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Two Dimensional Class Field Theory

Kazuya Kato and Shuji Saito

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Introduction

The classical class field theory studies abelian extensions of algebraic number fields. Since it was constructed by Takagi [23], it has been a fundamental tool in the study of algebraic number fields. In the modern number theory, it became recognized that finitely generated fields over the prime fields, which we call arithmetical fields, are important as their classical example, the algebraic number field. Recent development of algebraic K-theory enables us to begin the construction of the class field theory of general arithmetical fields. Is the generalized class field theory powerful as the classical one, in the study of arithmetical fields of higher dimensions? In this paper, we construct the class field theory of a two dimensional arithmetical field K, that is, an arithmetical field of transcendental degree one over $\mathbf{Q}$, or of transcendental degree two over $\mathbf{F}_p$. Take a regular connected scheme X with function field K which is proper over $\mathbf{Z}$.

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Then, the class field theory of K is stated as follows (cf. Chapter II Section 3 Theorem 4 and Section 4 Theorem 5). We do not explain here the definitions of the modulus and of the idele class groups $\bar{C}_m(X)$ and $C_m(X)$ with modulus m, but we remark that these groups are defined by using K_2-groups of various "local fields" of K. Let K^{ab} be the maximum abelian extension of K.

Theorem 1. (1) *Assume* $\mathrm{ch}(K)=0$. *Then, there exists a canonical isomorphism*

$$\varprojlim_m \bar{C}_m(X) \cong \mathrm{Gal}(K^{ab}/K)$$

where m ranges over all admissible moduli on X.

(2) *Assume* $\mathrm{ch}(K)=p>0$, *and let* $\mathrm{Gal}(K^{ab}/K)'$ *be the dense subgroup of* $\mathrm{Gal}(K^{ab}/K)$ *defined to be the inverse image of the subgroup of* $\mathrm{Gal}(\bar{F}_p/F_p)$ *consisting of all integral powers of the Frobenius automorphism. Then, there exists a canonical isomorphism*

$$\varprojlim_m C_m(X) \cong \mathrm{Gal}(K^{ab}/K)'$$

where m ranges over all moduli on X.

Let U be a two dimensional regular connected scheme of finite type over $\boldsymbol{Z}$. Then, the class field theory of unramified abelian coverings of U is described in a similar way. Take a regular scheme X proper over $\boldsymbol{Z}$ which contains U as a dense open subscheme. Then, we shall show that the theorem above also holds when we replace $\mathrm{Gal}(K^{ab}/K)$ by the abelian fundamental group $\pi_1^{ab}(U)$ of U, and restrict the moduli m to those with supports outside U.

For this global class field theory, we need one "purely local" theory and three "semi-global" theories. The first one is the class field theory of a complete discrete valuation field whose residue field is a usual local field. The three semi-global theories are, the class field theory of a complete discrete valuation field whose residue field is a usual global field, that of the field of fractions of a two dimensional arithmetical complete local ring, and that of a function field in one variable over a usual local field. These theories were studied in Bloch [3], Kato [5] and Saito [17] [18], and we shall review these theories in Chapter I adding some necessary complements and reformations. The global class field theory will be obtained in Chapter II by gluing these local theories together, and by applying the two dimensional unramified class field theory which was mainly accomplished by Bloch [3] and completed by our previous paper [8].

The K-theoretic generalization of class field theory was first studied by A. N. Paršin ([26], [27]). There have been various accomplishments without K-theory in the generalization of the class field theory. An important one is the class field theory of varieties over finite fields by Lang ([9] [10] [20]). In Chapter III, we study the relation of this theory with ours.

We express our sincere gratitudes to Professor Y. Ihara for his hearty encouragements.

Conventions

For a scheme X, X_i denotes the set of all points x of X such that the closure $\overline{\{x\}}$ is of dimension i. For $x \in X$, $\kappa(x)$ denotes the residue field of x. $H^*(X,\ )$ means the étale cohomology unless the contrary is explicitly stated, and $\pi_1^{\mathrm{ab}}(X)$ means $\mathrm{Hom}\,(H^1(X, \boldsymbol{Q}/\boldsymbol{Z}), \boldsymbol{Q}/\boldsymbol{Z})$. If $X = \mathrm{Spec}\,(A)$, $H^*(X,\ )$ is often denoted by $H^*(A,\ )$.

For a field k, $\mathrm{ch}\,(k)$ denotes the characteristic of k, k^{ab} denotes the maximum abelian extension of k, k_s denotes the separable closure of k, and $\bar{k}$ denotes the algebraic closure of k.

For a discrete valuation field (resp. an algebraic number field) k, O_k denotes the ring of integers in k.

Rings are always assumed to be commutative except graded rings. For a ring R, $R^\times$ denotes the group of all invertible elements in R.

Chapter I. Local theories

As is explained in the introduction, the nature of this chapter is a review of previous papers, and we give here only the proofs of new complementary results.

§ 1. Preliminaries

In the generalization of class field theory, two types of abelian groups associated with arbitrary fields play important roles.

The first is Milnor's K-group $K_q^M(k)$ $(q \geqq 0)$ of k. It is defined by

$$K_0^M(k) = \boldsymbol{Z}, \qquad K_1^M(k) = k^\times,$$
$$K_q^M(k) = (\underbrace{k^\times \otimes \cdots \otimes k^\times}_{q \text{ times}})/J \qquad \text{for } q \geqq 2$$

where J is the subgroup of the tensor product generated by elements $a_1 \otimes \cdots \otimes a_q$ $(a_1, \cdots, a_q \in k^\times)$ such that $a_i + a_j = 1$ for some indices $i \neq j$.

(Cf. Milnor [14].) An element $a_1 \otimes \cdots \otimes a_q$ mod J of $K_q^M(k)$ is denoted by $\{a_1, \cdots, a_q\}$. We denote the group law of $K_q^M(k)$ additively. The direct sum $\oplus_{q \geqq 0} K_q^M(k)$ has a natural ring structure and is called the Milnor ring of k. In the case $q \leqq 2$, we shall often use the notation K_q instead of K_q^M, for Milnor's K-group coincides with the standard K-group K_q of Quillen in this range of q.

The second is a sequence of torsion abelian groups $H^q(k)$ ($q \geqq 0$, k is a field), which contain as members important groups such as the Brauer group $\mathrm{Br}(k)$ of k. If $\mathrm{ch}(k)=0$, $H^q(k)$ is defined to be the Galois cohomology group $\varinjlim_n H^q(\mathrm{Gal}(k_s/k), \mu_n^{\otimes(q-1)})$, where μ_n denotes the group of all n-th roots of 1 in the separable closure k_s of k, $\mu_n^{\otimes(q-1)}$ denotes its $(q-1)$-th tensor power over Z/nZ, n ranges over all integers $\geqq 1$, and the transition maps of the inductive system are the homomorphisms induced by the canonical injections $\mu_n^{\otimes(q-1)} \to \mu_m^{\otimes(q-1)}$ given in the case $n|m$. If $\mathrm{ch}(k)=p>0$, let

$$H^q(k)=\varinjlim_n H^q(\mathrm{Gal}(k_s/k), \mu_n^{\otimes(q-1)}) \oplus \varinjlim_r H^q_{p^r}(k)$$

where n ranges over all integers $\geqq 1$ which are prime to p, r ranges over all integers $\geqq 0$, and where $H^q_{p^r}(k)$ is defined as follows. (Cf. [5] II Section 3.2. An alternative definition is $H^q_{p^r}(k)=H^1(\mathrm{Gal}(k_s/k), W_r\Omega^{q-1}_{k_s,\log})$). Let $W_r(k)$ be the group of all p-Witt vectors of length r over k (Serre [21] Chapter II Section 6). Let

$$H^q_{p^r}(k)=(W_r(k)\otimes \underbrace{k^\times \otimes \cdots \otimes k^\times}_{q-1 \text{ times}})/J$$

where J is the subgroup of the tensor product generated by elements of the following forms (1)—(3).

(1) $(w^{(p)}-w)\otimes b_1 \otimes \cdots \otimes b_{q-1}$ ($w \in W_r(k)$, $b_1, \cdots, b_{q-1} \in k^\times$) where for $w=(a_0. \cdots, a_{r-1})$, $w^{(p)}$ denotes $(a_0^p, \cdots, a_{r-1}^p)$.

(2) $(\underbrace{0, \cdots, 0}_{i \text{ times}}, b_1, 0, \cdots, 0)\otimes b_1 \otimes \cdots \otimes b_{q-1}$ ($0 \leqq i < r$, $b_1, \cdots, b_{q-1} \in k^\times$).

(3) $w \otimes b_1 \otimes \cdots \otimes b_{q-1}$ such that $b_i=b_j$ for some indices $i \neq j$ ($w \in W_r(k)$, $b_1, \cdots, b_{q-1} \in k^\times$).

We define $H^0_{p^r}(k)=0$. An element $w \otimes b_1 \otimes \cdots \otimes b_{q-1}$ mod J of $H^q_{p^r}(k)$ is denoted by $\{w, b_1, \cdots, b_{q-1}\}$. The transition maps are the homomorphisms $H^q_{p^r}(k) \to H^q_{p^{r+1}}(k)$;

$$\{(a_0, \cdots, a_{r-1}), b_1, \cdots, b_{q-1}\} \longmapsto \{(0, a_0, \cdots, a_{r-1}), b_1, \cdots, b_{q-1}\}.$$

For any field k, $H^1(k)$ is isomorphic to the group of all continuous homomorphisms $\mathrm{Gal}(k^{\mathrm{ab}}/k)\to \boldsymbol{Q}/\boldsymbol{Z}$ where k^{ab} denotes the maximum abelian extension of k, and $H^2(k)$ is isomorphic to the Brauer group of k.

The direct sum $\oplus_{q\geqq 0} H^q(k)$ has a structure of a graded right $\oplus_{q\geqq 0} K_q^M(k)$-module. Indeed, if n is an integer invertible in k, the canonical isomorphism

$$k^\times/(k^\times)^n \longrightarrow H^1(k, \mu_n)$$

induced by the exact sequence

$$0 \longrightarrow \mu_n \longrightarrow k_s^\times \xrightarrow{\ n\ } k_s^\times \longrightarrow 0$$

is uniquely extended to a homomorphism of graded rings

$$\bigoplus_{q\geqq 0} K_q^M(k) \longrightarrow \bigoplus_{q\geqq 0} H^q(k, \mu_n^{\otimes q}).$$

Via this homomorphism and the cup product from right, $\oplus_{q\geqq 0} K_q^M(k)$ acts on $\oplus_{q\geqq 0} \varinjlim_n H^q(k, \mu_n^{\otimes(q-1)})$ where n ranges over all natural numbers invertible in k. On the other hand, if $\mathrm{ch}(k)=p>0$, $\oplus_{q\geqq 0} H^q_{p^r}(k)$ has a structure of $\oplus_{q\geqq 0} K_q^M(k)$-module characterized by

$$\{w, b_1, \cdots, b_{q-1}\}\cdot\{c_1, \cdots, c_n\}=\{w, b_1, \cdots, b_{q-1}, c_1, \cdots, c_n\}.$$

(Cf. [5] II Section 3.2.)

Now, let K be a discrete valuation field with residue field F. In the following, we analyze the groups $K_q^M(K)$ and $H^q(K)$ by using various groups associated with F.

In this paper, we use the following notations. Let

ord_K be the normalized additive discrete valuation of K,
$O_K=\{x \in K;\ \mathrm{ord}_K(x)\geqq 0\}$ the valuation ring of K,
$m_K^i=\{x \in K;\ \mathrm{ord}_K(x)\geqq i\}$ for $i \in \boldsymbol{Z}$,
$m_K=m_K^1$ the maximal ideal of O_K,
$U_K^{(i)}=\mathrm{Ker}\,((O_K)^\times \longrightarrow (O_K/m_K^i)^\times)$ for $i\geqq 0$,
$U_K=U_K^{(0)}=(O_K)^\times$.

For $q\geqq 1$ and $i\geqq 1$, let $U^iK_q^M(K)$ be the image of

$$U_K^{(i)}\otimes K_{q-1}^M(K) \xrightarrow{\{\ ,\ \}} K_q^M(K).$$

On the other hand, let $U^0K_q^M(K)$ be the image of

$$\underbrace{U_K \otimes \cdots \otimes U_K}_{q \text{ times}} \xrightarrow{\{\ \}} K_q^M(K).$$

Then, we have inclusions

$$K_q^M(K) \supset U^0 K_q^M(K) \supset U^1 K_q^M(K) \supset U^2 K_q^M(K) \supset \cdots.$$

Proposition 1 (Bass and Tate [1] Chapter I Proposition 4.3). *Let K be a discrete valuation field with residue field F. Then,*

(1) $K_q^M(K)/U^0 K_q^M(K) \cong K_{q-1}^M(F)$.

(2) $U^0 K_q^M(K)/U^1 K_q^M(K) \cong K_q^M(F)$.

The isomorphism in (1) sends $\{a_1, \cdots, a_{q-1}\} \in K_{q-1}^M(F)$ to the class of $\{\tilde{a}_1, \cdots, \tilde{a}_{q-1}, \pi\} \in K_q^M(K)$ where $\tilde{a}$ denotes any lifting to O_K of an element a of F, and π denotes any prime element of K. The isomorphism in (2) sends $\{a_1, \cdots, a_q\} \in K_q^M(F)$ to the class of $\{\tilde{a}_1, \cdots, \tilde{a}_q\}$. The surjective homomorphism $\partial: K_q^M(K) \to K_{q-1}^M(F)$ given by (1) is called the tame symbol.

For a field k and $q \geqq 0$, let Ω_k^q be the q-th exterior power over k of the absolute differential module $\Omega_{k/Z}^q$. The following result is contained in Bloch [2] (in the positive characteristic case) and in [5] II Section 1.3.

Proposition 2. *Let K and F be as in Proposition 1 and let $i \geqq 1$. Fix a prime element π of K. Then, there is a well defined surjective homomorphism*

$$\Omega_F^{q-1} \oplus \Omega_F^{q-2} \longrightarrow U^i K_q^M(K)/U^{i+1} K_q^M(K)$$

$$\left(a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{q-1}}{b_{q-1}}, 0\right) \longmapsto \{1+\tilde{a}\pi^i, \tilde{b}_1, \cdots, \tilde{b}_{q-1}\}$$

$$\left(0, a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{q-2}}{b_{q-2}}\right) \longmapsto \{1+\tilde{a}\pi^i, \tilde{b}_1, \cdots, \tilde{b}_{q-2}, \pi\}$$

$(a \in F,\ b_1, \cdots, b_{q-1} \in F^\times)$.

Next we consider the group $H^q(K)$. If K is complete, there is a canonical injection $i: H^q(F) \xrightarrow{\subset} H^q(K)$ defined in a standard manner ([5] II Section 3.2 Definition 2). The following Theorem 1 (1) is a well known result, and (2) is proved in [6]. In the following, for a torsion abelian group A and a prime number p, $A\{p\}$ denotes the p-primary part of A.

Theorem 1. *Let K be a complete discrete valuation field with residue field F. Let p be a prime number.*

(1) *If* $\mathrm{ch}(F) \neq p$, *there is an exact sequence*

$$0 \longrightarrow H^q(F)\{p\} \xrightarrow{i} H^q(K)\{p\} \xrightarrow{\partial} H^{q-1}(F)\{p\} \longrightarrow 0$$

in which ∂ is characterized by the property that $\partial(\{i(\chi), \pi\})=\chi$ for any $\chi \in H^{q-1}(F)\{p\}$ and any prime element π of K.

(2) *Assume* $\mathrm{ch}(F)=p>0$ *and* $[F: F^p]=p^n<\infty$. *Then, $H^{q+1}(K)\{p\}$ and $H^q(F)\{p\}$ vanish for $q>n+1$, and there exists a canonical isomorphism*

$$\partial: H^{n+2}(K)\{p\} \cong H^{n+1}(F)\{p\}$$

characterized by $\partial(\{i(\chi), \pi\})=\chi$ for any $\chi \in H^{n+1}(F)\{p\}$ and any prime element π of K.

Remark 1. If $\mathrm{ch}(F)=p>0$ and $0\leqq q\leqq \log_p([F: F^p])$, $H^{q+1}(K)\{p\}$ is very big and it can not be presented in terms of the groups $H^*(F)$.

§ 2. Higher dimensional local fields

In this section, we call a field K an n-dimensional local field if a sequence of fields $k_0, \cdots, k_n$ are given satisfying the following conditions.

(1) k_0 is a finite field.

(2) For $i=1, \cdots, n$, k_i is a complete discrete valuation field with residue field k_{i-1}.

(3) $k_n=K$.

If $n=0$, K is a finite field. In this case, there is a canonical isomorphism

$$H^1(K) \cong \boldsymbol{Q}/\boldsymbol{Z}$$

in which an element χ of $H^1(K)$ regarded as a continuous homomorphism $\mathrm{Gal}(K^{\mathrm{ab}}/K) \to \boldsymbol{Q}/\boldsymbol{Z}$ corresponds to $\chi(F_K) \in \boldsymbol{Q}/\boldsymbol{Z}$, where F_K denotes the Frobenius automorphism in $\mathrm{Gal}(K^{\mathrm{ab}}/K)$. For n arbitrary, by Section 1 Theorem 1 and by induction on n, we obtain a canonical isomorphism

$$h_K: H^{n+1}(K) \cong \boldsymbol{Q}/\boldsymbol{Z}$$

for an n-dimensional local field K. This isomorphism induces the canonical pairing

$$H^{n+1-q}(K) \times K_q^M(K) \longrightarrow \boldsymbol{Q}/\boldsymbol{Z}$$

for $0\leqq q\leqq n+1$ by the right $\oplus_{q\geqq 0} K_q^M(K)$-module structure on $\oplus_{q\geqq 0} H^q(K)$.

In the following theorem, the word "continuous" is used in a topological sense only in the case $n\leqq 2$. In the case $n\geqq 3$, the group $K_q^M(K)$ seems to have no appropriate topology, and we define the continuity in this theorem from a new point of view ([7]).

Theorem 2 ([7]). *Let K be an n-dimensional local field. Then, $H^q(K)$*

$=0$ *for* $q>n+1$. *For* $0\leqq q\leqq n+1$, $H^{n+1-q}(K)$ *is isomorphic to the group of all "continuous" homomorphisms* $K_q^M(K)\to \boldsymbol{Q}/\boldsymbol{Z}$ *of finite orders.*

Proposition 3. ([5] II Section 3) *Let K be as above.*

(1) *For a finite extension L of K, the canonical map* $H^{n+1-q}(K)\to H^{n+1-q}(L)$ (*resp. the trace map* $H^{n+1-q}(L)\to H^{n+1-q}(K)$) *corresponds to the norm map* $N_{L/K}\colon K_q^M(L)\to K_q^M(K)$ (*resp. to the canonical map* $K_q^M(K)\to K_q^M(L)$) *in the duality of Theorem* 2.

