bar{M}} \|\bar{B}\| \, dm > c/2$ and $v_{\bar{M}} \geq c/(2\|\bar{B}\|_{\max}) > c/(4\|B\|_{\max})$. Also noting that $|K_{\bar{\sigma}}| \leq 2 \max |K_\sigma|$ we have our assertion from (5.5) and (1.3.1).

Now we put $\Phi := \mathrm{Exp}_{\bar{p}} \circ I \circ \mathrm{Exp}_p^{-1}$: $M \to \bar{M}$, where Exp_p^{-1} is some inverse of Exp_p. Although Exp_p^{-1} may not be continuous, we have from (5.4) and the definition of property *CM*,

(5.7) Let M have property *CM*. Then for any $\varepsilon > 0$ there exists δ such that $\rho(M, \bar{M}) < \delta$ implies $d(\Phi(m), \Phi(n)) < d(m, n) + \varepsilon$ for all $m, n \in M$.

Then Cheeger obtained

Theorem (5.8) ([C 2]). *Suppose that M has property CM and satisfy the condition in* (5.6). *Then there exists* δ *such that* $\rho(M, \bar{M}) < \delta$ *implies that for any field F with non-zero characteristic* $H^*(\bar{M}, F)$ *is isomorphic to a subring of* $H^*(M, F)$.

Proof. By Poincaré duality it suffices to construct a continuous map φ: $M \to \bar{M}$ with $\deg \varphi \neq 0$. From (5.6) there exists $i_0 > 0$ such that i_M, $i_{\bar{M}} > i_0$ if $\rho(M, \bar{M}) < \delta$. Now we put $\mathscr{I}_{\varepsilon_1} := \{x;\ x = (1+\varepsilon_1)u,\ u \in \mathscr{I}_p\}$, where $\mathscr{I}_p$ is the interior set defined in Chapter 1, Section 4, 1°. Then from R.C.T. there exists $\varepsilon_1 > 0$ such that $\mathrm{vol}\,(\mathrm{Exp}_p\, I(\mathscr{I}_{\varepsilon_1} - \mathscr{I}_{-\varepsilon_1})) < \mathrm{vol}\,(B_{i_0}(\bar{p}))$. Taking δ sufficiently small Φ defined above satisfies (5.7) and $\Phi_{|\mathrm{Exp}_p\,(\mathscr{I}_{-\varepsilon_1/4})}$ is a regular smooth map. Then we may approximate Φ by a continuous map φ: $M \to \bar{M}$ so that $\varphi_{|\mathscr{I}_{-\varepsilon_1/2}}$ and $\max d(\Phi(m), \varphi(m)) < 4\varepsilon$. Then taking ε sufficiently small (which is possible by taking φ small) we have $\varphi\, \mathrm{Exp}_p\, (\mathscr{I}_0 - \mathscr{I}_{-\varepsilon_1/2}) \subset \mathrm{Exp}_{\bar{p}}\, I(\mathscr{I}_{\varepsilon_1} - \mathscr{I}_{-\varepsilon_1})$.

Now we assert that there exists a point $\bar{p} \in \varphi(M)$ such that $\varphi^{-1}(\bar{p}) \subset \mathrm{Exp}_p\,(\mathscr{I}_{-\varepsilon_1/2})$. Then we are done because on this connected set $\mathrm{Exp}_p\,(\mathscr{I}_{-\varepsilon_1/2})$ φ is smooth and non-singular. It follows that all inverse images are counted with the same sign and this shows that $\deg \varphi \neq 0$. Now if there are no points $\bar{p} \in \varphi(M)$ with the above property we have $\varphi(\mathrm{Exp}_p\,(\mathscr{I}_0 - \mathscr{I}_{-\varepsilon_1/2})) = \varphi(M)$ and consequently $\mathrm{vol}\,(\varphi(\mathrm{Exp}_p\,(\mathscr{I}_0 - \mathscr{I}_{-\varepsilon_1/2}))) = \mathrm{vol}\,\varphi(M) > \mathrm{vol}\,(B_{i_0}(\bar{p}))$. On the other hand we get

$$\mathrm{vol}\,(\varphi(\mathrm{Exp}_p\,(\mathscr{I}_0 - \mathscr{I}_{-\varepsilon_1/2}))) \leq \mathrm{vol}\,(\mathrm{Exp}_{\bar{p}}\, I(\mathscr{I}_{\varepsilon_1} - \mathscr{I}_{-\varepsilon_1})) < \mathrm{vol}\,(B_{i_0}(p)),$$

a contradiction. q.e.d.

Cheeger furthermore refined the above argument especially for the estimate of i_M and got

Theorem (5.9) ([C 2]). *Let M be a compact simply connected riemannian symmetric space or have the property in Example* (5.2) *for all* $p \in M$. *Then there exists* $\delta > 0$ *such that* $\rho(M, M) < \delta$ *implies that M is PL-homeomorphic to M.*

2°. With respect to the above problem Ruh took another view point. Firstly we take as a model space compact simply connected semi-simple Lie group G with Lie algebra $\mathfrak{g}\cong T_eG$. Then we have the Maurer-Cartan form $\varpi: TG\to\mathfrak{g}$ defined by $\varpi(x):=L_g^{-1}x, x\in T_gG$, which satisfies the Maurer-Cartan equation $d\varpi+[\varpi, \varpi]=0$, where $[\ ,\ ]$ denotes the Lie bracket in $\mathfrak{g}$. Now Let P be a compact manifold, ω: $TP\to\mathfrak{g}$ a parallelization of P, i.e., $\omega: T_pP\to\mathfrak{g}$ is a vector space isomorphism for every $p\in P$. Then the curvature Ω of ω is defined to be a $\mathfrak{g}$-valued 2-form given by $\Omega=d\omega+[\omega, \omega]$. In our case since $\mathfrak{g}$ carries an inner product defined from the Killing form, ω induces a riemannian metric on P, by which we may define the norm $\|\Omega\|:=\max\{|\Omega(x_1, x_2)|;\ x_1, x_2\in TP$, unit vectors$\}$. Then Min-Oo and Ruh [MO-R1] solved the equation $d\varpi+[\varpi, \varpi]=0$ on P under the assumption that $\|\Omega\|$ is sufficiently small by the iteration method as in (3.1). But here we need more tools from P.D.E. Then for the universal covering $\tilde{P}$ of P with the pull back $\tilde{\omega}$ of ϖ via covering projection, vanishing of curvature of $\tilde{\omega}$ implies that $\tilde{\omega}$: $\tilde{\mathfrak{g}}$ $(:=\{$vector fields X on P with $\tilde{\omega}(X)=$const.$\})\to\mathfrak{g}$ is a Lie algebra isomorphism. From this we may get a diffeomorphism $\tilde{F}$: $\tilde{P}\to G$ with $d\tilde{F}=\tilde{\omega}$.

