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# Notes on Metaplectic Automorphic Functions and Zeta Functions

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The purpose of the present notes it to state, without detailed proofs, several miscellaneous facts which may slightly help to imagine properties of zeros of various $L$-functions including Riemann's zeta function.

Basic facts in our arguments in the sequel are that the Möbius function is connected with Gauss sums through the relation (28), and that fairly precise properties of Gauss sums can be derived by means of certain generalized theta functions which may be regarded as automorphic functions on a metaplectic group, that is, an $n$-fold covering group of the adelized $SL(2)$ over an algebraic number field $F$. Here, $n$ is a natural number with $n \geq 2$, but only odd $n$ are really useful for our purpose.

In Section 1 and Section 2, we will investigate usual real analytic Eisenstein series on the upper half plane to explain the main way of thinking, and in Section 3 we will turn to the three dimensional upper half space to consider cases of actual meaning.

## § 1. Fourier coefficients of Eisenstein series

Throughout the present notes, $(a, b; c, d)$ will denote a $2 \times 2$ matrix with the first line $a, b$ and the second line $c, d$, and an expression like $1 + a/b$ or $a/b + 1$ means exclusively $1 + ab^{-1}$ or $ab^{-1} + 1$, and never $(1+a)b^{-1}$ or $a(b+1)^{-1}$.

Let $H = \{z = x + iy \mid x \in \boldsymbol{R}, y > 0\}$ be the upper half plane, and let $\Gamma$ be a subgroup of $SL(2, \boldsymbol{R})$ acting on $H$ discontinuously. We assume that $\Gamma$ does not contain $(-1, 0; 0, -1)$, that $\Gamma \backslash H$ is of a finite volume, and that the stabilizer $\Gamma_\infty = \{\sigma \in \Gamma \mid \sigma\infty = \infty\}$ of $\infty$ in $\Gamma$ contains the group of translations by $\boldsymbol{Z}$.

For a complex number $s_0$ with $\operatorname{Re} s_0 > 1$, a real analytic Eisenstein series $E(z, s_0)$, $(z \in H)$, is defined by

$$E(z, s_0) = \sum_{c,d} y^{s_0} |cz + d|^{-2s_0} \tag{1}$$

where the sum ranges over all pairs $c$, $d$ such that there exists an element

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$\sigma$ in $\Gamma$ of the form $(*, *; c, d)$. As is known in the general theory, (1) has the Fourier expansion

$$(2)\qquad E(z, s_0)=y^{s_0}+\sum_m y^{1-s_0}\varphi_m(s_0)k(my, s_0)e(mx), \qquad (m \in \boldsymbol{Z}),$$

where $e(x)=\exp 2\pi ix$ for $x \in \boldsymbol{R}$,

$$(3)\qquad \varphi_m(s_0)=\sum_{c\neq 0}|c|^{-2s_0}(\sum_{d \bmod c} e(md/c)), \qquad ((*, *; c, d) \in \Gamma),$$

and

$$(4)\qquad k(t', s_0)=\int_{-\infty}^{\infty}(1+t^2)^{-s_0}e(-tt')dt, \qquad (t' \in \boldsymbol{R}).$$

Let $\kappa$ be a cusp of $\Gamma$ which is not equivalent to $\infty$, and let $\sigma'$ be an element of $SL(2, \boldsymbol{R})$ such that $\sigma'\infty=\kappa$. This $\sigma'$ is uniquely determined up to a triangular matrix. We choose the triangular factor so that $\sigma'^{-1}\Gamma'\sigma'$ contains the group of translations by $\boldsymbol{Z}$, where $\Gamma'$ denotes the stabilizer of $\kappa$ in $\Gamma$. The matrix $\sigma'$ has the form $\sigma'=(d', -b'; -c', a')$ with $\kappa=-d'/c'$. The Fourier expansion of $E(z, s_0)$ at the cusp $\kappa$ is then given by

$$(5)\qquad E(\sigma'z, s_0)=\sum_m y^{1-s_0}\varphi_{\kappa,m}(s_0)k(my, s_0)e(mx), \qquad (m \in \boldsymbol{Z}),$$

where

$$\varphi_{\kappa,m}(s_0)=\sum_{c\neq 0}|c|^{-2s_0}(\sum_{d \bmod c} e(md/c)), \qquad ((*, *; c, d) \in \Gamma\sigma').$$

