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Integrability of Infinitesimal Zoll Deformations

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1. A Riemannian metric on a sphere S^n ($n \geqq 2$) is called a *Zoll metric* when all the geodesics are closed and have a common length 2π. The metric of constant sectional curvature 1 is a well-known example of a Zoll metric, but we further know that this standard metric g_0 is deformable by Zoll metrics (Zoll [8], Guillemin [3]; see also Besse [1]).

A symmetric 2-form h on S^n which is a direction of a Zoll deformation of g_0 satisfies

$$\int_0^{2\pi} h(\dot{\gamma}_0(s), \dot{\gamma}_0(s))ds = 0 \tag{1.1}$$

for every geodesic γ_0 of g_0 parametrized by its arclength s, where $\dot{\gamma}_0$ is the tangent vector of γ_0. Conversely, if h is a symmetric 2-form satisfying (1.1) for every geodesic of g_0, then the geodesics of $g_t = g_0 + t \cdot h$ are nearly 2π-periodic in the first order of t. We call such a symmetric 2-form on S^n an *infinitesimal Zoll deformation*, which we abbreviate as *IZD*. We say an *IZD* h is *integrable* if there exists a family of Zoll metrics g_t with g_0 being the standard one such that $h = \partial g_t / \partial t|_{t=0}$.

V. Guillemin proved in [3] that every *IZD* on a 2-dimensional sphere is integrable. On the other hand, K. Kiyohara ([4], [5]) showed that the situation is quite different in higher dimensions; not all the *IZD* are integrable, and, moreover, the set of integrable *IZD* does not even form a linear subspace.

They both studied the *IZD* of conformal type. Up to trivial *IZD*, they are the only possible *IZD* on S^2 (Funk [2]). But there exists another type of *IZD* in higher dimensions, as we have seen in [7]. In this paper, we shall exhibit that this type of *IZD* are not integrable, using a representation theoretical counterpart of Kiyohara's argument. The problem to determine which *IZD* is integrable is not yet resolved for the mixture of these two types of *IZD*, though we get some information by our argument.

2. We first recall how the condition (1.1) is deduced. Let g_t be a family of metrics on S^n with g_0 being the standard metric. We fix a point

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$p \in S^n$ and a direction ℓ in T_pS^n. Let $\gamma_t(s)$ be the geodesic of g_t starting from p in the direction ℓ, parametrized by its arclength s with respect to g_t. We denote by $\dot{\gamma}_t(s)$ its tangent vector at $\gamma_t(s)$. Differentiating an obvious identity

$$\int_0^{2\pi} g_t(\dot{\gamma}_t(s), \dot{\gamma}_t(s))ds = 2\pi \tag{2.1}$$

with respect to the parameter t and setting $t=0$, we get

$$\int_0^{2\pi} h(\dot{\gamma}_0(s), \dot{\gamma}_0(s))ds + \frac{\partial}{\partial t}\left[\int_0^{2\pi} g_0(\dot{\gamma}_t(s), \dot{\gamma}_t(s))ds\right]\Bigg|_{t=0} = 0, \tag{2.2}$$

where we set $h = \partial g_t/\partial t|_{t=0}$. If g_t are Zoll metrics, γ_t are all 2π-periodic, and hence the second term in (2.2) vanishes, for it is a variation of energy of 2π-periodic curves around a geodesic γ_0 parametrized by its arclength. Thus a direction $h = \partial g_t/\partial t|_{t=0}$ of a Zoll deformation g_t satisfies (1.1) for every geodesic γ_0 of g_0.

We will always regard (S^n, g_0) as a unit sphere in a Euclidean space $\boldsymbol{R}^{n+1}$. Then the set of all oriented geodesics of (S^n, g_0), i.e., great circles, is identified with an Grassmann manifold of oriented 2-planes in $\boldsymbol{R}^{n+1}$, which we denote by GeodS^n. We define a mapping $\mathscr{A}$ from $\mathscr{S}^2(S^n)$, the space of symmetric 2-forms on S^n, to $\mathscr{F}(\mathrm{Geod}S^n)$, the space of functions on GeodS^n, by

$$\mathscr{A}(h)(\gamma_0) = (1/2\pi)\int_0^{2\pi} h(\dot{\gamma}_0(s), \dot{\gamma}_0(s))ds \quad (h \in \mathscr{S}^2(S^n),\ \gamma_0 \in \mathrm{Geod}S^n).$$

The space of *IZD* is nothing but the kernel of the mapping $\mathscr{A}$.