Theorem (5.10) ([MO–R1]) *Let $\mathfrak{g}$ be a compact semi-simple Lie algebra, ω: $TP\to\mathfrak{g}$ a parallelization of a compact manifold P. Then there exists $\delta>0$ such that $\|\Omega\|<\delta$ implies that P is diffeomorphic to a quotient $\Gamma\backslash G$, where Γ is a finite subgroup of G.*

This was extended to the symmetric case as follows: Let $\bar{M}=G/K$ be a compact simply connected irreducible symmetric space and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ be a Cartan decomposition. Then the $\mathfrak{k}$-valued part of ϖ is the Levi-Civita connection form of M and the $\mathfrak{m}$-valued part is the canonical soldering form given by the identification $T\tilde{M}\cong G\times_K\mathfrak{m}$.

Now let M be a compact riemannian manifold and π: $P\to M$ the bundle of frames of M. We assume that π: $P\to M$ has a reduction to the structure group K represented in $\mathfrak{m}$ via adjoint action. On P we have an $\mathfrak{m}$-valued form θ defined as $\theta(x):=u^{-1}\circ d\pi(x)$, $x\in T_uP$, u: $\mathfrak{m}\cong T_{\pi(u)}M$. Let η be a connection form on P, which is a metric connection since the structure group K of P is compact (We don't assume that η is a Levi-Civita connection). Combining θ and η to define a $\mathfrak{g}$-valued 1-form $\omega=\eta+\theta$ with curvature $\Omega=d\omega+[\omega, \omega]$, we get

Theorem (5.11) ([MO-R]). *Let $\bar{M}=G/K$, M, P, ω be as above. Then there exists a positive constant $\delta>0$ such that $\|\Omega\|<\delta$ implies that M is diffeomorphic to a quotient $\Gamma\backslash\bar{M}$, where Γ is a finite subgroup of G.*

Furthermore Ruh proposed to study "almost Lie group": A compact riemannian manifold with a metric connection D is called an ε-almost Lie group if $(\|DT\|+\|R\|)d_M^2<\varepsilon$, where R and T denote the curvature and torsin of D respectively. Then is almost Lie group diffeomorphic to $\Gamma\backslash G$ with Γ an extension of a lattice of a Lie group G? ([R 3])

Chapter 3. Finiteness Theorem

In the preceding chapter we compared riemannian manifold M with a model space $\bar{M}$ and asked whether M is topologically similar to $\bar{M}$ if M is similar to $\bar{M}$ in riemannian sense. More generally we may ask the problem to classify all the topological types of riemannian manifolds which satisfy some conditions given in terms of riemannian invariants, e.g., classify manifolds of positive (non-negative) or negative (non-positive) curvature. But usually these classification problems are very difficult and we treat here the problem whether there are only finitely many topological types of riemannian manifolds if their riemannian structures are restricted.

§ 1. Weinstein's homotopy type finiteness theorem

Weinstein ([W 1]) and Cheeger ([C 1,3]) attacked the above problem for the first time.

Theorem (1.1). *When $d\in Z^+$, Δ, ρ, $V\in R^+$ are given, there are only finitely many homotopy types of compact d-dimensional riemannian manifolds M with $|K_\sigma|\leq\Delta$, $d_M\leq\rho$ and $v_M\geq V$.*

Proof. The idea is to estimate the number of convex open balls which cover M. For that purpose we introduce the notion of ε-dense subset of M; $\{m_1, m_2, \cdots, m_N\}\subset M$ will be called ε-dense if $\bigcup_{i=1}^N B_\varepsilon(m_i)=M$. Now from the injectivity radius estimate (1.3.1) there exists $\varepsilon_1>0$ such that $i_M>\varepsilon_1$ if M satisfies the assumption of the theorem. Thus we see that for the convexity radius estimate $c_M\geq\varepsilon:=\min(\varepsilon_1/2, \pi/2\sqrt{\Delta})$ holds. To obtain an ε-dense subset we consider a maximal family of open balls $B_{\varepsilon/2}(m_i)$, $i=1, \cdots, N$, which are mutually disjoint in M. Then maximality implies that $\bigcup B_\varepsilon(m_i)=M$, i.e., $\{m_i\}_{i=1}^{i=N}$ is an ε-dense subset. Now by the volume comparison theorems (1.4.2), (1.4.3), we have

$$Nb_\Delta^d(\varepsilon/2)(:=Nv_{B_{\varepsilon/2}(M^d(\Delta))})\leq\sum_{i=1}^N v_{B_{\varepsilon/2}(m_i)}\leq v_M \tag{1.2}$$
$$\leq b_{-\Delta}^d(\rho)(:=v_{B_\rho(M^d(-\Delta))}).$$

Thus we have an open covering $\mathscr{U}=\{B_i:=B_\varepsilon(m_i)\}_{i=1}^{i=N}$ by strongly convex open subsets. We shall consider the simplicial complex K defined by the nerve of $\mathscr{U}$; we associate m_i the point $S_i:=(0, \cdots, 1, \cdots, 0) \in \boldsymbol{R}^N$ and we define $(S_{i_0}, \cdots, S_{i_k})$ is a k-simplex if and only if $B_{i_0} \cap \cdots \cap B_{i_k} \neq \phi$. We choose a continuous partition of unity $\{\varphi_i\}$ on M so that $\varphi_i(m)>0$ on B_i, $\varphi_i(m)=0$ outside B_i and $\sum \varphi_i(m)=1$.

