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Residue Homomorphisms in Milnor K-theory

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In this paper, we give a generalization of the residue homomorphism by using Milnor's K-group [9], and study its relation with class field theory. Our residue homomorphism provides a very plain definition of the p-primary part of the reciprocity map in the local class field theory in characteristic $p>0$. This definition was used in Brylinski [4] for the study of ramifications in abelian extensions of local fields of characteristic $p>0$ and those of surfaces over finite fields. Our residue homomorphism also provides a description of the relation between the class field theory of a higher local field and that of its constant field (Section 4 Theorem 2).

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§ 0. Notations and preliminaries

Here we fix our notations and review some properties of Milnor's K-group. For a ring R, let $R^\times$ be the multiplicative group of all invertible elements of R. For a field k, let $K_q(k)$ ($q\geqq 0$) be Milnor's K-group of k defined by generators $\{x_1, \cdots, x_q\}$ ($x_1, \cdots, x_q \in k^\times$) and certain relations (cf. [9]). For a discrete valuation field k, let ord_k be the normalized additive discrete valuation of k. Let

$$O_k=\{x \in k;\ \mathrm{ord}_k(x)\geqq 0\}, \qquad m_k=\{x \in k;\ \mathrm{ord}_k(x)\geqq 1\},$$
$$U_k=\{x \in k;\ \mathrm{ord}_k(x)=0\}.$$

The residue class field of k is denoted by $\bar{k}$. For $x \in O_k$, let $\bar{x}$ be the residue class of x in $\bar{k}$. Concerning the Milnor K-group of a discrete valuation field k, for $i\geqq 1$, let $\bigoplus_{q\geqq 0} U^iK_q(k)$ be the graded ideal of $\bigoplus_{q\geqq 0}K_q(k)$ generated by elements a of $k^\times=K_1(k)$ such that $\mathrm{ord}_k(a-1)\geqq i$. Let

$$\hat{K}_q(k)=\varprojlim_i K_q(k)/U^iK_q(k),$$

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$$U^i\hat{K}_q(k)=\varprojlim_{i'} U^iK_q(k)/U^{i'}K_q(k)\subset\hat{K}_q(k)\,.$$

The following lemma is proved easily, but it will play an essential role in the definition of the residue homomorphism in Section 1.

Lemma 1. *If k is a discrete valuation field and $\hat{k}$ is its completion, the canonical homomorphism $K_q(k)\to K_q(\hat{k})$ induces isomorphisms*

$$K_q(k)/U^iK_q(k)\cong K_q(\hat{k})/U^iK_q(\hat{k}),\qquad \hat{K}_q(k)\cong\hat{K}_q(\hat{k})$$

for any $q\geqq 0$ and $i\geqq 1$.

The following "boundary" homomorphism ∂ and the norm homomorphism of Milnor's K-group will be useful tools.

If k is a discrete valuation field, there exists a unique homomorphism

$$\partial\colon K_{q+1}(k)\longrightarrow K_q(\bar{k})\quad(q\geqq 0)$$

such that

$$\partial(\{x_1,\cdots,x_q,y\})=\mathrm{ord}_k(y)\cdot\{\bar{x}_1,\cdots,\bar{x}_q\}$$

for any $x_1,\cdots,x_q\in U_k$ and $y\in k^\times$ (cf. [9]). Note that

Lemma 2. *If $x_1,\cdots,x_q\in U_k$ and $y\in K_{r+1}(k)$,*

$$\partial(\{x_1,\cdots,x_q,y\})=\{\bar{x}_1,\cdots,\bar{x}_q,\partial(y)\}\qquad in\ K_{q+r}(\bar{k}).$$

In particular, ∂ annihilates $U^1K_*(k)$.

For a field k and any finite extension k' of k, there exists a canonical norm homomorphism $N_{k'/k}\colon K_q(k')\to K_q(k)$ (cf. Bass and Tate [1] Chapter I Section 5 and Kato [7] Section 1.7). It has the following properties. For a discrete valuation v of a field (resp. for a prime ideal $\mathfrak{p}$ of a ring), let $\kappa(v)$ (resp. $\kappa(\mathfrak{p})$) be the residue field of v (resp. $\mathfrak{p}$).

Lemma 3. *Let k be a discrete valuation field and let k' be a finite extension of k. If the integral closure of O_k in k' is a finitely generated O_k-module* (cf. *Bourbaki* [3] *Chapter* VI 8.5; *this is always the case if k is complete*), *the following diagram is commutative.*

$$\begin{array}{ccc} K_{q+1}(k') & \xrightarrow{(\partial_v)_v} & \bigoplus_v K_q(\kappa(v)) \\ \downarrow{\scriptstyle N_{k'/k}} & & \downarrow{\scriptstyle (N_{\kappa(v)/\bar{k}})_v} \\ K_{q+1}(k) & \xrightarrow{\partial} & K_q(\bar{k})\,. \end{array}$$

Here v ranges over all normalized additive discrete valuations of k' such *that* $\{x \in k;\, v(x) \geqq 0\} = O_k$.

Lemma 4. *Let k be a field and K an algebraic function field in one variable over k. Let $\mathfrak{P} = \mathfrak{P}(K/k)$ be the set of all normalized additive discrete valuations v of K such that $v(k^\times) = 0$. Then,*

$$\sum_{v \in \mathfrak{P}} N_{\kappa(v)/k}(\partial_v(x)) = 0 \qquad \text{in } K_q(k), \text{ for any } x \in K_{q+1}(K).$$

Indeed, Lemma 3 is reduced to the case where k is complete, and is proved in this case in [7] Section 1.7. By Lemma 3, Lemma 4 is reduced to the case $K = k(X)$, the rational function field in one variable over k, and in this case, this summation formula is essentially the very definition of the norm homomorphism (cf. [1] Chapter I Section 5).

§ 1. The definition of the residue homomorphism

Let k be a complete discrete valuation field, and let M be the field of fractions of $O_k[[X]]$. Let $\hat{M}$ be the completion of M with respect to the discrete valuation of M defined by the prime ideal $O_k[[X]]m_k$ of height one of $O_k[[X]]$. Then, $\hat{M}$ is the field of all formal Laurent series $\sum_{n \in Z} a_n X^n$ over k such that $\mathrm{ord}_k(a_n)$ is bounded below and $\lim_{n \to -\infty} a_n = 0$. The valuation $\mathrm{ord}_{\hat{M}}$ is given by $\inf_n \{\mathrm{ord}_k(a_n)\}$, and the residue field of $\hat{M}$ is $\bar{k}((X))$.

The aim of this section is to define a homomorphism

$$\mathrm{res}\colon \hat{K}_{q+1}(\hat{M}) \longrightarrow \hat{K}_q(k)$$

called *residue*. For the relation with the usual residue of a differential, cf. Section 3.

Let $\mathfrak{S}$ be the set of all prime ideals $\mathfrak{p}$ of $O_k[[X]]$ of height one such that $\mathfrak{p} \neq O_k[[X]]m_k$. Let $\mathfrak{P} = \mathfrak{P}(k(X)/k)$ be as in Section 0 Lemma 4. Then, by Weierstrass' preparation theorem (cf. [3] Chapter VII 3.8), each element of $\mathfrak{S}$ is generated by an irreducible polynomial over O_k of the form

$$X^n + a_1 X^{n-1} + \cdots + a_n \qquad (n \geqq 1,\ a_1, \cdots, a_n \in m_k)$$

and thus we can identify $\mathfrak{S}$ with the subset of $\mathfrak{P}$ which corresponds to the set of irreducible polynomials over k of this form. Let

$$A_+ = O_k[[X]] \otimes_{O_k} k,$$
$$A_- = \{f \in k(X);\, v(f) \geqq 0 \qquad \text{if } v \in \mathfrak{P} - \mathfrak{S} - \{\infty\}\}.$$

Here ∞ denotes the unique element of $\mathfrak{P}$ such that $\infty(X) = -1$.

When we consider the analogy with the analytic theory over the field of complex numbers $\boldsymbol{C}$, $\mathfrak{S}$ corresponds to the open disk $D^\circ=\{z \in \boldsymbol{C}; |z|<1\}$, A_+ corresponds to the ring of all holomorphic functions on D°, A_- corresponds to the ring of all rational functions on $\boldsymbol{C}$ whose poles (in $\boldsymbol{C}$) are concentrated in D°, and M corresponds to the ring of all meromorphic functions on D° which has only finite number of poles. Just as in the complex case, we have exact sequences

$$0 \longrightarrow k[X] \longrightarrow A_+ \oplus A_- \longrightarrow M \longrightarrow 0$$
$$0 \longrightarrow k^\times \longrightarrow (A_+)^\times \oplus (A_-)^\times \longrightarrow M^\times \longrightarrow 0.$$

The following Proposition 1 generalizes this latter sequence to Milnor's K-groups.

