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On the Zeta Function of an Abelian Scheme over the Shimura Curve II*)

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Introduction

This paper is a continuation of our previous work [17] under the same title. Let F be a totally real number field of finite degree g over $\boldsymbol{Q}$, and B a division quaternion algebra over F which is unramified at one infinite prime of F, and ramified at all the other infinite primes of F. In [17], we developed a theory which generalizes that of Kuga and Shimura [11], for F and B as above assuming that $g=[F:\boldsymbol{Q}]$ is odd. Namely, when g is odd, we constructed an abelian scheme A_S over the Shimura curve V_S attached to B, and expressed the Hasse-Weil zeta function of A_S^k (the k-fold fibre product of A_S over V_S) as a product of Dedekind zeta functions and automorphic L-functions associated with $B^\times$. Also, as its application, we proved the Ramanujan-Petersson conjecture for certain automorphic forms on $B^\times$ for almost all finite primes of F.

The aim of this paper is to supplement [17] in the following two points:

(I) To obtain results parallel to that in [17] when g is even.

(II) To prove the Ramanujan-Petersson conjecture for all "good primes" of F.

The construction of A_S is carried out in Section 2. We will redo the construction in the case when g is odd also, for the sake of completeness. The main result of Section 2 is (2.6.2), which immediately enables us to extend the results of [17] for general F. To construct A_S, we use the functoriality of the canonical models, due to Deligne [3], which generalizes that of Shimura [22] Section 8 (cf. also [22] 2.13). We recall necessary tools for this in the first preliminary section. The main results of this paper are (3.1.4) and (3.2.1). The proof of (3.1.4) goes exactly in the same way as in [17] Sections 3–4 after (2.6.2), and we omit it, refering to [17] for details. (3.2.1) and its corollaries give an answer to the above (II).

We note that, as for the Ramanujan-Petersson conjecture, *Morita* [14] *has recently shown that the assertion* (3.2.2) *is valid without our assumption*

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on $\{k_n\}$ there, by a different method. Although our result is weaker than his, we include in Section 3 a proof within the framework of our theory, which we hope to be of some interest.

Notation and terminology

For a field k and a Galois extension K of k, $\mathrm{Gal}(K/k)$ denotes the Galois group of K/k, which acts on K *on the right*. $\bar{k}$ (resp. k_{ab}) denotes an algebraic closure of k (resp. the maximal abelian extension of k in $\bar{k}$). For a ring R with unity, $M_n(R)$ stands for the ring of $n\times n$ matrices with entries in R. $GL_n(R)$ (resp. $R^\times$) denotes the group of invertible elements of $M_n(R)$ (resp. R). We usually let $M_n(R)$ act on R^n (row vectors) *on the right*.

For a number field F, F_A (resp. $F_A^\times$) denotes the adele ring (resp. the idele group) of F. We write $\boldsymbol{A}$ (resp. $\boldsymbol{A}^f$) for $\boldsymbol{Q}_A$ (resp. the finite part of $\boldsymbol{A}$). For a prime v of F, the subscript v will usually mean (except in 3.3 and 3.4) the completion with respect to v. $|-|_v$ denotes the valuation of F or F_v determined by v, normalized in the usual manner. The Artin symbol for F is denoted by $[-, F]$, whose sign convention is the usual one; a prime element of $F_{\mathfrak{p}}^\times \subset F_A^\times$ is mapped by $[-, F]$ to a Frobenius element at $\mathfrak{p}$, for a finite prime $\mathfrak{p}$ of F.

For an abelian group G and a positive integer a, we denote by ${}_aG$ the kernel of the multiplication by a on G. We use the similar notation for commutative group schemes. For a scheme X, $X_{\text{ét}}$ denotes the étale site of X. The cohomology groups of schemes are always the étale cohomology groups, in this paper.

§ 1. Canonical models

The purpose of this section is to recall known basic facts about canonical models (Deligne [3], [5], Shimura [22]) for our later use.

1.1. Let F be a totally real number field of finite degree g over $\boldsymbol{Q}$, and B a quaternion algebra over F. $B\otimes_{\boldsymbol{Q}}\boldsymbol{R}$ is isomorphic to $M_2(\boldsymbol{R})^r\times \boldsymbol{H}^{g-r}$ as an $\boldsymbol{R}$-algebra, with $\boldsymbol{H}$ the Hamilton quaternion algebra over $\boldsymbol{R}$. We fix such an isomorphism once for all, and hereafter identify them. *We also assume that* $r>0$ (except in the assertion (3.2.3)) in this paper. Fix a positive integer n, and define an affine algebraic group G over $\boldsymbol{Q}$ by requiring:

$$G(\boldsymbol{Q})=\{\alpha \in GL_n(B)\,|\,\alpha^t\alpha^\iota=\nu(\alpha)1_n,\ \nu(\alpha)\in F^\times\} \tag{1.1.1}$$

where ι denotes the canonical involution of B, ${}^t(a_{ij})^\iota={}^t(a_{ij}^\iota)$ for an element

$(a_{ij}) \in GL_n(B)$, and $1_n \in GL_n(B)$ is the unit matrix. There is also a natural isomorphism:

$$G(\boldsymbol{R}) \cong Gp_n(\boldsymbol{R})^r \times Gp_n^*(\boldsymbol{H})^{g-r}, \quad \text{with}$$

$$Gp_n(\boldsymbol{R}) = \{\alpha \in GL_{2n}(\boldsymbol{R}) \mid \alpha J_n {}^t\alpha = \nu(\alpha) J_n,\ \nu(\alpha) \in \boldsymbol{R}^\times\},$$

$$(1.1.2) \qquad J_n = \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix},$$

$$Gp_n^*(\boldsymbol{H}) = \{\alpha \in GL_n(\boldsymbol{H}) \mid \alpha {}^t\bar{\alpha} = \nu(\alpha) 1_n,\ \nu(\alpha) \in \boldsymbol{R}^\times\},$$

where the bar means the canonical involution of $\boldsymbol{H}$. We also fix an isomorphism (1.1.2), and hereafter identify the groups in both hand sides there.

Let $\boldsymbol{S} = R_{\boldsymbol{C}/\boldsymbol{R}}(\boldsymbol{G}_m)$ be the algebraic group over $\boldsymbol{R}$ whose set of real points is identified with $\boldsymbol{C}^\times$, where $\boldsymbol{G}_m$ is the multiplicative group in the usual sense and $R_{\boldsymbol{C}/\boldsymbol{R}}$ is the restriction of scalar functor. We define an $\boldsymbol{R}$-homomorphism h_0 of $\boldsymbol{S}$ to $G_{\boldsymbol{R}} = G \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ by:

$$(1.1.3) \qquad h_0(a+b\sqrt{-1}) = \left(\begin{bmatrix} a1_n & -b1_n \\ b1_n & a1_n \end{bmatrix}, \dots, \begin{bmatrix} a1_n & -b1_n \\ b1_n & a1_n \end{bmatrix}, 1_n, \dots, 1_n\right)$$

for $a+b\sqrt{-1} \in \boldsymbol{C}^\times = \boldsymbol{S}(\boldsymbol{R})$, where, in the right hand side above, the first r factors are $\begin{bmatrix} a1_n & -b1_n \\ b1_n & a1_n \end{bmatrix} \in Gp_n(\boldsymbol{R})$, and the other factors are $1_n \in Gp_n^*(\boldsymbol{H})$. Let X be the $G(\boldsymbol{R})$-conjugacy class of h_0. Then the pair (G, X) satisfies the axioms of Shimura varieties in Deligne [5] 2.1.1.