(2) *If* $n\geqq 1$ *and F is the residue field of K, the canonical injection* $i\colon H^{n+1-q}(F)\to H^{n+1-q}(K)$ *corresponds to the tame symbol* $\partial\colon K_q^M(K)\to K_{q-1}^M(F)$.

Corollary. *Let K and F be as in* (2) *above, and let* $\chi\in H^1(K)$. *Then,* χ *is unramified* (*i.e. the corresponding cyclic extension of K is unramified*) *if and only if the homomorphism* $K_n^M(K)\to \boldsymbol{Q}/\boldsymbol{Z}$ *induced by x factors through the tame symbol* $\partial\colon K_n^M(K)\to K_{n-1}^M(F)$.

Because only the case $n=2$ and $q=2$ of these results is used in this paper, we do not explain the definition of the continuity in Theorem 2 for the general case. In the following, we give the precise form of Theorem 2 in the case $n=2$ and $q=2$ (see Theorem 3 below). To treat similar fields $\boldsymbol{R}((T))$ and $\boldsymbol{C}((T))$ in a uniform manner, throughout the rest of this section, K denotes a complete discrete valuation field whose residue field F is a non-discrete locally compact field. We define the topologies of $K^\times/U_K^{(i)}$ and $K_2(K)/U^iK_2(K)$ ($i\geqq 0$) as follows.

Case 1. Let $i\geqq 1$ and assume that F is non-archimedean. Suppose that we are given a subring A of O_K/m_K^i satisfying the following conditions (1) (2) (3).

(1) A is a Noetherian complete local ring.

(2) The image of A in $O_K/m_K=F$ is an open subring of O_F.

(3) The total quotient ring of A is O_K/m_K^i.

Then, we endow $K^\times/U_K^{(i)}$ with the unique topology which is compatible with the group structure and for which $A^\times$ is an open subgroup with its m_A-adic topology (m_A denotes the maximal ideal of A). We endow $K_2(K)/U^iK_2(K)$ with the finest topology which is compatible with the group structure and for which the map

$$K^\times/U_K^{(i)}\times K^\times/U_K^{(i)}\longrightarrow K_2(K)/U^iK_2(K);\quad (a,b)\to\{a,b\}$$

is continuous. Such ring A always exists, and is often given canonically in each example treated in this paper.

Case 2. Next assume that we are given a subfield k of O_K such that F is a finite extension of k via $k\to O_K/m_K=F$. Then, O_K/m_K^i is a vector space over k of finite rank and hence has a natural topology. We endow $K^{\times}/U_K^{(i)}$ with the unique topology which is compatible with the group structure and for which $U_K/U_K^{(i)}$ is an open subgroup having the topology as a subspace of O_K/m_K^i. For $i \geqq 1$, we define the topology of $K_2(K)/U^iK_2(K)$ by the same method as in Case 1. Such field k exists if and only if $\mathrm{ch}(K) = \mathrm{ch}(F)$. If $k \xrightarrow{\cong} F$, the topology of $K^{\times}/U_K^{(i)}$ defined above has a very simple description; the map

$$Z \times k^{\times} \times \underbrace{k \times \cdots \times k}_{i-1 \text{ times}} \longrightarrow K^{\times}/U_K^{(i)}$$

$$(n, a, b_1, \cdots, b_{i-1}) \longmapsto \pi^n a(1 + b_1\pi + \cdots + b_{i-1}\pi^{i-1})$$

is a homeomorphism for any fixed prime element π of K.

In both Case 1 and Case 2, we endow $K_2(K)/U^0K_2(K)$ with the quotient topology. Then, the isomorphism $K_2(K)/U^0K_2(K) \cong F^{\times}$ of Section 1 Proposition 1 (1) becomes a homeomorphism.

We have to add remarks concerning the question whether the topologies of $K^{\times}/U_K^{(i)}$ and $K_2(K)/U^iK_2(K)$ defined above are independent of the choice of A or k. If $\mathrm{ch}(F) \neq 0$, it is easily seen that they coincide with the topologies defined in [5] I Section 7 and hence independent of the choice of A or k. If $\mathrm{ch}(F) = 0$ and $i \geqq 2$, they actually depend on the choice of A or k. However, in each example such that $\mathrm{ch}(F) = 0$ in the latter applications in this paper, we are always given a canonical choice of k, and furthermore, if A is also given in such example, A is finite over an open subring of O_k and the topologies given by A will coincide with those given by k. Even in case $\mathrm{ch}(F) = 0$, the topology induced on the quotient $K_2(K)/U^1K_2(K)$ is independent of the choice of A or k.

The precise form of the case $n = q = 2$ of Theorem 2 is the following Theorem 3. In the case $F = \boldsymbol{R}$ or $\boldsymbol{C}$, let h be the composite map $H^3(K) \xrightarrow{\partial} H^2(F) \xrightarrow{h_F} \boldsymbol{Q}/\boldsymbol{Z}$, where h_F is the well known Hasse invariant. This induces the canonical pairing $H^1(K) \times K_2(K) \to \boldsymbol{Q}/\boldsymbol{Z}$, which is the zero map in the case $F = \boldsymbol{C}$.

Theorem 3 ([5] I, II). *Let K be a complete discrete valuation field whose residue field F is a non-discrete locally compact field. Concerning the homomorphism $H^1(K) \to \mathrm{Hom}(K_2(K), \boldsymbol{Q}/\boldsymbol{Z})$ induced by the canonical parir-ing, we have;*

(1) *Assume* $\mathrm{ch}(F) \neq 0$. *Then, $H^1(K)$ is isomorphic to the group of all homomorphisms φ: $K_2(K) \to \boldsymbol{Q}/\boldsymbol{Z}$ such that $\varphi(U^iK_2(K)) = 0$ for some i and such that the induced map $K_2(K)/U^iK_2(K) \to \boldsymbol{Q}/\boldsymbol{Z}$ is continuous.*

(2) *Assume* $\mathrm{ch}(F) = 0$ *and $F \neq \boldsymbol{C}$. Then, $H^1(K)$ is isomorphic to the group of all homomorphisms $K_2(K) \to \boldsymbol{Q}/\boldsymbol{Z}$ of finite orders.*

Remark 2. In the case $\mathrm{ch}(F)=0$, the group $U^1K_2(K)$ is divisible and annihilated by any homomorphism $K_2(K)\to \boldsymbol{Q}/\boldsymbol{Z}$ of finite order. In this case, for $i\geqq 0$, any homomorphism $K_2(K)/U^iK_2(K)\to \boldsymbol{Q}/\boldsymbol{Z}$ of finite order is continuous for the topology on $K_2(K)/U^iK_2(K)$ defined by any choice of A or k.

The following lemma would be a slightly better description of the topology.

Lemma 1. *Assume F is non-archimedean, let $A\subset O_K/m_K^i$ be as in Case 1, and let π be a fixed prime element of K. Then, the topology on $G=K_2(K)/U^iK_2(K)$ defined above with respect to A coincides with the finest topology $\mathscr{T}$ which is compatible with the group structure and for which the maps $A^\times\times A^\times\to G$; $(a, b)\to\{a, b\}$ and $A^\times\to G$; $a\mapsto\{a, \pi\}$ are continuous.*

Proof. It is sufficient to prove that for each $g\in K^\times$ and for each neibourhood U of 0 in G for the topology $\mathscr{T}$, there is a neighbourhood V of 1 in $A^\times$ such that $\{V, g\}\subset U$. Since $K^\times/U_K^{(i)}$ is generated by π and $A\cap U_K/U_K^{(i)}\subset O_K/m_K^i$, we may assume $g\in A\cap U_K/U_K^{(i)}$. Then, the subgroups $1+g^nA (n\geqq 1)$ form a fundamental system of neibourhoods of 1 in $A^\times$. Let U' be a neighbourhood of 0 in G for the topology $\mathscr{T}$ such that $U'+U'+U'+U'\subset U$ and $U'=-U'$. Then, $\{1+g^nA, 1+g^nA\}\subset U'$ for some $n\geqq 1$. We claim $\{1+g^{2n+1}A, g\}\subset U$. Indeed, for each $a\in 1+g^{2n+1}A$, we can take elements $b_1, b_2, c_1, c_2\in 1+g^nA$ such that

$$a=(1-g^nb_1)(1-g^nb_2)=(1-g^{n+1}c_1)(1-g^{n+1}c_2).$$

Then, we have

$$\begin{aligned} n\{a, g\} &= n\{1-g^nb_1, g\}+n\{1-g^nb_2, g\} \\ &= \{1-g^nb_1, g^n\}+\{1-g^nb_2, g^n\} \\ &= -\{1-g^nb_1, b_1\}-\{1-g^nb_2, b_2\}\in U'+U'. \end{aligned}$$

Similarly,

$$\begin{aligned} (n+1)\{a, g\} &= (n+1)\{1-g^{n+1}c_1, g\}+(n+1)\{1-g^{n+1}c_2, g\} \\ &\in U'+U'. \end{aligned}$$

These prove $\{a, g\}\in U$ as is claimed.

§ 3. Local fields with global residue fields

From this section, we call the fields $\boldsymbol{R}((T))$ and $\boldsymbol{C}((T))$ also two dimensional local fields.

Let F be an A-field in the sense of Weil [25]. That is, F is an algebraic number field, or an algebraic function field in one variable over a finite field. Let P be the set of all places of F. For each $v \in P$, let F_v be the completion of F at v.

In this section, K denotes a complete discrete valuation field with residue field F. To describe the class field theory of K, we define a family $\{K_v\}_{v \in P}$ of two dimensional local fields which are called the "local fields of K".

First, assume $\mathrm{ch}(F)=0$. Then, O_K contains a unique subfield k such that $k \to O_K/m_K = F$ is a bijection. We identify k with F, and for each $v \in P$, let

$$O_{K_v} \underset{\mathrm{def}}{=} \varprojlim_i F_v \otimes_F O_K/m_K^i, \qquad K_v \underset{\mathrm{def}}{=} O_{K_v} \otimes_{O_K} K.$$

Then, K_v is a complete discrete valuation field with residue field F_v and with valuation ring O_{K_v}. Because F_v is canonically embedded in O_{K_v}, we obtain a canonical topology of $K_2(K_v)/U^i K_2(K_v)$ (Section 2 Case 2).

Next, assume $\mathrm{ch}(F) \neq 0$. Then, there exists a unique two dimensional local field K_v containing K having the following properties (1) (2).

(1) $O_K \subset O_{K_v}$ and $O_{K_v} \cdot m_K = m_{K_v}$.

(2) The residue field of K_v is identified with F_v as a complete discrete valuation field containing F.

(Cf. [5] III.) As is explained in Section 2, $K_2(K_v)/U^i K_2(K_v)$ has a canonical topology also in this case.

For $i \geqq 0$, we define the idele group $I_{(i)}(K)$ and the idele class group $C_{(i)}(K)$ of modulus i as follows. First, assume $i \geqq 1$. Take an irreducible scheme Y with generic point η which is of finite type over Z and which has an isomorphism $\mathcal{O}_{Y,\eta} \cong O_K/m_K^i$. (The existence of Y is clear.) If v is a closed point of Y such that the reduced part Y_{red} of Y is regular at v, we identify v with the corresponding place of F. For such v, we have a canonical isomorphism

$$\hat{\mathcal{O}}_{Y,v} \otimes_{\mathcal{O}_{Y,v}} O_K/m_K^i \cong O_{K_v}/m_{K_v}^i$$

where $\hat{\mathcal{O}}_{Y,v}$ denotes the completion of $\mathcal{O}_{Y,v}$. Furthermore, the image of $\hat{\mathcal{O}}_{Y,v}$ in $O_{K_v}/m_{K_v}^i$ satisfies the condition of A in Section 2 Case 1. Fix a prime element π of K and a non-empty regular open subset U of Y_{red}. For each $v \in U_0$(U_0 denotes the set of all closed points of U, see Conventions), let H_v be the subgroup of $K_2(K_v)/U^i K_2(K_v)$ generated by elements of the forms $\{a, b\}$ and $\{a, \pi\}$ such that $a, b \in \mathrm{Image}\,((\hat{\mathcal{O}}_{Y,v})^\times \to (O_{K_v}/m_{K_v}^i)^\times)$. By Section 2 Lemma 1, H_v is an open subgroup of $K_2(K_v)/U^i K_2(K_v)$. Now, let $I_{(i)}(K)$ be the subgroup of the direct product $\prod_{v \in P} K_2(K_v)/U^i K_2(K_v)$

consisting of all elements $(a_v)_{v\in P}$ such that $a_v \in H_v$ for almost all $v \in U_0$. We endow $I_{(i)}(K)$ with the unique topology which is compatible with the group structure and for which

$$\prod_{v\in U_0} H_v \times \prod_{v\in P-U_0} K_2(K_v)/U^iK_2(K_v)$$

is an open subgroup with the product topology. It is easily seen that the group $I_{(i)}(K)$ and its topology thus defined is independent of the choices of Y, π, and U.

The diagonal image of $K_2(K)$ in $\prod_{v\in P} K_2(K_v)/U^iK_2(K_v)$ is contained in $I_{(i)}(K)$. Let

$$C_{(i)}(K) = \mathrm{Coker}\,(K_2(K) \longrightarrow I_{(i)}(K)),$$

and define the topology of $C_{(i)}(K)$ to be the quotient of the topology of $I_{(i)}(K)$. Let $C(K) = \varprojlim_i C_{(i)}(K)$, and endow it with the inverse limit topology.

Let $I_{(0)}(K)$ be the idele group of F and let $C_{(0)}(K)$ be the idele class group of F with their natural topologies. Then, via tame symbols, $I_{(0)}(K)$ (resp. $C_{(0)}(K)$) is regarded as a quotient topological group of $I_{(i)}(K)$ (resp. $C_{(i)}(K)$) for $i \geqq 1$.

For $0 \leqq i \leqq j$, let $U^iI_{(j)}(K) = \mathrm{Ker}\,(I_{(j)}(K) \to I_{(i)}(K))$ and $U^iC_{(j)}(K) = \mathrm{Ker}\,(C_{(j)}(K) \to C_{(i)}(K))$. For $i \geqq 0$, let

$$U^iC(K) = \mathrm{Ker}\,(C(K) \longrightarrow C_{(i)}(K)) = \varprojlim_j U^iC_{(j)}(K).$$

Now, we state the class field theory of K. First, consider the commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & H^3(F) \arrow[r] \arrow[d, "\wr"'] & H^3(K) \arrow[r] \arrow[d] & H^2(F) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{v\in P} H^3(F_v) \arrow[r] & \bigoplus_{v\in P} H^3(K_v) \arrow[r] \arrow[dr, "(h_{K_v})_v"'] & \bigoplus_{v\in P} H^2(F_v) \arrow[r] \arrow[d, "(h_{F_v})_v"] & 0 \\
 & & & \boldsymbol{Q}/\boldsymbol{Z} \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

in which the two horizontal sequences are exact sequences given by Section 1 Theorem 1, and the vertical sequence in the right side is the well known Hasse's exact sequence. The left vertical arrow is bijective by Tate [24] Theorem 3.1 (c). From this diagram, we obtain an exact sequence

$$0 \longrightarrow H^3(K) \longrightarrow \bigoplus_{v \in P} H^3(K_v) \xrightarrow{(h_{K_v})_v} \boldsymbol{Q}/\boldsymbol{Z} \longrightarrow 0.$$

Let $\chi \in H^1(K)$. Then, there is an integer $i \geqq 0$ such that the image χ_{K_v} of χ in $H^1(K_v)$ annihilates $U^iK_2(K_v)$ for all $v \in P$ ([5] III Section 3 Lemma 10). Furthermore, if i is such integer and $(a_v)_{v \in P} \in I_{(i)}(K)$, $h_{K_v}(\{\chi_{K_v}, a_v\}) \in \boldsymbol{Q}/\boldsymbol{Z}$ is zero for almost all $v \in P$ by [5] III Section 3 Lemma 9. Hence χ defines a homomorphism

$$I_{(i)}(K) \longrightarrow \boldsymbol{Q}/\boldsymbol{Z}; \quad (a_v)_v \longmapsto \sum_{w \in P} h_{K_v}(\{\chi_{K_v}, a_v\}).$$

By the above exact sequence, this homomorphism factors through $C_{(i)}(K)$ and hence we obtain a canonical pairing

$$H^1(K) \times C(K) \longrightarrow \boldsymbol{Q}/\boldsymbol{Z}.$$

Theorem 4. *Let K be a complete discrete valuation field whose residue field is an A-field. Then, $H^1(K)$ is isomorphic to the group of all continuous homomorphisms $C(K) \longrightarrow \boldsymbol{Q}/\boldsymbol{Z}$ of finite orders.*

In this class field theory, the analogue of Section 2 Proposition 3 holds if we replace the K^M-groups by the idele class groups. In particular, $\chi \in H^1(K)$ is unramified if and only if it annihilates $U^0C(K)$, i.e. if and only if the homomorphism $C(K) \to \boldsymbol{Q}/\boldsymbol{Z}$ induced by χ factors through the idele class group of F.

The above Theorem 4 is proved in [5] III in the case $\mathrm{ch}(F) \neq 0$. In the case $\mathrm{ch}(F) = 0$, this result is easily deduced from the following theorem of Moore. For a field k, let $\mu(k)$ be the group of all roots of 1 in k.

Theorem (Moore [15]). (1) *Let k be a non-discrete locally compact field and assume $k \neq \boldsymbol{C}$. Then, the Hilbert symbol θ_k: $K_2(k) \to \mu(k)$ is surjective and its kernel is a divisible group. For any discrete abelian group G, a homomorphism f: $K_2(k) \to G$ factors through $\mu(k)$ if and only if the map*

$$k^\times \times k^\times \longrightarrow G; \quad (a, b) \longmapsto f(\{a, b\})$$

is continuous.

(2) *Let k be an A-field. Then,* $\mathrm{Ker}(\theta_v) = K_2(O_{k_v})$ *for almost all non-*

archimedean places v of k. Let $P'(k)$ be the set of all places v of k such that $k_v \neq C$. For each $v \in P'(k)$, let $n_v = [\mu(k_v) : \mu(k)]$ and let $n_v : \mu(k_v) \to \mu(k)$ be the homomorphism $x \mapsto x^{n_v}$. Then the sequence

$$K_2(k) \xrightarrow{(\theta_{k_v})_v} \bigoplus_{v \in P'(k)} \mu(k_v) \xrightarrow{(n_v)_v} \mu(k) \longrightarrow 0$$

is exact.