Let $\oplus_{q\geqq 0} K_q(A_+)$ (resp. $\oplus_{q\geqq 0} K_q(A_-)$) be the sub-graded ring of $\oplus_{q\geqq 0} K_q(M)$ generated over $\boldsymbol{Z}=K_0(M)$ by $(A_+)^\times \subset K_1(M)$ (resp. $(A_-)^\times \subset K_1(M)$). For $i\geqq 1$, let $\oplus_{q\geqq 0} U^iK_q(A_+)$ (resp. $\oplus_{q\geqq 0} U^iK_q(A_-)$) be the graded ideal of $\oplus_{q\geqq 0} K_q(A_+)$ (resp. $\oplus_{q\geqq 0} K_q(A_-)$) generated by elements $f \in (A_+)^\times$ (resp. $(A_-)^\times$) such that $\operatorname{ord}_{\mathfrak{M}}(f-1)\geqq i$.

Proposition 1. *For any q, the canonical map $K_q(k)\to K_q(M)$ is injective. If we regard $K_q(k)$ as a subgroup of $K_q(M)$, we have*

$$K_q(M)=K_q(A_+)+K_q(A_-), \qquad K_q(k)=K_q(A_+)\cap K_q(A_-),$$
$$U^iK_q(M)=U^iK_q(A_+)+U^iK_q(A_-), \qquad U^iK_q(k)=U^iK_q(A_+)\cap U^iK_q(A_-)$$

for any $q\geqq 0$ and $i\geqq 1$, where $U^iK_q(M)$ is defined with respect to the discrete valuation of M induced by $\operatorname{ord}_{\mathfrak{M}}$.

Now let res: $K_{q+1}(M)\to K_q(k)$ be the composite map

$$K_{q+1}(M) \xrightarrow{(\partial_{\mathfrak{p}})_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}\in\mathfrak{S}} K_q(\kappa(\mathfrak{p})) \xrightarrow{(N_{\kappa(\mathfrak{p})/k})_{\mathfrak{p}}} K_q(k).$$

Using the above proposition, we can prove;

Theorem 1. *The homomorphism* res: $K_{q+1}(M)\to K_q(k)$ *satisfies*

$$\operatorname{res}(U^iK_{q+1}(M))\subset U^iK_q(k) \qquad \text{for any } i\geqq 1,$$

and hence induces a homomorphism

$$\operatorname{res}\colon \hat{K}_{q+1}(\hat{M})\cong \hat{K}_{q+1}(M) \longrightarrow \hat{K}_q(k)$$

(cf. *Section* 0 *Lemma* 1). *The latter homomorphism is the unique continuous homomorphism* (*with respect to the topology defined by the filtrations* U^i) *which annihilates the image of $K_{q+1}(A_+)$ and elements of the form*

$$\{1+a_1X^{-1}+\cdots+a_nX^{-n}, y\} \quad (a_1, \cdots, a_n \in m_k,\ y \in K_q(A_-)),$$

and for which the composite

$$\hat{K}_q(k) \xrightarrow{\{\ ,X\}} \hat{K}_{q+1}(\hat{M}) \xrightarrow{\text{res}} \hat{K}_q(k)$$

coincides with the identity map.

Proof of Proposition 1. As is easily seen, res: $K_{q+1}(M) \to K_q(k)$ gives a left inverse of $K_q(k) \xrightarrow{\{\ ,X\}} K_{q+1}(M)$, and

$$\text{res}(\{U^iK_q(A_+), X\}) = U^iK_q(k) \qquad \text{for } i \geqq 1.$$

It follows that the canonical homomorphism $K_q(k) \to K_q(M)$ is injective and that

$$(\#) \qquad U^iK_q(A_+) \cap K_q(k) = U^iK_q(k).$$

Let $\oplus_{q\geqq 0} K'_q(A_-)$ be the subgraded ring of $\oplus_{q\geqq 0} K_q(k(X))$ generated over $Z = K_0(k(X))$ by $(A_-)^\times \subset K_1(k(X))$. There is clearly a surjection $K'_q(A_-) \to K_q(A_-)$. By [9], the sequence

$$(\#\#) \qquad 0 \longrightarrow K_q(k) \longrightarrow K_q(k(X)) \xrightarrow{(\partial_v)_v} \bigoplus_{v \in \mathfrak{P}-\{\infty\}} K_{q-1}(k(v)) \longrightarrow 0$$

is exact. Hence we have

$$K_q(k) \xrightarrow{\cong} \text{Ker}\,(K'_q(A_-) \longrightarrow \bigoplus_{v \in \mathfrak{S}} K_{q-1}(\kappa(v))).$$

This proves $K_q(A_+) \cap K_q(A_-) = K_q(k)$. By the above (#), we have $U^iK_q(A_+) \cap U^iK_q(A_-) = U^iK_q(k)$.

It remains to prove $K_q(M) = K_q(A_+) + K_q(A_-)$ and $U^iK_q(M) = U^iK_q(A_+) + U^iK_q(A_-)$. In the case $q=1$, these are deduced from Weierstrass' preparation theorem applied to the rings $(O_k/m_k^i)[[X]]$ $(i \geqq 1)$. For $q \geqq 2$, it is clearly sufficient to prove

$$\{K_1(A_+), K_1(A_-)\} \subset K_2(A_+) + K_2(A_-),$$
$$\{U^iK_1(A_+), K_1(A_-)\} \subset U^iK_2(A_+) + U^iK_2(A_-),$$
$$\{U^iK_1(A_-), K_1(A_+)\} \subset U^iK_2(A_+) + U^iK_2(A_-).$$

Let S be the set of all polynomials $f(X)$ over O_k such that $f=1$ or $f = X^n + a_1X^{n-1} + \cdots + a_n$ for some $n \geqq 1$ and $a_1, \cdots, a_n \in m_k$. Then, any element of $(A_-)^\times$ (resp. $U^iK_1(A_-)$) is written in the form $ch_1h_2^{-1}$ such that $c \in k^\times$, $h_1, h_2 \in S$ (resp. $c \in U^iK_1(k)$, $h_1, h_2 \in S$ and $h_1 \equiv h_2 \bmod O_k[X]m_k^i$).

On the other hand, if $h \in S$ and $h \neq 1$, $(A_+)^\times$ (resp. $U^iK_1(A_+)$) is generated by $k^\times$ (resp. $U^iK_1(k)$), elements of the form $1-fh$ such that $f \in O_k[[X]]$ (resp. $f \in O_k[[X]]m_k^i$), and elements of the form $1-Xf$ such that $f \in O_k[X]$ (resp. $f \in O_k[X]m_k^i$). Hence we are reduced to

Lemma 5. (1) *Let $f \in O_k[[X]]$, h_1, $h_2 \in S$, and let $i \geq 1$. Unless $f=h_1=h_2=1$, $\{1-fh_1h_2, h_1h_2^{-1}\}$ belongs to $K_2(A_+)$. If $f \equiv 0 \bmod O_k[[X]]m_k^i$ or if $h_1 \equiv h_2 \bmod O_k[X]m_k^i$, it belongs to $U^iK_2(A_+)$.*

(2) *Let $f \in O_k[X]$, h_1, $h_2 \in S$. Then, $\{1-Xf, h_1h_2^{-1}\}$ belongs to $K_2(A_+)+K_2(A_-)$. If $f \equiv 0 \bmod O_k[X]m_k^i$ or if $h_1 \equiv h_2 \bmod O_k[X]m_k^i$, it belongs to $U^iK_2(A_+)+U^iK_2(A_-)$.*

The following lemma is useful for the proof.

Lemma 6. *Let k be a field and let x, $y \in k$. If $x \neq 0, 1$, $y \neq 1$, and $xy \neq 1$, then*

$$\{1-x, 1-y\}=\{1-xy, -x\}+\{1-xy, 1-y\}-\{1-xy, 1-x\}.$$

Proof.

$$\begin{aligned}\{1-x, 1-y\}&=\{1-x, x(1-y)\}=\{1-x, -(1-x)+(1-xy)\}\\&=\{1-x, 1-(1-x)^{-1}(1-xy)\}=\{1-xy, 1-(1-x)^{-1}(1-xy)\}.\end{aligned}$$

Proof of Lemma 5. First, (1) follows from the equation

$$\{1-xy, -x\}=\{1-x, 1-y\}+\{1-xy, 1-x\}-\{1-xy, 1-y\}$$

of Lemma 6 applied to the case $x=h_1$ and $y=fh_2$, and to the case $x=h_2$ and $y=fh_1$.