1.2. In general, let $V_{\boldsymbol{R}}$ be a vector space over $\boldsymbol{R}$ of dimension $2n$, and ψ a non-degenerate alternating form on $V_{\boldsymbol{R}}$. Then there is an isomorphism $i\colon V_{\boldsymbol{R}} \xrightarrow{\sim} \boldsymbol{R}^{2n}$ (row vectors), which satisfies $\psi(x, y) = i(x) J_n {}^t i(y)$. The group $Gp(V_{\boldsymbol{R}}, \psi) = i^{-1} \circ Gp_n(\boldsymbol{R}) \circ i$ is the group of all the symplectic similitudes of $(V_{\boldsymbol{R}}, \psi)$ which acts on $V_{\boldsymbol{R}}$ *on the right*. Define an $\boldsymbol{R}$-homomorphism h_0 of $\boldsymbol{S}$ to $Gp(V_{\boldsymbol{R}}, \psi)$ by:

$$(1.2.1) \qquad h_0(a+b\sqrt{-1}) = i^{-1} \circ \begin{bmatrix} a1_n & -b1_n \\ b1_n & a1_n \end{bmatrix} \circ i, \qquad \text{for } a+b\sqrt{-1} \in \boldsymbol{C}^\times = \boldsymbol{S}(\boldsymbol{R})$$

and let X be the $Gp(V_{\boldsymbol{R}}, \psi)$-conjugacy class of h_0. If we denote by K_∞ (resp. K'_∞) the centralizer of h_0 (resp. $i \circ h_0(-) \circ i^{-1}$) in $Gp(V_{\boldsymbol{R}}, \psi)$ (resp. $Gp_n(\boldsymbol{R})$), then we have

$$(1.2.2) \qquad Gp(V_{\boldsymbol{R}}, \psi)/K_\infty \cong Gp_n(\boldsymbol{R})/K'_\infty \xrightarrow{\sim} H_n^\pm$$

where $H_n^{\pm}=H_n^+ \coprod H_n^-$, the disjoint union of the Siegel upper half space H_n^+ and the Siegel lower half space H_n^- of degree n. The right isomorphism is given by: $\begin{bmatrix} A & B \\ C & D \end{bmatrix} K'_\infty \mapsto (A\sqrt{-1}+B)(C\sqrt{-1}+D)^{-1}$.

The set $Gp(V_{\boldsymbol{R}}, \psi)/K_\infty$ may also be regarded as a set of homomorphisms of $\boldsymbol{S}$ to $Gp(V_{\boldsymbol{R}}, \psi)$ by the correspondence: $gK_\infty \mapsto gh_0(-)g^{-1}$.

Lemma (1.2.3) (Deligne [3] 1.6, [5] 1.3.1). *The above correspondence gives a bijective map from $Gp(V_{\boldsymbol{R}}, \psi)/K_\infty$ onto the set of homomorphisms h: $\boldsymbol{S}\to Gp(V_{\boldsymbol{R}}, \psi)$ of $\boldsymbol{R}$-algebraic groups satisfying:*

(1.2.3.1) *The Hodge structure of $V_{\boldsymbol{R}}$ defined by h is of type* $(-1, 0)+(0, -1)$.

(1.2.3.2) *The $\boldsymbol{R}$-bilinear form $\psi(x, yh(\sqrt{-1}))$ on $V_{\boldsymbol{R}}$ is symmetric and definite.*

Here the terminology for the Hodge structure is the same as in [5] (and not as in [3]). We note that, under the condition (1.2.3.1), $\psi(x, yh(\sqrt{-1}))$ is symmetric if and only if $V_{\boldsymbol{C}}^{0,-1}$ is totally isotropic for ψ.

1.3. Let the notation be as in 1.1, and let K be an open compact subgroup of $G(A^f)$. By definition, we put:

$$(1.3.1) \qquad {}_KM_{\boldsymbol{C}}(G, X)=G(\boldsymbol{Q})\backslash X\times G(A^f)/K.$$

Here the set X is equipped with the complex structure by the isomorphism (1.2.2). If $G(A)=\coprod_i G(\boldsymbol{Q})y_i(G(\boldsymbol{R})\times K)$ is a disjoint decomposition, ${}_KM_{\boldsymbol{C}}(G, X)$ is isomorphic to $\coprod_i G(\boldsymbol{Q})\cap y_i(G(\boldsymbol{R})\times K)y_i^{-1}\backslash X$, and hence it may be considered as the set of complex points of a quasi-projective algebraic variety. We also put:

$$(1.3.2) \qquad M_{\boldsymbol{C}}(G, X)=\varprojlim {}_KM_{\boldsymbol{C}}(G, X)$$

the projective limit being taken relative to natural projections. By the work of Shimura [22] (cf. also [3] § 6), ${}_KM_{\boldsymbol{C}}(G, X)$ (resp. $M_{\boldsymbol{C}}(G, X)$) admits a canonical model (cf. [5] 2.2.5) ${}_KM_E(G, X)$ (resp. $M_E(G, X)$) which is a scheme over the reflex field $E=E(G, X)$ of (G, X). In our case, if $\tau_1, \cdots, \tau_r$ are all the different embeddings of F into $\boldsymbol{R}$ which splits B, then $E(G, X)$ is the field generated by all $\sum_{n=1}^{r} a^{\tau_n} (a \in F)$ over $\boldsymbol{Q}$. In the following, when there is no fear of confusion, we will sometimes write ${}_KM_{\boldsymbol{C}}$, $M_{\boldsymbol{C}}$, ${}_KM_E$, and M_E for ${}_KM_{\boldsymbol{C}}(G, X)$ etc..

We will need the following result in the next section:

Theorem (1.3.3) (a specialized form of [3] Corollary 5.4). *Let (G^i, X^i) be as above with the reflex fields E^i $(i=1, 2)$. Let v: $G^1\to G^2$ be a $\boldsymbol{Q}$-rational* *homomorphism which sends the set X^1 of homomorphisms of $\boldsymbol{S}$ to $G^1_{\boldsymbol{R}}$ to the set X^2 of homomorphisms of $\boldsymbol{S}$ to $G^2_{\boldsymbol{R}}$. Then there is a unique morphism of the canonical models: $M_{E^1}(G^1, X^1) \to M_{E^2}(G^2, X^2)$ defined over $E^1(\supseteq E^2)$, whose base change to $\boldsymbol{C}$ gives the complex analytic morphism of $M_{\boldsymbol{C}}(G^1, X^1)$ to $M_{\boldsymbol{C}}(G^2, X^2)$ that is determined by v.*

1.4. We next recall the relation between the canonical model in the sense above, and Shimura's canonical system (Milne, Shih [12] § 4). For simplicity, we describe it here only when $r=1$ (see loc. cit. for the general case). Thus, assuming that the identity injection of F into $\boldsymbol{R}$ splits B, $E(G, X)=F$ in this case. The "reciprocity law" of the canonical model $M_F(G, X)$ ([3] Corollary 5.6) then implies that:

(1.4.1) The schemes ${}_K M_F(G, X)$ and $M_F(G, X)$ are irreducible.

(1.4.2) All the irreducible components of $M_F(G, X) \otimes_F \bar{F}$ are precisely F_{ab}-rational.

For an element $x \in G(A)$, we put

$$\sigma(x)=[\nu(x)^{-1}, F] \tag{1.4.3}$$

where $[-, F]$ is the Artin symbol for F (cf. Notation and terminology). Let $M^0(G, X)=M^0$ be the irreducible component of $M_F(G, X) \otimes_F F_{ab}$ that corresponds to the image of $H_n^+ \times \{e\}$ in $M_{\boldsymbol{C}}(G, X)$. Then it also follows that the natural right action of $x \in G(A^f)$ on $M_F(G, X)$ ([5] 2.1.4) sends M^0 to $M^{0\sigma(x)^{-1}}$. Put

$$G(\boldsymbol{R})_+ = \{x \in G(\boldsymbol{R}) \mid \nu(x) \text{ is totally positive}\}, \quad \text{and}$$
$$G(A)_+ = G(\boldsymbol{R})_+ \times G(A^f).$$

Then this group acts on M^0 as follows.

Definition (1.4.4). The left action $[x]$ of $x \in G(A)_+$ on M^0 is defined by the commutativity of the following diagram:

$$\begin{tikzcd}
M^0 \arrow[r, "x_f^{-1}"'] \arrow[dr] \arrow[rr, bend left, "{[x]}"] & M^{0\sigma(x)} \arrow[r] \arrow[d] & M^0 \arrow[d] \\
& \mathrm{Spec}\,(F_{ab}) \arrow[r, "\mathrm{Spec}\,(\sigma(x))"'] & \mathrm{Spec}\,(F_{ab})
\end{tikzcd}$$

where x_f is the finite part of x, the left morphism is the right action of x_f^{-1}, and the right cartesian square defines $M^{0\sigma(x)}$.