Now, we prove the case $\mathrm{ch}(F)=0$ of Theorem 4. By Section 1 Theorem 1 (1), we have an exact sequence

$$\text{(A)} \qquad 0 \longrightarrow H^1(F) \longrightarrow H^1(K) \longrightarrow H^0(F) \longrightarrow 0.$$

Note $H^0(F) = \mathrm{Hom}(\mu(F), \boldsymbol{Q}/\boldsymbol{Z})$. On the other hand, $U^1C(K)$ is a divisible group in this case, and is annihilated by any homomorphism $C(K) \to \boldsymbol{Q}/\boldsymbol{Z}$ of finite order. Furthermore, the group $C_{(1)}(K)$ has the following structure. Let $\prod'_{v \in P} K_2(F_v)$ be the subgroup of the product $\prod_{v \in P} K(F_v)$ consisting of all elements $(a_v)_{v \in P}$ such that $a_v \in K_2(O_{F_v})$ for almost all non-archimedean places v. Let R_F be the cokernel of $K_2(F) \to \prod'_{v \in P} K_2(F_v)$, and let C_F be the idele class group of F. Then, we have an exact sequence

$$\text{(B)} \qquad 0 \longrightarrow R_F \longrightarrow C_{(1)}(K) \longrightarrow C_F \longrightarrow 0.$$

Compare the exact sequences (A) and (B). By Moore's theorem, $H^0(F)$ is isomorphic to the group of all continuous homomorphisms $R_F \to \boldsymbol{Q}/\boldsymbol{Z}$, and by the class field theory, $H^1(F)$ is isomorphic to the group of all continuous homomorphisms $C_F \to \boldsymbol{Q}/\boldsymbol{Z}$ of finite orders. Theorem 4 for the case $\mathrm{ch}(F)=0$ follows easily from these facts.

The following lemma will be used in Chapter II.

Lemma 2. *If* $\mathrm{ch}(F) \neq 0$, *a discrete quotient group of* $U^0C(K)$ *is finite. If* $\mathrm{ch}(F)=0$, *a discrete quotient group of* $U^0C_{(1)}(K)$ *is finite.*

Proof. Let i be any non-negative integer in the case $\mathrm{ch}(F) \neq 0$, and let $i=0$ in the case $\mathrm{ch}(F)=0$. Let G be a discrete quotient group of $U^iC_{(i+1)}(K)$. It suffices to prove that G is finite.

First, we consider the case $i=0$. Let R_F be the "K_2-idele class group" of F defined as above. By Moore's theorem, the composite $R_F \cong U^0C_{(1)}(K) \to G$ factors through the canonical surjection $R_F \to \mu(F)$, and this shows that G is finite.

Next, assume $i \geqq 1$ and $\mathrm{ch}(F)=p>0$. Take an element g of U_K such that the residue class of g in F is not contained in F^p, and let π be a prime element of K. By Section 1 Proposition 2, we have a continuous surjection for each $v \in P$;

$$F_v \oplus F_v \longrightarrow U^i K_2(K_v)/U^{i+1}K_2(K_v)$$
$$(a, b) \longrightarrow \{1+\tilde{a}\pi^i, g\} + \{1+\tilde{b}\pi^i, \pi\}.$$

Take Y and U as before. Then, the image of g (resp. π) in $O_{K_v}/m^i_{K_v}$ is contained in the image of $(\hat{\mathcal{O}}_{Y,v})^\times$ (resp. $\hat{\mathcal{O}}_{Y,v}$) for almost all $v \in U_0$. This fact shows that there is a continuous surjection

$$\prod_{v\in P}{}' F_v \oplus \prod_{v\in P}{}' F_v \longrightarrow U^i I_{(i+1)}(K)$$
$$((a_v)_v, (b)_v) \longmapsto (\{1+\tilde{a}_v\pi^i, g\} + \{1+\tilde{b}_v\pi^i, \pi\})_v,$$

where $\prod'_{v\in P} F_v$ denotes the adele group of F. This induces a continuous surjection

$$(\prod_{v\in P}{}' F_v)/F \oplus (\prod_{v\in P}{}' F_v)/F \longrightarrow U^i C_{(i+1)}(K).$$

Since $(\prod'_{v\in P} F_v)/F$ is compact, a discrete quotient group of $U^i C_{(i+1)}(K)$ is isomorphic to a discrete quotient of a compact group, and hence is finite.

§ 4. Arithmetical two dimensional local rings

Let A be a normal complete Noetherian local domain with finite residue field k, and let K be the field of fractions of A. Let P be the set of all prime ideals of height one of A. For each $z \in P$, let K_z be the z-adic completion of K, which is a two dimensional local field.

The second semi-global theory is the class field theory of K. To describe this, we define the idele group $I_m(K)$ and the idele class group $C_m(K)$ with modulus m.

Definition 1. Let X be a normal Noetherian scheme. A modulus m on X is a family $(m(z))_z$ of non-negative integers $m(z)$ given for each point z of codimension one of X, such that $m(z)=0$ for almost all z.

For a modulus m on Spec(A), let

$$I_m(K) = \bigoplus_{z\in P} K_2(K_z)/U^{m(z)}K_2(K_z),$$
$$C_m(K) = \mathrm{Coker}\,(K_2(K) \to I_m(K))$$

(the map $K_2(K) \to I_m(K)$ is the diagonal map). The complex of Bloch-Gersten-Quillen

$$K_2(K) \longrightarrow \bigoplus_z K_1(\kappa(z)) \longrightarrow K_0(k) \longrightarrow 0$$

and the tame symbols $K_2(K_z) \to K_1(\kappa(z))$ give a canonical homomorphism $d_x\colon C_m(K) \to K_0(k) = \mathbf{Z}$, which is an isomorphism if $m=(0)$ and A is regular.

Let $C(K)=\varprojlim_{m} C_m(K)$ where m ranges over all moduli on $\mathrm{Spec}(A)$. We endow $C_m(K)$ with the discrete topology, and $C(K)$ with the topology as the inverse limit. These topologies are appropriate because;

Lemma 3. *For any modulus m and $z \in P$, the canonical homomorphism $K_2(K_z)/U^{m(z)}K_2(K_z)\to C_m(K)$ is continuous for the discrete topology on $C_m(K)$.*

Here, the topology of $K_2(K_z)/U^iK_2(K_z)$ is defined with respect to the image of A in $O_{K_z}/m^i_{K_z}$ (cf. Section 2 Case 1).

Proof. Fix a modulus m and $z \in P$. By Lemma 1, it suffices to prove the following fact: For each element h of $K^\times$, there is an open neighbourhood V of 1 in $A^\times$ such that all elements of $K_2(K_z)/U^iK_2(K_z)$ of the forms $\{a, b\}$ ($a \in V$, $b \in A^\times$) and $\{a, h\}$ ($a \in V$) have zero images in $C_m(K)$. Indeed, let J_1 be the ideal of A consisting of all elements a satisfying the following conditions (1) (2).

(1) $\mathrm{ord}_{K_{z'}}(a)\geqq m(z')$ if $z' \in P-\{z\}$.

(2) $\mathrm{ord}_{K_{z'}}(a)\geqq 1$ if $z' \in P-\{z\}$ and $\mathrm{ord}_{z'}(h)\neq 0$.

On the other hand, let J_2 be the ideal of A consisting of all elements a such that $\mathrm{ord}_{K_z}(a)\geqq \mathrm{Sup}\,(m(z), 1)$. Let $G_i=\mathrm{Ker}\,(A^\times\to(A/J_i)^\times)$ for $i=1, 2$. Then, for $z' \in P-\{z\}$,

$$\{a, b\}=\{a, h\}=0 \qquad \text{in } K_2(K_{z'})/U^{m(z')}K_2(K_{z'})$$

for any $a \in G_1$ and $b \in A^\times$. Hence, the images in $C_m(K)$ of

$$\{a, b\}, \{a, h\} \in K_2(K_z) \quad (a \in G_1G_2,\ b \in A^\times)$$

vanish by the definition of $C_m(K)$. But the subgroup G_1G_2 is open in $A^\times$.

Q.E.D.

We explain the class field theory of K. First, the group $H^3(K)$ satisfies the following reciprocity law: The image of $H^3(K)\to\prod_{z\in P} H^3(K_z)$ is contained in $\oplus_{z\in P} H^3(K_z)$ and the composite $H^3(K)\to\oplus_{z\in P} H^3(K_z)\xrightarrow{(h_{K_z})_z} \boldsymbol{Q}/\boldsymbol{Z}$ is the zero map (cf. Saito [24] Chapter I). Let $\chi \in H^1(K)$. Then, the image χ_{K_z} of χ in $H^1(K_z)$ induces a homomorphism $K_2(K_z)/U^iK_2(K_z)\to \boldsymbol{Q}/\boldsymbol{Z}$ for some i (which depends on z), and annihilates $U^0K_2(K_z)$ if χ_{K_z} is unramified (Section 2 Corollary to Proposition 3). Since χ_{K_z} is unramified at almost all z, there exists a modulus m such that χ defines a homomorphism $I_m(K)\to\boldsymbol{Q}/\boldsymbol{Z}$. The above reciprocity law shows that this homomorphism induces $C_m(K)\to\boldsymbol{Q}/\boldsymbol{Z}$. Thus we obtain the canonical pairing

$$H^1(K)\times C(K)\longrightarrow \boldsymbol{Q}/\boldsymbol{Z}.$$

The induced diagram

$$\begin{array}{ccc} C(K) & \longrightarrow & \mathrm{Gal}(K^{\mathrm{ab}}/K) \\ {\scriptstyle d}\downarrow & & \downarrow \\ \boldsymbol{Z} & \xrightarrow{\subset} & \mathrm{Gal}(\bar{k}/k)=\hat{\boldsymbol{Z}}. \end{array}$$

is commutative, where the right vertical arrow is induced by the restriction to the unramified part.

Theorem 5. *The image of* $H^1(K)\to\mathrm{Hom}(C(K), \boldsymbol{Q}/\boldsymbol{Z})$ *consists of all continuous homomorphisms of finite orders, and its kernel is isomorphic to the direct sum* $(\boldsymbol{Q}/\boldsymbol{Z})^r$ *of* r *copies of* $\boldsymbol{Q}/\boldsymbol{Z}$ *for some non-negative integer* r. *If* A *is regular, we have* $r=0$.

The part of this theorem concerning the kernel was proved in Saito [18], and the part concerning the image was proved under a certain condition on A also in [18]. (In [18], a restricted product $\prod'_{z\in P} K_2(K_z)$ is used instead of the above idele group $I_m(K)$.) This complete form of the theorem is proved by using a recent result of Merkuriev-Suslin [12]. In fact, the complete form of Theorem 5 is not necessary for the purpose of this paper. We need in Chapter II only the existence of the canonical pairing and some other facts explained in this section.

§ 5. Curves over local fields

Let k be a non-discrete locally compact field, X a regular proper connected curve over k, and let K be the function field of X. The third semi-global theory is the class field theory of K. (We call the fields K treated in Section 3, 4 and 5 semi-global fields.) Let P be the set of all closed points of X. For $u\in P$, let K_u be the u-adic completion of K, which is a two dimensional local field. Since the residue field of K_u is a finite extension of k, we obtain the topology of $K_2(K_u)/U^iK_2(K_u)$ with respect to k (Section 2 Case 2). Let m be a modulus on X in the sense of Section 4 Definition 1. We define

$$I_m(K)=\bigoplus_{u\in P} K_2(K_u)/U^{m(u)}K_2(K_u),$$

$$C_m(K)=\mathrm{Coker}\,(K_2(K)\longrightarrow I_m(K)).$$

Since $K_2(K_u)/U^0K_2(K_u)=\kappa(u)^\times$, $C_{(0)}(K)$ is isomorphic to the group which is usually denoted by $SK_1(X)$. We endow $C_m(K)$ with the finest topology which is compatible with the group structure and for which the canonical maps $K_2(K_u)/U^{m(u)}K_2(K_u)\to C_m(K)$ are continuous for all $u\in P$. Let

$C(K)=\varprojlim_m C_m(K)$ with the inverse limit topology.

The group $H^3(K)$ satisfies the following reciprocity law; for each $\chi \in H^3(K)$, $h_{K_u}(\chi_{K_u}) \in \boldsymbol{Q}/\boldsymbol{Z}$ are zero for almost all $u \in P$, and $\sum_{u \in P} h_{K_u}(\chi_{K_u}) = 0$ (χ_{K_u} denotes the image of χ in $H^3(K_u)$). This fact is proved in [17] in the case where k is non-archimedian and X is smooth over k. In the case $\mathrm{ch}\,(k)=0$ (which contains the cases $k=\boldsymbol{R}, \boldsymbol{C}$), this reciprocity law follows from the exact sequence

$$H^3(K) \longrightarrow \bigoplus_{u \in P} H^4_u(X, \boldsymbol{Q}/\boldsymbol{Z}(2)) \longrightarrow H^4(X, \boldsymbol{Q}/\boldsymbol{Z}(2))$$

((2) denotes the Tate twist) and from the fact that h_{K_u} coincides with the composite

$$\begin{aligned} H^3(K_u) &\longrightarrow H^4_u(X, \boldsymbol{Q}/\boldsymbol{Z}(2)) \longrightarrow H^4(X, \boldsymbol{Q}/\boldsymbol{Z}(2)) \\ &\longrightarrow H^2(k, H^2(X \otimes_k k_s, \boldsymbol{Q}/\boldsymbol{Z}(2))) \longrightarrow H^2(k) \subset \boldsymbol{Q}/\boldsymbol{Z}. \end{aligned}$$

If $\mathrm{ch}\,(k)=p>0$, by using the trace map of H^3 (cf. [5] II Section 3.2), the reciprocity law is reduced to the case X is smooth (even to the case $X=\boldsymbol{P}^1_k$).

Just as in Section 4, we obtain the canonical pairing

$$H^1(K) \times C(K) \longrightarrow \boldsymbol{Q}/\boldsymbol{Z}.$$

Theorem 6. *The image of the homomorphism* $H^1(K) \to \mathrm{Hom}\,(C(K), \boldsymbol{Q}/\boldsymbol{Z})$ *consists of all continuous homomorphisms of finite orders. If k is non-archimedian, its kernel is isomorphic to $(\boldsymbol{Q}/\boldsymbol{Z})^r$ for some interger $r \geqq 0$, and $r=0$ if X has good reduction.*

The part of Theorem 6 concerning the kernel was proved in [24]. (In the good reduction case, it is due to Bloch [4].) Partial results concerning the image were proved in [4] [14], and again the result of Merkuriev-Suslin enables us to complete them. We do not use this theorem in this paper though some partial precise results in Theorem 6 were necessary in [3] [8] for the study of the two dimensional unramified class field theory.

The following result is also not used in this paper, but is stated here to complete the description.

Theorem 7. *Let K be a field of the type considered in Section* 4 (*resp. in Section* 5 *and assume that k is non-archimedean*). *Then, we have an exact sequence*

$$0 \longrightarrow (\boldsymbol{Q}/\boldsymbol{Z})^r \longrightarrow H^3(K) \longrightarrow \bigoplus_{p \in P} H^3(K_p) \longrightarrow \boldsymbol{Q}/\boldsymbol{Z} \longrightarrow 0.$$

Here, r is the integer in Theorem 5 (*resp. Theorem* 6).

Chapter II. Global theory

§ 1. Localizations

Let k be an A-field, i.e. an algebraic number field or an algebraic function field in one variable over a finite field. In the former case, let S be a non-empty open subscheme of Spec (O_k). In the latter case, let S be a smooth connected (not necessarily proper) curve over a finite field with function field k. In both cases, denote by $P(k)$ the set of all places of k, and let Σ be the finite set of all places of k which are not contained in S.

Let X be a proper flat S-scheme which is normal, connected, and of dimension two. We assume that $X_k = X \otimes_S k$ is geometrically irreducible over k. The main purpose of our global class field theory is to describe the abelian fundamental groups $\pi_1^{\mathrm{ab}}(U)$ of regular open subschemes U of X, and the maximum abelian Galois group Gal (K^{ab}/K) of the function field K of X, by using the K_2-idele class groups of X. In this section, we observe how two dimensional local fields and semi-global fields of Chapter I are associated with X. As in Conventions, for a scheme T and $i \geqq 0$, T_i denotes $\{t \in T;\ \dim \overline{\{t\}} = i\}$.

1°. For each $y \in X_1$, let $\hat{\mathcal{O}}_{X,y}$ be the completion of the discrete valuation ring $\mathcal{O}_{X,y}$, and let K_y be the field of fractions of $\hat{\mathcal{O}}_{X,y}$. Then, K_y is a "semi-global field" of Chapter I Section 3 whose residue field is the global field $\kappa(y)$.

Next, for each $x \in X_0$, let $\hat{\mathcal{O}}_{X,x}$ be the completion of the two dimensional local ring $\mathcal{O}_{X,x}$, and let K_x be the field of fractions of $\hat{\mathcal{O}}_{X,x}$. Then, $\hat{\mathcal{O}}_{X,x}$ is normal by EGA., Chapter IV Section 7.8, and K_x is a semi-global field studied in Chapter I Section 4. Each $z \in \mathrm{Spec}\,(\hat{\mathcal{O}}_{X,x})_1$ defines the two-dimensional local field $K_{X,z}$, the z-adic completion of K_x.

Lastly, for each $v \in P(k)$, let K_v be the function field of $X_{k_v} = X \otimes_S k_v$. ($k_v$ denotes the completion of k at v.) It is a semi-global field of Chapter I Section 5. Each $u \in (X_{k_v})_0$ defines a two dimensional local field $K_{v,u}$, the u-adic completion of K_v.

2°. We next observe how the two dimensional local fields $K_{x,z}$ and $K_{v,u}$ are related to the semi-global field K_y ($y \in X_1$).

For $x \in X_0$ and $y \in X_1$, let the set $y(x)$ be the inverse image of y under the map Spec $(\hat{\mathcal{O}}_{X,x}) \to X$. This set $y(x)$ is not empty if and only if $x \in \overline{\{y\}}$. It is a finite subset of Spec $(\hat{\mathcal{O}}_{X,x})_1$, and is identified with the set of "branches" of $\overline{\{y\}}$ at x (Figure 1). For each $z \in y(x)$, the residue field $\kappa(z)$ of z is the completion of $\kappa(y)$ at some place, and in this way, we identify $y(x)$ with a subset of $P(\kappa(y))$ ($P(\kappa(y))$ is the set of all places of $\kappa(y)$). The field $K_{x,z}$ ($z \in y(x)$) is thus identified with the "local field" $K_{y,z}$ of K_y defined in

Chapter I Section 3.