Next, we prove (2) by induction on $(\deg(f), \max(\deg(h_1), \deg(h_2))) \in \boldsymbol{N}\times\boldsymbol{N}$, where we endow $\boldsymbol{N}\times\boldsymbol{N}$ with the lexicographic order. Note that any non-zero element of $O_k[X]$ is uniquely written in the form $c(1-Xf)h$ such that $c \in O_k-\{0\}$, $f \in O_k[X]$ and $h \in S$. For $f \in O_k[X]$ and $h \in S-\{1\}$, define $c_{f,h} \in U^1K_1(k)$, $f_h \in O_k[X]$ and $h_f \in S$ by

$$X^{n-1}-(X^n-h)f=c_{f,h}(1-Xf_h)h_f \qquad \text{where } n=\deg(h).$$

Then, $\deg(f_h)<\deg(f)$ if $f \neq 0$, and $\deg(h_f)=\deg(h)-1$. By Lemma 6 applied to the case $x=Xf$ and $y=1-X^{-n}h$, we have the following formula for $f \neq 0$ ($n=\deg(h)$).

$$\{1-Xf, h\}=n\{1-Xf, X\}+\{c_{f,h}(1-Xf_h)h_fX^{1-n}, -fhX^{1-n}(1-Xf)^{-1}\}.$$

If $f \equiv 0 \bmod O_k[X]m_k^i$ (resp. if h_1, $h_2 \in S$ and $h_1 \equiv h_2 \bmod O_k[X]m_k^i$), the preparation theorem for (O_k/m_k^i) $[[X]]$ shows

$$c_{f,h}\equiv 1,\quad f_h\equiv 0,\quad \text{and}\quad h_f\equiv X^{n-1}\mod O_k[X]m_k^i$$
$$(\text{resp. } c_{f,h_1}\equiv c_{f,h_2}, f_{h_1}\equiv f_{h_2}, \text{ and } (h_1)_f\equiv (h_2)_f \mod O_k[X]m_k^i).$$

Hence, by the above formula and by the hypothesis of our induction, we are reduced to the case $h_1=X$ and $h_2=1$ of Lemma 5 (1).

Proof of Theorem 1. By Proposition 1, $U^iK_{q+1}(M)$ is contained in the subgroup of $K_{q+1}(M)$ generated by $K_{q+1}(A_+)$, $\{U^iK_q(k), X\}$ and by elements z of the form $\{1+a_1X^{-1}+\cdots+a_nX^{-n}, y\}$ such that $n\geqq 0$, $a_1,\cdots,a_n\in m_k$, and $y\in K_q(A_-)$. But res: $K_{q+1}(M)\to K_q(k)$ satisfies $\mathrm{res}(K_{q+1}(A_+))=0$, $\mathrm{res}(\{U^iK_q(k), X\})=U^iK_q(k)$, and

$$\mathrm{res}(z)=-\sum_{v\in\mathfrak{P}-\mathfrak{S}} N_{\kappa(v)/k}(\partial(z))=0$$

by Lemma 4 and Lemma 2.

Remark 1. Each homomorphism $N_{\kappa(\mathfrak{p})/k}\circ\partial_{\mathfrak{p}}: K_{q+1}(M)\to K_q(k)$ $(\mathfrak{p}\in\mathfrak{S})$ is not continuous for the filtrations U^i. It is only the sum $\sum_{\mathfrak{p}\in\mathfrak{S}} N_{\kappa(\mathfrak{p})/k}\circ\partial_{\mathfrak{p}}$ that can be extended to $\hat{K}_{q+1}(\hat{M})\to\hat{K}_q(k)$.

The following result is deduced from Theorem 1, and implies that the norm homomorphism in Milnor K-theory is continuous for complete discrete valuation fields.

Proposition 2. *Let k be a complete discrete valuation field and k' a finite extension of k. Then,*

$$N_{k'/k}(U^{ie_{k'/k}}K_q(k'))\subset U^iK_q(k)\qquad \textit{for any } i\geqq 1,$$

where $e_{k'/k}$ is the index of the ramification of k'/k.

Let $c\in m_k^i$ $(i\geqq 1)$ and $a\in O_{k'}$. It is sufficient to prove $N_{k'/k}(1+ca, K_{q-1}(k')\})\subset U^iK_q(k)$ assuming $k'=k(a)$ and $a\neq 0$. Let v_0 be the element of $\mathfrak{P}=\mathfrak{P}(k(X)/k)$ corresponding to the irreducible polynomial of a^{-1} over k. Then, $a\in O_{k'}$ implies $v_0\notin\mathfrak{S}$. By the exactness of the sequence (##) in the proof of Proposition 2, for each $y\in K_{q-1}(k')=K_{q-1}(\kappa(v_0))$, there is an element $\tilde{y}$ of $K_q(k(X))$ such that

$$\partial_v(\tilde{y})=0\quad \text{if } v\in\mathfrak{P}-\{v_0\}-\{\infty\}, \text{ and}\qquad \partial_{v_0}(\tilde{y})=y.$$

We have

$$\begin{aligned}N_{k'/k}(\{1+ca, y\})&=N_{\kappa(v_0)/k}\circ\partial_{v_0}(\{1+cX^{-1}, \tilde{y}\})\\ &=-\mathrm{res}(\{1+cX^{-1}, \tilde{y}\})-\sum_{v\in\mathfrak{P}-\mathfrak{S}-\{v_0\}} N_{\kappa(v)/k}\circ\partial_v(\{1+cX^{-1}, \tilde{y}\})\end{aligned}$$

by Lemma 4. Since $\{1+cX^{-1}, \tilde{y}\} \in U^i K_{q+1}(M)$, the first term belongs to $U^i K_q(k)$ by Theorem 1. But the second term is zero by Lemma 2.

§ 2. The rigidity of the residue homomorphism

In Section 1, we defined the residue homomorphism $\hat{K}_{q+1}(\hat{M}) \to \hat{K}_q(k)$ using the variable X, but if the characteristic $\mathrm{ch}(\bar{k})$ of the residue field $\bar{k}$ of k is not zero, we can show that it is in fact independent of the choice of the "coordinates".

Let k and L be complete discrete valuation fields such that

(1) k is a subfield of L satisfying $O_k \subset O_L$ and $m_k \subset m_L$.

(2) The residue field $\bar{L}$ of L is a complete discrete valuation field such that $\bar{k} \subset O_{\bar{L}}$ and such that its residue field $\bar{\bar{L}}$ is of finite degree over $\bar{k}$.

A standard example of the pair (k, L) is the pair $(k, \hat{M})$ of Section 1.

We shall show that in the case $\mathrm{ch}(\bar{k}) \neq 0$, these data define a *canonical* homomorphism

$$\mathrm{res}_{L/k}\colon \hat{K}_{q+1}(L) \longrightarrow \hat{K}_q(k).$$

Assume $\mathrm{ch}(\bar{k}) \neq 0$, and let P be the set of all elements x of U_L such that $\bar{x} \in m_{\bar{L}}$. For each $x \in P$, let $\varphi_x\colon O_k[[X]] \to O_L$ be the unique homomorphism over O_k such that $\varphi_x(X) = x$ and such that the induced map $\bar{k}[[X]] \to \bar{L}$ is $\sum_{n \geqq 0} a_n X^n \mapsto \sum_{n \geqq 0} a_n x^n$ $(a_n \in \bar{k})$. The existence and the uniqueness of φ_x follows from the fact that $O_k[[X]]$ is formally étale over $O_k[X]$ with respect to the $O_k[X]m_k$-adic topology (Grothendieck [6] Chapter 0 Section 19) in the case $\mathrm{ch}(\bar{k}) \neq 0$. Let M and $\hat{M}$ be as in Section 1. Then, φ_x induces a homomorphism $\hat{M} \to L$, which we denote also by φ_x, and L becomes a finite extension of $\varphi_x(\hat{M})$. Let $\mathrm{res}_{x,L/k}$ (or simply res_x) be the composite

$$\hat{K}_{q+1}(L) \xrightarrow{\text{norm}} \hat{K}_{q+1}(\varphi_x(\hat{M})) \xrightarrow{\varphi_x^{-1}} \hat{K}_{q+1}(\hat{M}) \xrightarrow{\text{res}} \hat{K}_q(k).$$

Here the first arrow is defined by Section 1 Proposition 2.