For S_0 an open compact subgroup of $G(A^f)$, put $S=G(\boldsymbol{R})_+ \times S_0$ and

$$V_S = S \backslash M^0 \tag{1.4.5}$$

where the quotient is taken with respect to the action (1.4.4). V_S is defined over k_S, the field corresponding to $F^\times\nu(S)\subset F_A^\times$ by class field theory. If $xSx^{-1}\subseteq T$ with $x\in G(A)_+$ and S, T as above, the right action of x_f^{-1} induces, after taking the quotients, a morphism:

$$J_{TS}(x):\ V_S\longrightarrow V_T^{\sigma(x)} \tag{1.4.6}$$

which is defined over k_S ($\supseteq k_T$). There is a natural isomorphism φ_S: $\Gamma_S\backslash H_n^+ \rightrightarrows V_S(C)$, with $\Gamma_S=S\cap G(Q)$, and the system $\{V_S, \varphi_S, J_{TS}(x)\}$ thus obtained is the canonical system in the sense of Shimura [22].

In other words, for $S=G(R)_+\times S_0$ as above, all the geometric irreducible components of ${}_{S_0}M_F(G, X)$ are k_S-rational, and the irreducible component of ${}_{S_0}M_F(G, X)\otimes_F k_S$ which corresponds to $\Gamma_S\backslash H_n^+\subset {}_{S_0}M_C$ is V_S. The restriction of

$${}_{S_0}M_F\otimes_F k_S \xrightarrow[x_f^{-1}\otimes id]{} {}_{T_0}M_F\otimes_F k_S \xrightarrow[id\otimes \mathrm{Spec}\,(\sigma(x))]{} {}_{T_0}M_F\otimes_F k_S$$

with $T=xSx^{-1}$, to V_S etc. gives

$$\begin{tikzcd} V_S \arrow[r, "J_{TS}(x)"'] \arrow[rr, bend left, "{[x]}"] & V_T^{\sigma(x)} \arrow[r] & V_T \end{tikzcd} \tag{1.4.7}$$

where the morphism $[x]$ above is the one which was denoted by $[\tau(x)]$ in [16] 2.3.

1.5. We also have to recall the canonical family of abelian varieties on the Siegel modular varieties for later use ([3] §4). Let V be a vector space of dimension $2n$ over Q, and ψ a non-degenerate alternating form on V. We write G for the Q-algebraic group $Gp(V, \psi)$ of symplectic similitudes of (V, ψ), and X for the $G(R)$ ($=Gp(V_R, \psi)$)-conjugacy class of R-homomorphisms of S to G_R as in 1.2, in this subsection. We fix a Z-lattice V_Z in V, and a $Q^\times$-multiple ψ_Z of ψ which is Z-valued on V_Z. Let V_Z' be the maximal lattice in V which satisfies $\psi_Z(V_Z, V_Z')\subseteq Z$. For a positive integer N, put

$$K_N=\{k\in G(A^f)\mid V_{\hat{Z}}k=V_{\hat{Z}}, V_{\hat{Z}}(k-1)\subseteq NV_{\hat{Z}}\} \tag{1.5.1}$$

with $\hat{Z}=\prod_p Z_p$, the product being taken over all the rational primes, and $V_{\hat{Z}}=V_Z\otimes_Z\hat{Z}$. The points of ${}_{K_N}M_C(G, X)=G(Q)\backslash X\times G(A^f)/K_N$ then parametrizes abelian varieties as follows.

Let $z=[h, k]$ with $h\in X$ and $k\in G(A^f)$ be a point of ${}_{K_N}M_C$ determined by $(h, k)\in X\times G(A^f)$. Then h defines a Hodge structure of type $(-1, 0)$

$+(0,-1)$, or equivalently, a complex structure on $V_{\boldsymbol{R}}=V\otimes_{\boldsymbol{Q}}\boldsymbol{R}$ ([5]1.1.3). We write $(V_{\boldsymbol{R}}, h)$ for $V_{\boldsymbol{R}}$ with that complex structure. Put $V_{\boldsymbol{Z}}k^{-1}=V_{\hat{\boldsymbol{Z}}}k^{-1}\cap V$. It is a $\boldsymbol{Z}$-lattice in V, and we can find a unique multiple ψ_z of $\psi_{\boldsymbol{Z}}$ satisfying: (i) ψ_z is $\boldsymbol{Z}$-valued on $V_{\boldsymbol{Z}}k^{-1}$; (ii) $V'_{\boldsymbol{Z}}k^{-1}$ is the maximal lattice in V satisfying $\psi_z(V_{\boldsymbol{Z}}k^{-1}, V'_{\boldsymbol{Z}}k^{-1})\subseteq\boldsymbol{Z}$; and (iii) $\psi_z(x, yh(\sqrt{-1}))$ is a symmetric and positive definite $\boldsymbol{R}$-bilinear form on $V_{\boldsymbol{R}}$. We then put:

$$A_z=(V_{\boldsymbol{R}}, h)/V_{\boldsymbol{Z}}k^{-1}. \tag{1.5.2}$$

ψ_z determines a Riemann form on the complex torus A_z, and hence A_z is an abelian variety with

$$p_z=\text{the polarization of } A_z \text{ defined by } \psi_z. \tag{1.5.3}$$

We also have a natural level N-structure on A_z:

$$_NA_z\cong V_{\boldsymbol{Z}}k^{-1}/NV_{\boldsymbol{Z}}k^{-1}\xrightarrow{i_k}V_{\boldsymbol{Z}}/NV_{\boldsymbol{Z}} \text{ (the natural action of } k), \tag{1.5.4}$$

which can be extended to a symplectic isomorphism of $(\hat{T}(A)\overset{\text{dfn}}{=}\prod_l T_l(A_z), \psi_{p_z})$ onto $(V_{\hat{\boldsymbol{Z}}}, \psi_{\boldsymbol{Z}}\otimes id)$, where $\psi_{p_z}\colon \hat{T}(A_z)\times\hat{T}(A_z)\to\hat{\boldsymbol{Z}}(1)$ is the polarization form defined by p_z. The points of $_{K_N}M_{\boldsymbol{C}}$ corresponds bijectively with the set of isomorphism classes of the triple (A_z, p_z, i_k) satisfying the above conditions.

Consider the contravariant functor F from the category of locally noetherian $\boldsymbol{Q}$-schemes to that of sets which assigns to each S the isomorphism classes of the triple consisting of

(1.5.5) a projective abelian scheme A over S of relative dimension n;

(1.5.6) a polarization $\lambda\colon A\to\hat{A}=\underline{\mathrm{Pic}}^0(A/S)$ in the sense of Mumford [15] Definition 6.3;

(1.5.7) a level N-structure $_NA\xrightarrow{\sim}(V_{\boldsymbol{Z}}/NV_{\boldsymbol{Z}})_S$, with $(V_{\boldsymbol{Z}}/NV_{\boldsymbol{Z}})_S$ the constant group scheme over S defined by $V_{\boldsymbol{Z}}/NV_{\boldsymbol{Z}}$;

for which we require that

(1.5.8) for any geometric point s of S, let $\psi_s\colon \hat{T}(A_s)\times\hat{T}(A_s)\to\hat{\boldsymbol{Z}}(1)$ be the polarization form defined by λ. Then (1.5.7) can be extended to a symplectic isomorphism of $(\hat{T}(A_s), \psi_s)$ to $(V_{\hat{\boldsymbol{Z}}}, \psi_{\boldsymbol{Z}}\otimes id)$.

When $N\geqslant 3$, F is represented by a quasi-projective scheme $_{K_N}M_{\boldsymbol{Q}}(G, X)={}_{K_N}M_{\boldsymbol{Q}}$ over $\boldsymbol{Q}$, which is an open subscheme of Mumford's $A_{n,d,N}\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ with $d=\sqrt{\det\psi_{\boldsymbol{Z}}}$ ([15] Theorem 7.9). $_{K_N}M_{\boldsymbol{Q}}$ (resp. $\varprojlim {}_{K_N}M_{\boldsymbol{Q}}\overset{\text{dfn}}{=}M_{\boldsymbol{Q}}(G, X)$) is then a canonical model for $_{K_N}M_{\boldsymbol{C}}$ (resp. $M_{\boldsymbol{C}}$), by the main theorem of complex multiplication, by (the same reason as) [3] 4.21. Especially, when $N\geqslant 3$, there exists the universal family of abelian varieties

$$f\colon A\longrightarrow {}_{K_N}M_{\boldsymbol{Q}} \tag{1.5.9}$$

satisfying (1.5.5)–(1.5.8). If we consider $z \in {}_{K_N}M_{\boldsymbol{C}}$ as a $\boldsymbol{C}$-valued geometric point of ${}_{K_N}M_{\boldsymbol{Q}}$, then the fibre A_z of A at z with the canonical polarization and the level N-structure, is isomorphic to the one obtained by (1.5.2)–(1.5.4).