Now we define a map $f\colon M \to |K|$ by $f(m):=(\varphi_1(m), \cdots, \varphi_N(m))$, and show that f is in fact a homotopy equivalence. To see this we take the barycentric subdivision K' of K. Then any simplex σ of K' may be expressed by $((S_{i_l}, \cdots, S_{i_k}), (S_{i_{l-1}}, \cdots, S_{i_k}), \cdots, (S_{i_0}, \cdots, S_{i_k}))$, where $(S_{i_0}, \cdots, S_{i_k})$ etc. also denotes the barycenter of the simplex of K. We define a map $g\colon |K| \to M$ inductively so that for above σ $g(\sigma) \subset B_{i_l} \cap \cdots \cap B_{i_k}$: firstly to vertices $(S_{i_j}, \cdots, S_{i_k})$ of K' we assign any point in $B_{i_j} \cap \cdots \cap B_{i_k}$. Next assuming that we have defined a map g on the l-skelton of K' we define g on any $(l+1)$-simplex $\tau=((S_{i_{l+1}}, \cdots, S_{i_k}), \cdots, (S_{i_0}, \cdots, S_{i_k}))$ as follows. Taking a point p in $B_{i_0} \cap \cdots \cap B_{i_k}$, we map segments joining the barycenter $(S_{i_0}, \cdots, S_{i_k})$ and the points of $\partial\tau$ to the minimal geodesics joining p to the corresponding points of $g(\partial\tau)$. Then $g \circ f$ is homotopic to the identity because m and $g \circ f(m)$ are contained in a convex ball and it is not so difficult to see that $f \circ g$ is also homotopic to the identity (see e.g. ([dR])) for details). Thus M has a homotopy type of a simplicial complex with at most $b^d_{-\Delta}(\rho)/b^d_{\Delta}(\varepsilon/2)$ vertices. q.e.d.

Corollary (1.3). *Let d be an even positive integer and $0<\delta<1$. Then there are finitely many homotopy types of compact simply connected riemannian manifolds of dimension d with $\delta \leq K_\sigma \leq 1$.*

In fact we have in this case $d_M \leq \pi/\sqrt{\delta}$ ((1.2.21)) and $i_M \geq \pi$ (1.3.2)). On the other hand at least seven dimensional case there are infinitely many homotopy types of compact simply connected riemannian manifolds with $\delta \leq K_\sigma \leq 1$, where δ is some positive constant ([Hu]).

§ 2. Cheeger's finiteness theorem

Cheeger ([C 3]) has proved the following which improves the previous result (1.1).

Theorem (2.1). *For given $d \in \boldsymbol{Z}^+$, Δ, ρ, $V \in \boldsymbol{R}^+$, there are only finitely many homeomorphism classes of compact d-dimensional riemannian manifolds such that $|K_\sigma| \leq \Delta$, $d_M \leq \rho$, $v_M \geq V$.*

Let M be a compact riemannian manifold satisfying the assumption of theorem. There exists $r=c_d(\Delta, \rho, V)>0$ such that $c_M \geq c$ for such M. Considering as before a maximal mutually disjoint open balls $\{B_{r/2}(p_i)\}_{i=1}^{i=N}$ $(r:=c/8)$ we have an open covering $\{B_r(p_i)\}_{i=1}^{i=N}$ of M with $N \leq$

$b^d_{-\Delta}(\rho)/b^d_{\Delta}(r/2)$. Thus considering the exponential mappings and homotheties in the tangent spaces we have the family of embeddings

$$\phi_k\colon \bar{B}_8(0),\ \bar{B}_1(0),\ \bar{B}_{1/2}(0)\ (\subset \boldsymbol{R}^d) \longrightarrow \bar{B}_{16r}(p_i),\ \bar{B}_{2r}(p_i),\ \bar{B}_r(p_i)\ (\subset M)$$

$(k=1, \cdots, N)$ with the following properties:

(2.2) $\phi_k(\bar{B}_8(0))$ are strongly convex and $\bigcup_{k=1}^N \phi_k(B_{1/2}(0))=M$.

(2.3) $\phi_l(\bar{B}_1(0)) \cap \phi_j(\bar{B}_1(0)) \neq \phi \Longrightarrow \phi_l(\bar{B}_1(0)) \subset \phi_j(\bar{B}_4(0))$.

(2.4) There exists a $\xi(\Delta, r)$ such that the C^1-norm of the coordinates transformations satisfy $\|(\phi_k^{-1}\circ\phi_i)_{|B_1(0)}\|_{C^1} \leq \xi(\Delta, r)$.

In fact for (2.4) put $(y^u)=\phi_k^{-1}\circ\phi_i(x^a)$, $g_{ab}=g(\partial/\partial x^a, \partial/\partial x^b)$ and $\bar{g}_{uv}=g(\partial/\partial y^u, \partial/\partial y^v)$. Let $\lambda(x)$ be the maximum eigenvalue of $(g_{ab}(x))$, $\mu(y)$ the minimum eigenvalue of $(\bar{g}_{uv}(y))$. Then we have $\lambda(x) \geq \mu(y)\cdot\sum|\partial y^a/\partial x^i|^2$. On the other hand from R.C.T. (1.2.20) we get $\lambda(x) \leq s_{-\Delta}(2r)/(2r)$ and $\mu(y) \geq 2/\pi$, and consequently $|\partial y^a/\partial x^i| \leq \pi/2\cdot s_{-\Delta}(2r)/(2r)$.

2°. Now for the proof of theorem we need a tool to show that two manifolds are homeomorphic. We consider the following situation: let M be a compact (topological) manifold, $\phi_l\colon \bar{B}_2(0)(\subset \boldsymbol{R}^d) \to M$ $(l=1, \cdots, N)$ (topological) embeddings. Put $B_j^l := \phi_l(\bar{B}_{1-j/2N}(0))$, $j=0, \cdots, N$ and

$$(2.5)\qquad \begin{cases} K_l := B_l^1 \cup \cdots \cup B_l^l, & \\ H_l := B_l^1 \cup \cdots \cup B_l^{l-1}, & (K_{l-1} \supset H_l) \\ L_l := \phi_l^{-1}(H_l \cap B_l^l), & \\ J_l := \phi_l^{-1}(K_{l-1} \cap B_{l-1}^l) \subset B_1(0), & (L_l \subset J_l). \end{cases}$$

We assume that $\bigcup_{l=1}^N \operatorname{int} B_N^l = M$. Now the main tool is

Lemma (2.6). *For any $\varepsilon>0$ there exists $\varepsilon_1>0$ with the following property: let $\Psi_j\colon B_0^j\ (=\phi_j(\bar{B}_1(0)) \to \bar{M}\ (j=1, \cdots, N)$ be embeddings into a riemannian manifold $\bar{M}$ such that*

$$(2.7)\qquad \Psi_l(B_{j+2}^l \cap B_{j+2}^j) \subset \Psi_j(B_0^j),$$

$$(2.8)\qquad d_0(\phi_j^{-1}\circ\Psi_j^{-1}\circ\Psi_l\phi_j, \iota) < \varepsilon_1 \quad on \quad \phi_j^{-1}(B_{j+2}^l \cap B_{j+2}^j) \quad for\ all\ j,$$

where we denote by d_0 the uniform C^0-topology, i.e., $d_0(\phi, \psi);=Sup_{x,y}\, d(\phi(x), \psi(y))$ and ι denotes the inclusion map. Then we have an immersion $\Psi\colon M \to \bar{M}$ such that $d_0(\Psi_{j|B_j^j}, \Psi_{|B_j^j}) < \varepsilon$.