Proposition 3. *The homomorphism* $\mathrm{res}_{x,L/k}$ *is independent of* $x \in P$.

Corollary 1. *Let α be an automorphism of $\hat{M}$ over O_k such that* $\mathrm{ord}_{\hat{M}} \circ \sigma = \mathrm{ord}_{\hat{M}}$ *and such that the induced map on the residue field $\bar{k}((X))$ also preserves the valuation. Then, if* $\mathrm{ch}(k) \neq 0$,

$$\mathrm{res} \circ \sigma = \mathrm{res}\,;\ \hat{K}_{q+1}(\hat{M}) \longrightarrow \hat{K}_q(k).$$

To prove this proposition, we need define similar homomorphisms res_A and $\mathrm{res}_{A,L/k}$. Let k be a complete discrete valuation field and let A be a ring over O_k such that

(3) A is a Noetherian normal complete local ring of dimension 2.

(4) A has only one prime ideal which is of height one and contains m_k.

A standard example of the pair (k, A) is $(k, O_k[[X]])$. Let M_A be the field of fractions of A, $\hat{M}_A$ the completion of M_A with respect to the discrete valuation defined by the above prime ideal, and $\mathfrak{S}_A$ the set of all prime ideals $\mathfrak{p}$ of A of height one such that $\mathfrak{p} \not\ni m_k$. Then,

Lemma 7. *There is a unique continuous homomorphism*

$$\mathrm{res}_A\colon \hat{K}_{q+1}(\hat{M}_A) \longrightarrow \hat{K}_q(k)$$

such that the induced composite

$$K_{q+1}(M_A) \longrightarrow \hat{K}_{q+1}(\hat{M}_A) \xrightarrow{\mathrm{res}_A} \hat{K}_q(k)$$

coincides with $\sum_{\mathfrak{p} \in \mathfrak{S}_A} N_{\kappa(\mathfrak{p})/k} \circ \partial_{\mathfrak{p}}$.

Proof. We can regard A as a finite extension of $O_k[[X]]$. By applying Section 0 Lemma 3 to the extension M_A/M, we see that the desired homomorphism is the composite

$$\hat{K}_{q+1}(M_A) \xrightarrow{\mathrm{norm}} \hat{K}_{q+1}(M) \xrightarrow{\mathrm{res}} \hat{K}_q(k).$$

Now let k and L be complete discrete valuation fields satisfying (1), (2), and let A be a subring of O_L containing O_k which satisfies (3) (4) and the following condition.

(5) Let $h\colon A \to O_L/m_L = \bar{L}$ be the canonical map. Then, $h(A) \subset O_{\bar{L}}$ and $h(m_A) \subset m_{\bar{L}}$, where m_A is the maximal ideal of A.

We denote by $\mathrm{res}_{A,L/k}$ (or simply by res_A) the composite

$$\hat{K}_{q+1}(L) \xrightarrow{\mathrm{norm}} \hat{K}_q(\hat{M}_A) \xrightarrow{\mathrm{res}_A} \hat{K}_q(k)$$

regarding L as a finite extension of $\hat{M}_A$.

Now we prove Proposition 3. Assume $\mathrm{ch}(\bar{k}) = p > 0$. From the above construction of res_A, we have,

Claim 1. $\mathrm{res}_A = \mathrm{res}_x\colon \hat{K}_{q+1}(L) \to \hat{K}_q(k)$ for any $x \in A \cap P$. In particular, $\mathrm{res}_x = \mathrm{res}_{x^n}$ for any $x \in P$ and $n \geqq 1$.

Claim 2. Fix $x \in P$, and assume that L is separable over $\varphi_x(\hat{M})$. Then, for each $i \geqq 1$, there exists an integer $n \geqq 1$ such that if $y \in P$ and $y \equiv x \bmod m_L^n$, then the composite

$$\hat{K}_{q+1}(L) \xrightarrow{\mathrm{res}_x - \mathrm{res}_y} \hat{K}_q(k) \longrightarrow \hat{K}_q(k)/U^i \hat{K}_q(k)$$

is zero for any $q \geqq 1$.

Proof. Let α be an element of O_L such that $L=\varphi_x(\hat{M})(\alpha)$, and let $a_1, \cdots, a_m$ be elements of $O_{\hat{M}}$ such that $X^m+\varphi_x(a_1)X^{m-1}+\cdots+\varphi_x(a_m)$ is the irreducible polynomial of α over $\varphi_x(\hat{M})$. If y is sufficiently near to x with respect to the valuation of L, $\varphi_y(a_j)$ $(1 \leqq j \leqq m)$ becomes sufficiently near to $\varphi_x(a_j)$. Since α is separable over $\varphi_x(\hat{M})$, the equation $X^m+\varphi_y(a_1)X^{m-1}+\cdots+\varphi_y(a_m)$ becomes having a solution β in L which is sufficiently near to α. We can define a homomorphism $\tau: L \to L$ such that $\tau \circ \varphi_x=\varphi_y: \hat{M} \to L$ and $\tau(\alpha)=\beta$. If y is sufficiently near to x, τ satisfies $\tau(b)b^{-1} \in U^{(ie_{L/k})}K_1(L)$ for all $b \in L^{\times}$, where $e_{L/k}$ is the integer such that $m_k O_L=m_L^{e_{L/k}}$. This implies $\tau(b)-b \in U^{(ie_{L/k})}K_{q+1}(L)$ for any q and $b \in K_{q+1}(L)$. We have, for any $b \in K_{q+1}(L)$,

$$\mathrm{res}_x(b)=\mathrm{res}_y(\tau(b))=\mathrm{res}_y(\tau(b)-b)+\mathrm{res}_y(b) \in U^i\hat{K}_q(k)+\mathrm{res}_y(b)$$

by Theorem 1 and Proposition 2.

Claim 3. Assume that $x \in P$ and L is separable over $\varphi_x(\hat{M})$. If A is a subring of O_L containing O_k which satisfies the conditions (3) (4) (5) and the condition $\hat{M}_A=L$, we have $\mathrm{res}_A=\mathrm{res}_x$.

Proof. It suffices to prove $\mathrm{res}_A \equiv \mathrm{res}_x \bmod U^i\hat{K}_q(k)$ for any $i \geqq 1$. Choose $n \geqq 1$ in Claim 2 for a fixed i. As is easily seen, for sufficiently large $N \geqq 1$, we have $x^N \in A+m_L^n$. Take N prime to p, and write $x^N=a+b$, $a \in A$, $b \in m_L^n$. Then, there exists an element y of O_L such that $y^N=a$ and $y \equiv x \bmod m_L^n$. But $\mathrm{res}_A=\mathrm{res}_a=\mathrm{res}_y$ by Claim 1, and $\mathrm{res}_x \equiv \mathrm{res}_y \bmod U^i\hat{K}_q(k)$ by Claim 2.

Now we can finish the proof of Proposition 3. Let $x, y \in P$. Let L' be the separable closure of $\varphi_x(\hat{M})$ in L and let $[L: L']=p^c$. Let $y'=y^{p^c}$. Then, $y' \in L'$ and hence $\varphi_{y'}(\hat{M}) \subset L'$. Let L'' be the separable closure of $\varphi_{y'}(\hat{M})$ in L', and let $[L': L'']=p^d$. Let $x'=x^{p^d}$. Then, $\varphi_{x'}(\hat{M}) \subset L''$. Since $[\varphi_x(\hat{M}): \varphi_{x'}(\hat{M})]=p^d$, L'' is separable over $\varphi_{x'}(\hat{M})$. Thus we may assume that L is separable over both $\varphi_x(\hat{M})$ and $\varphi_y(\hat{M})$.

Let α be an element of L generating L over $\varphi_y(\hat{M})$. If a monic polynomial f over $\varphi_y(M)$ is sufficiently near to the monic irreducible polynomial of α over $\varphi_y(\hat{M})$, f has a root β in L and thus we have

$$\varphi_y(M)(\beta) \otimes_{\varphi_y(M)} \varphi_y(\hat{M})=L.$$

Let A be the integral closure of $\varphi_y(O_k[[X]])$ in $\varphi_y(M)(\beta)$. Then, A satisfies the conditions (3) (4) (5) and $\hat{M}_A=L$. We have $\mathrm{res}_y=\mathrm{res}_A=\mathrm{res}_x$ by Claim 1 and Claim 3.