For $v \in P(k)$ and $y \in X_1$, let the set $y(v)$ be the inverse image of y under the map $X_{k_v} \to X$. It is not empty if and only if $\overline{\{y\}}$ is a horizontal divisor of X with respect to the fibration $X \to S$. For each $z \in y(v)$, $\kappa(z)$ is a completion of $\kappa(y)$ at some place, and we thus identify $y(v)$ with the finite subset of $P(\kappa(y))$ consisting of all extensions of v to $\kappa(y)$. The field $K_{v,u}$ ($u \in y(v)$) is thus identified with the "local field" $K_{y,u}$ of K_y defined in Chapter I Section 3.

For example, let $k = \boldsymbol{Q}$, $S = \operatorname{Spec}(\boldsymbol{Z})$, and $X = \boldsymbol{P}^1_{\boldsymbol{Z}} = \operatorname{Spec}(\boldsymbol{Z}[T]) \cup \operatorname{Spec}(\boldsymbol{Z}[T^{-1}])$. If y is the point in X_1 corresponding to the prime ideal (T) of $\boldsymbol{Z}[T]$, we have $K_y = \boldsymbol{Q}((T))$. If p is a prime number and x is the point in X_0 corresponding to the maximal ideal (p, T) of $\boldsymbol{Z}[T]$, K_x is the field of fractions of $\boldsymbol{Z}_p[[T]]$ and the T-adic completion of K_x is the local field $\boldsymbol{Q}_p((T))$ of K_y. If v is the place of $\boldsymbol{Q}$, $K_v = \boldsymbol{Q}_v(T)$ and the T-adic completion of K_v is the local field $\boldsymbol{Q}_v((T))$ of K_y.

For $y \in Y_1$, the set $P(\kappa(y))$ is the disjoint union of the subsets $y(x)$ and $y(v)$, where x ranges over all closed points of $\overline{\{y\}}$ and v ranges over $\Sigma = P(k) - S_0$. If $\overline{\{y\}}$ is horizontal, $P(\kappa(y))$ is also the disjoint union of subsets $y(v)$ where v ranges over $P(k)$. If $\overline{\{y\}}$ is vertical, $P(\kappa(y))$ is the disjoint union of $y(x)$ ($x \in \overline{\{y\}}_0$). In any case, we obtain a complete system $K_{y,z}$ ($z \in P(\kappa(y))$) of local fields of the semi-global field K_y.

3°. Lastly, we observe that for $v \in S_0$, some of the two dimensional local fields $K_{x,z}$ are regarded as local fields of the semi-global field K_v. Let Y_v be the fiber $X \otimes_S \kappa(v)$ of X on v, which we identify with the closed fiber of the scheme $X \otimes_S O_{k_v}$ over O_{k_v}. For $x \in (Y_v)_0$, $\hat{\mathcal{O}}_{X,x}$ is isomorphic to the completion of the local ring of $X \otimes_S O_{k_v}$ at x, and hence, we have a morphism $\operatorname{Spec}(\hat{\mathcal{O}}_{X,x}) \to X \otimes_S O_{k_v}$. Let $v(x)_{\mathrm{ver}}$ (resp. $v(x)_{\mathrm{hor}}$) be the inverse image in $\operatorname{Spec}(\hat{\mathcal{O}}_{X,x})_1$ of the closed fiber Y_v (resp. the generic fiber X_{k_v}) of $X \otimes_S O_{k_v}$. The set $v(x)_{\mathrm{ver}}$ is finite, and we have

$$v(x)_{\mathrm{ver}} = \coprod_{y \in (Y_v)_1} y(x).$$

On the other hand, we have a canonical bijection

$$\coprod_{x \in (Y_v)_0} v(x)_{\mathrm{hor}} \longrightarrow (X_{k_v})_0.$$

If $z \in v(x)_{\mathrm{hor}}$ and u is the corresponding point in $(X_{k_v})_0$, the natural homomorphisms $K_v \to K_x \to K_{x,z}$ induce a canonical isomorphism

$$K_{v,u} \cong K_{x,z} \qquad \text{(Figure 2).}$$

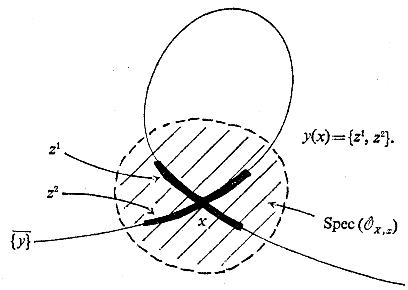

Fig. 1

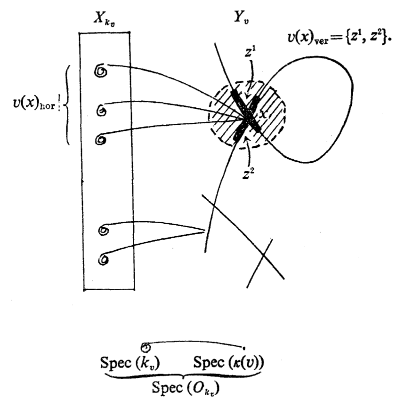

Fig. 2

§ 2. The idele class group

Let k, S and X be as in Section 1. Let $m=(m(y))_{y\in X_1}$ be a modulus on X in the sense of Chapter I Section 4 Definition 1. We define the support Supp(m) of m to be the closure in X of the finite set $\{y \in X_1; m(y)\neq 0\}$. In this section, we define the K_2-idele class group $C_m(X/S)$ of X with modulus m. This group will be related to the class field theory of abelian coverings of X which are unramified outside Supp (m).

For each $x \in X_0$, m induces a modulus m_x on Spec $(\hat{\mathcal{O}}_{X,x})$ as follows. If the image of $z \in$ Spec $(\hat{\mathcal{O}}_{X,x})_1$ in X is $y \in X_1$, let $m_x(z)=m(y)$. If the image of z in X is the generic point of X, let $m_x(z)=0$. (The second situation actually occurs. For example, if $S=$ Spec $(\boldsymbol{Z})$, $X=\boldsymbol{P}^1_{\boldsymbol{Z}}$ and $\hat{\mathcal{O}}_{X,x}=\boldsymbol{Z}_p[[T]]$ as in the example in Section 1, the image in X of the prime ideal $(T-a)$ of $\hat{\mathcal{O}}_{X,x}$ for $a \in p\boldsymbol{Z}_p$ is the generic point of X if a is transcendental over $\boldsymbol{Q}$.) Define

$$C_m(x)=C_{m_x}(K_x),$$

which is the idele class group of K_x with modulus m_x defined in Chapter I Section 4. In the case $\mathcal{O}_{X,x}$ is regular and $x \notin$ Supp (m), we identify $C_m(x)$ with $\boldsymbol{Z}=K_0(x)$ as in Chapter I Section 4.

For $v \in P(k)$, m induces a modulus m_v on X_{k_v} as follows. For $u \in (X_{k_v})_0$, let $m_v(u)=m(y)$ if the image of u in X is $y \in X_1$, and $m_v(u)=0$ if the image of u in X is the generic point of X. Let

$$C_m(v)=C_{m_v}(K_v)$$

(see Chapter I Section 5). We have $C_m(v)=SK_1(X_{k_v})$ if Supp (m) has only vertical components.

Let $y \in \overline{\{x\}}$. As was explained in Section 1,

$$P(\kappa(y)) = (\coprod_{x \in X_0} y(x)) \coprod (\coprod_{v \in \Sigma} y(v)) \qquad (\Sigma = P(k) - S_0).$$

This gives a homomorphism

$$\begin{aligned} &\prod_{v \in P(\kappa(y))} K_2(K_{y,v})/U^{m(y)}K_2(K_{y,v}) \\ &= \prod_{x \in X_0} \prod_{z \in y(x)} K_2(K_{x,z})/U^{m_x(z)}K_2(K_{x,z}) \\ &\qquad \oplus \prod_{v \in \Sigma} \prod_{u \in y(v)} K_2(K_{v,u})/U^{m_v(u)}K_2(K_{v,u}) \\ &\to \prod_{x \in X_0} C_m(x) \oplus \prod_{v \in \Sigma} C_m(v). \end{aligned}$$

Lemma 4. *The image of the idele group $I_{(m(y))}(K_y)$ (Chapter I Section 3) in $\prod_{x \in X_0} C_m(x) \oplus \prod_{v \in \Sigma} C_m(v)$ is contained in $\bigoplus_{x \in X_0} C_m(x) \oplus \bigoplus_{v \in \Sigma} C_m(v)$. The first projection $I_{(m(y))}(K_y) \to \bigoplus_{x \in X_0} C_m(x)$ is continuous if we endow $\bigoplus_{x \in X_0} C_m(x)$ with the discrete topology.*

Definition 1. Let $C_m(X/S)$ be the cokernel of the composite map

$$\bigoplus_{y \in X_1} K_2(K_y) \longrightarrow \bigoplus_{y \in X_1} I_{(m(y))}(K_y) \longrightarrow (\bigoplus_{x \in X_0} C_m(x)) \oplus (\bigoplus_{v \in \Sigma} C_m(v)).$$

We endow $C_m(X/S)$ with the finest topology which is compatible with the group structure and for which the canonical maps $C_m(v) \to C_m(X)$ are continuous for all $v \in \sum$ ($C_m(v)$ has the topology defined in Chapter I Section 5). We obtain canonical homomorphisms

$$C_{(m(y))}(K) \longrightarrow C_m(X) \qquad (y \in X_1)$$

which are continuous by Lemma 4.

Proof of Lemma 4. Let I be the ideal of $\mathcal{O}_X$ defining the reduced closed subscheme $\overline{\{y\}}$ of X. Then, the closed subscheme $Y = \mathrm{Spec}\,(\mathcal{O}_X/I^i)$ of X satisfies the condition of Y in Chapter I Section 3. Take an element π of K which is a prime element in K_y, and take a sufficiently small non-empty regular open subset U of Y_{red} satisfying the following condition: If $y' \in X_1 - \{y\}$ and if y' satisfies either $\mathrm{ord}_{K_{y'}}(\pi) \neq 0$ or $m(y') \neq 0$, then $U \cap \overline{\{y'\}} = \phi$. For any $x \in U_0$ and the unique element v of $y(x)$, the elements

$$\{a, b\}, \{a, \pi\} \in K_2(K_{y,v})/U^{m(y)}K_2(K_{y,v}) \qquad (a, b \in (\hat{\mathcal{O}}_{X,x})^\times)$$

have zero images in $C_m(x)$ (see the proof of Chapter I Section 4 Lemma 3).

By the definitions of $I_{(m(y))}(K_y)$ and its topology, and by Chapter I Lemma 3, this fact shows that the image of $I_{(m(y))}(K_y)$ in $\prod_{x \in X_0} C_m(x)$ is contained in $\bigoplus_{x \in X_0} C_m(x)$, and that the image of some open subgroup of $I_{(m(y))}(K_y)$ in $\bigoplus_{x \in X_0} C_m(x)$ is zero. Q.E.D.

Lastly, we vary the base S. Let S' be a non-empty open subscheme of S, let $X_{S'} = X \times_S S'$, and denote the restriction of m (as a function on X_1) to $(X_{S'})_1$ also by m. Then, we obtain a surjective continuous homomorphism

$$C_m(X_{S'}/S') \longrightarrow C_m(X/S)$$

as follows. For each $v \in S_0$, let $Y_v = X \otimes_S \kappa(v)$, and let $\tilde{C}_m(v)$ be the cokernel of

$$\bigoplus_{y \in (Y_v)_1} K_2(K_y) \longrightarrow \bigoplus_{x \in (Y_v)_0} C_m(x).$$

Then, $C_m(X/S)$ is isomorphic to the cokernel of

$$\bigoplus_{y \in (X_1)_{\mathrm{hor}}} K_2(K_y) \longrightarrow (\bigoplus_{v \in S_0} \tilde{C}_m(v)) \oplus (\bigoplus_{v \in P(k) - S_0} C_m(v))$$

and $C_m(X_{S'}/S')$ is isomorphic to the cokernel of

$$\bigoplus_{y \in (X_1)_{\mathrm{hor}}} K_2(K_y) \longrightarrow (\bigoplus_{v \in (S')_0} \tilde{C}_m(v)) \oplus (\bigoplus_{v \in P(k) - (S')_0} C_m(v)),$$

where $(X_1)_{\mathrm{hor}}$ denotes the subset of X_1 consisting of all points y such that $\overline{\{y\}}$ are horizontal divisors. The desired homomorphism $C_m(X_{S'}/S') \to C_m(X/S)$ is induced by the surjective homomorphisms $C_m(v) \to \tilde{C}_m(v)$ defined as follows.

Let $v \in S_0$. For $x \in (Y_v)_0$, let

$$\mathrm{Spec}\,(\hat{\mathcal{O}}_{X,x})_1 = v(x)_{\mathrm{ver}} \coprod v(x)_{\mathrm{hor}}$$

be the decomposition given in Section 1, 3°. Since $v(x)_{\mathrm{ver}}$ is a finite set, the projection

$$K_2(K_x) \longrightarrow \bigoplus_{z \in v(x)_{\mathrm{ver}}} K_2(K_{x,z})/U^{m_x(z)} K_2(K_{x,z})$$

is surjective by the usual approximation theorem for a finite family of valuations. Hence,

$$\bigoplus_{z \in v(x)_{\mathrm{hor}}} K_2(K_{x,z}) \longrightarrow C_m(x)$$

is surjective. The commutative diagram

$$\begin{CD}
K_2(K_v) @>>> \bigoplus_{u\in(X_{k_v})_0} K_2(K_{v,u})/U^{m_v(u)}K_2(K_{v,u}) \\
@VVV @| \wr \\
@. \bigoplus_{x\in(Y_v)_0}\bigoplus_{z\in v(x)_{\mathrm{hor}}} K_2(K_{x,z})/U^{m_x(z)}K_2(K_{x,z}) \\
@. @VVV \\
\bigoplus_{y\in(Y_v)_1} K_2(K_y) @>>> \bigoplus_{x\in(Y_v)_0} C_m(x)
\end{CD}$$

defines a surjective homomorphism between the cokernels of the two horizontal arrows; $C_m(v)\to\tilde{C}_m(v)$. It is continuous by Chapter I Section 4 Lemma 3.

§ 3. The class field theory

In this section, we prove main results of this paper. Let S and X be as in Section 1. For a commutative topological group G, let G^* be the group of all continuous homomorphisms $G\to\boldsymbol{Q}/\boldsymbol{Z}$ of finite orders.

Theorem 1. *Let U be a non-empty regular open subscheme of X. Then, there exists a canonical isomorphism*

$$H^1(U, \boldsymbol{Q}/\boldsymbol{Z})\cong\bigcup_m C_m(X/S)^*$$

where m ranges over all moduli on X such that $U\cap$ Supp $(m)=\phi$.

Theorem 2. *Let K be the function field of X. Then, there exists a canonical isomorphism*

$$H^1(K)\cong\bigcup_m C_m(X/S)^*$$

where m ranges over all moduli on X.

In these results, if m and m' are moduli such that $m'\geqq m$, we regard $C_m(X/S)^*$ as a subgroup of $C_{m'}(X/S)^*$ in the natural way.

1°. First, we define a homomorphism

$$H^1(U, \boldsymbol{Q}/\boldsymbol{Z})\longrightarrow\bigcup_{U\cap\mathrm{Supp}(m)=\phi} C_m(X/S)^*,$$

or equivalently, a continuous homomorphism

$$\varprojlim_{U\cap\mathrm{Supp}(m)=\phi} C_m(X/S)\longrightarrow\pi_1^{\mathrm{ab}}(U)$$

for a non-empty open subscheme U of X. The definition of

$$H^1(K)\longrightarrow\bigcup_{\text{all } m} C_m(X/S)^*$$

will follow from this, for $H^1(K)=\bigcup_U H^1(U, \boldsymbol{Q}/\boldsymbol{Z})$.

Let $\chi \in H^1(U, \boldsymbol{Q}/\boldsymbol{Z})$. For $y \in X_1$, $x \in X_0$, and $v \in \Sigma = P(k)-S_0$, by the class field theories of K_y, K_x, and K_v (Chapter I Sections 3, 4, 5), χ induces elements of $C(K_y)^*$, $C(K_x)^*$ and $C(K_v)^*$ respectively. Since χ is unramified over U, the induced element of $C(K_y)^*$ annihilates $U^0C(K_y)$ for any $y \in U_1$. For $y \in U_1$, let $m(y)=0$, and for $y \in X_1 - U_1$, let $m(y)$ be any positive integer such that the induced element of $C(K_y)^*$ factors through $C_{(m(y))}(K_y)$. Let m be the modulus $(m(y))_{y\in X_1}$. Then, by the identifications of "local fields" of K_y, K_x and K_v explained in Section 1, 2°, the induced elements of $C(K_x)^*$ and $C(K_v)^*$ factor through $C_m(x)$ and $C_m(v)$, respectively. The induced homomorphism $\bigoplus_{x\in X_0} C_m(x) \oplus \bigoplus_{v\in\Sigma} C_m(v) \to \boldsymbol{Q}/\boldsymbol{Z}$ annihilates the images of $K_2(K_y)$ ($y \in X_1$), and hence it defines an element of $C_m(X/S)^*$.

By this definition, we see

Lemma 2. *For $x \in U_0$ such that $\mathcal{O}_{X,x}$ is regular, the image of $1 \in \boldsymbol{Z}$ under*

$$\boldsymbol{Z} = \varprojlim_{U\cap \mathrm{Supp}(m)=\phi} C_m(x) \longrightarrow \varprojlim_{U\cap \mathrm{Supp}(m)=\phi} C_m(X/S) \longrightarrow \pi_1^{\mathrm{ab}}(U)$$

coincides with the Frobenius substitution of x.

2°. Let U be a non-empty open subscheme of X. We prove that the homomorphism

$$H^1(U, \boldsymbol{Q}/\boldsymbol{Z}) \longrightarrow \bigcup_{U\cap \mathrm{Supp}(m)=\phi} C_m(X/S)^*$$

is injective. Indeed, an element χ of $H^1(U, \boldsymbol{Q}/\boldsymbol{Z})$ of order r defines a cyclic étale connected covering U_χ of U of degree r. If χ is in the kernel of this homomorphism, Lemma 2 shows that the covering U_χ/U splits completely over any closed point of a non-empty regular open subscheme U' of U. This implies that the Hasse-Zeta functions of $U'_\chi = U_\chi \times_U U'$ and U' (cf. Serre [22]) satisfy

$$Z(U'_\chi, s) = Z(U', s)^r \qquad \text{in } \mathrm{Re}\,(s)>2.$$

Since both $Z(U'_\chi, s)$ and $Z(U', s)$ are analytically continued to the range $\mathrm{Re}\,(s)>3/2$ and have simple poles at $s=2$ ([22] Theorem 2, Theorem 3), we have $r=1$ and hence $\chi=0$. (Cf. Lang [10] for this method using Zeta functions.)