We don't give a proof of (2.6), but only mention that it follows by the successive applications of the isotopy extension theorem ([E-K]): let C_1, C_2 be closed sets in $\boldsymbol{R}^d$ such that $C_1 \subset \operatorname{int} C_2 \subset B_1(0)$. For any $\varepsilon>0$ there exists $\delta>0$ such that if the inclusion ι: $C_2 \to \bar{B}_1(0)$ and a (topological) immersion h: $C_2 \to \bar{B}_1(0)$ satisfy $d_0(h, \iota)<\delta$, then there exists a homeomorphism $\tilde{h}$: $\bar{B}_1(0) \to \bar{B}_1(0)$ with $\tilde{h}_{|C_1}=h_{|C_1}$ and $d_0(\tilde{h}, \mathrm{id})<\varepsilon$. To apply this to (2.7) we need closed sets C_l^1, C_l^2 $(l=1, \cdots, N)$ with $L_l \subset C_l^1 \subset \operatorname{int} C_l^2 \subset J_l \subset B_1(0)$.

3°. Assuming (2.7) we proceed as follows. By Ascoli-Arzela theorem the set $\mathscr{E}_1$ of embeddings f: $\bar{B}_1(0) \to \bar{B}_4(0)$ whose C^1-norm $\leq \xi(\Delta, r)$ is totally bounded with respect to the uniform C^0-topology. Namely for any $\delta>0$ there exists a δ-dense subset $\{f_1, \cdots, f_{n(\delta)}\}$ of $\mathscr{E}_1$. From (2.4) $\phi_j^{-1} \circ \phi_{l|\bar{B}_1(0)} \in \mathscr{E}_1$. Now there exists $\delta_1=\delta_1(\Delta, r)$ such that

$$d(\phi_j^{-1} \circ \phi_l(\partial B_{1-3k/6N}(0)),\ \phi_j^{-1} \circ \phi_l(\partial B_{1-(3k-1)/6N}(0)))>\delta_1,$$

because $\phi_j^{-1} \circ \phi_l \in \mathscr{E}_1$. For a fixed l $(1 \leq l \leq N)$ we choose $f_{i_1}, \cdots, f_{i_{l-1}} \in \mathscr{E}_1$ so that $d_0(f_{i_j}, \phi_l^{-1} \circ \phi_j)<\delta_1/4$ $(j=1, \cdots, l-1)$ and define

$$C_l^1:=\{f_{i_1}(\bar{B}_{1-(3l-1)/6N}(0)) \cup \cdots \cup f_{i_{l-1}}(\bar{B}_{1-(3l-1)/6N}(0))\} \cap \bar{B}_{1-(3l-1)/6N}(0)$$
$$C_l^2:=\{f_{i_1}(\bar{B}_{1-(3l-2)/6N}(0)) \cup \cdots \cup f_{i_{l-1}}(\bar{B}_{1-(3l-2)/6N}(0))\} \cap \bar{B}_{1-(3l-2)/6N}(0).$$

Then we may check that C_l^1, C_l^2 satisfy the above mentioned property.

Now we prove the theorem by contradiction. Suppose that there are infinitely many compact riemannian manifolds M_i $(i=1, 2, \cdots)$ satisfying the assumption of the theorem which are not mutually homeomorphic. We may further assume that all M_i have the atlases consisting of the same number of local charts $\{(\phi_j^{(i)}, \bar{B}_8(0)), j=1, \cdots, N\}$ with (2.2), (2.3) and (2.4) and that $\phi_j^{(i)}(B_1(0)) \cap \phi_l^{(i)}(B_1(0)) \neq \phi$ if and only if $\phi_j^{(k)}(B_1(0)) \cap \phi_l^{(k)}(B_1(0)) \neq \phi$ for all pairs (i, k). Now from Ascoli's theorem there exists (i, k) $(i \neq k)$ such that

$$d_0(\phi_j^{(k)-1} \circ \phi_{l|\bar{B}_1(0)}^{(k)},\ \phi_j^{(i)-1} \circ \phi_{l|B_1(0)}^{(i)})<\varepsilon/\xi(\Delta, v) \text{ for all } i, l=1, \cdots, N.$$

Put $\Psi_j:=\phi_j^{(i)} \circ \phi_j^{(k)-1}$: $B_0^j(\subset M_k) \to M_i$. Then we have by taking $\varepsilon<1/N$, $\Psi_l(B_{j-2}^l \cap B_{j-2}^j) \subset \Psi_j(B_0^j)$ and

$$\begin{aligned} (2.9) \qquad & d_0(\phi_j^{(k)-1} \circ \Psi_j^{-1} \circ \Psi_l \circ \phi_j^{(k)}, \iota) \\ & =d_0(\phi_j^{(i)-1} \circ \phi_l^{(i)}(\phi_l^{(k)-1} \circ \phi_j^{(k)}),\ \phi_j^{(k)-1} \circ \phi_l^{(k)}(\phi_l^{(k)-1} \circ \phi_j^{(k)}))<\varepsilon. \end{aligned}$$

Thus applying (2.6) to $(M=M_k, \bar{M}=M_i, \Psi_j)$ we have an immersion ψ_k: $M_k \to M_i$ and also an immersion ψ_i: $M_i \to M_k$ reversing i and k. Taking ε sufficiently small $\psi_k \circ \psi_i(m)$ and m (resp. $\psi_i \circ \psi_k(n)$ and n) are in a small convex neighborhood and we see that $\psi_k \circ \psi_i$ and $\psi_i \circ \psi_k$ are homotopic to the identity. Then since ψ_k is a covering map and a homotopy equivalence, ψ_k is in fact a homeomorphism. This contradiction completes the proof of (2.1). Then from the results of differential topology we get

Corollary (2.10). *If $d\neq 4$ then there are only finitely many diffeomorphism classes of riemannian manifolds which satisfy the assumption of the theorem.*

Remark (2.11). In general dimension, under the additional assumption $\max\|\nabla R\|\leq \Lambda_1$, Cheeger directly proved that there are only finitely many diffeomorphism classes. For this we need in (2.8) of lemma (2.6) the estimate of uniform C^1-topology to apply Thom's isotopy extension theorem. We need the estimate on $\|\nabla R\|$ to get (2.4) in terms of C^2-norm, which suffices to get the estimate (2.9) w.r.t. the C^1-topology. Now recently T. Yamaguchi obtained a more explicit result, namely the estimate of the number of diffeomorphism classes of riemannian manifolds satisfying the above conditions, by using center of mass technique ([Y 1], [Y 2]).