Since we have proved Proposition 3, we denote $\mathrm{res}_{x,L/k}$ $(x \in P)$ by $\mathrm{res}_{L/k}$. It has the following properties.

Corollary 2. $\mathrm{res}_{L/k}$: $\oplus_{q\geq 1} \hat{K}_q(L) \to \oplus_{q\geq 0} \hat{K}_q(k)$ *is a homomorphism of left* $\oplus_{q\geq 0} \hat{K}_q(k)$ *-modules. The homomorphism* $\mathrm{res}_{L/k}$: $\hat{K}_1(L) = L^\times \to \hat{K}_0(k) = \boldsymbol{Z}$ *is the unique homomorphism such that* $\mathrm{res}_{L/k}(k^\times)=0$ *and such that*

$$\mathrm{res}_{L/k}(x) = e_{L/k}[\bar{\bar{L}}: \bar{k}]\ \mathrm{ord}_{\bar{L}}(\bar{x})$$

for any $x \in U_L$. *Here* $e_{L/k}$ *is defined by* $m_k O_L = m_L^{e_{L/k}}$.

Remark. We can show that if $\mathrm{ch}(\bar{k})=0$, res_A: $\hat{K}_{q+1}(L) \to \hat{K}_q(k)$ actually depends on the choice of A, and Corollary 1 to Proposition 3 does not hold any more. Indeed, suppose $\mathrm{ch}(\bar{k})=0$, and let f be an element of $O_k[[X]]$ such that f mod $O_k[[X]]m_k$ is transcendental over $\bar{k}(X)$. Then, for any $g \in m_{\hat{M}}$, there is an O_k-automorphism σ of $\hat{M}$ which satisfies the assumption of Corollary 1 such that $\sigma(X)=X$ and $\sigma(f)=f+g$. Then,

$$\begin{aligned}&\mathrm{res}_{\sigma^{-1}(O_k[[X]])}(\{f, X\}) - \mathrm{res}_{O_k[[X]]}(\{f, X\})\\ &\quad = \mathrm{res} \circ \sigma(\{f, X\}) - \mathrm{res}(\{f, X\}) = \mathrm{res}(\{1+gf^{-1}, X\})\end{aligned}$$

need not be zero.

§ 3. The relation with the residues of differentials

For a field F, let $\oplus_{q\geq 0} \Omega_F^q$ be the exterior algebra of the F-module $\Omega_{F/Z}^1$ of absolute differentials. For $q, i \geq 1$, there exists a well defined homomorphism (cf. Bloch [2])

$$\rho_i\colon \Omega_F^{q-1} \longrightarrow U^i K_q(F((T)))/U^{i+1}K_q(F((T)))$$

$$\rho_i\left(x\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{q-1}}{y_{q-1}}\right) = \{1+xT^i, y_1, \cdots, y_{q-1}\}.$$

Let K be a complete discrete valuation field such that $F \subset O_K$ and such that O_K/m_K is of finite degree over F. By Section 2 Lemma 7 applied to the case $k=F((T))$ and $A=O_K[[T]]$, there is a unique homomorphism

$$\mathrm{res}\colon \hat{K}_{q+1}(K((T))) \longrightarrow \hat{K}_q(F((T)))$$

such that the composite

$$K_{q+1}(M) \to \hat{K}_{q+1}(K((T))) \to \hat{K}_q(F((T))) \quad \text{is} \quad \sum_{\mathfrak{p} \in \mathfrak{S}} N_{\kappa(\mathfrak{p})/F((T))} \circ \partial_{\mathfrak{p}},$$

where $\hat{K}_*$ are defined for the T-adic valuations, M is the field of fractions of $O_K[[T]]$, and $\mathfrak{S}$ is the set of all prime ideals of $O_K[[T]]$ of height one distinct from (T). As is easily seen, the diagram

$$\begin{tikzcd} \Omega^1_K \arrow[r] \arrow[d, "\rho_i"'] & F \arrow[d, "\rho_i", "\wr"'] \\ U^i\hat{K}_2(K((T)))/U^{i+1}\hat{K}_2(K((T))) \arrow[r, "\mathrm{res}"] & U^i\hat{K}_1(F((T)))/U^{i+1}\hat{K}_1(F((T))) \end{tikzcd}$$

commutes, where the upper horizontal arrow is the usual residue homomorphism.

Next, let C be a proper regular irreducible curve over F, and K the function field of C. For each $v \in \mathfrak{P}(K/F)$ (cf. Section 0 Lemma 4), let K_v be the completion of K at v. As above, we have the residue homomorphism res_v: $\hat{K}_{q+1}(K_v((T))) \longrightarrow \hat{K}_q(F((T)))$. We can prove the residue formula:

Proposition 4. *For each $y \in \hat{K}_{q+1}(K((T)))$, the infinite sum $\sum_{v \in \mathfrak{P}(K/F)} \mathrm{res}_v(y)$ converges to zero in the topological group $\hat{K}_q(F((T)))$.*

Proof. Let $\mathfrak{X} = C \otimes_F F[[T]]$, $X = C \otimes_F F((T))$, and let J be the function field of X. Since J is dense in $K((T))$, it is sufficient to prove that for each $y \in K_{q+1}(J)$, $\mathrm{res}_v(y) = 0$ for almost all v, and $\sum_{v \in \mathfrak{P}(K/F)} \mathrm{res}_v(y) = 0$.

Each $w \in \mathfrak{P}(J/F((T)))$ corresponds to a closed points of X and hence to a closed integral subscheme $\mathfrak{X}(w)$ of $\mathfrak{X}$ of codimension one. Since $\mathfrak{X}$ is proper over $F[[T]]$, $\mathfrak{X}(w)$ is finite over $F[[T]]$, and hence contains only one closed point $\bar{w}$. Since the closed points of $\mathfrak{X}$ are contained in $C = \mathfrak{X} \otimes_{F[[T]]} F$, we have a map

$$s;\ \mathfrak{P}(J/F((T))) \longrightarrow \mathfrak{P}(K/F);\ w \longmapsto \bar{w}.$$

On the other hand, for each $v \in \mathfrak{P}(K/F)$,

$$\mathrm{res}_v = \sum_{w \in s^{-1}(v)} N_{\kappa(w)/F((T))} \circ \partial_w :\ K_{q+1}(J) \longrightarrow \hat{K}_q(F((T))).$$

Thus,

$$\sum_{v \in \mathfrak{P}(K/F)} \mathrm{res}_v = \sum_{w \in \mathfrak{P}(J/F((T)))} N_{\kappa(w)/F((T))} \circ \partial_w = 0 \qquad \text{on } K_{q+1}(J)$$

by Section 0 Lemma 4.

§ 4. Relations with local class field theory

A field K is called a local field of dimension n, if a sequence of fields $k_0, \cdots, k_n$ is given satisfying the following conditions: (i) k_0 is a finite field, (ii) for $i = 1, \cdots, n$, k_i is a complete discrete valuation field with residue field k_{i-1}, and (iii) $k_n = K$. Let K^{ab} be the maximum abelian extension of an n-dimensional local field K. Then, as in [7] [8], there is a canonical homomorphism

$$\Psi_K: K_n(K) \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K),$$

which generalizes the reciprocity map in the usual local class field theory.

In this Section 4, we first show that in the case $\mathrm{ch}(K) = p > 0$, the p-primary part of this reciprocity map is given by using the residue homomorphism of this paper. We shall next show that the residue homomorphism describes the relation of the class field theories of two higher local fields k and L which satisfy the conditions (1) (2) of Section 2 (cf. Theorem 2).