3°. Let U be as above and let $U_k = U\otimes_S k$. We obtain a canonical homomorphism

$$H^1(U_k, \boldsymbol{Q}/\boldsymbol{Z}) \longrightarrow \bigcup_{m,S'} C_m(X_{S'}/S')^*$$

where $X_{S'}=X\times_S S'$, m ranges over all moduli on X such that $U\cap \mathrm{Supp}(m)=\phi$, S' ranges over all non-empty open subschemes of S, and where we define the inclusion $C_m(X_{S'}/S')^*\subset C_m(X_{S''}/S'')^*$ for $S''\subset S'$ as at the end of Section 2. Since the group $\bigcup_{m,S'} C_m(X_{S'}/S')^*$ depends only on the curve U_k over k, we denote it simply by $D(U_k)$. We now prove

Theorem 3. $H^1(U_k, Q/Z)\xrightarrow{\cong} D(U_k)$.

By 2°, it suffices to prove the surjectivity.

First, we consider the case $U_k=X_k$. The unramified class field theory of X_k is studied in Bloch [3] Section 4 and Kato-Saito [8] Section 6. We proved in [8] that there exists a canonical isomorphism

$$H^1(X_k, Q/Z)\cong (C_{X_k})^*$$

assuming that X_k is smooth over k (this condition is automatically satisfied in the case $\mathrm{ch}(k)=0$), where C_{X_k} is the idele class group defined as follows. We shall soon see that this isomorphism is extended to the case where X_k is not necessarily smooth. Let $\prod'_{v\in P(k)} SK_1(X_{k_v})$ be the subgroup of the direct product $\prod_{v\in P(k)} SK_1(X_{k_v})$ consisting of all elements $(a_v)_{v\in P(k)}$ such that a_v belongs to the kernel of the (surjective) boundary map $SK_1(X_{k_v})\to CH_0(Y_v)$ for almost all $v\in S_0$, where $Y_v=X\otimes_S\kappa(v)$ and

$$CH_0(Y_v)=\mathrm{Coker}\,(\bigoplus_{y\in(Y_v)_1}\kappa(y)^\times\longrightarrow\bigoplus_{x\in(Y_v)_0}Z).$$

The definition of C_{X_k} is

$$C_{X_k}=\mathrm{Coker}\,(SK_1(X_k)\longrightarrow\prod_{v\in P(k)}{}' SK_1(X_{k_v})).$$

Assume $\mathrm{ch}(k)=p>0$ and X_k is not necessarily smooth. For some $r\geqq 0$, the integral closure X' of X_k in the composite field $K\cdot k^{p^{-r}}$ is smooth over $k^{p^{-r}}$. We have a commutative diagram

$$\begin{array}{ccc} H^1(X_k, Q/Z) & \longrightarrow & (C_{X_k})^* \\ \downarrow & & \downarrow{\scriptstyle \text{by norm}} \\ H^1(X', Q/Z) & \longrightarrow & (C_{X'})^*. \end{array}$$

The left vertical arrow is bijective by SGA 1, Chapter IX Theorem 4.10, and the right vertical arrow is injective since the norm homomorphism $C_{X'}\to C_{X_k}$ is surjective as is easily seen. Thus, the bijectivity of the upper horizontal arrow is reduced to that of the lower horizontal arrow.

Now, we show that in the case $U_k=X_k$, Theorem 3 is deduced from the above unramified class field theory of X_k. Take sufficiently small S' such that $X_{S'}$ is regular. Then, we have (cf. the last part of Section 2)

$$C_{(0)}(X_{S'}/S') = \mathrm{Coker}\,(\bigoplus_{y\in (X_k)_0} \kappa(y)^\times \longrightarrow (\bigoplus_{v\in (S')_0} \tilde{C}_{(0)}(v)) \oplus (\bigoplus_{v\in P(k)-(S')_0} C_{(0)}(v)))$$
$$= \mathrm{Coker}\,(SK_1(X_k) \longrightarrow (\bigoplus_{v\in (S')_0} CH_0(Y_v)) \oplus (\bigoplus_{v\in P(k)-(S')_0} SK_1(X_{k_v}))).$$

This gives a canonical homomorphism

$$C_{X_k} \longrightarrow \varprojlim_{S'} C_{(0)}(X_{S'}/S')$$

having dense image. Thus, we obtain homomorphisms

$$H^1(X_k, Q/Z) \longrightarrow D(X_k) = \bigcup_{S'} C_{(0)}(X_{S'}/S')^* \longrightarrow (C_{X_k})^*$$

whose composite is bijective and the second arrow is injective. This proves the bijectivity of the first arrow.

Now, we consider the general case of Theorem 3. Let Y be the finite set $X_k - U_k$ which is regarded as a subset of X_1. For each $y \in Y$, we have a complete discrete valuation field K_y with residue field $\kappa(y)$. If we take S' sufficiently small so that $Y = (X_{S'})_1 - (U_{S'})_1$, we have an exact sequence

$$\bigoplus_{y\in Y} U^0C(K_y) \longrightarrow C_m(X_{S'}/S') \longrightarrow C_{(0)}(X_{S'}/S') \longrightarrow 0$$

for any modulus m such that $U \cap \mathrm{Supp}\,(m) = \phi$. Consider the following commutative diagram, in which the above exact sequence gives the lower horizontal complex except the part $\mu(k)^*$.

$$\text{(A)}\quad \begin{array}{ccccccccc} 0 \to & H^1(X_k, Q/Z) & \to & H^1(U_k, Q/Z) & \to & \bigoplus_{y\in Y} H^2_y(X_k, Q/Z) & \xrightarrow{j} & H^2(X_k, Q/Z) \\ & \wr\Vert\downarrow & & \downarrow & & a\downarrow & & b\downarrow \\ 0 \longrightarrow & D(X_k) & \longrightarrow & D(U_k) & \xrightarrow{i} & \bigoplus_{y\in Y} (U^0C(K_y))^* & \xrightarrow{c} & \mu(k)^* \end{array}$$

Here, the upper horizontal sequence is the localization sequence in étale cohomology theory, and is exact. The lower horizontal sequence is exact at $D(X_k)$ and $D(U_k)$, and $\mu(k)$ denotes the group of all roots of 1 in k. The definitions of the homomorphisms a, b and c are as follows. First, the class field theory of K_y defines a homomorphism $H^1(K_y) \to C(K_y)^*$ and the unramified part $H^1(\kappa(y))$ of $H^1(K_y)$ annihilates $U^0C(K_y)$. Hence, the exact sequence

$$0 \longrightarrow H^1(\kappa(y)) \longrightarrow H^1(K_y) \longrightarrow H^2_y(X_k, Q/Z) \longrightarrow 0$$

defines the homomorphism a. Next, the spectral sequence

$$H^q(k, H^r(X_{k_s}, Q/Z)) \Longrightarrow H^*(X_k, Q/Z)$$

gives the homomorphism b;

$$H^2(X_k, Q/Z) \longrightarrow H^0(k, H^2(X_{k_s}, Q/Z)) \cong H^0(k) \cong \mu(k)^* .$$

Lastly, let $\prod'_{v \in P(k)} K_2(k_v)$ and $R_k = \mathrm{Coker}\,(K_2(k) \to \prod'_{v \in P(k)} K_2(k_v))$ be as in Chapter I Section 3. For $y \in Y$, the family of diagonal homomorphisms

$$\{K_2(k_v) \longrightarrow \bigoplus_{u \in y(v)} U^0 K_2(K_{v,u})\}_{v \in P(k)}$$

(y is regarded as an element of X_1) induces a homomorphism $\prod'_{v \in P(k)} K_2(k_v) \to U^0 I_{(i)}(K_y)$ ($i \geqq 0$), and furthermore, a homomorphism $R_k \to U^0 C(K_y)$. By Moore's theorem introduced in Chapter I Section 3, this last homomorphism induces

$$c\colon (U^0 C(K_y))^* \longrightarrow \mu(k)^*.$$

Lemma 3. *An element in the kernel of b is annihilated by the homomorphism $H^2(X_k, Q/Z) \longrightarrow H^2(X_{k'}, Q/Z)$ for some finite Galois extension k' of k, where $X_{k'} = X \otimes_k k'$.*

By using the spectral sequence, this is deduced from the fact that each element of $H^q(k, H^r(X_{k_s}, Q/Z))$ with $q > 0$ is annihilated by a finite Galois extension of k.

Now, we prove the surjectivity of the homomorphism of Theorem 3 by using the diagram (A). Let $\varphi \in D(U_k)$. For $y \in Y$, via the homomorphism $C(K_y) \longrightarrow C_m(X_{S'}/S')$ (cf. Section 2), φ induces an element of $C(K_y)^*$. By the class field theory of K_y (Chapter I Section 3), it comes from $H^1(K_y)$. Hence, $i(\varphi) = a(\gamma)$ for some $\gamma \in \bigoplus_{y \in Y} H^2_y(X_k, Q/Z)$. By Lemma 3 and the diagram (A), we can take a finite Galois extension k' of k such that the image of $j(\gamma)$ in $H^2(X_{k'}, Q/Z)$ is zero. Since the diagram

$$\begin{tikzcd} H^1(U_k, Q/Z) \arrow[r] \arrow[d] & H^1(U_{k'}, Q/Z) \arrow[d] \\ D(U_k) \arrow[r, "\text{by norm}"] & D(U_{k'}) \end{tikzcd}$$

is commutative ("by norm" means "induced by norm homomorphisms of K-groups", see Chapter I Section 2 Proposition 3), the diagram (A) for $X_{k'}$ shows that the image $\varphi_{k'} = \varphi \circ \mathrm{norm}_{k'/k}$ of φ in $D(U_{k'})$ coincides with the image of an element χ of $H^1(U_{k'}, Q/Z)$. To prove that φ itself comes from $H^1(U_k, Q/Z)$, we may assume that the order of φ is a power of a prime number p. Let H be a p-Sylow subgroup of $\mathrm{Gal}(k'/k)$, and let k_H be the fixed field of H in k'. By using the commutative diagram

$$\begin{tikzcd} & H^1(U_{k_H}, Q/Z) \arrow[r, "\mathrm{Tr}"] \arrow[d] & H^1(U_k, Q/Z) \arrow[d] \\ D(U_k) \arrow[r, "\text{by norm}"] \arrow[rr, bend right, "{[\mathrm{Gal}(k'/k): H]}"'] & D(U_{k_H}) \arrow[r] & D(U_k) \end{tikzcd}$$

(Tr means the trace map), we are easily reduced to the case $k_H = k$. So, assume that $\mathrm{Gal}(k'/k)$ is a p-group. We can take fields $k = k_0 \subset k_1 \subset \cdots \subset k_r = k'$ such that k_{i+1} is a cyclic extension of degree p of k_i for $0 \leqq i \leqq r-1$. We show by descending induction on i that φ_{k_i} comes from $H^1(U_{k_i}, Q/Z)$. Let $G = \mathrm{Gal}(k_{i+1}/k_i)$. Since G is cyclic, the spectral sequence

$$H^q(G, H^r(U_{k_{i+1}}, Q/Z)) \Longrightarrow H^*(U_{k_i}, Q/Z)$$

induces the upper horizontal exact sequence of the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & G^* \arrow[r] & H^1(U_{k_i}, Q/Z) \arrow[r] \arrow[d] & H^1(U_{k_{i+1}}, Q/Z)^G \arrow[r] \arrow[d] & 0 \\ & & D(U_{k_i}) \arrow[r, "\text{by norm}"] & D(U_{k_{i+1}})^G & \end{tikzcd}$$

Here, for a G-module M,

$$M^G = \{x \in M; gx = x \text{ for all } g \in G\}.$$

By the hypothesis of the induction, $\varphi_{k_{i+1}}$ is the image of an element χ of $H^1(U_{k_{i+1}}, Q/Z)$. Since $\varphi_{k_{i+1}}$ is fixed by G and $H^1(U_{k_{i+1}}, Q/Z) \to D(U_{k_{i+1}})$ is injective, χ is also fixed by G. By this fact and by the diagram above, we see it suffices to prove that the sequence

$$\text{(B)} \qquad 0 \longrightarrow G^* \longrightarrow D(U_{k_i}) \longrightarrow D(U_{k_{i+1}})$$

is exact. Let $Y' = X_{k_i} - U_{k_i}$. For $y \in Y'$, let K'_y be the field of fractions of $\hat{\mathcal{O}}_{X_{k_i}, y}$. Let ψ be an element of $D(U_{k_i})$ which vanishes in $D(U_{k_{i+1}})$. Then, ψ induces an element of $C(K'_y)^*$ for each $y \in Y'$. Since this element is annihilated by the unramified extension $K'_y k_{i+1}$ of K'_y, the class field theory of K'_y shows that ψ annihilates $U^0C(K'_y)$ for all $y \in Y'$. By the lower horizontal sequence of (A) applied to X_{k_i}, this implies that ψ comes from $D(X_{k_i})$. Now, the exactness of (B) follows from the exactness of

$$0 \longrightarrow G^* \longrightarrow D(X_{k_i}) \longrightarrow D(X_{k_{i+1}})$$

which follows from the case $U_k = X_k$ of Theorem 3. This completes the proof of Theorem 3.

4°. We deduce Theorem 2 from Theorem 3. By making U smaller, we have

$$H^1(K) \cong \bigcup_{U \neq \phi} D(U_k).$$

But this isomorphism is the composite of two injective homomorphisms

$$H^1(K) \longrightarrow \bigcup_{\text{all } m} C_m(X/S)^* \longrightarrow \bigcup_{U \neq \phi} D(U_k).$$

Hence, the first homomorphism is bijective.

5°. Lastly, we prove Theorem 1. Let $\varphi \in C_m(X/S)^*$, U a non-empty regular open subscheme U of X, and assume $U \cap \operatorname{Supp}(m) = \phi$. We prove that φ comes from $H^1(U, \boldsymbol{Q}/\boldsymbol{Z})$. By Theorem 2, the image of φ in $\bigcup_{\text{all } m} C_m(X/S)^*$ is induced by an element χ of $H^1(K)$. It remains to prove that χ comes from $H^1(U, \boldsymbol{Q}/\boldsymbol{Z})$. Since U is regular, it suffices to prove χ is unramified at any $y \in U_1$. We have a commutative diagram

$$\begin{array}{ccccc} \chi \in & H^1(K) & \longrightarrow & H^1(K_y) \\ \downarrow & \downarrow & & \downarrow \\ \varphi \in & \bigcup_{\text{all } m} C_m(X/S)^* & \longrightarrow & C(K_y)^*. \end{array}$$

Since $m(y) = 0$ for $y \in U_1$, the image of φ annihilates $U^0 C(K_y)$. By the class field theory of K_y, this proves that χ is unramified at $y \in U_1$.

Remark 1. If $\operatorname{ch}(k) = 0$, any homomorphism of finite order $C_m(v) \to \boldsymbol{Q}/\boldsymbol{Z}$ is continuous, and hence any homomorphism of finite order $C_m(X/S) \to \boldsymbol{Q}/\boldsymbol{Z}$ comes from H^1.

Remark 2. What relation exists between the modulus of a ramified abelian covering X' of X and the conductors of associated L-functions? It seems that the conductor of an L-function contains terms concerning closed points of X, besides the terms $m(y)$ which concern only the points of codimension one.

Lastly, in the case k is a number field, $S = \operatorname{Spec}(O_k)$, and X is regular, we give more precise forms of Theorem 1 and Theorem 2. For a modulus m on X, let $\overline{C}_m(X)$ be the quotient of $C_m(X)$ defined to be the cokernel of

$$\bigoplus_{y \in X_1} K_2(K_y) \longrightarrow \Big(\bigoplus_{x \in X_0} C_m(x)\Big) \oplus \Big(\bigoplus_{v \text{ real}} C_m(v)/2C_m(v)\Big)$$

where "real" means the real places of k. We call a modulus m on X admissible if $m(y) \leqq 1$ for any y such that $\operatorname{ch}(\kappa(y)) = 0$.

Theorem 4. *Assume that k is an algebraic number field, $S=\mathrm{Spec}\,(O_k)$, and X is regular.*

(1) *For any admissible modulus m on X, the group $\overline{C}_m(X)$ is finite.*

(2) *Let U be a non-empty open subscheme of X. Then, as a topological group, $\pi_1^{\mathrm{ab}}(U)$ is canonically isomorphic to $\varprojlim_m \overline{C}_m(X)$ where m ranges over all admissible muduli such that $U\cap \mathrm{Supp}\,(m)=\phi$.*

(3) *As a topological group, $\mathrm{Gal}\,(K^{\mathrm{ab}}/K)$ is canonically isomorphic to $\varprojlim_m \overline{C}_m(X)$ where m ranges over all admissible moduli.*

This result is similar to the classical class field theory of Takagi-Artin. We have the ray class group $\overline{C}_m(X)$ which is a finite group generated by canonical generators corresponding to closed points outside $\mathrm{Supp}\,(m)$, the corresponding ray class field K_m which is a finite abelian extension of K such that $\bigcup_m K_m=K^{\mathrm{ab}}$, and the Artin isomorphism $\overline{C}_m(X)\cong \mathrm{Gal}\,(K_m/K)$.

Proof of Theorem 4. If v is a complex place of k, $C_m(v)$ is a divisible group for any modulus m. If v is a real place of k, $C_m(v)/2C_m(v)$ is discrete as is easily seen, and the norm argument for the extension $\boldsymbol{C}/\boldsymbol{R}$ shows that $2C_m(v)$ is divisible, for any modulus m. Furthermore, if $y\in X_1$ and $\mathrm{ch}\,(\kappa(y))=0$, $U^1C(K_y)$ is a divisible group. By cutting off these divisible groups, Theorem 4 is reduced to Theorem 1 and Theorem 2 if we prove the finiteness of $\overline{C}_m(X)$ for m admissible. For $m=(0)$, the regularity of X implies

$$\mathrm{Coker}\,(\bigoplus_{y\in X_1} K_2(K_y)\longrightarrow \bigoplus_{x\in X_0} C_{(0)}(x))\cong \mathrm{Coker}\,(\bigoplus_{y\in X_1}\kappa(y)^{\times}\longrightarrow \bigoplus_{x\in X_0}\boldsymbol{Z})$$

and the latter group is finite by Bloch [3]. On the other hand, for a real place v of k, $C_{(0)}(v)/2C_{(0)}(v)=SK_1(X_{k_v})/2SK_1(X_{k_v})$ is finite as follows. If the residue fields of all closed points of X_{k_v} are the complex number fields, $SK_1(X_{k_v})$ is divisible. If X_{k_v} has a k_v-rational point, there is an exact sequence

$$J(k_v)\otimes k_v^{\times}\longrightarrow SK_1(X_{k_v})/2SK_1(X_{k_v})\longrightarrow k_v^{\times}/(k_v^{\times})^2\longrightarrow 0$$

([3] Section 1), where $J(k_v)$ is the group of all k_v-rational points of the Jacobian variety of X_k. The finiteness of $C_{(0)}(v)/2C_{(0)}(v)$ follows from the finiteness of $J(k_v)/2J(k_v)$ and $k_v^{\times}/(k_v^{\times})^2$. Thus, we have shown that $\overline{C}_{(0)}(X)$ is finite.