If we put $x=a'/c'$ in (5), then we have $\sigma'z=\kappa+ic'^{-2}y^{-1}$ and

$$E(\kappa+ic'^{-2}y^{-1}, s_0)=\sum_m y^{1-s_0}\varphi_{\kappa,m}(s_0)k(my, s_0)e(ma'/c')$$

or equivalently

$$(6)\qquad E(\kappa+ic'^{-2}y, s_0)=\sum_m y^{-1+s_0}\varphi_{\kappa,m}(s_0)k(my^{-1}, s_0)e(ma'/c').$$

Let us now consider the Dirichlet series corresponding to $E(z, s_0)$ in (2), which is given by usual Mellin transformation as follows:

$$(7)\qquad \begin{aligned}&\int_0^{\infty}(E(iy, s_0)-(y^{s_0}+y^{1-s_0}k(0, s_0)))y^{s-1}dy\\ &\qquad =\sum_{m\neq 0}\varphi_m(s_0)|m|^{-s-1+s_0}\cdot M(k(*, s_0), s+1-s_0)=\Lambda(s, s_0),\end{aligned}$$

where $M$ means the Mellin transformation and the variable of the transformed function is shown by $*$; namely

$$M(f(*), s) = \int_0^\infty f(y) y^{s-1} dy. \tag{8}$$

Since our discontinuous group $\Gamma$ does not contain the transformation $z \to -z^{-1}$, functional equation of $\Lambda(s, s_0)$ in a simple form is not expected. But, if anyway 0 is a cusp, (2) and (6) are enough to prove the analytic continuation of $\Lambda(s, s_0)$ to the whole $s$-plane, and to determine singularities of $\Lambda(s, s_0)$ which are only poles at $-s_0$, $-s_0+1$, and $s_0-1$, provided that 0 is not equivalent to $\infty$. In case 0 is equivalent to $\infty$, one more term of the form $(\alpha y)^{-s_0}$, $(\alpha > 0)$, is necessary in (6), and there is one more pole of $\Lambda(s, s_0)$ at $s_0$.

These arguments are valid in some variations. If we wish to investigate a Dirichlet series of the same form as in (7) in which, however, $m$ is restricted to an arithmetic progression, say, by the condition $m \equiv 1 \pmod N$, $N$ being a natural number, we have only to consider the linear combination $\sum_j e(-j/N) E(z+j/N, s)$, $(j=0, 1, \cdots, N-1)$, provided that $j/N$ are all cusps.

On the other hand, we know by the general theory that $E(z, s_0)$ has, as a function of $s_0$, a meromorphic analytic continuation to the whole complex plane, and has a finite number of poles of first order on the interval (1/2, 1]. Let $\alpha_0$ be one such pole; then $\lim_{h \to 0} hE(z, \alpha_0 + h)$ is an automorphic function called "residual form". Let $\rho_m$ be the residue of $\varphi_m$ in (3) at the same pole; then going over to residues in every formula, we can obtain the analytic continuation of the Dirichlet series $\sum_{m \neq 0} \rho_m |m|^{-s-1+\alpha_0}$ as well as the determination of its poles.

Moreover it is possible to generalize these results to the case of Eisenstein series containing a character $\chi$ of the discontinuous group $\Gamma$. For our purpose, it is enough to assume that $(\Gamma : \ker \chi) < \infty$. Under this assumption, no essential change takes place; the only difference is that one has to take $\sigma'$ so that $\sigma'^{-1} (\Gamma' \cap \ker \chi) \sigma'$ contains the group of translations by $\boldsymbol{Z}$. To explain the situation more precisely, assume that $\chi$ is trivial on $\Gamma_\infty$, and start with the Eisenstein series

$$E(z, s_0, \chi) = \sum_{c,d} \chi(c, d) y^{s_0} |cz+d|^{-2s_0},$$

where $\chi(c, d) = \chi(\sigma)$ for $\sigma = (*, *; c, d) \in \Gamma$, and other notations are as in (1). Then, we have only to replace $\varphi_m(s_0)$ in (2) by

$$\varphi_m(s_0, \chi) = \sum_{c \neq 0} |c|^{-2s_0} \Big( \sum_{d \bmod c} \chi(c, d) e(md/c) \Big),$$

and $\varphi_{\kappa, m}(s_0)$ in (5) and (6) by

$$\varphi_{\kappa, m}(s_0, \chi) = \sum_{c \neq 0} |c|^{-2s_0} \Big( \sum_{d \bmod c} \chi_\kappa(c, d) e(md/c) \Big),$$

where $\chi_\kappa(c, d)=\chi((*, *; c, d)\sigma'^{-1})$ for an element $(*, *; c, d)$ in $\Gamma\sigma'$.