A Lie derivative of the standard metric g_0 by a vector field X, denoted by $\mathscr{L}_X g_0$, is an integrable *IZD*, since it is a direction of a trivial Zoll deformation $\varphi_t^* g_0$, where φ_t is a family of diffeomorphisms generated by X. We call such an *IZD* *trivial* and denote the space of trivial *IZD* by $\mathscr{T}$.

Let σ be the antipodal mapping on (S^n, g_0): $\sigma(x) = -x$ $(x \in S^n \subset \boldsymbol{R}^{n+1})$. Another type of symmetric 2-forms which are easily seen to be *IZD* are those of the form $f \cdot g_0$, where f is an odd function on S^n with respect to σ. We call them *of conformal type* and denote the space of *IZD* of conformal type by $\mathscr{C}$.

We shall study what happens to the geodesics γ_t of $g_t = g_0 + t \cdot h$ for an *IZD* h, in order to see that an *IZD* is in fact worth its name. By the way, we prepare notations for the second order condition for integrability.

We take an orthonormal basis $X_1 = \dot{\gamma}_0(0), X_2, \cdots, X_n$ in T_pS^n and extend them parallel with respect to g_0 to vector fields along γ_0. They turn out to be 2π-periodic. We define a normal coordinate near γ_0 by

$$(s^1=s, s^2, \cdots, s^n) \longmapsto \exp_{\gamma_0(s)}(\textstyle\sum_{i=2}^n s^i X_i).$$

Writing down the equations of geodesics with respect to g_t in this coordinate and taking the differential at $t=0$, we get an equation of a variation vector field $\xi(s)=(\partial\gamma_t/\partial t)(s)|_{t=0}$ along γ_0:

$$\text{(2.3)} \qquad \begin{cases} d^2\xi^1/ds^2 = -\beta^1 = -(1/2)dh_{11}/ds, \\ d^2\xi^i/ds^2 + \xi^i = -\beta^i \ (i \neq 1), \end{cases}$$

where the indices mean the component with respect to our normal coordinate and we set

$$\text{(2.4)} \qquad \begin{cases} \beta^i(s) = (\partial h_{1i}/\partial s - (1/2)\partial h_{11}/\partial s^i)(s, 0, \cdots, 0), \\ h_{11}(s) = h_{11}(s, 0, \cdots, 0)(=h(\dot\gamma_0(s), \dot\gamma_0(s))). \end{cases}$$

Notice that $\{\beta^i\}$ corresponds to the variation of Levi-Civita connections.

Since every γ_t starts from the same point in the same direction, the initial condition for ξ is given by

$$\begin{cases} \xi^i(0)=0, \\ (d\xi^1/ds)(0) = -(1/2)h_{11}(0), \\ (d\xi^i/ds)(0)=0 \quad (i\neq 1). \end{cases}$$

The solution of (2.3) with this initial condition is explicitly written as

$$\begin{cases} \xi^1(s) = -(1/2)\displaystyle\int_0^s h_{11}(u)du, \\ \xi^i(s) = -\displaystyle\int_0^s \beta^i(u)\sin(s-u)du \quad (i\neq 1). \end{cases}$$

When a family of metrics g_t consists of Zoll metrics, the variation vector field ξ is 2π-periodic, and hence so is each component ξ^i. The component ξ^1 is 2π-periodic if and only if

$$\xi^1(2\pi) = -(1/2)\int_0^{2\pi} h_{11}(u)du = 0,$$

which is the same condition with (1.1). The component ξ^i $(i\neq 1)$ is 2π-periodic if and only if

$$\text{(2.5)} \qquad \begin{cases} \xi^i(2\pi) = \displaystyle\int_0^{2\pi} \beta^i(u)\sin u\, du = 0, \\ (d\xi^i/ds)(2\pi) = -\displaystyle\int_0^{2\pi} \beta^i(u)\cos u\, du = 0. \end{cases}$$

It is easily seen that the above conditions for the periodicity of ξ do not depend on the choice of a starting point p of a geodesic γ_0, but only on the symmetric 2-form h and the geodesic γ_0 as an element of GeodS^n. Now the following lemma holds.