§ 2. An abelian scheme over the Shimura curve

In this section, we fix a division quaternion algebra B over F such that $B \otimes_{\boldsymbol{Q}} \boldsymbol{R} \cong M_2(\boldsymbol{R}) \times \boldsymbol{H}^{g-1}$. G denotes the algebraic group over $\boldsymbol{Q}$ defined by $B^\times$, i.e. $G(\boldsymbol{Q}) = B^\times$. The aim of this section is to construct an abelian scheme A_S over the Shimura curve V_S (2.5.5), and then to study its property (2.6.2). When $g = [F: \boldsymbol{Q}]$ is odd, it was done in [17] by a slightly different method.

2.1. Let us first recall fundamental properties of the homomorphism β defined by Shimura [22] § 8;

$$\beta: B^\times \longrightarrow GL_q(C), \qquad q = 2^{g-1} \tag{2.1.1}$$

with C a quaternion algebra over $\boldsymbol{Q}$. Let $p_1, \cdots, p_g$ be inequivalent absolutely irreducible $\boldsymbol{Q}$-linear representations of B into $M_2(\boldsymbol{C})$. Then the following result characterizes β:

Proposition (2.1.2) ([22] 8.3). *Let L be a subfield of $\boldsymbol{C}$, and E a central simple algebra over L. Suppose that we are given an L-rational homomorphism β': $(B \otimes_{\boldsymbol{Q}} L)^\times \to E^\times$, and an absolutely irreducible L-linear representation γ: $E \to M_{2q}(\boldsymbol{C})$ such that $\gamma \circ \beta'$ is equivalent to $p_1 \otimes \cdots \otimes p_g$ on $B^\times$. Then there exists an L-algebra isomorphism ε: $M_q(C) \otimes_{\boldsymbol{Q}} L \rightleftarrows E$, and it holds that $\beta' = \varepsilon \circ \beta$.*

Let ι (resp. $'$) denote the canonical involution of B (resp. C). We also recall that there is an involution π of $M_q(C)$ satisfying $\beta(x^\iota) = \beta(x)^\pi$ for all $x \in B^\times$. Thus if we take an element $v \in GL_q(C)$ so that $y^\pi = v^t y' v^{-1}$ for all $y \in M_q(C)$, β is actually a homomorphism into a subgroup

$$G(C, v) = \{\alpha \in GL_q(C) \mid \alpha v^t \alpha' = \nu(\alpha) v,\ \nu(\alpha) \in \boldsymbol{Q}^\times\}$$

of $GL_q(C)$. Note that v as above is unique up to a scalar multiple.

In [22] 8.6, Shimura also proved that

$$ {}^t v' = (-1)^{g-1} v \tag{2.1.3}$$

(2.1.4) C is indefinite if and only if g is odd.

Therefore if g is odd, changing β by an inner automorphism of $GL_q(C)$ if necessary, we may (and do so hereafter) assume that $v=1_q$ (cf. [22] 2.1). We also recall that

$$\nu(\beta(\alpha))=N_{F/Q}(\nu(\alpha)) \qquad \text{for all } \alpha \in B^\times. \tag{2.1.5}$$

2.2. Put $V=C^q$, which is isomorphic to $\boldsymbol{Q}^{2^{g+1}}$ (row vectors) as vector spaces over $\boldsymbol{Q}$. The group $GL_q(C)$ acts on V $\boldsymbol{Q}$-linearly on the right. We define a $\boldsymbol{Q}$-bilinear form ψ on V by:

$$\psi(x, y)=\mathrm{tr}_{C/\boldsymbol{Q}}(xv\,{}^t y') \qquad \text{for } x, y \in C^q, \text{ when } g \text{ is even,} \tag{2.2.1}$$

where $\mathrm{tr}_{C/\boldsymbol{Q}}$ denotes the reduced trace of C over $\boldsymbol{Q}$. When g is odd, we choose and fix an element $\rho \in C$ whose square is a negative rational number, and put

$$\psi(x, y)=\mathrm{tr}_{C/\boldsymbol{Q}}(x\,{}^t y'\rho), \qquad \text{for } x, y \in C^q, \text{ when } g \text{ is odd.} \tag{2.2.2}$$

Then in both cases, ψ is a non-degenerate alternating form on V, and β defines a homomorphism of $\boldsymbol{Q}$-algebraic groups which we write by the same letter

$$\beta\colon G \longrightarrow Gp(V, \psi) \tag{2.2.3}$$

$Gp(V, \psi)$ being as before the algebraic group of symplectic similitudes of (V, ψ) defined over $\boldsymbol{Q}$.

Proposition (2.2.4). *Let h_0: $\boldsymbol{S}\to G_{\boldsymbol{R}}$ be the homomorphism* (1.1.3). *Then the Hodge structure of $V_{\boldsymbol{R}}$ defined by $h=\beta\circ h_0$ is of type* $(-1, 0)+(0, -1)$, *and the $\boldsymbol{R}$-bilinear form $\psi(x, yh(\sqrt{-1}))$ on $V_{\boldsymbol{R}}$ is symmetric and definite.*

Before giving the proof in 2.3 and 2.4, we state here the immediate corollaries of (1.3.3) and (2.2.4):

Corollary (2.2.5). *Let X (resp. X') be the $G(\boldsymbol{R})$-(resp. $Gp(V, \psi)(\boldsymbol{R})$-) conjugacy class of homomorphisms of $\boldsymbol{S}$ to $G_{\boldsymbol{R}}$ (resp. $Gp(V, \psi)_{\boldsymbol{R}}$) as in* 1.1 (*resp.* 1.2). *Then $\beta(X)\subseteq X'$, and β defines an F-rational morphism of the canonical models*: $M_F(G, X)\to M_{\boldsymbol{Q}}(Gp(V, \psi), X')$.

Identifying X (resp. X') with $H_1^{\pm}$ (resp. $H_{2g}^{\pm}$) by the isomorphism (1.2.2), we may assume that β maps H_1^+ to H_{2g}^+. Let $\{V_S, \varphi_S, J_{TS}(x), (S, T\in\mathscr{Z}, x\in G(A)_+)\}$ and $\{V_M^*, \varphi_M^*, J_{NM}^*(y), (M, N\in\mathscr{Z}^*, y\in Gp(V, \psi)(A)_+)\}$ be Shimura's canonical systems for G and $Gp(V, \psi)$, respectively. We will use the following notation for them in the following: L_S (resp. L_M^*) denotes the field of rational functions of V_S (resp. V_M^*), and we put $L=$

$\varinjlim L_S$ (resp. $L^*=\varinjlim L_M^*$). k_S (resp. k_M^*) denotes the constant field of L_S (resp. L_M^*). τ: $G(A)_+\to\mathrm{Aut}\,(L/F)$ (resp. τ^*: $Gp(V, \psi)(A)_+\to\mathrm{Aut}\,(L^*/\boldsymbol{Q})$) is the homomorphism defined in [22] 2.7.

Corollary (2.2.6). *Let the notation be as above. If $\beta(S)\subseteq M$ with $S\in\mathscr{X}$ and $M\in\mathscr{X}^*$, then β induces a k_S $(\supseteq k_M^*)$-rational morphism E_{MS}: $V_S\to V_M^*$. For $x\in G(A)_+$, S, $T\in\mathscr{X}$, M, $N\in\mathscr{X}^*$ satisfying $xSx^{-1}\subseteq T$, $\beta(x)M\beta(x)^{-1}\subseteq N$, $\beta(S)\subseteq M$ and $\beta(T)\subseteq N$, we have $E_{NT}^{\sigma(x)}\circ J_{TS}(x)=J_{NM}^*(\beta(x))\circ E_{MS}$. Also, E_{MS} commute with the left actions* (1.4.4) $[x]$ *and* $[\beta(x)]$ *for all $x\in G(A)_+$.*