Very recently S. Peters ([P]) gave a direct proof of corollary (2.10) for all dimensions. Consider two compact d-dimensional riemannian manifolds M, $\bar{M}$ with i_M, $i_{\bar{M}}\geq i_0$ and $|K_M|$, $|K_{\bar{M}}|\leq \Lambda^2$. Suppose that M ($\bar{M}$) is covered by N convex balls $\{B_{R/2}(x_i)\}_{i=1}^{i=N}$ ($\{B_{R/2}(\bar{x}_i)\}_{i=1}^{i=N}$) such that $B_{R/4}(x_i)$ ($B_{R/4}(\bar{x}_i)$)'s are mutually disjoint, where we have put $R:=i_0/10$. Then we have normal coordinates $\phi_i:=\mathrm{Exp}_{x_i}\circ u_i$: $B_R(0)$ $(\subset T_{x_i}M)\to B_R(x_i)$ ($\bar{\phi}_i:=\mathrm{Exp}_{\bar{x}_i}\circ \bar{u}_i$: $B_R(0)\to B_R(\bar{x}_i)$), where u_i: $R^d\to T_{x_i}M$ etc. is an orthonormal basis at x_i. For $B_{R/2}(x_i)\cap B_{R/2}(x_j)\neq\phi$ we get embeddings $\phi_j^{-1}\circ\phi_i$: $B_R(0)\to B_{3R}(0)$. We denote by P_{ij} the parallel translation along the shortest geodesic from x_i to x_j. Then Peters gave instead of (2.6)

Proposition (2.12). *Let M, $\bar{M}$ be as above and $\varepsilon_0\leq i_0\cdot 6^{2-n}$, $\varepsilon_1\leq \Lambda i_0/70$. Then*

$$d_0(\phi_j^{-1}\phi_i,\ \bar{\phi}_j^{-1}\bar{\phi}_i)<2/3\cdot\varepsilon_0, \tag{2.13}$$

$$\|u_j^{-1}\circ P_{ij}\circ u_i-\bar{u}_j^{-1}\circ\bar{P}_{ij}\circ\bar{u}_i\|<\varepsilon_1^2 \qquad \textit{for all } i,j \tag{2.14}$$

imply that M and $\bar{M}$ are diffeomorphic.

Then Corollary (2.10) follows from (2.12) as in 3° of the proof of (2.1). For infinitely many mutually non-diffeomorphic riemannian manifolds $\{M_k\}$ satisfying the assumptions of (2.1), (2.14) holds for some two of them because the set $\{(u_j^{(k)})^{-1}\circ P_{ij}^{(k)}\circ u_i^{(k)}\}$ is contained in the compact group $O(n)$, which admits a finite covering by balls of radius $\varepsilon_{1/2}^2$. For the proof of (2.12) we glue up local maps $F_i:=\bar{\phi}_i\phi_i^{-1}$: $B_R(x_i)\to B_R(\bar{x}_i)$ to a smooth map F: $M\to\bar{M}$ by center of mass technique, which turns out to be regular by R.C.T. and an estimate of the area of geodesic triangles in terms of curvature.

§ 3. Gromov's approach ([G 7])

Now Gromov ([G 7]) considered to embed a compact d-dimensional riemannian manifold M with $|K_\sigma| \leq \Delta$, $d_M \leq \rho$ and $v_M \geq V$ into euclidean space $\boldsymbol{R}^N$ of dimension N, where N may be estimated in terms of Δ, ρ, V and d. Take $0 < r < c_M := \min\{\pi/2\sqrt{\Delta}, i_M/2\}$. We choose an ε-dense subset $\boldsymbol{N} = \{m_i\}_{i=1}^{i=N}$ with $\varepsilon < \min\{r/8, s_\Delta(r)/2 \cdot (1-(1/\sqrt{2}+\alpha)^2)^{1/2}, 1/(2(r\Delta + 16/r)) \cdot (1 - r/(8s_\Delta(r/4))\}$, where α will be determined later. From this ε we may estimate $N \leq b^d_{-\Delta}(\rho)/b^d_\Delta(\varepsilon/2)$ (see (1.1)).

Taking a C^∞ cut function $h: \boldsymbol{R} \to \boldsymbol{R}^+$ such that $h(t)=1$ if $t \leq 0$, $h(t)=0$ if $t \geq r$, and $h'(t) < 0 (0 < t < r)$, we define a smooth mapping $f: M \to \boldsymbol{R}^N$ as

$$f(p) := (h(d(m_1, p)), \cdots, h(d(m_N, p)). \tag{3.1}$$

Note that there exists $k(r) > 0$ such that $|h'(t)|, |h''(t)| < k(r)$.

1°. Firstly we see that f is immersive at any point $p \in M$. Take an orthonormal basis $\{e_i\}$ of T_pM and choose $m_{i_1}, \cdots, m_{i_d} \in \boldsymbol{N}$ such that $d(m_{i_j}, \mathrm{Exp}_p\, r/2 \cdot e_j) < \varepsilon$ $(j=1, \cdots, d)$. Note that $3r/8 < d(p, m_{i_j}) < 5r/8$. We take $u_j \in U_pM$ $(j=1, \cdots, N)$ such that $m_{i_j} = \mathrm{Exp}_p\, t_j u_j$, $t_j := d(p, m_j)$. Then from R.C.T. (1.2.20) we have

$$\frac{s_\Delta(r)}{r}|r/2 \cdot e_j - t_j u_j| \leq d(m_{i_j}, \mathrm{Exp}_p\, r/2 \cdot e_j) < \varepsilon < s_\Delta(r)/2 \cdot (1-(1/\sqrt{2}+\alpha)^2)^{1/2},$$

from which we easily see that $g(e_j, u_j) > 1/\sqrt{2} + \alpha$. This implies that $\{u_j\}$ are linearly independent. Now remark that rank $df(p) \geq \mathrm{rank}\,(d \cdot h(d(m_{i_j}, \cdot)) (p)), \cdots, d \cdot h(d(m_{i_d}, \cdot)(p))) = \mathrm{rank}\,(h'(d(m_{i_1}, p))u_1, \cdots, h'(d(m_{i_d}, p))u_d) = d$, because grad $d_{m_{i_j}} = -u_j$ ((1.4.4)).