## § 2. Some tentative investigations on the powers of the Fourier coefficients

In Section 1, we investigated Dirichlet series of the form $\sum_{m\neq 0}\varphi_m(s_0)|m|^{-s-1+s_0}$ as appeared in (7). The aim of this section is to give some comments on Dirichlet series whose coefficients are $n$-th powers $\varphi_m(s_0)^n$ with a natural number $n$.

Define $\Lambda_n(s, s_0)$ by

$$\begin{aligned}(9)\qquad &\int_0^\infty \sum_{m\neq 0}(y^{1-s_0}\varphi_m(s_0)k(my, s_0))^n y^{s-1}dy\\ &=\sum_{m\neq 0}\varphi_m(s_0)^n|m|^{-s-n+ns_0}M(k(*, s_0)^n, s+n-ns_0)\\ &=\Lambda_n(s, s_0), \qquad (m\in Z).\end{aligned}$$

Then, $k(y, s_0)$ is rapidly decreasing as $y\to+\infty$, so the analytic properties of $\Lambda_n(s, s_0)$ depends mainly on the behavior of the series

$$\sum(y^{1-s_0}\varphi_m(s_0)k(my, s_0))^n$$

as $y\to 0$. This is very difficult to study, because it concerns the $n$-fold convolution of $E(z, s_0)$ as a function of $x$ with period 1. Accordingly we can give here merely rough comments about it.

Suppose first $n=2$, and, for the sake of simplicity, $\varphi_m(s_0)=\varphi_{-m}(s_0)$ for all $m\in Z$. In this case, it is possible to handle $\Lambda_n(s, s_0)$ directly by means of Rankin's method. Since the general idea seems to be well-known, let us recall only a special case where the cusps of our discontinuous group $\Gamma$ are all equivalent to $\infty$. Furthermore, we consider, again for the simplicity, a residual form

$$(10)\qquad r(z)=\sum_m y^{1-\alpha_0}\rho_m k(my, \alpha_0)e(mx), \qquad (m\in Z),$$

instead of $E(z, s_0)$; $\alpha_0$ is therefore supposed to be a constant at which the residual form is derived.

Denote by $(1, Z; 0, 1)$ the group of all matrices of the form $(1, b; 0, 1)$, $(b\in Z)$, and let $S$ be a set of representatives of $(1, Z; 0, 1)\backslash\Gamma$. Then, there exists a connected domain $D$ such that i) $D$ is contained in the strip $0\leq \mathrm{Re}\, z\leq 1$, ii) $D$ contains the strip $D'_Y=\{z\in C\,|\,0<\mathrm{Re}\, z<1,\ \mathrm{Im}\, z>Y\}$ for a positive constant $Y$, and iii) if $c_\infty$ denotes the index $(\Gamma_\infty : (1, Z; 0, 1))$, then $\bigcup_{\sigma\in S}\sigma D$ fills up the strip $\{z\in H\,|\,0<\mathrm{Re}\, z<1\}$ $c_\infty$-times modulo $(1, Z; 0, 1)$; $D$ is a finite sum of fundamental domains of $\Gamma$, and does not touch the real axis by the assumption. $D$ may be supposed to be open, but the discussion of the boundary of $D$ is not necessary for our purpose. Write $D_Y=D-D'_Y$, and put

$$E^Y(z, s)=\begin{cases} E(z, s) & (z \in D_Y), \\ E(z, s)-y^s, & (z \in D'_Y), \end{cases}$$

and

$$\tilde{E}^Y(z, s)=\begin{cases} E(z, s), & (z \in D_Y), \\ E(z, s)-(y^s+y^{1-s}\varphi_0(s)k(0, s)), & (z \in D'_Y), \end{cases}$$