2.3. Let us prove (2.2.4) when g is even. Let p_i' be the projection to the *i-th* factor of $B_{\boldsymbol{R}}=M_2(\boldsymbol{R})\times \boldsymbol{H}^{g-1}(1\leqslant i\leqslant g)$. Apply (2.1.2) to $L=\boldsymbol{R}$, $E=M_2(\boldsymbol{R})\otimes\overset{g-1}{\otimes}\boldsymbol{H}$, and $\beta'=p_1'\otimes\cdots p_g'$ to obtain an isomorphism ε: $M_q(C_{\boldsymbol{R}})\xrightarrow{\sim}E$ satisfying $\beta'=\varepsilon\circ\beta$. By (2.1.4) (or rather by ε), $C_{\boldsymbol{R}}$ is definite, and there is an $\boldsymbol{R}$-algebra isomorphism: $M_q(C_{\boldsymbol{R}})\xrightarrow{\sim}M_q(\boldsymbol{H})$. On the other hand, let us fix an isomorphism of E onto $M_q(\boldsymbol{H})$ as follows. Let

$$\delta_0\colon \overset{g-2}{\otimes}\boldsymbol{H}\xrightarrow{\sim}M_{q/2}(\boldsymbol{R}) \tag{2.3.1}$$

be an isomorphism of $\boldsymbol{R}$-algebras satisfying $\delta_0(\bar{x}_2\otimes\cdots\otimes\bar{x}_{g-1})={}^t\delta_0(x_2\otimes\cdots\otimes x_{g-1})$ (cf. [22] 8.10). We define an isomorphism ξ of E onto $M_q(\boldsymbol{H})$ by:

$$\xi(x_1\otimes\cdots\otimes x_g)=\begin{bmatrix} a\delta_0(x')x_g & b\delta_0(x')x_g \\ c\delta_0(x')x_g & d\delta_0(x')x_g\end{bmatrix} \tag{2.3.2}$$

$$\text{with}\quad x_1=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\quad\text{and}\quad x'=x_2\otimes\cdots\otimes x_{g-1}.$$

Then there exists an element $S\in GL_q(\boldsymbol{H})$, and the following diagram commutes with int (S) the inner automorphism of $GL_q(\boldsymbol{H})$ defined by S:

$$\begin{tikzcd} B_{\boldsymbol{R}}^{\times} \arrow[r, "\beta"] \arrow[dr, "\beta'"'] & GL_q(C_{\boldsymbol{R}}) \arrow[r, "\sim"] \arrow[d, "\varepsilon"] & GL_q(\boldsymbol{H}) \arrow[d, "\mathrm{int}\,(S)"] \\ & E^{\times} \arrow[r, "\sim", "\xi"'] & GL_q(\boldsymbol{H}) \end{tikzcd}$$

Put $\beta_1=\xi\circ\beta'$, and $h_1=\beta_1\circ h_0$. There is an element $v_1\in GL_q(\boldsymbol{H})$ satisfying $\beta_1(x^\iota)=v_1{}^t\overline{\beta_1(x)}v_1^{-1}$ for all $x\in B_{\boldsymbol{R}}^{\times}$. It is then easy to see that (2.2.4) is equivalent to the assertion that (i) the Hodge structure of $\boldsymbol{H}^q\cong\boldsymbol{R}^{4q}$ defined by h_1 is of type $(-1, 0)+(0, -1)$; and (ii) the $\boldsymbol{R}$-bilinear form $\mathrm{tr}_{\boldsymbol{H}/\boldsymbol{R}}(xv_1{}^t\overline{(yh_1(\sqrt{-1}))})$ $(x, y\in\boldsymbol{H}^q)$ is symmetric and definite. But from the construction, $h_1(a+b\sqrt{-1})=\begin{bmatrix} a1_{q'} & -b1_{q'} \\ b1_{q'} & a1_{q'} \end{bmatrix}$, and v_1 is a scalar multiple of $\begin{bmatrix} 0 & -1_{q'} \\ 1_{q'} & 0 \end{bmatrix}$ with $q'=q/2$. Our assertion is now obvious.

2.4. We next prove (2.2.4) when g is odd. Although this case was treated in [17] (in a slightly different manner; cf. also [22] § 8), we sketch the proof for completeness. As in (2.3.1), there exists an isomorphism

$$\delta_0 : \overset{g-1}{\otimes} \boldsymbol{H} \xrightarrow{\sim} M_q(\boldsymbol{R}) \tag{2.4.1}$$

satisfying $\delta_0(\bar{x}_2 \otimes \cdots \otimes \bar{x}_g)={}^t\delta_0(x_2\otimes \cdots \otimes x_g)$. Define an isomorphism ξ of $E=M_2(\boldsymbol{R})\otimes \overset{g-1}{\otimes} \boldsymbol{H}$ onto $M_{2g}(\boldsymbol{R})=M_q(M_2(\boldsymbol{R}))$ by sending $x_1\otimes \cdots \otimes x_g$ to an element of $M_q(M_2(\boldsymbol{R}))$ whose (i,j)-entry is $x_1 y_{ij} \in M_2(\boldsymbol{R})$ if $\delta_0(x_2 \otimes \cdots \otimes x_g) =(y_{ij})$. Let $\beta' : B_{\boldsymbol{R}}^{\times} \rightarrow E^{\times}$ be as in 2.3, and put $\beta_1=\xi\circ\beta'$. Then, considering $\beta_1(x)$ as an element of $M_q(M_2(\boldsymbol{R}))$, we have $\beta_1(x^{\iota})={}^t\beta_1(x)'$, with $'$ the canonical involution of $M_2(\boldsymbol{R})$. Also, $\beta_1\circ h_0(a+b\sqrt{-1}) \in M_q(M_2(\boldsymbol{R}))$ is a matrix whose diagonal entries are $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in M_2(\boldsymbol{R})$ and whose other entries are all 0. The first assertion of (2.2.4) easily follows from this. The second assertion is equivalent to saying that the $\boldsymbol{R}$-bilinear form $\mathrm{tr}_{M_2(\boldsymbol{R})/\boldsymbol{R}} \left(x\,{}^t y \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} i(\rho)\right)$ $(x, y \in M_2(\boldsymbol{R}))$ is symmetric and definite on $M_2(\boldsymbol{R})$, where i is an $\boldsymbol{R}$-algebra isomorphism of $C_{\boldsymbol{R}}$ onto $M_2(\boldsymbol{R})$. This follows from the argument of Shimura [21] pp. 132–133.

2.5. Now let S be a member of $\mathscr{Z}$ with respect to G. There exists a maximal order $\mathfrak{o}$ of B such that S_0, the finite part of S, is contained in $\prod \mathfrak{o}_{\mathfrak{p}}^{\times}$, the product being over all the finite primes of F. In the following, we assume that

(2.5.1) *S_0 is contained in $\prod_{\mathfrak{p}\nmid n} \mathfrak{o}_{\mathfrak{p}}^{\times} \times \prod_{\mathfrak{p}\mid n} (1+n\mathfrak{o}_{\mathfrak{p}})$, with an integer $n \geqslant 3$.*

There also exists a $\boldsymbol{Z}$-lattice $V_{\boldsymbol{Z}}$ in $V=\boldsymbol{C}^q$ satisfying:

(2.5.2) *$V_{\boldsymbol{Z}}$ is $\beta(\mathfrak{o}\cap B^{\times})$-stable.*

(2.5.3) *For $\alpha \in \mathfrak{o}\cap B^{\times}$ satisfying $\alpha-1 \in a\mathfrak{o}$ with a positive integer a, we have $V_{\boldsymbol{Z}}(\beta(\alpha)-\beta(1))\subseteq aV_{\boldsymbol{Z}}$.*

(cf. [16] § 5). Thus $\beta(S_0)$ is contained in:

$$K_n=\{k \in Gp(V, \psi)(A^f) \mid V_{\hat{\boldsymbol{Z}}} k=V_{\hat{\boldsymbol{Z}}},\ V_{\hat{\boldsymbol{Z}}}(k-1)\subseteq nV_{\hat{\boldsymbol{Z}}}\}. \tag{2.5.4}$$

Put $M=Gp(V, \psi)(\boldsymbol{R})_+ \times K_n$. By (2.2.6), β induces a $k_S(\supseteq k_n^*)$-morphism $E_{MS}: V_S \rightarrow V_M^*$, the notation being the same as in 2.2. On the other hand, V_M^* is identified with a connected component of ${}_{K_n}M_{\boldsymbol{Q}}(Gp(V, \psi), H_{2g}^{\pm})\otimes_{\boldsymbol{Q}} k_M^*$

by 1.4.

Definition (2.5.5). The notation being as above, let S be a member of $\mathscr{X}$ satisfying (2.5.1). We then define the projective abelian scheme $f_S\colon A_S \to V_S$ as the base change by: $V_S \xrightarrow{E_{MS}} V_M^* \hookrightarrow {}_{K_n}M_Q \otimes_Q k_M^*$ of the universal family (1.5.9) on ${}_{K_n}M_Q(Gp(V, \psi), H_{2g}^{\pm})$.