First, assume that K is an n-dimensional local field of characteristic $p > 0$, and take a homomorphism $s_i: k_{i-1} \to O_{k_i}$ for each i, such that the composite $k_{i-1} \to O_{k_i}/m_{k_i} = k_{i-1}$ is the identity map. By Section 3, the embeddings $s_i: k_{i-1} \overset{\subset}{\to} O_{k_i}$ $(1 \leqq i \leqq n)$ give residue homomorphisms

$$\hat{K}_{*+1}(k_i((T))) \longrightarrow \hat{K}_*(k_{i-1}((T))).$$

We denote by Res_K the composite

$$\hat{K}_{n+1}(K((T))) \xrightarrow{\mathrm{res}} \hat{K}_{n+1}(k_{n-1}((T))) \xrightarrow{\mathrm{res}} \cdots \xrightarrow{\mathrm{res}} \hat{K}_1(k_0((T)))$$
$$\xrightarrow{\mathrm{norm}} \hat{K}_1(\boldsymbol{F}_p((T))) \quad (\boldsymbol{F}_p = \boldsymbol{Z}/p\boldsymbol{Z}).$$

On the other hand, for a field k of characteristic $p > 0$, let $W(k)$ (resp. $W_r(k)$) be the ring of p-Witt vectors over k of infinite length (resp. of length r). Then, $W(k)$ is embedded as a topological group in $k((T))^\times$ by the Artin-Hasse exponential

$$W(k) \longrightarrow k((T))^\times;\ (a_0, a_1, \cdots) \longmapsto \prod_{i \geqq 0} E_i(a_i)$$

where $E_i(x) = \prod_{(n,p)=1} (1 - (xT^{p^i})^n)^{-\mu(n)/n}$ ($\mu(n)$ is the Möbius' function). For $q, r \geqq 1$, let $T_q^{(r)}(k)$ be the closure in $\hat{K}_q(k((T)))$ of the subgroup generated by elements of the forms

$$\{E_i(x), y_1, \cdots, y_{q-1}\}, \qquad \{E_i(x), y_1, \cdots, y_{q-2}, T\}$$

such that $i \geqq r$, $x \in k$ and $y_1, \cdots, y_{q-1} \in k^\times$. In particular, $T_1^{(0)}(k) \cong W(k)$ and $T_1^{(0)}(k)/T_1^{(r)}(k) \cong W_r(k)$.

Proposition 5. (1) *Let F be a field of characteristic $p > 0$, and let K be a complete discrete valuation field such that $F \subset O_K$ and such that O_K/m_K*

is of finite degree over F. Then,

$$\mathrm{res}\,(T^{(r)}_{q+1}(K))\subset T^{(r)}_{q}(F) \textit{ for any } q \textit{ and } r.$$

(2) *Let K be a local field of dimension n of characteristic* $p>0$, *and let* Res_K: $\hat{K}_{n+1}(K((T)))\to\hat{K}_1(\boldsymbol{F}_p((T)))$ *be the homomorphism defined above. Then we have isomorphisms*

$$\begin{array}{ccc}
U^1\hat{K}_{n+1}(K((T)))/(F-1)U^1\hat{K}_{n+1}(K((T))) & \xrightarrow[\cong]{\mathrm{Res}_K} & U^i\hat{K}_1(\boldsymbol{F}_p((T))) \\
\cup\uparrow & & \cup\uparrow \\
T^{(0)}_{n+1}(K)/(F-1)T^{(0)}_{n+1}(K) & \xrightarrow[\cong]{\text{by } \mathrm{Res}_K} & W(\boldsymbol{F}_p)=\boldsymbol{Z}_p,
\end{array}$$

where F: $\hat{K}_{n+1}(K((T)))\to\hat{K}_{n+1}(K((T)))$ *is the norm associated with the K-homomorphism of degree p*; $K((T))\to K((T))$; $T\longmapsto T^p$.

(An explicit computation shows that $T^{(r)}_n$ is stable under the action of F.)

By this proposition, we have a canonical pairing

$$\begin{aligned}
W_r(K)/(F-1)W_r(K)\times K_n(K) &\xrightarrow{\ j\ } T^{(0)}_{n+1}(K)/(T^{(r)}_{n+1}(K)+(F-1)T^{(0)}_{n+1}(K)) \\
&\xrightarrow{\mathrm{Res}_K} T^{(0)}_1(\boldsymbol{F}_p)/T^{(r)}_1(\boldsymbol{F}_p)\cong W_r(\boldsymbol{F}_p)=\boldsymbol{Z}/p^r\boldsymbol{Z}\,,
\end{aligned}$$

in which the first map j is $((x_0,\cdots,x_{r-1}),y)\mapsto\prod_{i=0}^{r-1}\{E_i(x_i),y\}$, and F: $W_r(K)\to W_r(K)$ is the homomorphism $(x_0,\cdots,x_{r-1})\mapsto(x_0^p,\cdots,x_{r-1}^p)$. By the theory of Artin-Schreier and Witt, for any field k of characteristic $p>0$, the pro-p-part $\mathrm{Gal}\,(k^{\mathrm{ab}}/k)\,(p)$ of $\mathrm{Gal}\,(k^{\mathrm{ab}}/k)$ is isomorphic to

$$\varprojlim_r \mathrm{Hom}\,(W_r(k)/(F-1)W_r(k),\ \boldsymbol{Z}/p^r\boldsymbol{Z}).$$

Hence the above pairing induces a homomorphism

$$\Psi_K(p)\colon K_n(K)\longrightarrow\mathrm{Gal}\,(K^{\mathrm{ab}}/K)\,(p)\,.$$

Proposition 6 below shows that this homomorphism coincides with the pro-p-part of the reciprocity map Ψ_K: $K_n(K)\to\mathrm{Gal}\,(K^{\mathrm{ab}}/K)\,(p)$ defined in [7] by using the residue homomorphism in Quillen's K-theory.

Remark 3. From our point of view, the p-primary part of the reciprocity law in the global class field theory in characteristic $p>0$ follows from the residue formula Proposition 4.

Proof of Proposition 5. Though this proposition can be proved using the explicit definition of the residue homomorphism, we prove it here using the result of [7] for the brevity, relating the residue homomorphism in Milnor K-theory to that in Quillen's K-theory defined in [7] Section 2. We use

Proposition 6. (1) *Let k be a field of characteristic $p>0$, and let $S\hat{C}K_q(k)$ and* filt^r $T\hat{C}K_q(k)$ $(q, r\geqq 0)$ *be the groups defined in Bloch* [2] *Chapter* II *using Quillen's K-theory. Then, the canonical homomorphism from Milnor's K-group to Quillen's K-group for the field $k((T))$ induces isomorphisms*

$$U^1\hat{K}_q(k((T)))\xrightarrow{\cong} S\hat{C}K_q(k),\qquad T_q^{(r)}(k)\xrightarrow{\cong}\mathrm{filt}^r\, T\hat{C}K_q(k).$$

(2) *Let K and F be as in the hypothesis of Proposition* 5 (1). *Then the following diagram is commutative, where the left* (*resp. the right*) *vertical arrow is the residue homomorphism defined in this paper* (*resp. in* [7] Section 2).

$$\begin{array}{ccc} U^1\hat{K}_{q+1}(K((T))) & \xrightarrow{\cong} & S\hat{C}K_{q+1}(K) \\ {\scriptstyle\mathrm{res}}\downarrow & & \downarrow{\scriptstyle\mathrm{res}} \\ U^1\hat{K}_q(F((T))) & \xrightarrow{\cong} & S\hat{C}K_q(F) \end{array}$$

Proof. The assertion (1) follows from the determination of the structures of $S\hat{C}K_q(k)$ and $T\hat{C}K_q(k)$ in Bloch [2] Chapter II (the hypothesis $q\leqq p$ in [2] is eliminated by [7] Section 2.2 Proposition 2.) Indeed, let φ_i be the surjective homomorphism

$$\begin{aligned} \Omega_k^{q-1}\oplus\Omega_k^{q-2} &\longrightarrow U^i\hat{K}_q(k((T)))/U^{i+1}\hat{K}_q(k((T))) \\ (w, w') &\longmapsto \rho_i(w)+\{\rho_i(w'), T\}\quad \text{(cf. Section 3).} \end{aligned}$$

Let $U^iS\hat{C}K_q(k)$ be the image of $U^i\hat{K}_q(k((T)))$ in $S\hat{C}K_q(k)$. Then, by the structure theorem of $U^iS\hat{C}K_q(k)/U^{i+1}S\hat{C}K_q(k)$ in [2], we can easily verify that the kernel of the composite map

$$\Omega_k^{q-1}\oplus\Omega_k^{q-2}\longrightarrow U^iS\hat{C}K_q(k)/U^{i+1}S\hat{C}K_q(k)$$

is contained in the kernel of φ_i, and hence we have

$$U^i\hat{K}_q(k((T)))/U^{i+1}\hat{K}_q(k((T)))\xrightarrow{\cong} U^iS\hat{C}K_q(k)/U^{i+1}S\hat{C}K_q(k).$$

This proves the first isomorphism of (1), and the second is proved in the same way.