2°. Secondly we show that f is injective. Suppose that $f(m) = f(n)$ for $m, n \in M$ $(m \neq n)$. We have then $d(m_i, m) = d(m_i, n)$ for all $m_i \in \boldsymbol{N} \cap B_r(m) = \boldsymbol{N} \cap B_r(n)$. Note that $d := d(m, n) < 2\varepsilon (< r/4)$. Let γ be the minimal geodesic from m to n and $z := \gamma\,(r/2 + d/2)$. Then $z \in B_{r/2}(n) \backslash \bar{B}_{r/2}(m)$ and $B_\varepsilon(z) \subset B_r(n) \backslash \bar{B}_{r/4}(m)$. Now there exists a point $p \in \boldsymbol{N} \cap B_\varepsilon(z)$. Let λ be the minimal geodesic from n to p and put $u := \dot{\lambda}(0) \in U_nM$ and $d' := d(p, n)$. We estimate $g(u, \dot{\gamma}(d))$. From R.C.T. (1.2.20) we get

$$\begin{aligned} &|(r/2 - d/2)\dot{\gamma}(d) - d'u| \\ &\quad = |\mathrm{Exp}_n^{-1} z - \mathrm{Exp}_n^{-1} p| \leq d'/s_\Delta(d') \cdot d(p, z) < d'/s_\Delta(d') \cdot \varepsilon \end{aligned}$$

from which follows

$$\begin{aligned} r/2\cdot g(\dot\gamma(d), u) &> (r/2-d/2)g(\dot\gamma(d), u) \\ &= g((r/2-d/2)\dot\gamma(d)-d'u, u)+d' \geq d'(1-\varepsilon/s_\Delta(d')) \\ &\geq (r/2-d/2-\varepsilon)(1-\varepsilon/s_\Delta(r/2-d/2-\varepsilon)) \geq r/4(1-r/(8s_\Delta(r/4))), \end{aligned}$$

namely

$$g(\dot\gamma(d), u) \geq 1-r/(8s_\Delta(r/4)).$$

On the other hand note that $d(p, \gamma(t))<r$ $(0<t<d)$. From Rolle's theorem the distance function d_m has a point $m_0=\gamma(t_1)$, $0<t_1<d$ with $g(\dot\gamma(t_1), u_{t_1}(0))=0$, where u_t is the initial direction of the minimal geodesic from $\gamma(t)$ to p. Then we have from the hessian estimate (1.4.4) that

$$\begin{aligned} g(\dot\gamma(d), u) &= \int_{t_1}^{d} d/dt\cdot g(\dot\gamma(t), u)\,dt = \int_{t_1}^{d} \text{Hess}\, d_p(\dot\gamma(t), \dot\gamma(t))\,dt \\ &\leq \int_{t_1}^{d} \{(1/d(p, \gamma(t))+d(p, \gamma(t))\Delta/2\}dt < 2\varepsilon(8/r+r\Delta/2). \end{aligned}$$

Thus we get $2\varepsilon(r\Delta+16/r)>1-r/(8s_\Delta(r/4))$, a contradiction.

3°. Here we remark that there exists $D(\alpha, d, r)>0$ such that $\text{dil}_m f^{-1} < D(\alpha, d, r)$ (see remark after (1.2.20) for the definition of the dilatation). In fact taking a basis $\{u_1, \cdots, u_d\}$ it suffices to consider a linear map l: $T_pM \to R^d$ defined by $l(\xi):=(a_1 g(u_1, \xi), \cdots, a_d g(u_d, \xi))$ with $a_i=h'(d(p, m_i))$. We have the following properties: $g(u_j, e_j)>1/\sqrt{2}+\alpha$, $|g(u_i, e_j)|<(1-(1/\sqrt{2}+\alpha)^2)^{1/2}$ $(i\neq j)$, and there exists $c(r)>0$ such that $|a_i|=|h'(d(p, m_i))|<c(r)$ (recall that $3r/8<d(p, m_i)<5r/8$). Then we can show that

$$\begin{aligned} &\text{Min}\,\{|l(\xi)|; |\xi|=1\} \\ &\quad \geq c(r)/\sqrt{d}\cdot\{(1/\sqrt{2}+\alpha)/d-(d-1)(1-(1/\sqrt{2}+\alpha)^2)^{1/2}\}. \end{aligned}$$

α is chosen so that the last quantity is positive.

Now we extend the above embedding to a tubular neighborhood of $f(M)$. Let ν: $TM^\perp \to M$ be the normal bundle of M and Exp_ν: $TM^\perp \to \boldsymbol{R}^N$ the normal exponential mapping. Then Exp_ν is a diffeomorphism on $B_\delta(TM^\perp):=\{u\in TM^\perp; |u|<\delta\}$ for some $\delta>0$. We estimate the value of δ for which $\text{Exp}_{\nu|B_\delta(TM^\perp)}$ is a local diffeomorphism. For that purpose suppose that $n\in\boldsymbol{R}^N$ is a critical value of Exp_ν. Namely there exist a curve $s\to c_s=f(m_s)$ in $f(M)$, normal vector field n_s along c_s such that $n=c_0+n_0$, $\dot c_0+\dot n_0=0$. Then we have from $g(n_s, \dot c_s)=0$,

$$g(n_0, \ddot c_0)=-g(\dot n_0, \dot c_0)=|\dot c_0|^2.$$

Since $c_s=(h(d(m_i, m_s)))$ we have

$$\ddot{c}_0=(h''(d(m_i, m_0))(d/ds_{|s=0}d(m_i, m_s))^2+h'(d(m_i, m_0))d^2/ds^2_{|s=0}d(m_i, m_0)).$$

Recall that

$$|d/ds_{|s=0}d(m_i, m_s)|=|g(\text{grad } d_{m_i}, \dot{m}_0)|\leq|\dot{m}_0|,$$
$$|d^2/ds^2_{|s=0}d(m_i, m_s)|=|\text{Hess } d_{m_i}(\dot{m}_0, \dot{m}_0)|\leq|\dot{m}_0|^2\psi(d(m_i, m_0), \Delta).$$