Thus the relative dimension of A_S over V_S is 2^g.

2.6. Let S be a member of $\mathscr{X}$, and let $\overline{(S \cap F^\times)_0}$ denote the closure in $G(A^f)$ of the projection to the finite part $(S \cap F^\times)_0$ of $S \cap F^\times$. Shimura's homomorphism τ then induces an isomorphism

$$\tau\colon S_0/\overline{(S \cap F^\times)_0} \xrightarrow{\sim} \mathrm{Gal}\,(L/L_S)\,. \tag{2.6.1}$$

Fix a rational prime l, and denote by p_l the projection from $G(A)_+$ to the l-factor $G(Q_l) = (B \otimes_Q Q_l)^\times$. In the argument of the next section, the following result takes the place of [17] (3.5.4).

Theorem (2.6.2). *Suppose that* $S \in \mathscr{X}$ *satisfies* (2.5.1), *and let* $f_S\colon A_S \to V_S$ *be the abelian scheme constructed above. Let* η *be the generic point of* V_S, *and* $A_{S,\eta}$ *the generic fibre of* A_S *at* η. *Then, identifying the residue field of* η *with* L_S, *the natural l-adic representation*:

$$\mathrm{Gal}\,(\bar{L}_S/L_S) \longrightarrow \mathrm{Aut}_{Q_l}\,(V_l(A_{S,\eta}))\,,$$

with $V_l(A_{S,\eta}) = (\varprojlim {}_{l^n}A_{S,\eta}(\bar{L}_S)) \otimes_{Z_l} Q_l$, *factors through* $\mathrm{Gal}\,(L/L_S)$. *If we combine the resulting l-adic representation of* $\mathrm{Gal}\,(L/L_S)$ *with the isomorphism* (2.6.1), *then the representation of* $S_0/\overline{(S \cap F^\times)_0}$ *thus obtained is equivalent to* $\beta_{Q_l} \circ p_l\colon S_0/\overline{(S \cap F^\times)_0} \to GL_{2g+1}(Q_l)$.

Proof. Let m be a positive multiple of n. Then by (2.5.3), the image of $S_0 \cap (\prod_{\mathfrak{p} \nmid m} \mathfrak{o}_\mathfrak{p}^\times \times \prod_{\mathfrak{p} \mid m} (1 + m\mathfrak{o}_\mathfrak{p})) \overset{\mathrm{dfn}}{=} S_{m,0}$ under β is contained in K_m. Put $S_m = G(R)_+ \times S_{m,0}$, and $M_m = Gp(V, \psi)(R)_+ \times K_m$. The projection morphism of ${}_{K_m}M_Q(Gp(V, \psi), H_{2g}^{\pm}) = {}_{K_m}M_Q$ to ${}_{K_n}M_Q$ may be viewed as the correspondence: (a level m-structure) $\mapsto$ (the underlying level n-structure), for the abelian varieties parametrized by them. Hence if A (resp. A') denotes the universal family (1.5.9) on ${}_{K_n}M_Q$ (resp. ${}_{K_m}M_Q$), then A' is canonically isomorphic to the base change of A by the above projection morphism. Since $E_{M_m S_m}$, E_{MS} and the natural projections commute (2.2.6), the description in 1.4 implies that ${}_mA_S \times_{V_S} V_{S_m}$ is isomorphic to a constant group scheme, which proves the first assertion. Moreover by loc. cit., it is enough to show the commutativity of the following diagram for all $k \in K_n$ to complete the proof (cf. [17] (3.4.3)):

$$\begin{array}{ccccc}
(V_Z/mV_Z) & \xrightarrow{k} & (V_Z/mV_Z) & \xrightarrow{id} & (V_Z/mV_Z) \\
\uparrow\wr & & \uparrow\wr & & \uparrow\wr \\
{}_mA'\otimes_Q k^*_{M_m} & \longrightarrow & {}_mA'\otimes_Q k^*_{M_m} & \longrightarrow & {}_mA'\otimes_Q k^*_{M_m} \\
\downarrow & & \downarrow & & \downarrow \\
{}_{K_m}M_Q\otimes_Q k^*_{M_m} & \longrightarrow & {}_{K_m}M_Q\otimes_Q k^*_{M_m} & \longrightarrow & {}_{K_m}M_Q\otimes_Q k^*_{M_m}
\end{array}$$

where the left lower morphism is the base change of the right action of $k \in K_n \subset Gp(V, \psi)(A^f)$, the right lower morphism is $id\otimes\mathrm{Spec}\,(\sigma^*(k))$ with σ^* the map (1.4.3) for $Gp(V, \psi)$, the middle horizontal morphisms are the canonical isomorphisms via $A' \cong$(the base change of A by ${}_{K_m}M_Q \to {}_{K_n}M_Q$), and the upper vertical morphisms are the canonical level m-structures. The commutativity of the right hand side is obvious, and to prove that of the left hand side, it is enough to show the corresponding commutativity for C-valued geometric points of ${}_{K_m}M_Q\otimes_Q k^*_{M_m}$. This easily follows from the explicit description of the fibres in 1.5. Q.E.D.

§ 3. Main results

3.1. Let F, B, and G be as in the previous section. For a g-tuple $\{k_1, \cdots, k_g\}$ of non-negative integers, a real number w, and $S \in \mathscr{X}$ with respect to G, we can define the space $\mathfrak{S}(S; \{k_n\}; w)$ of automorphic forms and the Hecke operators acting on it (cf. [17] § 1). Let $H_\mathfrak{p}(T; \mathfrak{S}(S; \{k_n\}; w))$ be the "$\mathfrak{p}$-*th* Hecke polynomial" for the space $\mathfrak{S}(S; \{k_n\}; w)$, for a finite prime $\mathfrak{p}$ of F which is prime to the discriminant $D(B/F)$ of B and the "level" $L(S)$ of S (cf. [17] (1.3.5)). The Dirichlet series

$$(3.1.1) \qquad D(s; S; \{k_n\}; w) = \prod H_\mathfrak{p}(N_{F/Q}(\mathfrak{p})^{-s}; \mathfrak{S}(S; \{k_n\}; w))^{-1}$$

the product being over all the finite primes of F which are prime to $D(B/F)L(S)$, converges for $\mathrm{Re}\,(s)$ large, and can be continued to the whole complex plane.

When $w \in Z$, $k_n \leqslant w$, and $k_1 \equiv \cdots \equiv k_g \equiv w \pmod 2$, let $a(k; \{k_n\}; w)$ be the numbers defined in [17] (4.2.3). Let $S \in \mathscr{X}$ satisfy the conditions:

(3.1.2) *S is stable under the canonical involution of B (or $B_A^\times$).*

(3.1.3) *$x^{-1}Sx \cap B^\times$ have no non-trivial elliptic elements for all $x \in G(A)_+$.*

We then obtain the following theorem, which was proved in [17] (4.5.3) when g is odd.

Theorem (3.1.4). *The notation being as above, let S be a member of $\mathscr{X}$ satisfying the conditions* (2.5.1), (3.1.2), *and* (3.1.3). *Let $f_S\colon A_S \to V_S$*

be the abelian scheme (2.5.5), *and* A_S^k *the k-fold fibre product of* A_S *over* V_S $(k\geqslant 1)$. *Then the i-dimensional part of the Hasse-Weil zeta function* $Z^{(i)}(s; A_S^k/F)$ *of* A_S^k $(0\leqslant i\leqslant 2\dim A_S^k)$ *can be determined, up to the* $\mathfrak{p}$*-factors with* $\mathfrak{p}$ *dividing* $D(B/F)L(S)$, *as follows*:

(i) $Z^{(0)}(s; A_S^k/F)=\zeta(s; k_S)$, *the Dedekind zeta function of* k_S.

(ii) *When i is odd,* $Z^{(i)}(s; A_S^k/F)=\prod D(s; S; \{k_n\}; i-1)^{a(2k;\{k_n\};i-1)}$, *where the product ranges over all the g-tuples of non-negative integers* $\{k_n\}$ *satisfying* $k_1\equiv\cdots\equiv k_g\equiv i-1 \pmod 2$ *and* $k_n\leqslant i-1$ $(1\leqslant n\leqslant g)$.