$(z=x+iy)$. Then, since

$$\int_0^\infty \int_0^1 f(z)d\mu(z) = c_\infty^{-1}\int_D \sum_{\sigma \in S} f(\sigma z)d\mu(z)$$

with the invariant measure $d\mu(z)=y^{-2}dxdy$ holds for any function $f$ of period 1 on $H$, we have

$$\begin{aligned}
&\sum_{m\neq 0} \rho_m^2|m|^{-(s+2-2\alpha_0)}M(k(*, \alpha_0)^2, s+2-2\alpha_0) \\
&\quad =\int_0^\infty \int_0^1 (r(z)^2-y^{2-2\alpha_0}\rho^2)y^{s+1}d\mu(z) \\
&\quad =\int_{D'_Y} (r(z)^2-y^{2-2\alpha_0}\rho^2)y^{s+1}d\mu(z) \\
&\qquad +\int_D (r(z)^2E^Y(z, s+1)-\rho^2E^Y(z, s+3-2\alpha_0))d\mu(z) \\
&\quad =\int_{D'_Y} (r(z)^2-y^{2-2\alpha_0}\rho^2)y^{s+1}d\mu(z) \\
&\qquad +\int_{D_Y} (r(z)^2E^Y(z, s+1)-\rho^2E^Y(z, s+3-2\alpha_0))d\mu(z) \\
&\qquad +\int_{D'_Y} (r(z)^2\tilde{E}^Y(z, s+1)-\rho^2\tilde{E}^Y(z, s+3-2\alpha_0))d\mu(z) \\
&\qquad +\int_{D'_Y} r(z)^2\varphi(s+1)y^{-s}d\mu(z)-\int_{D'_Y} \rho^2\varphi(s+3-2\alpha_0)y^{1-(s+3-2\alpha_0)}d\mu(z)
\end{aligned}$$

with $\rho=\rho_0 k(0, \alpha_0)$ and $\varphi(s)=\varphi_0(s)k(0, s)$, and consequently

$$\begin{aligned}
&\sum_{m\neq 0} \rho_m|m|^{-(s+2-2\alpha_0)}M(k(*, \alpha_0)^2, s+2-2\alpha_0) \\
&\quad =\int_{D'_Y} (r(z)^2-y^{2-2\alpha_0}\rho^2)y^{s+1}d\mu(z) \\
&\qquad +\int_{D_Y} (r(z)^2E^Y(z, s+1)-\rho^2E^Y(z, s+3-2\alpha_0))d\mu(z) \qquad (11)
\end{aligned}$$

$$+\int_{D_Y'} r(z)^2\tilde{E}^Y(z, s+1)d\mu(z)$$
$$+\int_{D_Y'} (r(z)^2 - y^{2-2\alpha_0}\rho^2)\varphi(s+1)y^{-s}d\mu(z)$$
$$+\rho^2\varphi(s+1)Y^{-s+1-2\alpha_0}/(s-1+2\alpha_0)$$
$$-\rho^2\varphi(s+3-2\alpha_0)Y^{-s-3+2\alpha_0}/(s+3-2\alpha_0).$$

This formula, obtained under the assumptions $\mathrm{Re}(s+2\alpha_0)>1$ and $\mathrm{Re}(s+4-2\alpha_0)>1$, now provides a meromorphic, analytic continuation of the Dirichlet series $\sum_{m\neq 0}\rho_m^2|m|^{-(s+2-2\alpha_0)}$, a residual version of the Dirichlet series in (9), to the whole $s$-plane, and supplies many important properties of it.

We now propose to investigate the same problem from another point of view which is somewhat similar to the circle method. Let, in general, $f(z)$ be an automorphic $C^\infty$-function for our group $\Gamma$ whose Fourier expansion is of the form

$$f(z)=\sum_m a_m(y)e(mx), \qquad (m\in \boldsymbol{Z}),$$

and assume that $a_0(y)-y^{t_0}$ is rapidly decreasing as $y\to\infty$ for a $t_0\in \boldsymbol{R}$ with $t_0>1/2$, and that $a_m(y)$ is, as a function of $y$ and $m$, rapidly decreasing as $|m|y\to\infty$. Define a neighborhood $U_\infty$ of $\infty$ by

$$U_\infty=\bigcup_b (1, b; 0, 1)D, \qquad (b\in \boldsymbol{Z}),$$

where $D$ is determined as above by the conditions i), ii) and iii), and put $U_\kappa=\sigma U_\infty$ for a cusp $\kappa=-d/c$ with a $\sigma=(d, -b; -c, a)\in\Gamma$. Then, $\{\boldsymbol{R}+iy\}\cap D=\phi$ suffices for