($\psi(t, \Delta):=1/t+\Delta t/2$). There exists $k(r)>0$ such that $|h'(t)\psi(t, \Delta)|$, $|h''(t)| <k(r)$ $(0\leq t\leq r)$. Then we have

$$|\dot{c}_0|^2\leq|n_0||\ddot{c}_0|\leq 2|n_0||\dot{m}_0|^2\sqrt{N}\ k(r), \quad \text{namely}$$
$$d(n, f(M))=|n_0|\geq 1/(2\sqrt{N}k(r))\cdot|\dot{c}_0|^2/|\dot{m}_0|^2$$
$$\geq 1/(2\sqrt{N}k(r))\cdot(\text{dil}_{c0}(f^{-1}))^{-2}\geq D^{-2}/(2\sqrt{N}k(r)).$$

This gives an estimate for δ such that $\text{Exp}_{\nu|B\delta(TM^\perp)}$ is a local diffeomorphism. Gromov further asserts that we may have $\delta\approx\text{const}\,(r, \Delta, d)N\sqrt{N}$.

Next suppose that we have another compact riemannian manifold M' with $|K'_\sigma|\leq\Delta$, $d_{M'}\leq\rho$, $v_{M'}\geq V$, which carries an ε-dense subset $\{m'_i\}_{i=1}^{i=N}$ such that

$$1-a\leq\frac{d(m'_i, m'_j)}{d(m_i, m_j)}\leq 1+a. \tag{3.2}$$

Then we get from the definition of f and f' that $d(f(m_k), f'(m'_k))\leq k(r)a\sqrt{N}\rho$ and similarly $d(f(m), f'(M'))\leq k(r)\sqrt{N}\rho(a+\varepsilon)$. Namely for sufficiently small a, ε, $f(M)\subset B_\delta(f'(M'))$. From this Gromov asserts that M is diffeomorphic to M' (it seems to the author that we need some more arguments for this). Anyway this implies finiteness of diffeomorphism types of compact riemannian manifolds with $|K_\sigma|\leq\Delta$, $d_M\geq\rho$, $v_M\geq V$. In fact if there are infinitely many such M_k which are not mutually diffeomorphic we have a sequence of points $(d(m_i^{(k)}, m_j^{(k)}))\in \boldsymbol{R}^{N(N+1)/2}$, which are obtained from ε-dense subsets and lie in a bounded subset of $\boldsymbol{R}^{N(N+1)/2}$. Thus we have two M_k and $M_{k'}$ for which (3.2) holds.

5°. Now Gromov proposed much more general scheme. Namely he considered the Hausdorff metric on the space of metric structures. Firstly for metric spaces X, Y assume that there exists a bijection $f\colon X\to Y$ such that $\text{dil} f$, $\text{dil} f^{-1}<+\infty$. We define the Lipschitz distance $d_L(X, Y)$ between X, Y as $d_L(X, Y):=\inf\{|\log\text{dil}\,g|, |\log\text{dil}\,g^{-1}|; g\colon X\to Y, \text{bijection}\}$. Secondly we set for subsets A, B of a metric space Z

$$d_H(A, B):=\inf\{R>0;\ A\subset B_R(B), \text{ and } B\subset B_R(A)\},$$

where $B_R(A):=\{z \in Z; d(z, A)<R\}$ etc. Then for metric spaces X, Y, we define the Hausdorff distance between them as

(3.3) $d_H(X, Y):=\inf\{d_H^Z(f(X), g(Y)); Z$, metric space, $f: X\to Z$, $g: Y\to Z$, isometric injections$\}$.

In the case when X, Y are compact $d_H(X, Y)<+\infty$ and $d_H(X, Y)=0$ holds if and only if X is isometric to Y. In particular d_H determines a distance on the set of isometry classes of compact riemannian manifolds. Gromov considered in [G 7] what is the limit of sequences of riemannian structures w.r.t. Hausdorff distance. In general such a limit may not be a differentiable manifold. Nevertheless using the above arguments he asserts the following:

Gromov's convergence theorem (3.4). *Let* $\mathfrak{M}:=\{(M, g); \dim M=d, d_M\le\rho, |K_\sigma|\le\Lambda, v_M\ge V\}$. *Then a sequence* g_k *of riemannian structures in* $\mathfrak{M}$ *admits a limit g which is a weak riemannian structure.*

By weak riemannian manifold we mean a differentiable manifold which admits continuous metric, notion of geodesics, exponential mapping and injectivity radius etc. In the above the fact that i_{g_k} has a positive lower bound is essential. Applying the above to pinching problem Berger asserts the following:

Theorem (3.5). *For even* $d \in \mathbf{Z}^+$ *there exists* $\delta(d)<1/4$ *such that all compact simply connected riemannian manifolds of dimension d with* $\delta(d)\le K_\sigma\le 1$ *are either homeomorphic to a sphere or diffeomorphic to one of GROSS's.*

Remark. In [G 1] Gromov also asserts that the number of diffeomorphism types of compact riemannian manifolds with $d_M\le 1$, $|K_\sigma|\le\Lambda$, $v_M\ge\Lambda^{-1}$, $\dim M=d$, is less than $\mathrm{ex}_6\,(d+\Lambda)$, where

$$\mathrm{ex}_6\cdot=\underbrace{\exp(\exp(\cdots(\exp}_{6}\cdot))).$$

It will be also very nice if we get finiteness theorems assuming bounded Ricci curvature instead of sectional curvature.

§ 4. Curvature, diameter and Betti numbers

As we have seen in (1.2.21), for a compact riemannian manifold M with positive Ricci curvature its fundamental group $\pi_1(M)$ is finite and first Betti number $b_1(M)$ equals zero. In the case of non-negative Ricci curvature Milnor obtained the following (see also [Wo 2], [G 4,7] for further informations):

Theorem (4.1) ([Mi 2]). *Let M be a compact riemannian manifold with $r(v)\geq 0$. Then $\pi_1(M)$ has a polynomial growth* (*Let $\pi_1(M)$ be generated by $\{g_1, \cdots, g_k\}$ and we put $m(s):=\#\{g\in G;\ g=g_{i1}^{p_1}\cdots g_{i1}^{p_1}$ with $|g|:=|p_1|+|p_2|+\cdots+|p_1|\leq s\}$. Then by definition $\pi_1(M)$ has polynomial growth if $m(s)\leq$ Const. s^e for some $e\in Z^+$. This is independent of the choice of the generators*).