For any modulus m, we have an exact sequence

$$\bigoplus_{y\in X_1\cap \mathrm{Supp}(m)} U^0C_m(K_y)\longrightarrow \overline{C}_m(X)\longrightarrow \overline{C}_{(0)}(X)\longrightarrow 0.$$

Since $\overline{C}_m(X)$ is discrete, the image of the first arrow is finite by Chapter I Section 3 Lemma 2 if m is admissible. This reduces the finiteness of $\overline{C}_m(X)$ to that of $\overline{C}_{(0)}(X)$.

§ 4. The case of surfaces over a finite field

Let k be a field, and let X be a proper normal surface over k. Then, we define the idele class group $C_m(X)$ for a modulus m on X as follows in the way similar to Section 2.

For $y \in X_1$, let K_y be the field of fractions of $\hat{\mathcal{O}}_{X,y}$. For $x \in X_0$, let K_x be the field of fractions of $\hat{\mathcal{O}}_{X,x}$, and for $z \in \mathrm{Spec}(\hat{\mathcal{O}}_{X,x})_1$, let $K_{x,z}$ be the z-adic completion of K_x. For a modulus m on X, the groups $C_m(x)$ ($x \in X_0$) and the homomorphisms $K_2(K_y) \to \bigoplus_{x \in X_0} C_m(x)$ ($y \in X_1$) are defined in the same way as in Section 2. Let $C_m(X)$ be the cokernel of the induced homomorphism

$$\bigoplus_{y \in X_1} K_2(K_y) \longrightarrow \bigoplus_{x \in X_0} C_m(x).$$

If X is regular and $m=(0)$,

$$C_{(0)}(X) \cong \mathrm{Coker}\,(\bigoplus_{y \in X_1} \kappa(y)^{\times} \longrightarrow \bigoplus_{x \in X_0} \boldsymbol{Z}) = CH_0(X),$$

where $CH_0(X)$ is the Chow group of zero cycles on X modulo rational equivalence.

Most of topological arguments in the previous sections can be applied to this situation. We define the topology of $K_2(K_{x,z})/U^i K_2(K_{x,z})$ by using the image A of $\hat{\mathcal{O}}_{X,x}$ in $O_{K_{x,z}}/m^i_{K_{x,z}}$ (see Chapter I Section 2 Case 1). We define the idele group of K_y and its topology by using $Y = \mathrm{Spec}\,(\mathcal{O}_X/I^i)$ where I is the ideal of $\mathcal{O}_X$ defining the reduced closed subscheme $\overline{\{y\}}$ (the condition "of finite type over $\boldsymbol{Z}$" on Y in Chapter I Section 3 is replaced here by "of finite type over k"). We regard $C_m(X)$ and $C_m(x)$ as discrete groups, and then, all the homomorphisms $K_2(K_{x,z})/U^{m_x(z)} K_2(K_{x,z}) \to C_m(x)$ and $C_{(m(y))}(K_y) \to C_m(X)$ are continuous.

Assume now that k is a finite field. Then, as in Section 3, we obtain a canonical homomorphism

$$\varprojlim_{U \cap \mathrm{Supp}(m) = \phi} C_m(X) \longrightarrow \pi_1^{\mathrm{ab}}(U)$$

for any dense open subscheme U of X. If X is connected and K is the function field of X, we obtain a canonical homomorphism

$$\varprojlim_{\text{all } m} C(X) \longrightarrow \mathrm{Gal}\,(K^{\mathrm{ab}}/K).$$

Theorem 5. *Let X be a smooth projective geometrically irreducible surface over a finite field k with function field K. Then,*

(1) *For any modulus m, the kernel $C_m(X)^0$ of the degree map $C_m(X) \to Z$ is finite.*

(2) *Let U be a non-empty open set of X, and let $\pi_1^{ab}(U)^0$ be the kernel of $\pi_1^{ab}(U) \to \mathrm{Gal}(\bar{k}/k)$. We have commutative diagrams of exact sequences*

$$\begin{array}{ccccccccc}
0 \longrightarrow & \varprojlim_{U \cap \mathrm{Supp}(m)=\phi} C_m(X)^0 & \longrightarrow & \varprojlim_{U \cap \mathrm{Supp}(m)=\phi} C_m(X) & \xrightarrow{\deg} & Z & \longrightarrow 0 \\
& \wr\!\| \downarrow & & \downarrow & & \cap \downarrow & \\
0 \longrightarrow & \pi_1^{ab}(U)^0 & \longrightarrow & \pi_1^{ab}(U) & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 \longrightarrow & \varprojlim_{\text{all } m} C_m(X)^0 & \longrightarrow & \varprojlim_{\text{all } m} C_m(X) & \longrightarrow & Z & \longrightarrow 0 \\
& \wr\!\| \downarrow & & \downarrow & & \cap \downarrow & \\
0 \longrightarrow & \mathrm{Gal}(K^{ab}/\bar{k}K) & \longrightarrow & \mathrm{Gal}(K^{ab}/K) & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow 0.
\end{array}$$

in which the vertical arrows in the left sides are isomorphisms of profinite groups.

Here, the degree map deg: $C_m(X) \to Z$ is the composite

$$C_m(X) \longrightarrow C_{(0)}(X) \cong CH_0(X) \xrightarrow{\deg} Z.$$

We begin with some elementary properties of $C_m(X)$.

Lemma 3. *Let X be a proper normal surface over a field k, m a modulus on X, and let U be any regular dense open subset of X such that $U \cap \mathrm{Supp}(m) = \phi$. Then,*

$$\bigoplus_{x \in U_0} Z \cong \bigoplus_{x \in U_0} C_m(x) \longrightarrow C_m(X)$$

is surjective.

Proof. Fix $x \in X$. Let T be the subset of $\mathrm{Spec}\,(\hat{\mathcal{O}}_{X,x})_1$ consisting of all points whose images in X are the generic point of X. Let M (resp. N) be the subset of $\mathrm{Spec}\,(\hat{\mathcal{O}}_{X,x})_1 - T$ consisting of all points z such that $m_x(z) \geqq 1$ (resp. $m_x(z) = 0$). Since M is a finite set, we see easily that

$$\bigoplus_{z \in N \amalg T} \kappa(z)^\times \longrightarrow C_m(x)$$

is surjective. We prove further

Clain. $\bigoplus_{z \in N} \kappa(z)^\times \to C_m(x)$ is surjective.

Indeed, fix $t \in T$ and let a be any element of $\kappa(t)^\times$. We show that the image of a in $C_m(x)$ is contained in the image of $\bigoplus_{z \in N} \kappa(z)^\times$. Take an element h of K_x which is a prime element in $K_{x,t}$ and which satisfies

$$\mathrm{ord}_{K_{x,z}}(h-1) \geqq m_x(z)$$

for any $z \in M$. Let $D = \{z \in \mathrm{Spec}\,(\hat{\mathcal{O}}_{X,x})_1;\ \mathrm{ord}_{K_{x,z}}(h) \neq 0\}$, and let s be any non-zero element of $\mathcal{O}_{X,x}$ (not $\hat{\mathcal{O}}_{X,x}$) such that $\mathrm{ord}_{K_{x,z}}(s) = 0$ if $z \in D$. Since $\hat{\mathcal{O}}_{X,x}[s^{-1}]$ is a Dedekind domain, there are $b \in \hat{\mathcal{O}}_{X,x}$ and $n \in \boldsymbol{Z}$ such that the image of $s^n b$ in $\kappa(t)$ is a, and its image in $\kappa(z)$ is 1 if $z \in D - \{t\}$. Let J be the intersection of all the prime ideals in D. Since $J + b\hat{m}_{X,x}$ ($\hat{m}_{X,x}$ is the maximal ideal of $\hat{\mathcal{O}}_{X,x}$) is an open ideal of $\hat{\mathcal{O}}_{X,x}$, there is an element c of $\mathcal{O}_{X,x}$ such that

$$c \equiv b \bmod (J + b\hat{m}_{X,x}).$$

This shows that

$$c \equiv bu \bmod J \qquad \text{for some } u \in (\hat{\mathcal{O}}_{X,x})^\times.$$

For $z \in M \coprod T$, the image of $\{s^n cu^{-1}, h\} \in K_2(K_x)$ in $K_2(K_{x,z})/U^{m_x(z)}K_2(K_{x,z})$ is $a \in \kappa(t)^\times$ if $z = t$, and is zero if $z \neq t$. By the definition of $C_m(x)$, this proves the claim.

Now, we complete the proof of Lemma 3. By making m large and U small, we may assume $U = X - \mathrm{Supp}\,(m)$. By the claim, $C_m(X)$ is generated by the images of the groups $C_{(0)}(K_y)$ such that $y \in X_1$ and $m(y) = 0$. For such $y \in X_1$, if x is a closed point of $\overline{\{y\}} \cap U$ at which the reduced scheme $\overline{\{y\}}$ is regular, the image in $C_m(X)$ of the local factor of $I_{(0)}(K_y)$ corresponding to x coincides with the image of $\boldsymbol{Z} = C_m(x) \to C_m(X)$. Since $C_{(0)}(K_y)$ is isomorphic to the idele class group of $\kappa(y)$ and $C_{(0)}(K_y) \to C_m(X)$ is continuous, the image of $C_{(0)}(K_y)$ in $C_m(X)$ is generated by the images of $C_m(x)$ where x ranges over all regular closed points of $\overline{\{y\}} \cap U$.

Lemma 4. *Let X be as in Lemma 3, and let $f\colon X' \to X$ be a proper birational morphism with X' normal. Let m be a modulus on X, and let m' be any modulus on X' satisfying $m'(y) = m(f(y))$ for any $y \in (X')_1$ such that $f(y) \in X_1$. Then, there is a unique surjective homomorphism $f^*\colon C_m(X) \to C_{m'}(X')$ such that; if U is a regular dense open subscheme of X such that $f^{-1}(U) \cong U$ via f, the following diagram is commutative.*

$$\begin{array}{ccc} \bigoplus_{x \in U_0} \boldsymbol{Z} & \longrightarrow & C_m(X) \\ \wr\Vert & & \downarrow f^* \\ \bigoplus_{x \in f^{-1}(U)_0} \boldsymbol{Z} & \longrightarrow & C_{m'}(X'). \end{array}$$

Proof. Fix $x \in X_0$. Let $T=\hat{\mathcal{O}}_{X,x}\otimes_X X'$. Since $T\to\operatorname{Spec}(\hat{\mathcal{O}}_{X,x})$ is proper and birational, each $z \in \operatorname{Spec}(\hat{\mathcal{O}}_{X,x})_1$ determines a unique element t of T_1 having z as its image in $\operatorname{Spec}(\hat{\mathcal{O}}_{X,x})$. The closure $\overline{\{t\}}$ in T is finite over $\operatorname{Spec}(\hat{\mathcal{O}}_{X,x})$ and hence has a unique closed point s. Let x' be the image of s in X'. Then, $\hat{\mathcal{O}}_{X',x'}\cong\hat{\mathcal{O}}_{T,s}$. Since $\hat{\mathcal{O}}_{T,s}$ is isomorphic to the t-adic completion of $\mathcal{O}_{T,s}$, we can regard t as an element of $\operatorname{Spec}(\hat{\mathcal{O}}_{T,s})_1$. Let z' be the image of $t \in \operatorname{Spec}(\hat{\mathcal{O}}_{T,s})_1$ in $\operatorname{Spec}(\hat{\mathcal{O}}_{X',x'})_1$. We have

$$K_{x',z'}\xrightarrow[\cong]{}K_{s,t}\xleftarrow[\cong]{}K_{x,z}.$$

The homomorphism f^* is defined by collecting the isomorphisms

$$K_2(K_{x,z})/U^{m_x(z)}K_2(K_{x,z})\xrightarrow[\cong]{}K_2(K_{x',z'})/U^{m_{x'}(z')}K_2(K_{x',z'}).$$

The well-definedness of f^* is proved easily, and the uniqueness and the surjectivity of f^* follow from Lemma 3.

Lemma 5. *Let f': $X'\to X$ and m, m' be as in Lemma* 4. *Assume that there is an open set W of X which contains* Supp (m) *and all points of X at which X is not regular, and which satisfies $f^{-1}(W)\cong W$ via f. Then, f^*: $C_m(X)\to C_{m'}(X')$ is bijective.*

Remark 3. It is probable that the group $\varprojlim_{\text{all } m} C_m(X)$ is a birational invariant of X. This Lemma 5 is too weak to deduce this. If k is a finite field, the class field theory affirms this fact for X regular.

Proof of Lemma 5. We define a homomorphism

$$f_*\colon C_{m'}(X')\to C_m(X)$$

as follows. For $x \in f^{-1}(W)_0$, let f_x: $C_{m'}(x)\xrightarrow[\cong]{} C_m(f(x))$ be the canonical isomorphism. For $x \in (X')_0-f^{-1}(W)_0$, let d_x: $C_{m'}(x)\to \boldsymbol{Z}=K_0(x)$ be the homomorphism defined by the Bloch-Gersten-Quillen complex for $\hat{\mathcal{O}}_{X',x}$, and let f_x: $C_{m'}(x)\to C_m(f(x))$ be the composite

$$C_{m'}(x)\xrightarrow{d_x}\boldsymbol{Z}\xrightarrow{[\kappa(x):\,\kappa(f(x))]}\boldsymbol{Z}=C_m(f(x)).$$

The homomorphism f_* is defined by collecting these homomorphisms f_x $(x \in (X')_0)$. It is easily seen that this homomorphism f_* is well defined. By applying Lemma 3 to U and $f^{-1}(U)$ where U is any regular dense open subscheme of $X-\operatorname{Supp}(m)$ such that $f^{-1}(U)\cong U$, we see that $f^*\circ f_*$ and $f_*\circ f^*$ are the identity maps.

Now, we prove Theorem 5. Since X is regular, $C_{(0)}(X)=CH_0(X)$. By Milne [13] and Kato-Saito [8], $CH_0(X)^0$ is a finite group. By the proof of Section 3 Theorem 4, we see that the kernel of $C_m(X)\to C_{(0)}(X)$ is finite for any modulus m on X. Hence, Theorem 4 is a consequence of

Proposition 1. *Let X be a projective normal connected surface over a finite field k with function field K. Let U be a non-empty regular open subscheme of X. Then, we have canonical isomorphisms*

$$H^1(U, \boldsymbol{Q}/\boldsymbol{Z}) \xrightarrow{\cong} \bigcup_{U\cap \operatorname{Supp}(m)=\phi} C_m(X)^*,$$

$$H^1(K) \xrightarrow{\cong} \bigcup_{\text{all } m} C_m(X)^*.$$

Proof. The injectivity of these homomorphisms are proved just as in the proofs of Theorem 1 and Theorem 2. We prove the surjectivity. It suffices to treat the first homomorphism. Let $\varphi\colon C_m(X)\to \boldsymbol{Q}/\boldsymbol{Z}$ be a homomorphism of finite order such that $U\cap \operatorname{Supp}(m)=\phi$. To show that φ comes from $H^1(U, \boldsymbol{Q}/\boldsymbol{Z})$, we may assume that the order of φ is a power of a prime number p, and we may replace k by any finite extension k' of k such that $[k'\colon k]$ is prime to p. Take a projective embedding $X \overset{\subset}{\to} \boldsymbol{P}^n_k$, and let Y be a closed subset of X such that $Y\neq X$ and $X-U\subset Y$, so that all the singular points of X are contained in Y. By taking an extension of the constant field of degree prime to p, we may assume that there is a linear subvariety L of $\boldsymbol{P}^n_k$ of codimension 2 which does not meet Y. The hyperplanes H in $\boldsymbol{P}^n$ which contain L form a variety T which is isomorphic to $\boldsymbol{P}^1_k$. Let X' be the subvariety of $X\times_k T$ of points (x, H) such that $x\in H$. Then, the first projection $X'\to X$ is proper and birational, and satisfies the condition of Lemma 5 with respect to the modulus m. Let m' be the modulus on X' defined by $m'(y)=m(f(y))$ if $f(y)\in X_1$ and $m'(y)=0$ if $f(y)\in X_0$. Then, we have $f_*\colon C_{m'}(X')\cong C_m(X)$ by Lemma 5. Take the integral closure S of T in K. Then, the projection $X'\to T$ induces a morphism $X'\to S$ which satisfies the conditions at the beginning of Section 1. We have $C_{m'}(X'/S)=C_{m'}(X')$. Hence, in the commutative diagram

$$\begin{tikzcd}
H^1(U, \boldsymbol{Q}/\boldsymbol{Z}) \arrow[r] \arrow[d] & \displaystyle\bigcup_{U\cap \operatorname{Supp}(m)=\phi} C_m(X)^* \arrow[d, "\cap"', "\text{by } f_*"] \\
H^1(f^{-1}(U), \boldsymbol{Q}/\boldsymbol{Z}) \arrow[r] & \displaystyle\bigcup_{f^{-1}(U)\cap \operatorname{Supp}(m)=\phi} C_m(X')^*,
\end{tikzcd}$$

the lower horizontal arrow is bijective by Theorem 1. Since $f^{-1}(U)\to U$ is proper and birational and U is regular, $H^1(U, Q/Z)\to H^1(f^{-1}(U), Q/Z)$ is bijective. This proves that φ comes from $H^1(U, Q/Z)$.

Chapter III. Relation with the class field theory of Lang

Let V be a variety over a finite field k with function field K. The class field theory of Lang [9] [10] [20] constructs a group $A_k(V)$ by using commutative algebraic groups over k, and an isomorphism

$$\mathrm{Gal}\,(K^{\mathrm{ab}}/K)'\cong A_k(V) \qquad \text{([20] Chapter VI } n^\circ \text{ 16 Theorem 1)}$$

where $\mathrm{Gal}\,(K^{\mathrm{ab}}/K)'$ is the dense subgroup of $\mathrm{Gal}\,(K^{\mathrm{ab}}/K)$ defined to be the inverse image of the subgroup Z of $\mathrm{Gal}\,(\bar{k}/k)\cong\hat{Z}$ ($1\in Z\subset\hat{Z}$ corresponds to the Frobenius). In this chapter, in the case where V is a smooth proper surface over k, we shall show that the composite of this isomorphism and the isomorphism in Chapter II Section 4

$$\varprojlim_m C_m(V)\cong\mathrm{Gal}\,(K^{\mathrm{ab}}/K)'\cong A_k(V)$$

is given by the two dimensional version of the local symbol of [20] Chapter III.