$$\int_0^1 f(z)dx=\sum_\kappa \int_{(\kappa)} f(z)dx, \qquad (0\leq\kappa\leq 1), \tag{12}$$

where $(\kappa)$ means the integral restricted on $U_\kappa$. Furthermore, if we denote by $C(x, y)$ the circle in the upper half plane $H$ with diameter $y^{-1}$ touching the real axis at $x$, and by $dl'$ the invariant distance $y'^{-1}(dx'^2+dy'^2)^{1/2}$, then

$$\int_{(\kappa)} f(z)dx=c^{-2}\int_{(a/c,\, c^2y)} c^2yf(x'+iy')dl', \tag{13}$$

where $(a/c, c^2y)$ means that the integral should be taken on the intersection of $U_\infty$ and the circle $C(a/c, c^2y)$, but, if $\kappa=0$ or 1, the integral on the left hand side should be replaced by $\int_{(0)}+\int_{(1)}$.

We consider the integral

$$\int_{(x,y)} yf(x'+iy')dl' = F(x, y) \tag{14}$$

which is a function of $x+iy \in H$ having period 1 with respect to $x$, and therefore allows a Fourier expansion:

$$F(x, y) = \sum_m A_m(y)e(mx), \qquad (m \in \boldsymbol{Z}). \tag{15}$$

Denoting by $U_\infty(z)$ the characteristic function of $U_\infty$ with its Fourier expansion

$$U_\infty(z) = \sum_m D_m(y)e(mx), \qquad \left(D_m(y) = \int_0^1 U_\infty(z)e(-mx)dx\right),$$

we have

$$\begin{aligned} A_m(y) &= y\int_0^1\int_{C(x,y)} f(x'+iy')U_\infty(x'+iy')dl'\cdot e(-mx)dx \\ &= y\int_0^1\int_{C(x,y)} f(x+x'+iy')U_\infty(x+x'+iy')dl'\cdot e(-m(x+x'))e(mx')dx. \end{aligned}$$

$(m \in \boldsymbol{Z})$. So,

$$A_m(y) = y\int_{C(0,y)} e(mx')\sum_{m'} a_{m'}(y')D_{m-m'}(y')dl'. \tag{16}$$

If $y' < Y_0$, where $Y_0$ is a positive constant such that $y' < Y_0$ implies $\{\boldsymbol{R}+iy'\} \cap D = \phi$, then it is clear that $D_m(y') = 0$ for all $m$, and there exists a constant $Y_1 (> Y_0)$ such that $D_m(y') = \delta_{0,m}$ (Kronecker's $\delta$) for all $y' > Y_1$. If $Y_0 \leq y' \leq Y_1$, then $D_m(y')$ does not decrease very rapidly as $|m| \to \infty$, which may cause difficulties in evaluating $A_m$ in (16). But, this difficulty is removed by using a "smooth fundamental domain" instead of an ordinary one $\hat{D}$. Namely, denote by $\hat{D}(z)$ the characteristic function of $\hat{D}$, and let $\psi$ be a non-negative valued $C^\infty$-function with compact support on $H$. Both $\hat{D}$ and $\psi$ can be viewed as functions on $SL(2, \boldsymbol{R})$ through the identification $H \cong SL(2, \boldsymbol{R})/SO(2)$, and, if $\psi$ satisfies an additional condition

$$\int_{SL(2,\boldsymbol{R})} \psi(g)dg = 1,$$

$dg$ being a Haar measure, then

$$\hat{D}_\psi(z) = \int_{SL(2,\boldsymbol{R})} \hat{D}(g\xi^{-1})\psi(\xi)d\xi$$

is a function on $H$, and has the property $\sum_\sigma \hat{D}_\psi(\sigma z) = 1$, $(\sigma \in \Gamma)$, for all

$z \in H$. Therefore, $\hat{D}_\psi$ can be considered to determine a "smooth" fundamental domain $\hat{D}_\psi$. Changing all formulas which we hitherto obtained suitably to fit $\hat{D}_\psi$, we have a new $D_m(y')$ which is rapidly decreasing uniformly for all $y'>0$ as $|m|\to\infty$.