Next we prove (2). Recall the definition of the residue homomorphism for Quillen's K-group in [7]. Let K_*^Q be Quillen's K-group. Let B

be a (commutative) ring, S a multiplicatively closed set of non-zero divisors of B, and let H be the category of all B-modules X having a resolution of length one by finitely generated projective B-modules such that $sX=0$ for some $s \in S$. By Grayson [6], we have an exact sequence

$$\cdots \longrightarrow K_{q+1}^{Q}(B) \longrightarrow K_{q+1}^{Q}(S^{-1}B) \longrightarrow K_{q}^{Q}(H) \longrightarrow K_{q}^{Q}(B) \longrightarrow \cdots.$$

Assume further that B is a flat ring over a ring R and that for any $s \in S$, B/sB is finitely generated and projective as an R-module. Then, all objects of H are finitely generated and projective over R, and the restriction of scalars defines a homomorphism $K_q^Q(H) \to K_q^Q(R)$. Let $\partial_{B/R,S}$ be the composite $K_{q+1}^Q(S^{-1}B) \to K_q^Q(H) \to K_q^Q(R)$. We consider the following cases. Let K and F be as in (2).

(i) $R=F[T]/(T^n)$, $B=O_K[T]/(T^n)$ $(n \geqq 1)$ and $S=O_K-\{0\}$.

(ii) $R=F((T))$, $B=O_K[[T]][T^{-1}]$ and $S=B-\{0\}$.

(iii) $R=F[[T]]$, $B=O_K[[T]]$ and $S=B-(T)$ ((T) denotes the ideal of B generated by T).

Let M be the field of fractions of $O_K[[T]]$, and I the local ring of $O_K[[T]]$ at the prime ideal (T), and let $k=F((T))$. Let

$$\begin{aligned} \partial_n &: K_{q+1}^Q(K[T]/(T^n)) \longrightarrow K_q^Q(F[T]/(T^n)), \\ f &: K_{q+1}^Q(M) \longrightarrow K_q^Q(k) = K_q^Q(F((T))), \\ g &: K_{q+1}^Q(I) \longrightarrow K_q^Q(F[[T]]) \end{aligned}$$

be the homomorphism $\partial_{B/A,S}$ in the above cases (i) (ii) (iii), respectively. The residue homomorphism in [7] is defined as the inverse limit of ∂_n. We have a commutative diagram

$$\begin{array}{ccccccc} K_{q+1}(M) & \longrightarrow & K_{q+1}^Q(M) & \longleftarrow & K_{q+1}^Q(I) & \longrightarrow & K_{q+1}^Q(K[T]/(T^n)) \\ \Big\downarrow {\scriptstyle \sum_{\mathfrak{p} \in \mathfrak{S}} N_{\kappa(\mathfrak{p})/k} \circ \partial_{\mathfrak{p}}} & & \Big\downarrow {\scriptstyle f} & & \Big\downarrow {\scriptstyle g} & & \Big\downarrow {\scriptstyle \partial_n} \\ K_q(k) & \longrightarrow & K_q^Q(k) & \longleftarrow & K_q^Q(F[[T]]) & \longrightarrow & K_q^Q(F[T]/(T^n)), \end{array}$$

which proves (2).

By Proposition 6, Proposition 5 follows from the corresponding results for Quillen's K-group proved in [7] Section 2 and Section 3.

The next aim of this section is to prove

Theorem 2. *Let k and L be fields which satisfy the conditions* (1) (2) *in Section* 2, *and assume that k is a local field of dimension $n \geqq 1$. Then, L is a local field of dimension $n+1$. If we denote by Ψ_L and Ψ_k the reciprocity maps in the class field theory of L and k respectively, the following diagram is commutative for any element x of U_L such that $\bar{x} \in m_{\bar{L}}$.*

$$\begin{array}{ccc} K_{n+1}(L) & \xrightarrow{\Psi_L} & \mathrm{Gal}\,(L^{\mathrm{ab}}/L) \\ {\scriptstyle -\mathrm{res}_{A,L/k}}\downarrow & & \downarrow \\ K_n(k) & \xrightarrow{\Psi_k} & \mathrm{Gal}\,(k^{\mathrm{ab}}/k). \end{array}$$

Here the right vertical arrow is the natural restriction. Recall that if $\mathrm{ch}\,(\bar{k})\neq 0$, $\mathrm{res}_{A,L/k}$ is independent of the choice of A by Proposition 3, and is written as $\mathrm{res}_{L/k}$.

Proof. By using the commutative diagram

$$\begin{array}{ccc} \hat{K}_{n+1}(L) & \xrightarrow{\Psi_L} & \mathrm{Gal}\,(L^{\mathrm{ab}}/L) \\ {\scriptstyle N_{L/\hat{M}}}\downarrow & & \downarrow \\ \hat{K}_{n+1}(\hat{M}) & \xrightarrow{\Psi_{\hat{M}}} & \mathrm{Gal}\,(\hat{M}^{\mathrm{ab}}/\hat{M}) \end{array}$$

defined by the finite extension $L/\hat{M}$ corresponding to the choice of A ([7] Section 3 Cor. 1 to Proposition 1), we are reduced to the case $L=\hat{M}$. Since the image of $K_{n+1}(M)$ is dense in $\hat{K}_{n+1}(\hat{M})$ for the filtration U^i, it is sufficient to prove that the following diagram is commutative.

$$\begin{array}{ccc} K_{n+1}(M) & \xrightarrow{\Psi_{\hat{M}}} & \mathrm{Gal}\,(\hat{M}^{\mathrm{ab}}/\hat{M}) \\ {\scriptstyle -\sum_{\mathfrak{p}\in\mathfrak{S}} N_{\kappa(\mathfrak{p})/k}\circ\partial_{\mathfrak{p}}}\downarrow & & \downarrow \\ K_n(k) & \xrightarrow{\Psi_k} & \mathrm{Gal}\,(k^{\mathrm{ab}}/k). \end{array}$$

Let $\tilde{\mathfrak{S}}=\mathfrak{S}\cup\{O_k[[X]]m_k\}$ be the set of all prime ideals of height one of $O_k[[X]]$. For $\mathfrak{p}\in\tilde{\mathfrak{S}}$, let $M_{\mathfrak{p}}$ be the completion of M with respect to the $\mathfrak{p}$-adic valuation. If $\mathfrak{p}\neq O_k[[X]]m_k$ (i.e. if $\mathfrak{p}\in\mathfrak{S}$), the abelian extension k^{ab}/k induces an unramified abelian extension of $M_{\mathfrak{p}}$, and the unramified part of the class field theory of $M_{\mathfrak{p}}$ shows that the diagram

$$\begin{array}{ccc} K_{n+1}(M) & \xrightarrow{\Psi_{M_{\mathfrak{p}}}} & \mathrm{Gal}\,((M_{\mathfrak{p}})^{\mathrm{ab}}/M_{\mathfrak{p}}) \\ {\scriptstyle N_{\kappa(\mathfrak{p})/k}\circ\partial_{\mathfrak{p}}}\downarrow & & \downarrow \\ K_n(k) & \xrightarrow{\Psi_k} & \mathrm{Gal}\,(k^{\mathrm{ab}}/k) \end{array}$$

is commutative (cf. [7] Section 3 Corollary 2 to Proposition 2). Thus, it is sufficient to prove that for each $x\in K_{n+1}(M)$, the sum of the images of x under the composite maps

$$K_{n+1}(M)\xrightarrow{\Psi_{M_{\mathfrak{p}}}}\mathrm{Gal}\,((M_{\mathfrak{p}})^{\mathrm{ab}}/M_{\mathfrak{p}})\longrightarrow\mathrm{Gal}\,(k^{\mathrm{ab}}/k)$$

converges to zero in $\mathrm{Gal}\,(k^{\mathrm{ab}}/k)$ where $\mathfrak{p}$ ranges over $\tilde{\mathfrak{S}}$. This fact is contained in the following reciprocity law (take F to be the residue field of k, and $N=n-1$).