First, we review the class field theory of Lang, following [20] with slight modifications.

Let k be an arbitrary field. Let G be a commutative algebraic group over k (i.e. a commutative group scheme of finite type over k). Then, a principal homogeneous space H over G defines an exact sequence of commutative group schemes over k;

$$0\longrightarrow G\longrightarrow E_H\longrightarrow Z\longrightarrow 0.$$

Precisely, E_H is the disjoint union $\coprod_{n\in Z}H^{(n)}$ where $H^{(n)}$ is the principal homogeneous space $H\overset{G}{\times}\cdots\overset{G}{\times}H$ (n times) over G defined in the well-known way for any $n\in Z$. (In particular, $H^{(1)}=H$ and $H^{(0)}=G$). The group law of E_H is defined by the canonical morphisms $H^{(m)}\times H^{(n)}\to H^{(m+n)}$ ($m, n\in Z$). The homomorphism $E_H\to Z$ is the morphism which is constant on each $H^{(n)}$ with value n. (Another definition of the above exact sequence is that H defines an element of

$$H^1(\mathrm{Spec}\,(k)_{\mathrm{fppf}}, G)\cong\mathrm{Ext}^1_{\mathrm{Spec}(k)_{\mathrm{fppf}}}(Z, G)\,,$$

where fppf means the fppf topology.)

Let K be a field over a field k. We define the category $M_{K/k}$ as follows. An object of $M_{K/k}$ is a triple (G, H, α) where G is a commutative algebraic group over k, H is a principal homogeneous space over G defined over k, and α is a K-rational point of H such that the image of α: Spec (K) $\to H$ is dense. A morphism $(G, H, \alpha)\to(G', H', \alpha')$ is a pair (f, g) of a k-homomorphism $f\colon G\to G'$ and a k-morphism $g\colon H\to H'$ such that $g\circ\alpha=\alpha'$ and such that the diagram

$$\begin{array}{ccc} G\times H & \xrightarrow{(f,g)} & G'\times H' \\ \downarrow & & \downarrow \\ H & \xrightarrow{g} & H' \end{array}$$

is commutative. This category $M_{K/k}$ is essentially small (i.e. isomorphism classes of objects in $M_{K/k}$ form a set) and co-filtered (its dual category is filtered in the sense of Schubert [19] 9.3.4). Hence we can regard any covariant (resp. contravariant) functor $F\colon M_{K/k}\to\mathscr{C}$ from $M_{K/k}$ to a category $\mathscr{C}$, as a filtered inverse (resp. inductive) system $\{F(G, H, \alpha)\}$ in $\mathscr{C}$ with the index category $M_{K/k}$. A general result of [20] is;

$$\varinjlim_{(G,H,\alpha)\in M_{K/k}} \mathrm{Ext}^1(E_H, \boldsymbol{Q}/\boldsymbol{Z})\cong H^1(K)$$

where $\mathrm{Ext}^1(E_H, \boldsymbol{Q}/\boldsymbol{Z})$ is the group of all classes of short exact sequences $0\to\boldsymbol{Q}/\boldsymbol{Z}\to * \to E_H\to 0$ of fppf sheaves of abelian groups over Spec (k), and $\varinjlim$ means the inductive limit of the contra-variant functor $(G, H, \alpha)\mapsto \mathrm{Ext}^1(E_H, \boldsymbol{Q}/\boldsymbol{Z})$.

Let

$$A_{K/k}=\varprojlim_{(G,H,\alpha)\in M_{K/k}} E_H(k),$$

where $E_H(k)$ denotes the group of all k-valued points of E_H which we regard as a covariant functor $(G, H, \alpha)\mapsto E_H(k)$. If K is the function field of a variety V over k, this group $A_{K/k}$ coincides with the group $A_k(V)$ in [20] as is easily seen.

Assume that k is a finite field. Then, an exact sequence $0\to\boldsymbol{Q}/\boldsymbol{Z}\to * \to E_H\to 0$ gives a homomorphism $E_H(k)=H^0(k, E_H)\to H^1(k, \boldsymbol{Q}/\boldsymbol{Z})=\boldsymbol{Q}/\boldsymbol{Z}$, and this correspondence induces an isomorphism $\mathrm{Ext}^1(E_H, \boldsymbol{Q}/\boldsymbol{Z})\cong \mathrm{Hom}(E_H(k), \boldsymbol{Q}/\boldsymbol{Z})$. Combining this fact with the above general result, we have

Theorem. ([20] Chepter VI n° 16 Theorem 1) *Let k be a finite field, K a field over k, and let* $\mathrm{Gal}(K^{\mathrm{ab}}/K)'\subset\mathrm{Gal}(K^{\mathrm{ab}}/K)$ *be the inverse image of* $\boldsymbol{Z}\subset\hat{\boldsymbol{Z}}=\mathrm{Gal}(\bar{k}/k)$. *Then, there exists a canonical isomorphism*

$$\mathrm{Gal}\,(K^{\mathrm{ab}}/K)' \cong A_{K/k}.$$

Our aim is to prove

Proposition 1. *Let X be a projective smooth connected surface over a field k, and let K be the function field of X. Then,*

(1) *There is a canonical homomorphism*

$$\gamma_X\colon \varprojlim_{\text{all } m} C_m(X) \longrightarrow A_{K/k}$$

having the following characterization. Let (G, H, α) be an object of $M_{K/k}$ and assume that α comes from a morphism $\tilde{\alpha}\colon U \to H$ for a non-empty open subscheme U of X. Then, the corresponding homomorphism

$$\varprojlim_{\text{all } m} C_m(X) \longrightarrow E_H(k)$$

induced by γ_X factors through $C_m(X)$ for some modulus m such that $U \cap \mathrm{Supp}\,(m) = \phi$, and the composite

$$Z = C_m(x) \longrightarrow C_m(X) \xrightarrow{\text{by } \gamma_X} E_H(k) \qquad \textit{for } x \in U_0$$

sends $1 \in Z$ to $\mathrm{Tr}_{\kappa(x)/k}(\tilde{\alpha}(x))$. Here, $\tilde{\alpha}(x)$ is the composite morphism

$$\mathrm{Spec}\,(\kappa(x)) \longrightarrow U \xrightarrow{\tilde{\alpha}} H = H^{(1)} \subset E_H$$

regarded as a $\kappa(x)$-rational point of E_H, and $\mathrm{Tr}_{\kappa(x)/k}$ is the trace map $E_H(\kappa(x)) \to E_H(k)$.

(2) *If k is finite, this homomorphism γ_X is bijective, and the following diagram is commutative.*

$$\begin{tikzcd}
\varprojlim\limits_{\text{all } m} C_m(X) \arrow[r, "\text{Ch. II §4}", "\cong"'] \arrow[dr, "\gamma_X"', "\cong"] & \mathrm{Gal}\,(K^{\mathrm{ab}}/K)' \arrow[d, "\text{Lang}", "\cong"'] \\
& A_{K/k}
\end{tikzcd}$$

We must explain the definition of the trace map. Let G be a commutative group scheme over a field k and let E be a finite extension of k. Then, we define the trace map

$$\mathrm{Tr}_{E/k}\colon G(E) \longrightarrow G(k)$$

as follows. Let E' be the maximum separable subextension of k in E.

Let $\mathrm{Tr}_{E'/k}: G(E')\to G(k_s)$ be the homomorphism $x\mapsto \sum_\sigma \sigma(x)$, where σ ranges over all k-homomorphisms $E'\to k_s$. Then, the image is fixed by $\mathrm{Gal}(k_s/k)$ and hence $\mathrm{Tr}_{E'/k}$ is in fact a homomorphism $G(E')\to G(k)$. If $\mathrm{ch}(k)=p>0$ and $[E: E']=p^r$ $(r\geqq 0)$, we have

$$p^r x \in G(E')\subset G(E) \qquad \text{for any } x \in G(E).$$

Indeed, let B be the fixed subring of $A=E\otimes_k\cdots\otimes_k E$ (p^r times) by the actions $a_1\otimes\cdots\otimes a_{p^r}\mapsto a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(p^r)}$ of the permutations σ on the set $\{1, \cdots, p^r\}$. For $x \in G(E)$, let x_i be the image of x in $G(A)$ induced by

$$E\longrightarrow A;\, a\longmapsto \underbrace{1\otimes\cdots\otimes 1}_{i-1 \text{ times}}\otimes a\otimes 1\otimes\cdots\otimes 1.$$

Then, $x_1+\cdots+x_{p^r} \in G(A)$ is contained in $G(B)$. On the other hand, by Serre [20] Chepter III n° 14 Lemma 11, the image of B under

$$\theta\colon A\longrightarrow E;\; a_1\otimes\cdots\otimes a_{p^r}\longmapsto a_1\cdots a_{p^r}$$

is contained in kE^{p^r}. Write the induced map $B\to E'$ by θ'. In $G(E)$, we have

$$p^r x=\theta(x_1+\cdots+x_{p^r})=\theta'(x_1+\cdots+x_{p^r}) \in G(E').$$

Now we define $\mathrm{Tr}_{E/k}\colon G(E)\to G(k)$ by

$$x\longmapsto \mathrm{Tr}_{E'/k}([E: E']\cdot x).$$

To prove Proposition 1, we define the local symbols for higher dimensional local fields. Let k be a perfect field, and let $k_0, \cdots, k_n$ be a sequence of fields over k satisfying the following conditions (i) (ii).

(i) k_0 is a finite extension of k.

(ii) For $i=1, \cdots, n$, k_i is a complete discrete valuation field with residue field k_{i-1}. Furthermore, $k\subset O_{k_i}$, and the reduction map $O_{k_i}\to k_{i-1}$ is a k-homomorphism.

Let $K=k_n$. We define a valuation ring V_K as follows. Let $V_0=K$ and define V_i $(1\leqq i\leqq n)$ inductively to be the inverse image of $O_{k_{n-i+1}}$ under $V_{i-1}\to k_{n-i+1}$. Then, $V_0\supset V_1\supset\cdots\supset V_n$ and each V_i is a valuation ring of rank i with field of fractions K. We define $V_K=V_n$. Then, the residue field of V_K is k_0.

In the following, for a commutative algebraic group G over k, we define a canonical pairing

$$(\ ,\ )\colon G(K)\times K_n^M(K)\longrightarrow G(k),$$

which is a generalization of the local symbol of [20] Chapter III (see Lemma 3 below). First, we assume that G is affine. We need some facts from the localization theory in algebraic K-theory.

Let B be a ring and S a multiplicatively closed subset of B consisting of non-zero divisors of B. Let H be the exact category of all the B-module M such that $S^{-1}M=0$ having a resolution of length one by finitely generated projective B-modules. Then, we have a long exact sequence

$$K_{q+1}(B) \longrightarrow K_{q+1}(S^{-1}B) \xrightarrow{\partial} K_q(H) \longrightarrow K_q(B) \longrightarrow K_q(S^{-1}B)$$

for $q \geqq 0$ (cf. Grayson [4]), where K_* means Quillen's K-group. If B is flat over a ring A and if B/sB is a finitely generated projective A-module for any $s \in S$, any object of H is finitely generated and projective as an A-module, and we obtain a homomorphism $K_q(H) \to K_q(A)$.

Definition 1. Let I be a field, and let J be a complete discrete valuation field containing I such that $I \subset O_J$ and such that the residue field of J is a finite extension of I. Then, for any ring A over I and any $q \geqq 0$, we denote by $\mathrm{Res}_{J/I}$ the composite

$$K_{q+1}(A \otimes_I J) \xrightarrow{\partial} K_q(H) \longrightarrow K_q(A)$$

defined by taking $B = A \otimes_I O_J$ and $S = O_J - \{0\}$.

Let k and K be as above. For each $i=1, \cdots, n$, choose a k-homomorphism $f_i\colon k_{i-1} \to O_{k_i}$ such that the composite $k_{i-1} \xrightarrow{f_i} O_{k_i} \to O_{k_i}/m_{k_i}$ is the identity map (such f_i exists, for k_{i-1} is formally smooth over the perfect field k in the sense of EGA., Chapter 0 Section 19). For any ring R over k, let $\mathrm{Res}_{(f_1, \cdots, f_n)}$ be the composite map

$$K_{n+1}(R \otimes_k K) \xrightarrow{\mathrm{Res}_{k_n/k_{n-1}}} K_n(R \otimes_k k_{n-1}) \longrightarrow \cdots$$
$$\xrightarrow{\mathrm{Res}_{k_1/k_0}} K_1(R \otimes_k k_0) \xrightarrow{\text{norm}} K_1(R) \xrightarrow{\det} R^\times,$$

where we regard k_{i-1} as a subfield of k_i via f_i for each i.

For a ring R over k which is a k-vector space of finite rank, let G_m^R be the algebraic group over k having the characterization

$$G_m^R(A) = (R \otimes_k A)^\times \qquad \text{for any ring } A \text{ over } k.$$

For an affine commutative algebraic group G, there is an exact sequence $0 \to G \to G_m^R \to G_m^{R'}$ for some R and R'. This shows that for any embedding $G \subset G_m^R$, the image of

$$G(K)\times K_n(K)\longrightarrow G_m^R(K)\times K_n(K)$$
$$=(R\otimes_k K)^\times\times K_n(K)\xrightarrow{\{\ ,\ \}}K_{n+1}(R\otimes_k K)\xrightarrow{\mathrm{Res}_{(f_1,\cdots,f_n)}}R^\times=G_m^R(k)$$

is contained in $G(k)$ and the induced pairing

$$(\ ,\ )_K\colon G(K)\times K_n(K)\longrightarrow G(k)$$

is independent of the choices of R and the embedding $G\subset G_m^R$.

It is probable that the homomorphism

$$\begin{aligned}&\mathrm{Res}_{(f_1,\cdots,f_n)}\colon K_{n+1}(R\otimes_k K)\longrightarrow R^\times\\ &\text{and hence}\qquad (\ ,\ )_K\colon G(K)\times K_n(K)\longrightarrow G(k)\end{aligned}$$

are independent of the choices of $f_1,\cdots,f_n$. In the positive characteristic case, this can be proved by the same method in the proof of [5] II Section 2 Lemma 12, which treated the case $R=k[T]/(T^i)$ $(i\geqq 1)$ and used the p-th power homomorphism. In the case $\mathrm{ch}(k)=0$, we can prove at least that the composite

$$G(K)\times K_n^M(K)\longrightarrow G(K)\times K_n(K)\longrightarrow G(k),$$

which we shall denote also by $(\ ,\ )_K$, is independent of the choices of $f_1,\cdots,f_n$. In this case, we may assume that k is algebraically closed, and then we are reduced to the cases $G=G_m$ and $G=G_a$. In the case $G=G_m$, $(\ ,\ )_K$ is induced by the composite of tame symbols which are independent of $f_1,\cdots,f_n$. In the case $G=G_a$, we have a commutative diagram

$$\begin{array}{ccc}(a,\{b_1,\cdots,b\}_n) & \qquad & G_a(K)\times K_n^M(K)\xrightarrow{(\ ,\ )_K}G_a(k)\\ \downarrow & & \downarrow\qquad\qquad\qquad \Vert\\ a\dfrac{db_1}{b_1}\wedge\cdots\wedge\dfrac{db_n}{b_n} & & \Omega_K^n\qquad\xrightarrow{\mathrm{res}}\qquad k\end{array}$$

where the lower horizontal arrow "res" is characterized by the following properties and is independent of $f_1,\cdots,f_n$.

(i) $\mathrm{res}\,(d\Omega_K^{n-1})=0$.

(ii) Let V_K be the valuation ring of rank n with residue field k_0 defined above. Then,

$$\mathrm{res}\left(a\frac{db_1}{b_1}\wedge\cdots\wedge\frac{db_n}{b_n}\right)=\partial^n(\{b_1,\cdots,b_n\})\cdot\mathrm{Tr}_{k_0/k}(\bar a)$$

for all $a\in V_K$ and $b_1,\cdots,b_n\in K^\times$. Here, ∂^n denotes the composite of tame symbols $K_n^M(K) \to K_0^M(k_0) = \boldsymbol{Z}$ and $\bar{a}$ denotes the residue class of a.

Next, we extend the definition of $(\ ,\ )_K: G(K) \times K_n^M(K) \longrightarrow G(k)$ to any commutative algebraic group G. By Chevalley's theorem, there is an exact sequence

$$0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$$

such that G' is smooth affine and G'' is proper. Let V_K be the valuation ring defined above. Since G'' is proper, we have $G''(V_K) = G''(K)$. Since V_K is henselian and G' is smooth, an element of $G''(V_K)$ is contained in the image of $G(V_K k') \to G''(V_K k')$ after a finite Galois extension k'/k. These facts show that an element α of $G(K)$ is written in $G(Kk')$ in the form $\alpha' + \beta$ such that $\alpha' \in G'(Kk')$ and $\beta \in G(V_K k')$. If $\alpha = \alpha' + \beta$ with $\alpha' \in G'(K)$ and $\beta \in G'(V_K)$, we define

$$(\alpha,\ )_K: K_n^M(K) \longrightarrow G(k)$$

to be the sum of the following two maps

$$(\alpha',\ )_K: K_n^M(K) \longrightarrow G'(k) \subset G(k),$$
$$K_n^M(K) \longrightarrow G(k); \quad x \longrightarrow \partial^n(x) \cdot \mathrm{Tr}_{k_0/k}(\bar{\beta})$$

where ∂^n is the composite of tame symbols $K_n^M(K) \to K_0^M(k_0) = \boldsymbol{Z}$, $\mathrm{Tr}_{k_0/k}$ is the trace map $G(k_0) \to G(k)$, and $\bar{\beta}$ denotes the image of β in $G(k_0)$. For general α, $(\alpha,\ )_K: K_n^M(K) \to G(k')$ is defined after a finite Galois extension k'/k, but the image is invariant by $\mathrm{Gal}(k'/k)$ and hence contained in $G(k)$. It is easily seen that this determines a well defined pairing

$$(\ ,\ )_K: G(K) \times K_n^M(K) \longrightarrow G(k).$$

Lemma 1. *Let G be a commutative algebraic group over k, and let $\alpha \in G(K)$.*

(1) *There is an $i \geqq 0$ such that $(\alpha,\ )_K$ annihilates $U^i K_n^M(K)$.*

(2) *If $\alpha \in G(O_K)$, we can take $i = 0$. If $\alpha \in G(V_K)$, we have $(\alpha, x)_K = \partial^n(x) \cdot \mathrm{Tr}_{k_0/k}(\bar{\alpha})$.*

(3) *If G is of multiplicative type, we can take $i = 1$.*

(4) *Let R be a ring over k which is a k-vector space of finite rank, and let $G = G_m^R$. Assume α is the unipotent element $1 - c^{-1}a$, where a is a nilpotent element of $R \otimes_k O_K$ and $c \in O_K - \{0\}$. Then, if $a^N = 0$, we can take $i = N \cdot \mathrm{ord}_K(c) + 1$.*

Proof. The assertion (2) follows easily from the definition of $(\ ,\ )_K$. For (3), we may replace k by its any finite extension, and we are reduced to the case $G=G_m$. Then, the residue map is induced by tame symbols which annihilate U^1.