By the assumption on $a_m$, the evaluation of $A_m(y)$ in (16) reduces to the evaluation of the integral of the individual summand, and for this purpose, it is enough to handle

$$B_{m,m'}(y)=y\int_{C(0,y)} e(mx')a_{m'}(y')D_{m-m'}(h')dl'. \tag{17}$$

The evaluation of this integral as $y\to 0$ is not difficult unless very precise results are required. Suppose first $m=m'=0$ and $y<Y_1^{-1}$. Then, (17) is equal to

$$\int_0^\infty a_0(y(x^2+y^2)^{-1})D_0(y(x^2+y^2)^{-1})dx,$$

while $D_0(y(x^2+y^2)^{-1})=1$ for $x<(yY_1^{-1}-y^2)^{1/2}$, and $D_0(y(x^2+y^2)^{-1})=0$ for $x>(yY_0^{-1}-y^2)^{1/2}$; between the two bounds, $D_0(y(x^2+y^2)^{-1})$ is bounded. From this fact and from the assumption on $a_0(y)$, it follows that

$$B_{0,0}(y)=k(0, t_0)y^{1-t_0}+O(y^{1/2})$$

as $y\to 0$. The case of $m'\neq 0$ is similar but simpler, because $D_{-m'}(y(x^2+y^2))\neq 0$ can hold only when $(yY_1^{-1}-y^2)^{1/2}<x<(yY_0^{-1}-y^2)^{1/2}$, and we have

$$B_{0,m'}(y)=O(y^{1/2}), \qquad (m'\neq 0),$$

as $y\to 0$. Hence,

$$A_0(y)=k(0, t_0)y^{1-t_0}+O(y^{1/2}) \tag{18}$$

for $y<Y_0^{-1}$. Similar investigations show

$$A_m(y)=O(y^{1/2}), \qquad (m\neq 0, y<Y_0^{-1}). \tag{19}$$

If $y\geqq Y_0^{-1}$, then all $A_m(y)$ are 0, and, for a fixed $y$, $A_m(y)$ decreases rapidly as $|m|\to\infty$, due to the assumption on $a_m$, and due to the fact that $D_m(y')$ is also rapidly decreasing as $|m|\to\infty$, which is the case as far as we take a smooth fundamental domain explained above. Therefore, in particular, we may say that the implied constant in (19) is rapidly decreasing as $|m|\to\infty$.

Denote now by $A(y)$ the right hand side of (12). Then, formulas (13), (14) and (15) show that

$$A(y)=\sum_{c\neq 0}\sum_{d \bmod c} c^{-2}F(a/c, c^2y)=\sum_{c\neq 0}\sum_{d \bmod c} c^{-2}\sum_m A_m(c^2y)e(ma/c),$$

$(ad-bc=1)$, and therefore that the Mellin transform of $A(y)$ in the sense of (8) is given by

$$\begin{aligned} M(A, s)&=\sum_{c\neq 0}|c|^{-2-2s}\sum_{d \bmod c}\sum_m e(ma/c)M(A_m, s) \\ &=\sum_m \varphi_{-m}(s+1)M(A_m, s), \qquad (m\in \boldsymbol{Z}). \end{aligned} \tag{20}$$

By virtue of (18) and (19), the above equality (20) has a meaning for $\operatorname{Re} s>-1/2$, and consequently determines the behavior of $A(y)$ as $y\to 0$. Namely, $M(A, s)$ is holomorphic in $\operatorname{Re} s>-1/2$ except possible poles at $t_0-1$ and $\alpha_1-1, \alpha_2-1, \cdots$, where $\alpha_1, \alpha_2, \cdots$ are finite number of poles of the Eisenstein series on the interval $(1/2, 1]$. If we specialize these results to the case of $f(z)=r(z)^2$, $r(z)$ being as in (10), we will be able to obtain similar consequences to those derived from (11) at least in the region $\operatorname{Re} s>-1/2$, since $A(y)=a_0(y)$ whenever $y<Y_0$, (c.f. (12)).