Proof. We consider elements $g\in\pi_1(M)$ as deck transformations of the universal covering $\pi\colon \tilde{M}\to M$. Then $g\in\pi_1(M)$ is an isometry of $\tilde{M}$ w.r.t. the complete induced metric. For a fixed $\tilde{m}\in\tilde{M}$, there exists an $\varepsilon>0$ such that $\|g\|:=d(\tilde{m}, g\tilde{m})>\varepsilon$ for all $g\in\pi_1(M)\backslash\{e\}$ and consequently $\{B_{\varepsilon/2}(g\tilde{m});\ g\in\pi_1(M)\}$ are mutually disjoint. Taking $R\geq\max_{1\leq i\leq k} d(g_i\tilde{m}, \tilde{m})$ we see that if $|g|\leq s$, $B_{\varepsilon/2}(gm)$ is contained in $B_{\varepsilon/2+Rs}(\tilde{m})$. Then we have from (1.4.2) and (1.4.3)

$$b_0^d(\varepsilon/2+Rs)\geq \mathrm{vol}\,(B_{\varepsilon/2+Rs}(\tilde{m}))\geq \sum_{g;\,|g|\leq s} \mathrm{vol}\,(B_{\varepsilon/2}(g\tilde{m}))\geq m(s)b_\Delta^d(\varepsilon/2),$$

where Δ is an upper bound of K_σ. Namely we get $m(s)\leq$ Const. $(\varepsilon/2+Rs)^d$.
q.e.d.

On the other hand for the estimate of the first Betti number we know that for a compact manifold M with $r(v)\geq 0$, $b_1(M)\leq d(=\dim M)$ holds, where the equality holds just for flat tori ([Bo]). Now Gromov ([G 7]) took the following approach: let M be a compact riemannian manifold and $\pi\colon\tilde{M}\to M$ the universal covering. Take $\tilde{m}\in\tilde{M}$ and $\varepsilon>0$. We put for $g\in\pi_1(M)$, $\|g\|:=d(\tilde{m}, g\tilde{m})$. Now let $h_1, \cdots, h_p$ be a maximal family of elements of $\pi_1(M)$ with the following properties;

(i) $\|h_i h_j^{-1}\|>\varepsilon$ if $i\neq j$,

(ii) $\|h_i\|<2d_M+\varepsilon$.

Let Γ be the normal subgroup of $\pi_1(M)$ generated by $\{h_i\}$. We show that Γ is of finite index in $\pi_1(M)$. To see this we consider the Galois covering $\pi'\colon M'\to M$ corresponding to Γ. Namely the group of deck transformations of M' is isomorphic to $\pi_1(M)/\Gamma$. Now suppose that $\#\pi_1(M)/\Gamma=\infty$. Let $m'\in M'$ correspond to $\tilde{m}$. Then there exists $n'\in M'$ such that $d(m', n')=d_M+\varepsilon$ because M' is not compact. On the other hand there exists $h'\in\pi_1(M)/\Gamma$ such that $d(n', h'm')\leq d_M$ because of $d(m, \pi'n')\leq d_M$. Then we get $\varepsilon\leq d(m', h'm')\leq d(m', n')+d(n', h'm')\leq 2d_M+\varepsilon$. Namely choosing $h\in\pi_1(M)$ such that $h'=h\Gamma$ we see that $\|h\|\leq 2d_M+\varepsilon$, $\|hh_i^{-1}\|>\varepsilon$, which contradicts the maximality.

Now considering the Hurewicz map $\varphi\colon\pi_1(M)\to H_1(M, Z)$ the subgroup G of $H_1(M, Z)$ generated by $\varphi(h_1), \cdots, \varphi(h_p)$ is of finite index. This shows that $b_1(M) \leq p$. Then (i) means that $B_{\varepsilon/2}(g_i m)$ $(i=1, \cdots, p)$ are mutually disjoint and (ii) means that they are contained in $B_{2d_M+3\varepsilon/2}(m)$. Thus we have as before

$$b_1(M) \leq p \leq \mathrm{vol}\,(B_{2d_M+3\varepsilon/2}(m))/\mathrm{vol}\,(B_{\varepsilon/2}(m)).$$

Now assuming that $r(v) \geq -(d-1)r$ for all $v \in UM$ we get from (1.4.1)

$$b_1(M) \leq \int_0^{5d_M} (sh\ rt)^{d-1} dt \Big/ \int_0^{d_M} (sh\ rt)^{d-1} dt,$$

which depends only on r, d, d_M. Refining the above argument Gromov got

Theorem (4.2) ([G 7]). *There exists an integer-valued function* $\varphi(d, r, d_M)$ *such that for all compact riemannian manifolds* M *of dimension* d *with* $r(v) \geq -(d-1)r$ *we have* $b_1(M) \leq \varphi(d, r, d_M)$. $\varphi = d$ *when* rd_M^2 *is sufficiently small.*

Remark. Gallot ([G 1,2], [B 9]) gave a proof of the above result by analytic tools.

Now assuming that curvature is bounded below, Gromov obtained the estimate of all Betti numbers.

Theorem (4.3) ([G 3]). *There exists a constant* $C = C(d)$ *such that for all compact d-dimensional riemannian manifolds* M *with* $K_\sigma \geq -\Lambda^2$ *and* $d_M \leq \rho$ *we get* $\sum_{i=0}^{d} b_i(M) \leq C^{1+\Lambda\rho}$.

From this we see that $\sum b_i(M) \leq C$ if $K_\sigma \geq 0$ and that connected sum of sufficiently many copies of $S^p \times S^{d-p}$ $(0<p<d)$ can not admit riemannian metric of non-negative sectional curvature. Note also that there are infinitely many homotopy types of riemannian manifolds satisfying the assumption of the theorem. If we can estimate the number of convex open balls which cover M then we easily have such an estimate for $\sum b_i(M)$ by Mayer-Vietoris sequence. Such an estimate follows from the injectivity radius estimate which is impossible in this case because we assume nothing about the volume. Gromov overcame the difficulties by many brilliant ideas including isotopy lemma (1.4.11) (see [G 3]).

Gallot ([G 1,2], [B 9]) also got such an estimate using analytic methods refining Weitzenböck's formula, Sobolev's inequality, etc. His methods are also applied for the estimate of eigenvalues of Laplacian, dimension of harmonic spinors, dimension of moduli of Einstein metrics etc. It will be very interesting if we have similar estimate for $\sum b_i(M)$ assuming bounded Ricci curvature instead of sectional curvature.

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