We prove (4). Let $r=\mathrm{ord}_K(c)$. As is easily seen, $U^{rN+1}K_n^M(K)$ is contained in the subgroup of $K_n^M(K)$ generated by elements of the form $\{x, y_1, \cdots, y_{n-1}\}$ such that $x \in U_K^{(rN)}$ and $y_1, \cdots, y_{n-1} \in U_K$. Since $\alpha=(c-a)c^{-1}$, we are reduced to proving that

$$\{c-a, U_K^{(rN)}\} \in \mathrm{Im}\,(K_2(R\otimes_k O_K)\longrightarrow K_2(R\otimes_k K)).$$

Let $f \in O_K$ and let

$$g=\sum_{j=0}^{N-1} c^j a^{N-1-j} \in R\otimes_k O_K.$$

Then,

$$\{c-a, 1-c^N f\}=\{c-a, 1-(c-a)fg\}.$$

The subgroup $1-(c-a)(R\otimes_k O_K)$ of $(R\otimes_k K)^\times$ is generated by elements $1-(c-a)h$ such that $h \in (R\otimes_k O_K)^\times$. But, for $h \in (R\otimes_k O_K)^\times$,

$$\{c-a, 1-(c-a)h\}=-\{h, 1-(c-a)h\} \in \mathrm{Im}\,(K_2(R\otimes_k O_K)).$$

Lastly, an element $\alpha \in G(K)$ becomes a sum of elements of the types (2) (3) (4) after a finite extension of k (by Chevalley's thoerem introduced above), which proves (1).

Lemma 2. *Let K' be a finite extension of K.*

(1) $(\alpha_{K'}, x)_{K'}=(\alpha, N_{K'/K}(x))_K \quad (\alpha \in G(K), x \in K_n^M(K'))$.

(2) $(\alpha, x_{K'})_{K'}=(\mathrm{Tr}_{K'/K}(\alpha), x)_K \quad (\alpha \in G(K'), x \in K_n^M(K))$.

Here, for $\alpha \in G(K)$ (resp. $x \in K_n^M(K)$), $\alpha_{K'} \in G(K')$ (resp. $x_{K'} \in K_n^M(K')$) denotes its canonical image.

This follows from formal properties of the norm homomorphism in K-theory.

Lemma 3. *Let F be an algebraic function field in one variable over a perfect field k. Let $\alpha \in G(F)$ (G is a commutative algebraic group over k). Then, for any place v of F over k, the homomorphism $(\alpha,\ )_{F_v}$: $(F_v)^\times \to G(k)$ defined above coincides with the local symbol of Serre* [20] *Chapter III.*

Proof. It is sufficient to prove that the family $\{(\ ,\ )_{F_v}\}_v$ satisfies the characterizing condition of the local symbol in [20] Chapter III. By virtue of Lemma 1 (2), what we must check reduces to

(A) $$\sum_{\text{all } v} (\alpha, x)_{F_v}=0 \qquad \text{for } x \in F^\times.$$

By Lemma 2, we are reduced to the case where x is transcendental over k and $F=k(x)$. Let $0\to G'\to G\to G''\to 0$ be an exact sequence such that G' is affine and G'' is proper. By [20] Chapter III n° 8 Lemma 7, we have $G''(k(x))=G''(k)$. From this, it follows that $G(k(x))$ is generated by $G'(k(x))$ and $G(k)$. For $\alpha\in G(k)$, the formula (A) above is nothing but the well known formula

$$\sum_{\text{all } v}[\kappa(v)\colon k]\cdot \mathrm{ord}_{F_v}=0 \qquad \text{on } F^\times.$$

Thus, we are reduced to the case G is affine, and hence to the case $G=G_m^R$. In this case, the formula (A) follows from the residue formula

$$\sum_{\text{all } v}\mathrm{Res}_{F_v/k}(x)=0 \qquad \text{in } K_q(R) \text{ for all } x\in K_{q+1}(R\otimes_k F).$$

This formula is proved as follows. Let C be the regular proper curve over k with function field F. Let H be the category of coherent sheaves $\mathscr{F}$ on $R\otimes_k C$ having a resolution of length one by vector bundles such that $\mathscr{F}\otimes_{\mathcal{O}_C}F=0$. Then, the sum $\sum_{\text{all } v}\mathrm{Res}_{F_v/k}$ is the composite

$$K_{q+1}(R\otimes_k F)\xrightarrow{\partial}K_q(H)\longrightarrow K_q(R),$$

but the second arrow factors as

$$K_q(H)\xrightarrow{i}K_q(R\otimes_k C)\xrightarrow{\text{norm}}K_q(R\otimes_k \boldsymbol{P}_k^1)\longrightarrow K_q(R),$$

and $i\circ\partial=0$.

Now, we return to the smooth proper surface X over a field k and prove Proposition 1. The uniqueness of γ_X follows from Chapter II Section 4 Lemma 3. To prove the existence of γ_X, an easy reduction shows that we may assume that k is algebraically closed.

If k is algebraically closed, any principal homogeneous space H over G has a k-rational point, and hence isomorphic to G itself. We have $E_H=G\times \boldsymbol{Z}$. Our task becomes to show that an element α of $G(K)$ defines a canonical homomorphism $\varprojlim_{\text{all } m} C_m(X)\longrightarrow G(K)$ having the property stated in Proposition 1 (1) in which H and E_H are replaced by G. For $x\in X_0$ and $z\in \mathrm{Spec}(\hat{\mathcal{O}}_{X,x})_1$, we have already a canonical homomorphism

$$(\alpha,\ )_{K_{x,z}}\colon K_2(K_{x,z})\longrightarrow G(k).$$

We show that the collection $\{(\alpha,\ )_{K_{x,z}}\}$ defines a homomorphism $C_m(X)\to G(k)$ for some modulus m. The fact that the obtained homomorphism γ_X has the required property in Proposition 1 (1) will then follow from

Lemma 1 (2). It suffices to prove the following Lemma 4 and Lemma 5, in which G denotes a commutative algebraic group over a perfect field k.

Lemma 4. *Let F be a function field in one variable over k. Let $K=F((T))$, and $K_v=F_v((T))$ for each place v of F over k. Let $\alpha \in G(K)$. Then,*

(1) *There exists an integer $i \geqq 0$ such that $(\alpha, \ )_{K_v}: G(K_v) \to G(k)$ annihilate $U^i K_2(K_v)$ for all v.*

(2) *For $b \in K_2(K)$, $(\alpha, b)=0$ for almost all v, and $\sum_{\text{all } v} (\alpha, b)_{K_v}=0$.*

Lemma 5. *Let $x \in X_0$ and $\alpha \in G(K)$. (Here K is the function field of X as before.) Then, for $b \in K_2(K_x)$, $(\alpha, b)_{K_{x,z}}=0$ for almost all $z \in$ Spec $(\hat{\mathcal{O}}_{X,x})_1$ and $\sum_{\text{all } z} (\alpha, b)_{K_{x,z}}=0$.*

Proof of Lemma 4. We may assume that k is algebraically closed. By Chevalley's theorem, for some finite separable extension F' of F, the image of α in $G(F'((T)))$ is expressed as a sum $\alpha'+\beta$ such that $\alpha' \in G'(F'((T)))$ for an affine subgroup G' of G and $\beta \in G(F'[[T]])$. Let $K'=F'((T))$. An easy study of the norm maps $(K')^\times \to K^\times$ and $(K'_{v'})^\times \to K_v^\times$ shows that

$$K_2(K)=N_{K'/K}K_2(K') \quad \text{and} \quad U^i K_2(K_v) \subset N_{K'_{v'}/K_v}(U^i K_2(K'_{v'})) \quad (i \geqq 0),$$

where v' is any place of F' over k lying over v and $K'_{v'}=F'_{v'}((T))$ (k is assumed to be algebraically closed). Hence, by Lemma 2, we may assume $K'=K$. Thus, we may treat separately the case G is affine and the case $\alpha \in G(O_K)$. In the affine case, we may assume $G=G_m^R$, and α is an element of $K^\times \subset G_m^R(K)$ or an element of $G_m^R(K)$ of the form $1-c^{-1}a$ for some nilpotent element a of $R \otimes_k O_K$ and for some $c \in O_K-\{0\}$. Hence, Lemma 4 (1) for the affine case follows from Lemma 1. For $\alpha \in G_m^R(K)$ and $b \in K_2(K)$, $(\alpha, b)_{K_v}$ is the image of $\{\alpha, b\}$ under

$$K_3(R \otimes_k K) \xrightarrow{\mathrm{Res}_{K/F}} K_2(R \otimes_k F) \xrightarrow{\mathrm{Res}_{F_v/k}} K_1(R)=R^\times.$$

Hence, Lemma 4 (2) for the affine case follows from the residue formula $\sum_{\text{all } v} \mathrm{Res}_{F_v/k}= 0$ (see the proof of Lemma 3). In the case $\alpha \in G(O_K)$, we can take $i=0$ by Lemma 1. Let $b \in K_2(K)$. Let $\bar{\alpha}=\alpha \bmod T \in G(F)$, and let $\partial(b) \in F^\times$ be the image of b under the tame symbol. We have,

$$\sum_{\text{all } v} (\alpha, b)_{K_v}= \sum_{\text{all } v} (\bar{\alpha}, \partial(b))_{F_v}=0$$

where the last equation follows from Lemma 3.

Proof of Lemma 5. We may again assume that k is algebraically closed. Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence such that G' is smooth affine and G'' is proper. Then, $G''(\mathcal{O}_{X,x})=G''(K)$ by the proof of Lang [11] Chapter II Section 1 Theorem 2. Since k is algebraically closed and G' is smooth, $G(\hat{\mathcal{O}}_{X,x})\to G''(\hat{\mathcal{O}}_{X,x})$ is surjective, and hence α is the sum of $\alpha'\in G'(K_x)$ and $\beta\in G(\hat{\mathcal{O}}_{X,x})$ in $G(K_x)$. The complex of Bloch-Gersten-Quillen for the ring $\hat{\mathcal{O}}_{X,x}$ and Lemma 1 (2) show $\sum_{\text{all } z}(\beta, b)_{K_{x,z}}=0$ for all $b\in K_2(K_x)$. So, we may assume that G is affine and hence that $G=G_m^R$. Since $\hat{\mathcal{O}}_{X,x}$ is a finite extension of $k[[T_1, T_2]]$, the norm argument reduces us to proving the following

Lemma 6. *Let k be a field, $A=k[[T_1, T_2]]$, K the field of fractions of A. Let p be the point of* Spec $(A)_1$ *corresponding to the prime ideal (T_2) of A, and let $D=\mathrm{Spec}\,(A)_1-\{p\}$. Let $J=k((T_2))$, and denote T_1 by T. Then, for any ring R over k and any $q\geqq 0$, the homomorphism $K_{q+2}(R\otimes_k K)\to K_q(R)$;*

$$\mathrm{Res}_{k((T))/k}\circ\mathrm{Res}_{K_p/k((T))}+\mathrm{Res}_{J/k}\circ(\sum_{z\in D}\mathrm{Res}_{K_z/J})$$

is the zero map.

(Note $K_p=k((T_1))((T_2))$.) To prove this we use

Lemma 7. *Let S be the multiplicatively closed subset of $J[T^{-1}]$;*

$$\{1+a_1T^{-1}+\cdots+a_nT^{-n};\ n\geqq 0,\ a_1,\cdots,a_n\in m_J\},$$

and let $I_+=\kappa[[T_1, T_2]][T_2^{-1}]$, $I_-=S^{-1}J[T^{-1}]$. ($T_1=T$ as above.) Then, $K_q(R\otimes_k K)$ is generated by the images of

$$K_q(R\otimes_k I_+),\quad K_q(R\otimes_k I_-),\quad \text{and}\quad \{K_{q-1}(R\otimes_k J), T\}.$$

Proof. Let H (resp. H') be the category of all coherent sheaves $\mathcal{F}$ on $R\otimes_k \boldsymbol{P}_J^1$ (resp. on Spec $(R\otimes_k I_+)$) having a resolution of length one by vector bundles such that $I_-\otimes_{\boldsymbol{P}_J^1}\mathcal{F}=0$ (resp. $K\otimes_{I_+}\mathcal{F}=0$). Here, $\boldsymbol{P}_J^1$ is the projective line Spec $(J[T])\cup$ Spec $(J[T^{-1}])$. Then, the natural functor $H\to H'$ is an equivalence of categories, and hence

$$K_q(H)\cong K_q(H')\qquad \text{for all } q.$$

Hence, this lemma follows from the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
\cdots\longrightarrow & K_q(H) & \longrightarrow & K_q(R\otimes_k \boldsymbol{P}_J^1) & \longrightarrow & K_q(R\otimes_k I_-) & \longrightarrow & K_{q-1}(H) & \longrightarrow\cdots \\
 & \wr\Vert\downarrow & & \downarrow & & \downarrow & & \wr\Vert\downarrow & \\
\cdots\longrightarrow & K_q(H') & \longrightarrow & K_q(R\otimes_k I_+) & \longrightarrow & K_q(R\otimes_k K) & \longrightarrow & K_{q-1}(H') & \longrightarrow\cdots
\end{array}$$

and from the structure theorem of the K-group of the projective line (Quillen [16] Section 8.3).

Now, we prove Lemma 6. Note that $D=\operatorname{Spec}(I_+)_0$ and $(P_J^1)_0=\operatorname{Spec}(I_+)_0 \coprod \operatorname{Spec}(I_-)_0$. Let $a \in K_{q+1}(R\otimes_k J)$. We have,

$$\operatorname{Res}_{K_z/J}(K_{q+2}(R\otimes_k I_+))=0$$

for $z \in D$ as is easily seen,

$$(\sum_{z\in D}\operatorname{Res}_{K_z/J})(K_{q+2}(R\otimes_k I_-))=-(\sum_{v\in\operatorname{Spec}(I_-)_0}\operatorname{Res}_{J(T)_v/J})(K_{q+2}(R\otimes_k I_-))=0$$

by the residue formula of $J(T)/J$ where $J(T)_v$ is the completion of $J(T)$ at v,

$$\operatorname{Res}_{K_z/J}(\{a, T\})=0 \qquad \text{for } z \in D-\{(T)\},$$
$$\operatorname{Res}_{J/k}\circ\operatorname{Res}_{K_z/J}(\{a, T\})=\operatorname{Res}_{J/k}(a) \qquad \text{for } z=(T).$$

On the other hand,

$$\operatorname{Res}_{k((T))/k}\circ\operatorname{Res}_{K_p/k((T))}(K_{q+2}(R\otimes_k I_+))\subset\operatorname{Res}_{k((T))/k}(K_{q+1}(R\otimes_k k[[T]]))=0,$$
$$\operatorname{Res}_{k((T))/k}\circ\operatorname{Res}_{K_p/k((T))}(K_{q+2}(R\otimes_k I_-))\subset\operatorname{Res}_{k((T))/k}(K_{q+1}(R\otimes_k k[T^{-1}]))=0$$

where the last identity follows from the residue formula of $k(T)/k$,

$$\operatorname{Res}_{k((T))/k}\circ\operatorname{Res}_{K_p/k((T))}(\{a, T\})=-\operatorname{Res}_{J/k}(a).$$

By Lemma 7, these affirm the formula in Lemma 6.

Thus, we have proved Proposition 1 (1).

Proposition 1 (2) follows from the following facts. Assume k is finite. Let U be a non-empty open set of X. Just as in Chapter II Section 3 Lemma 2, for $x \in U_0$, the image of $1 \in \boldsymbol{Z}$ under

$$\boldsymbol{Z}=\varprojlim_{U\cap\operatorname{Supp}(m)=\phi} C_m(x)\longrightarrow\varprojlim_{U\cap\operatorname{Supp}(m)=\phi} C_m(X)\longrightarrow\pi_1^{\mathrm{ab}}(U)$$

is the Frobenius substitution over x. On the other hand, for a morphism α: $U\to H$, the homomorphism of Lang induced by α; $\operatorname{Gal}(K^{\mathrm{ab}}/K)'\to E_H(k)$ is unramified on U, and sends the Frobenius substitution of $x \in U_0$ to $\operatorname{Tr}_{\kappa(x)/k}(\alpha(x))$ ([20] Chapter VI n° 24 Theorem 2).

Remark 1. If k is not assumed to be a finite field, the map

$$\gamma_X\colon \varprojlim_m C_m(X)\longrightarrow A_{K/k}$$

may have a big kernel in general. For example, let k be a usual local field. Then, for any proper normal connected surface X over k with function field K, there is a canonical pairing

$$H^2(K)\times \varprojlim_m C_m(X) \longrightarrow \mathbf{Q}/\mathbf{Z},$$

which is an analogue of the pairing $H^1(K)\times \varprojlim_m C_m(X)\to \mathbf{Q}/\mathbf{Z}$ of the case k is finite studied in this paper. Let $X=\mathbf{P}_k^2$, T_1 and T_2 are the canonical variables on X, $x\in X_0$ the point corresponding to the maximal ideal (T_1, T_2) of $k[T_1, T_2]$, and $z\in \operatorname{Spec}(\hat{\mathcal{O}}_{X,x})_1$ the point corresponding to the prime ideal (T_2) of $k[[T_1, T_2]]$. Assume that k contains a primitive n-th root ζ of 1, and let χ be the element of $H^2(K)=\mathrm{Br}(K)$ represented by the K-algebra

$$\bigoplus_{0\leq i<n,\, 0\leq j<n} K\alpha^i\beta^j \quad \text{with} \quad \alpha^n=T_1,\quad \beta^n=T_2,\quad \alpha\beta=\zeta\beta\alpha.$$

Then, the composite

$$K_2(k)\subset K_2(k((T_1))((T_2)))=K_2(K_{x,z})\longrightarrow \varprojlim_m C_m(X) \xrightarrow{\text{by } \chi} \frac{1}{n}\mathbf{Z}/\mathbf{Z}$$

is nothing but the Hilbert symbol and hence is surjective. On the other hand, $\gamma_X\colon \varprojlim_m C_m(X)\to A_{K/k}$ annihilates the image of $K_2(k)\subset K_2(K_{x,z})$. Thus, γ_X annihilates an important element of $\varprojlim_m C_m(X)$.

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Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Bunkyo-ku
Tokyo 113, Japan