Lemma 2.1. *Let h be an IZD. Take any geodesic γ_0 of g_0 and define β^i by* (2.4) *using a prescribed normal coordinate associated with γ_0. Then the condition* (2.5) *is satisfied.*

Proof. Suppose an *IZD h* is even with respect to σ ($\sigma^* h=h$). Then h is considered as a symmetric 2-form on a real projective space $P^n(\boldsymbol{R})$ and satisfies

$$\int_0^\pi h(\dot{\gamma}_0(s), \dot{\gamma}_0(s))\,ds=0$$

for every geodesic γ_0 of g_0. By Michel's theorem ([6]), such an *IZD h* is trivial; it is written as $\mathscr{L}_X g_0$ on $P^n(\boldsymbol{R})$, where X is a vector field on $P^n(\boldsymbol{R})$ and g_0 is considered as a metric on $P^n(\boldsymbol{R})$. Obviously h is also trivial on S^n, and therefore h satisfies (2.5) because a trivial *IZD* is integrable. On the other hand, if h is odd with respect to σ ($\sigma^* h=-h$), then β^i ($i\neq 1$) are even functions on S^1 ($\beta^i(s+\pi)=\beta^i(s)$), and hence (2.5) is satisfied by h. Since each *IZD h* is a sum of even and odd parts with respect to σ and since the condition (2.5) is linear in h, our lemma follows.

Thus, if h is an *IZD*, the variation vector field for the metric deformation $g_t=g_0+t\cdot h$ is always 2π-periodic, which implies that a family of curves γ_t (g_t-geodesics) are nearly 2π-periodic in the first order of t.

We now examine the second order condition for integrability. Differentiating (2.1) twice in t and setting $t=0$, we get a formula satisfied by $h=\partial g_t/\partial t|_{t=0}$ and $k=\partial^2 g_t/\partial t^2|_{t=0}$ when g_t is a Zoll deformation:

$$(2.6)\qquad \begin{aligned}(1/2\pi)\int_0^{2\pi} k(\dot{\gamma}_0(s), \dot{\gamma}_0(s))\,ds &= (1/4\pi)\int_0^{2\pi}\{h_{11}(s)\}^2 ds\\ &\quad -(1/\pi)\sum_{i=2}^{n}\int_0^{2\pi} ds\,\beta^i(s)\int_0^s \beta^i(u)\sin(s-u)\,du.\end{aligned}$$

We define for two *IZD* $h^{(1)}$ and $h^{(2)}$ a function $\mathscr{B}(h^{(1)}, h^{(2)})$ on GeodS^n by

$$(2.7)\qquad \begin{aligned}\mathscr{B}(h^{(1)}, h^{(2)})(\gamma_0) &= (1/4\pi)\int_0^{2\pi} h_{11}^{(1)}(s)h_{11}^{(2)}(s)\,ds\\ &\quad -(1/\pi)\sum_{i=2}^{n}\int_0^{2\pi} ds\,\beta_{(1)}^i(s)\int_0^s \beta_{(2)}^i(u)\sin(s-u)\,du\\ &\qquad (\gamma_0\in \mathrm{Geod}S^n),\end{aligned}$$

where $\beta^i_{(1)}[\beta^i_{(2)}]$ is defined by (2.4) for $h^{(1)}[h^{(2)}]$ using a normal coordinate associated with γ_0. It is easily seen from Lemma 2.1 that the right hand side of (2.7) does not depend on the choice of a starting point p of γ_0. We also notice that the bilinear mapping $\mathscr{B}$ is symmetric, i.e., $\mathscr{B}(h^{(1)}, h^{(2)}) = \mathscr{B}(h^{(2)}, h^{(1)})$.

The formula (2.6) means that, if an *IZD* h is integrable, then there exists a symmetric 2-form k such that $\mathscr{A}(k)=\mathscr{B}(h, h)$ ($\in \mathscr{F}(\mathrm{Geod}S^n)$). It is a non-trivial condition for integrability, for the mapping $\mathscr{A}$ is not surjective for $n\geqq 3$.