(iii) *When i is even and* $\geqslant 2$,

$$Z^{(i)}(s; A_S^k/F)=\prod D(s; S; \{k_n\}; i-1)^{a(2k;\{k_n\};i-1)} \times\zeta(s-i/2; k_S)^{a(2k;\{0\};i)+a(2k;\{0\};i-2)},$$

where the product has the same meaning as in (ii).

Indeed, after (2.6.2), the proof given in [17] works without assuming that g is odd. Precisely, the proof of [17] (3.3.3) was solely based on the existence of A_S satisfying the property in (2.6.2), while, in [17] § 4, we only used [17] (3.3.3) to establish the main theorem (4.5.3) there.

Remark (3.1.5). In a subsequent paper, we will show that a main theorem (5.4.6) of [16] is valid under a weaker assumption that S' is conjugate to S, than (3.1.2) above. Consequently, the theorem (3.1.4) above will also hold under that assumption, instead of (3.1.2).

3.2. We next apply the theorem above to the Ramanujan-Petersson conjecture. In 3.3–3.4, we will prove the following

Theorem (3.2.1). *The notation being as above, let S be a member of* $\mathscr{X}$ *satisfying* (2.5.1) *and* (3.1.3), *and* $\mathfrak{p}$ *a finite prime of F which is prime to* $D(B/F)L(S)$. *Let* $\sigma_{\mathfrak{p}}\in \mathrm{Gal}(\bar{F}/F)$ *be a Frobenius element at* $\mathfrak{p}$. *Then for* $0\leqslant i\leqslant 2\dim A_S^k$ *and a rational prime l which is prime to* $\mathfrak{p}$, *the* $\mathrm{Gal}(\bar{F}/F)$*-module* $H^i(A_S^k\otimes_F\bar{F}, \mathbf{Q}_l)$ *is unramified at* $\mathfrak{p}$, *and the characteristic roots of* $\sigma_{\mathfrak{p}}^{-1}$ *acting on it are algebraic integers whose absolute values are* $N_{F/\mathbf{Q}}(\mathfrak{p})^{i/2}$.

We note that the unramifiedness in the above is already contained in the course of the proof of (3.1.4) (cf. [16] (3.5.5); cf. also below). As a consequense, we obtain the following

Corollary (3.2.2). *Let* $k_1, \cdots, k_g$ *be non-negative integers satisfying* $k_1\equiv\cdots\equiv k_g \pmod 2$, *and w a real number. Let S be a member of* $\mathscr{X}$ *for the quaternion algebra B as above. Then for a finite prime* $\mathfrak{p}$ *of F, which is prime to* $D(B/F)L(S)$, *all the roots of the equation* $H_{\mathfrak{p}}(T; \mathfrak{S}(S; \{k_n\}; w))=0$ *have absolute value* $N_{F/\mathbf{Q}}(\mathfrak{p})^{-(w+1)/2}$.

Proof. We may assume that w is an integer satisfying $k_n \leqslant w\,(1 \leqslant n \leqslant g)$ and $k_n \equiv w \pmod 2$ (cf. [17] (1.3.6)). Let $\mathfrak{o}$ be a maximal order of B such that $S_0 \subseteq \prod \mathfrak{o}_{\mathfrak{p}}^{\times}$. For a fixed $\mathfrak{p} \nmid D(B/F)L(S)$, take a positive integer n which is prime to $\mathfrak{p}$. Then the assertion above will follow from that for $G(\boldsymbol{R})_+ \times S_0 \cap S_0' \cap (\prod_{\mathfrak{p} \nmid n} \mathfrak{o}_{\mathfrak{p}}^{\times} \times \prod_{\mathfrak{p} | n}(1+n\mathfrak{o}_{\mathfrak{p}}))$ (cf. [16] (1.6.2)). Thus taking n sufficiently large, we may suppose that S satisfies (2.5.1), (3.1.2), and (3.1.3). In this case, the assertion follows from (3.1.4), (3.2.1), and [17] (4.6.2).

Q.E.D.

As we noted in the introduction, Morita [14] has recently shown that the assertion above holds without assuming that $k_1 \equiv \cdots \equiv k_g \pmod 2$. Accordingly, the following variant would also hold without the assumption on the parity of $\{k_n\}$.

Corollary (3.2.3). *Let $v_1, \cdots, v_g$ be the archimedean primes of F, and let C be an arbitrary quaternion algebra over F. Suppose that we are given a real number w and a g-tuple of non-negative integers $\{k_n\}$ which are congruent* mod 2. *Let $\pi = \otimes_v \pi_v$ be an infinite dimensional irreducible constituent of the space of automorphic forms $\mathscr{A}(C_A^{\times})$ (resp. cusp forms $\mathscr{A}_0(GL_2(F_A))$ if $C = M_2(F)$) on $C_A^{\times}$, satisfying:*

(i) *π_{v_n} is isomorphic to the discrete series representation*

$$\sigma(|-|_{v_n}^{(k_n+1-w)/2}\operatorname{sgn}(-)^{k_n},\ |-|_{v_n}^{(-k_n-1-w)/2}) \qquad \textit{if } C_{v_n} \cong M_2(\boldsymbol{R})\,.$$

(ii) *π_{v_n} is isomorphic to the representation: $\boldsymbol{H}^{\times} \ni x \mapsto \det(x)^{-(k_n+w)/2} \rho_{k_n}(x)$ if $C_{v_n} \cong \boldsymbol{H}$, where we consider $\boldsymbol{H}^{\times}$ as a subgroup of $(\boldsymbol{H} \otimes_{\boldsymbol{R}} \boldsymbol{C})^{\times} \cong GL_2(\boldsymbol{C})$, and ρ_a denotes the symmetric tensor representation of degree a of $GL_2(\boldsymbol{C})$.*

(iii) *When g is even and $D(C/F) = (1)$, $\pi_{\mathfrak{p}}$ is not isomorphic to a principal series representation for at least one finite prime $\mathfrak{p}$ of F.*

Then, for $\mathfrak{p}$ not dividing $D(C/F)$, if $\pi_{\mathfrak{p}}$ is isomorphic to an unramified principal series representation $\pi(|-|_{\mathfrak{p}}^{s_1}, |-|_{\mathfrak{p}}^{s_2})$, we have $\operatorname{Re}(s_i) = -w/2$ $(i = 1, 2)$.

Proof (cf. [17] § 5). When $F = \boldsymbol{Q}$, the above assertion is known and due to Deligne [2], and hence we assume that $F \neq \boldsymbol{Q}$. By virtue of the correspondence of Jacquet and Langlands [10] Theorem 14.4 (cf. also Shimizu [20] Theorem 1), it is enough to prove the assertion for $C = M_2(F)$. On the other hand, an irreducible constituent of $\mathscr{A}_0(GL(F_A))$ satisfying (i) and (iii) comes from a constituent of $\mathscr{A}(B_A^{\times})$ satisfying (i) and (ii) with a division quaternion algebra B over F with $r = 1$, by the above correspondence ([10] "Theorem" 16.1). For such a B, our conclusion is equivalent to (3.2.2) (cf. [17] § 5). Q.E.D.

3.3. We now go on to prove (3.2.1). Thus we fix a member S of $\mathscr{X}$ satisfying (2.5.1) and (3.1.3), and a finite prime $\mathfrak{p}$ of F which is prime to $D(B/F)L(S)$. Also, l is a fixed prime number which is prime to $\mathfrak{p}$. We denote by $\mathfrak{r}_{F,\mathfrak{p}}$ (resp. $\mathfrak{r}_{S,\mathfrak{p}}$) the localization of $\mathfrak{r}_F$ at $\mathfrak{p}$ (resp. the normalization of $\mathfrak{r}_{F,\mathfrak{p}}$ in k_S). We first recall:

(3.3.1) (Morita [13] Main Theorem 1) *There exists a* (*unique*) *smooth projective scheme W_S over $\mathfrak{r}_{S,\mathfrak{p}}$ whose general fibre is isomorphic to V_S.*

(3.3.2) (a consequence of Ihara, Miki [9] Theorem 2B) *Let $T\subseteq S$ be another member of $\mathscr{X}$ for which $\mathfrak{p}\nmid L(T)$. Then the normalization of W_S in L_T is étale over W_S, and isomorphic to W_T.*

Remark (3.3.3). Morita's original result is slightly weaker than (3.3.1) above; one needs (3.3.2) to strengthen it in the form above (cf. the remark after (3.4.3) in [16] for the relation between (3.3.1) and (3.3.2)).