Proposition 7. *Let $N\geqq 0$, and let F be a local field of dimension N. Let A be a two dimensional complete normal noetherian local ring with residue field F, and let M be the field of fractions of A. Let $x\in K_{N+2}(M)$. Then, when $\mathfrak{p}$ range over all prime ideals of height one of A, the sum of the images of x under the composite maps*

$$K_{N+2}(M)\xrightarrow{\Psi_{M_{\mathfrak{p}}}}\mathrm{Gal}\,((M_{\mathfrak{p}})^{\mathrm{ab}}/M_{\mathfrak{p}})\longrightarrow\mathrm{Gal}\,(M^{\mathrm{ab}}/M)$$

converges to zero in $\mathrm{Gal}\,(M^{\mathrm{ab}}/M)$.

This reciprocity law is stated in Paršin [10] (in the case $N=0$) without proof. My proof will be introduced in Saito [11] in the case $N=0$, and that proof is valid for any N without essential change. Here we assume $\mathrm{ch}\,(M)=p>0$, and by using the residue, we give a proof of the fact that for each $x\in K_{N+2}(M)$, the sum of the images of x under $K_{N+2}(M)\to\mathrm{Gal}\,((M_{\mathfrak{p}})^{\mathrm{ab}}/M_{\mathfrak{p}})\to\mathrm{Gal}\,(M^{\mathrm{ab}}/M)\ (p)$ converges to zero. By the definition of the pro-p part of the reciprocity map given in this section, it is sufficient to prove the following lemma.

Lemma 8. *Let F, A, and M be as in Proposition 7, and assume $\mathrm{ch}\,(M)=p>0$. Then, for $x\in U^1\hat{K}_{N+3}(M((T)))$, when $\mathfrak{p}$ ranges over all prime ideals of height one of A, $\sum_{\mathfrak{p}}\mathrm{Res}_{M_{\mathfrak{p}}}(x)$ converges to zero in $U^1\hat{K}_1(\boldsymbol{F}_p((T)))$.*

Proof. By using the norm homomorphism, we may assume $A=F[[X, Y]]$. Let $k=F((Y))$, and identify A with $O_k[[X]]$. Let $\mathfrak{S}$ and $\tilde{\mathfrak{S}}$ be as before. Then, if $\mathfrak{p}=(Y)=Am_k$ (i.e. if $\mathfrak{p}\notin\mathfrak{S}$), $\mathrm{Res}_{M_{\mathfrak{p}}}\colon U^1\hat{K}_{N+3}(M_{\mathfrak{p}}((T)))\to U^1\hat{K}_1(\boldsymbol{F}_p((T)))$ is written as $\mathrm{Res}_F\circ r^2_{\mathfrak{p}}\circ r^1_{\mathfrak{p}}$, where

$$r^1_{\mathfrak{p}}=\mathrm{res}_{M_{\mathfrak{p}}((T))/F((X))((T))}\colon U^1\hat{K}_{N+3}(M_{\mathfrak{p}}((T)))\longrightarrow U^1\hat{K}_{N+2}(F((X))((T))),$$
$$r^2_{\mathfrak{p}}=\mathrm{res}_{F((X))((T))/F((T))}\colon U^1\hat{K}_{N+2}(F((X))((T)))\longrightarrow U^1\hat{K}_{N+1}(F((T))).$$

(Note $M_{\mathfrak{p}}=F((X))((Y))$.) If $\mathfrak{p}\in\mathfrak{S}$, $\mathrm{Res}_{M_{\mathfrak{p}}}=\mathrm{Res}_F\circ r^2_{\mathfrak{p}}\circ r^1_{\mathfrak{p}}$ where

$$r^1_{\mathfrak{p}}=\mathrm{res}_{M_{\mathfrak{p}}((T))/k((T))}\colon U^1\hat{K}_{N+3}(M_{\mathfrak{p}}(T)))\longrightarrow U^1\hat{K}_{N+2}(k((T))),$$
$$r^2_{\mathfrak{p}}=\mathrm{res}_{k((T))/F((T))}\colon U^1\hat{K}_{N+2}(k((T)))\longrightarrow U^1\hat{K}_{N+1}(F((T)))$$

(cf. [7] Section 3 Lemma 12). So it suffices to prove that for any q and any $x\in U^1K_{q+2}(M((T)))$, $\sum_{\mathfrak{p}\in\tilde{\mathfrak{S}}}r^2_{\mathfrak{p}}\circ r^1_{\mathfrak{p}}(x)$ converges to zero in $U^1\hat{K}_q(F((T)))$.

Let $b_1, \cdots, b_N$ be a p-base of F over $\boldsymbol{F}_p$ (cf. [6] Chapter 0 Section 12). Then, the M-vector space Ω^1_M has a base (db_i $(1\leqq i\leqq N)$, dX, dY). Since

$$\Omega_M^{q+1} \oplus \Omega_M^q \longrightarrow U^i \hat{K}_{q+2}(M((T)))/U^{i+1}\hat{K}_{q+2}(M((T)))$$
$$(w, w') \longmapsto \rho_i(w) + \{\rho_i(w'), T\}$$

is surjective for any $i \geqq 1$, and since

$$M = A_+ + A_- = A_+ + (A_-)_0 \qquad \text{where } (A_-)_0 = \{f \in A_-;\ \mathrm{ord}_\infty(f) > 0\},$$

it follows that $U^1\hat{K}_{q+2}(M((T)))$ is topologically generated by elements of the forms

$$\{c, 1+aT^i, X, Y\}, \quad \{c', 1+aT^i, X\}, \quad \{c', 1+aT^i, Y\}$$

such that $c \in K_{q-1}(F((T)))$, $c' \in K_q(F((T)))$, $i \geqq 1$, and a is either in A_+ or $(A_-)_0$. Since $r_\mathfrak{p}^2 \circ r_\mathfrak{p}^1$ is a homomorphism of left $\oplus_{q \geqq 0} K_q(F((T)))$ -modules and

$$r_\mathfrak{p}^2 \circ r_\mathfrak{p}^1(U^1\hat{K}_2(M((T))) \subset U^1\hat{K}_0(F((T))) = 0$$

for any $\mathfrak{p} \in \tilde{\mathfrak{S}}$, we are reduced to proving

$$\sum_{\mathfrak{p} \in \tilde{\mathfrak{S}}} r_\mathfrak{p}^2 \circ r_\mathfrak{p}^1(\{1+aT^i, X, Y\}) = 0 \qquad \text{in } \hat{K}_1(F((T)))$$

for $a \in A_+$ and $a \in (A_-)_0$. First, assume $a \in A_+$ and write a in the form

$$a = \sum_{j \gg -\infty} a_j(X) Y^j \qquad \text{with } a_j(X) \in F[[X]].$$

Then, $r_\mathfrak{p}^i(\{1+aT^i, X, Y\}) = 0$ unless $\mathfrak{p} = (X)$ or (Y). If $\mathfrak{p} = (Y)$,

$$\begin{aligned} r_\mathfrak{p}^2 \circ r_\mathfrak{p}^1(\{1+aT^i, X, Y\}) &= r_\mathfrak{p}^2(\{1+a_0(X)T^i, X\}) \\ &= \{1+a_0(0)T^i\} \in \hat{K}_1(F((T))). \end{aligned}$$

If $\mathfrak{p} = (X)$,

$$\begin{aligned} r_\mathfrak{p}^2 \circ r_\mathfrak{p}^1(\{1+aT^i, X, Y\}) &= -r_\mathfrak{p}^2(\{1+\sum_{j \gg -\infty} a_j(0) Y^j T^i, Y\}) \\ &= -\{1+a_0(0)T^i\} \in \hat{K}_1(F((T))) \end{aligned}$$

(here for $u \in F((T))^\times$, $\{u\}$ denotes the corresponding element of $\hat{K}_1(F((T)))$, so, $\{uv\} = \{u\} + \{v\}$). Next, assume $a \in (A_-)_0$, and write

$$a = \sum_{j \gg -\infty} a_j(X) Y^j \qquad \text{with } a_j(X) \in X^{-1}k[X^{-1}].$$

We have, for $\mathfrak{p} = (Y)$,

$$r_\mathfrak{p}^2 \circ r_\mathfrak{p}^1(\{1+aT^i, X, Y\}) = r_\mathfrak{p}^2(\{1+a_0(X)T^i, X\}) = 0$$

(cf. Theorem 1). On the other hand,

$$\sum_{\mathfrak{p}\in\mathfrak{S}} r^1_{\mathfrak{p}}(\{1+aT^i, X, Y\}) = -\sum_{v\in\mathfrak{P}(k(X)/k)-\mathfrak{S}} \mathrm{res}_v(1\{+aT^i, X, Y\})$$
$$=0 \qquad \text{in } \hat{K}_2(k((T)))$$

by the residue formula.

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