Proposition 2.2. *Let h be an IZD. If $\mathscr{B}(h, h)$ is not contained in* Im $\mathscr{A}$, *then it is not integrable.*

3. Our argument on integrability of *IZD* is based on the fact that (S^n, g_0) is a compact rank one symmetric space. The special orthogonal group $SO(n+1)$ acts on (S^n, g_0) transitively and isometrically, and on $\mathrm{Geod}S^n$ transitively. The actions induce $SO(n+1)$-module structures on $\mathscr{S}^2(S^n)$ and $\mathscr{F}(\mathrm{Geod}S^n)$, and they are endowed with natural $SO(n+1)$-invariant inner products. The spaces Ker $\mathscr{A}$ and Im $\mathscr{A}$ are considered as their $SO(n+1)$-submodules, since the mapping $\mathscr{A}$ is an $SO(n+1)$-homomorphism. The spaces $\mathscr{T}$ and $\mathscr{C}$ are considered as $SO(n+1)$-submodules of Ker $\mathscr{A}$. For convenience's sake, we complexify all the modules appearing above and denote them by the same symbols in the following.

The $SO(n+1)$-module structure of Ker $\mathscr{A}$, etc., has been studied in [7]. We denote by $V(\Lambda)$ an irreduicible $SO(n+1)$-module over $\boldsymbol{C}$ with the highest weight Λ; see [7] for the notation of weights.

Proposition 3.1. *Assume $n\geqq 4$.*

i) *The $SO(n+1)$-module* Ker $\mathscr{A}$ *includes densely an orthogonal sum $M_0\oplus M_1\oplus M_2$ of $SO(n+1)$-submodules, where M_i ($i=0, 1, 2$) is isomorphic to a direct sum of irreducible $SO(n+1)$-modules;*

$$M_0\cong\sum_{k=1}^{\infty} V(k\lambda_1)\oplus\sum_{k=0}^{\infty} V(k\lambda_1+(\lambda_1+\lambda_2)),$$

$$M_1\cong\sum_{k=1}^{\infty} V((2k+1)\lambda_1),$$

$$M_2\cong\sum_{k=0}^{\infty} V((2k+1)\lambda_1+2(\lambda_1+\lambda_2)).$$

ii) *M_0 is a dense submodule of $\mathscr{T}$.*

iii) *$M_0\oplus M_1$ is a dense submodule of $\mathscr{T}+\mathscr{C}$.*

Remark. i) The sum $\mathscr{T}+\mathscr{C}$ is not orthogonal nor direct. But we can still consider M_1 as a representative of $\mathscr{C}$, because integrability of an *IZD* does not change if we add an element of $\mathscr{T}$ to it.

ii) In order to get the results for $n=3$, we have only to add in the above formulas the terms with $\lambda_1+\lambda_2$ changed to $\lambda_1-\lambda_2$. In the following we always assume $n\geqq 4$, but the reasoning works equally well with small changes for $n=3$, contrary to the case $n=2$, when the terms containing λ_2 disappear.

Since $\mathscr{A}$ is a continuous $SO(n+1)$-homomorphism, Schur's lemma enables us to compute the $SO(n+1)$-irreducible decomposition of Im $\mathscr{A}$, using those of Ker $\mathscr{A}$ and $\mathscr{S}^2(S^n)$.

Proposition 3.2. *The* $SO(n+1)$*-module* Im $\mathscr{A}$ *densely includes an* $SO(n+1)$*-submodule isomorphic to the following direct sum of irreducible* $SO(n+1)$*-modules,*

$$\sum_{k=0}^{\infty} V(2k\lambda_1)\oplus\sum_{k=0}^{\infty} V(2k\lambda_1+2(\lambda_1+\lambda_2)).$$

We denote by V_k the irreducible $SO(n+1)$-submodule of M_2 that is isomorphic to $V((2k+1)\lambda_1+2(\lambda_1+\lambda_2))$. Let $\{x_1, \cdots, x_{n+1}\}$ be a Cartesian coordinate of $\boldsymbol{R}^{n+1}$ and set $z_1=x_1+\sqrt{-1}x_2$ and $z_2=x_3+\sqrt{-1}x_4$. The restriction to S^n of a ($\boldsymbol{C}$-valued) symmetric 2-form on $\boldsymbol{R}^{n+1}$,

$$(z_1)^{2k+1}((z_2)^2dz_1dz_1+(z_1)^2dz_2dz_2-z_1z_2(dz_1dz_2+dz_2dz_1)),$$

is a maximal vector in V_k, which we denote by v_k. We extend $\boldsymbol{C}$-bilinearly the mapping $\mathscr{B}$: Ker $\mathscr{A}\times$Ker $\mathscr{A}\to\mathscr{F}$ (GeodS^n).