Proposition (3.3.4). *Let the notation and the assumption be as above, and $f_S\colon A_S\to V_S$ the abelian scheme* (2.5.5). *Then the $\boldsymbol{Q}_l$-sheaf $R^1f_{S*}(\boldsymbol{Q}_l)$ on $V_{S,\text{ét}}$ extends uniquely to a smooth* (=*constant tordu constructible, in the terminology of* [8]) *$\boldsymbol{Q}_l$-sheaf G on $W_{S,\text{ét}}$.*

Proof. Let η denote the generic point of V_S, and also that of W_S, and let $\bar{\eta}$ be the geometric point above η which corresponds to $\bar{L}_S$ via the identification $\kappa(\eta)=L_S$. The fundamental group $\pi_1(V_S;\bar{\eta})$ is canonically isomorphic to a quotient of $\mathrm{Gal}(\bar{L}_S/L_S)$ ([7] V 8.2), and we hereafter consider $\pi_1(V_S;\bar{\eta})$ (resp. $\pi_1(W_S;\bar{\eta})$) as a quotient of $\mathrm{Gal}(\bar{L}_S/L_S)$ (resp. $\pi_1(V_S;\bar{\eta})$). The assertion is then equivalent to saying that the natural l-adic representation: $\pi_1(V_S;\bar{\eta})\to\mathrm{Aut}_{\boldsymbol{Q}_l}(R^1f_{S*}(\boldsymbol{Q}_l)_{\bar{\eta}})$ (cf. [8] VI 1.4.2), with $R^1f_{S*}(\boldsymbol{Q}_l)_{\bar{\eta}}$ the "fibre at $\bar{\eta}$" of $R^1f_{S*}(\boldsymbol{Q}_l)$, factors through $\pi_1(W_S;\bar{\eta})$.

Let L' be the union in L of L_T for all T satisfying $T\subseteq S$ and $\mathfrak{p}\nmid L(T)$. Under (3.1.3), the canonical map: $\mathrm{Gal}(\bar{L}_S/L_S)\to\mathrm{Gal}(L/L_S)$ factors through $\pi_1(V_S;\bar{\eta})$, and the theorem of Ihara and Miki (3.3.2) says that the resulting $\pi_1(V_S;\bar{\eta})\to\mathrm{Gal}(L'/L_S)$ factors through $\pi_1(W_S;\bar{\eta})$. We therefore easily obtain the following commutative diagram:

$$\begin{array}{ccccc} S_0/(\overline{S\cap F^\times})_0 & \xrightarrow[\sim]{\tau} & \mathrm{Gal}(L/L_S) & \longleftarrow & \pi_1(V_S;\bar{\eta}) \\ \downarrow & & \downarrow & & \downarrow \\ S_0/(\overline{S\cap F^\times})_0\mathfrak{o}_{\mathfrak{p}}^\times & \xrightarrow[\sim]{\tau} & \mathrm{Gal}(L'/L_S) & \longleftarrow & \pi_1(W_S;\bar{\eta}) \end{array}$$

where all the vertical maps are the canonical projections. Now since $\mathfrak{p}$ is prime to l, the map $\pi_1(V_S;\bar{\eta})\to\mathrm{Aut}_{\boldsymbol{Q}_l}(R^1f_{S*}(\boldsymbol{Q}_l)_{\bar{\eta}})$ factors through $\mathrm{Gal}(L'/L_S)$ by (2.6.2) and the Kummer theory ([1] IX 3). Q.E.D.

3.4. Let the situation be as in 3.3. We denote by $\kappa(\mathfrak{p})$ the residue field of $\mathfrak{p}$ in the following

Proposition (3.4.1). *There exists an abelian scheme* $g\colon B \to W_S \otimes_{\mathfrak{r}_{F,\mathfrak{p}}} \kappa(\mathfrak{p}) \overset{\mathrm{dfn}}{=} W_S \otimes \kappa(\mathfrak{p})$ *for which* $R^1 g_*(\boldsymbol{Q}_l)$ *is isomorphic to the inverse image* $G|_{W_S \otimes \kappa(\mathfrak{p})}$ *of* G *to* $W_S \otimes \kappa(\mathfrak{p})$.

Proof. Let $\mathfrak{p} = \mathfrak{P}_1 \cdots \mathfrak{P}_t$ be the prime decomposition of $\mathfrak{p}$ in k_S ($\mathfrak{p}$ is unramified in k_S because we assumed that $\mathfrak{p} \nmid L(S)$). The assertion is equivalent to saying that there is an abelian scheme $g_i\colon B_i \to W_S \otimes_{\mathfrak{r}_{S,\mathfrak{p}}} \kappa(\mathfrak{P}_i) \overset{\mathrm{dfn}}{=} W_S \otimes \kappa(\mathfrak{P}_i)$ satisfying $R^1 g_{i*}(\boldsymbol{Q}_l) \simeq G|_{W_S \otimes \kappa(\mathfrak{p}_i)}$ for each i. Let ξ_i be the generic point of $W_S \otimes \kappa(\mathfrak{P}_i)$. Then the local ring $\mathcal{O}_{W_S, \xi_i}$ is a discrete valuation ring. By (3.3.4) and the Kummer theory, the group schemes ${}_{l^n}A_{S,\eta}$ on $\mathrm{Spec}(L_S)$ extend to finite étale group schemes on $\mathrm{Spec}(\mathcal{O}_{W_S, \xi_i})$. Hence the "criterion of Néron-Ogg-Šafarevič" (Serre, Tate [18] Theorem 1; cf. also the footnote in p. 493) applies, and there exists an abelian scheme $f_i'\colon A_i' \to \mathrm{Spec}(\mathcal{O}_{W_S, \xi_i})$ whose general fibre is isomorphic to $A_{S,\eta}$. It is easy to see that $R^1 f'_{i*}(\boldsymbol{Q}_l)$ is isomorphic to the inverse image of G to $\mathrm{Spec}(\mathcal{O}_{W_S, \xi_i})$. Let $g_i'\colon B_i' \to \mathrm{Spec}(\kappa(\xi_i))$ be the closed fibre of f_i'. $R^1 g'_{i*}(\boldsymbol{Q}_l)$ is then isomorphic to the inverse image of G to $\mathrm{Spec}(\kappa(\xi_i))$. Let $g_i\colon B_i \to W_S \otimes \kappa(\mathfrak{P}_i)$ be the global Néron model of B_i'. Applying [18] Theorem 1 again to the restrictions of g_i to the local rings of $W_S \otimes \kappa(\mathfrak{P}_i)$, we conclude that g_i is proper, i.e. B_i is an abelian scheme over $W_S \otimes \kappa(\mathfrak{P}_i)$. This g_i has the required property. Q.E.D.

We can now complete the proof of (3.2.1). As was shown in [17] § 4, the étale cohomology group $H^i(A_S^k \otimes_F \bar{F}, \boldsymbol{Q}_l)$ is canonically isomorphic to $\bigoplus_{j=0}^{i'} H^j(V_S \otimes_F \bar{F}, \bigwedge^{i-j}(\bigoplus^k R^1 f_{S*}(\boldsymbol{Q}_l)))$ with $i' = \min(i, 2)$. The specialization theorem of the cohomology groups ([1] XVI 2.2) implies that the characteristic polynomial of $\sigma_{\mathfrak{p}}^{-1}$ on the above cohomology group is equal to that of the geometric Frobenius acting on $\bigoplus_{j=0}^{i'} H^j(\overline{W_S \otimes_{\mathfrak{r}_{F,\mathfrak{p}}} \kappa(\mathfrak{p})}, \bigwedge^{i-j}(\bigoplus^k G|_{W_S \otimes \kappa(\mathfrak{p})}))$. But by (3.4.1) and the same reason as above, this group is isomorphic to $H^i(B^k \otimes_{\kappa(\mathfrak{p})} \overline{\kappa(\mathfrak{p})}, \boldsymbol{Q}_l)$ with B^k the k-fold fibre product of B over $W_S \otimes \kappa(\mathfrak{p})$. By Raynaud [19] Theorem XI 1.4, we know that B^k is projective over $W_S \otimes \kappa(\mathfrak{p})$, and hence so over $\mathrm{Spec}(\kappa(\mathfrak{p}))$. Our conclusion therefore follows from Deligne [4] Theorem 1.6 (the Weil conjecture). Alternatively, we could directly appeal to Deligne [6] Corollary (3.3.9).

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