Lemma 3.3. *A function* $\mathscr{B}(v_k, v_k)$ *on* GeodS^n *is not zero.*

Proof. Its value at a geodesic $\gamma_0(s)=(\cos s, 0, \sin s, 0, \cdots, 0)$, which is computed by (2.7), is

$$((4k+3)/4\pi)\int_0^{2\pi}(\cos s)^{2k+2}ds \quad (>0).$$

We notice that $\mathscr{B}$ is considered as a linear mapping from the symmetric tensor product of Ker $\mathscr{A}$ to $\mathscr{F}$(GeodS^n), and that as such it is an $SO(n+1)$-homomorphism. Let us observe its restriction to the symmetric tensor product of V_k, denoted by S^2V_k. The $SO(n+1)$-module S^2V_k includes the unique $SO(n+1)$-submodule V_k^2 isomorphic to $V((4k+2)\lambda_1+4(\lambda_1+\lambda_2))$, which is generated by $v_k\cdot v_k$. We denote by R the sum of other irreducible components; $S^2V_k=V_k^2\oplus R$. The image $\mathscr{B}(V_k^2)$ is not zero by Lemma 3.3, and hence the restriction of $\mathscr{B}$ to V_k^2 is an isomorphism by

Schur's lemma. Because Proposition 3.1 implies that $\operatorname{Im}\mathscr{A}$ does not include an $SO(n+1)$-submodule isomorphic to $V((4k+2)\lambda_1+4(\lambda_1+\lambda_2))$, the image $\mathscr{B}(V_k^2)$ is orthogonal to $\operatorname{Im}\mathscr{A}$. It is also orthogonal to $\mathscr{B}(R)$ by the same reason. For any non-zero element v in V_k, there exists an element a of $SO(n+1)$ such that $a\cdot v_k$ is not orthogonal to v, since V_k is irreducible. Then $v\cdot v$ is not orthogonal to $a\cdot(v_k\cdot v_k)=(a\cdot v_k)\cdot(a\cdot v_k)$, which shows that the V_k^2-component of $v\cdot v$ does not vanish. Therefore $\mathscr{B}(v, v)$, whose $\mathscr{B}(V_k^2)$-component does not vanish, is not contained in $\operatorname{Im}\mathscr{A}$.

The situation does not change if we add to v some elements in V_i $(i<k)$ in the above, since V_k^2 remains the unique $SO(n+1)$-module isomorphic to $V((4k+2)\lambda_1+4(\lambda_1+\lambda_2))$ in $S^2(\sum_{i=0}^k V_i)$. Thus we proved

Theorem 3.4. *For a non-zero element h is M_2, $\mathscr{B}(h, h)$ is not contained in* $\operatorname{Im}\mathscr{A}$. *That is, an IZD in M_2 is not integrable.*

Remark. The above argument also holds if we add some elements in $SO(n+1)$-submodules of M_1 isomorphic to $V(i\lambda_1)$ $(i\leqq k)$. This implies non-integrability of some elements in $M_1\oplus M_2$.

Unfortunately, it would be difficult to study integrability of an element in M_1 or general one in $M_1\oplus M_2$ by our method. The following would illustrate the situation.

Let W_k be the unique irreducible $SO(n+1)$-submodule of M_1 isomorphic to $V((2k+1)\lambda_1)$. Its symmetric tensor product S^2W_k decomposes as follows:

$$\begin{aligned} &S^2W_k=W_k^2\oplus W_k^{2,1}\oplus W_k^{2,2}\oplus R';\\ &W_k^2\cong V((4k+2)\lambda_1),\\ &W_k^{2,1}\cong V((4k-2)\lambda_1+2(\lambda_1+\lambda_2)),\\ &W_k^{2,2}\cong V((4k-6)\lambda_1+4(\lambda_1+\lambda_2)). \end{aligned}$$

The image $\mathscr{B}(W_k^2)$ or $\mathscr{B}(W_k^{2,1})$ may be included in $\operatorname{Im}\mathscr{A}$. It is the image $\mathscr{B}(W_k^{2,2})$ and some of the images of irreducible components of R' that cannot be included in $\operatorname{Im}\mathscr{A}$ if non-zero. The $W_k^{2,2}$-component of $v\cdot v$ vanishes for some element v in M_1, and not for another.

We are left with the problem: Determine the integrable *IZD*. Or, at least, determine the elements in $M_1\oplus M_2$ that satisfy the second order condition for integrability. The author now feels that the latter is no easier than the former.

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