This method, although it gives weaker results than Rankin's, seems to have some possibility to work in the case of $n\geq 3$ in (9), too. In the case of $n=2$, corresponding to the two-fold convolution of a function of period 1, the square of a function was enough to investigate the value on the imaginary axis of the convolution, provided, for instance, that $\kappa$ and $-\kappa$ are simultaneously a cusp. But, if $n\geq 3$, then we have really to deal with convolution assuming that $-(\kappa_1+\kappa_2)$ is a cusp, whenever $\kappa_1$ and $\kappa_2$ are cusps. In addition, a new type of Dirichlet series will arise, which can be more difficult than $\varphi_m(s)$, especially when a character of the discontinuous group is considered. If, nevertheless, one could obtain a result, such as $A_n(s, s_0)$ is holomorphic in a region, say, $\operatorname{Re} s>\gamma_n$, except a finite number of poles, then a consequence would be that the non-real poles of $\sum_{m\neq 0}\varphi_m(s_0)^n|m|^{-s-n+ns_0}$ in the same region is a zero of $M(k(*, s_0)^n, s+n-ns_0)$.

If this kind of fact, which is equivalent to the determination of the behavior of the function

$$\sum_{m\neq 0}(y^{1-s_0}\varphi_m(s_0)k(my, s_0))^n$$

as $y\to 0$, were really attained, it would not be very absurd to imagine that a similar fact would be true for

$$\sum_{m\neq 0}(y^{1-s_0}\varphi_m(s_0)k(my, s_0))^n e(mxy), \qquad (x\in \boldsymbol{R}),$$

as $y\to 0$. Then, taking a function $\eta(x)$ such that

$$\int_{-\infty}^{\infty} \eta(x)e(xy)dx = k(y, s_0)^{-n}|y|^h \exp(-y^2)$$

where $h>0$ should be chosen so that $\eta(x)$ decreases sufficiently rapidly as $|x|\to\infty$, one would come to the conclusion that $\sum_{m\neq 0} \varphi_m(s_0)^n|m|^{-s-n+ns_0}\cdot \Gamma((h+s)/2)$ has no non-real poles in $\operatorname{Re} s>\gamma_n$, and therefore the Dirichlet series itself has no non-real poles in the same region. For $n=1$ and 2, the function $M(k(*, s_0)^n, s)$ is a simple composition of gamma factors, but, for $n\geq 3$, the function is a more complicated higher transcendental function related to generalized hypergeometric functions.

## § 3. Metaplectic automorphic functions on the upper half space

The contents of this section are practically explanatory remarks to indicate how metaplectic automorphic functions are combined with zeta functions of standard type. In this section, $H$ stands for the three dimensional upper half space, i.e., the space of all points $u=(z, v)=(x+iy, v)$ with $x, y \in \boldsymbol{R}$ and $v>0$. If $u$ is identified with the matrix $(z, -v; v, \bar{z})$, and if $\tilde{w}$ stands for $(w, 0; 0, \bar{w})$ for $w \in \boldsymbol{C}$, then $SL(2, \boldsymbol{C})$ operates on $H$ by $\sigma u=(\tilde{a}u+\tilde{b})(\tilde{c}u+\tilde{d})^{-1}$, $(\sigma=(a, b; c, d) \in SL(2, \boldsymbol{C}))$. In this way, the relationship between the upper half space and $SL(2, \boldsymbol{C})$ becomes completely similar to the relationship between the upper half plane and $SL(2, \boldsymbol{R})$.

Patterson [1] proved that the Fourier coefficients of a special type of automorphic functions on $H$, which are in fact metaplectic automorphic functions related to the field $F=\boldsymbol{Q}((-3)^{1/2})$, are explicitly expressed by Gauss sums. More precisely speaking, put $e(z)=\exp(\pi i(z+\bar{z}))$ for $z \in \boldsymbol{C}$, let $K_{1/3}$ be a modified Bessel function, and let $\mathfrak{o}$ be the ring of integers of $F$, then

$$\theta(u)=c_0v^{2/3}+v^{2/3}\sum_m c_m(v|m|)^{1/3}K_{1/3}(4\pi|m|v)e(2mz), \qquad (m \in (-3)^{-3/2}\mathfrak{o}),$$

with $c_0=3^{5/2}/2$ and with the coefficients $c_m$ related to Gauss sums gives rise to a metaplectic automorphic function in the sense that $\theta(\sigma u)=\chi(\sigma)\theta(u)$ for any $\sigma$ in the principal congruence subgroup $\Gamma_3$ mod 3 of $SL(2, \mathfrak{o})$, where $\chi(\sigma)=1$ or $=(c/d)_3$, cubic residue symbol, according as $c=0$ or not for $\sigma=(a, b; c, d) \in \Gamma_3$. This function is obtained as the residual form at $s_0=4/3$ of an Eisenstein series which is roughly of the form

$$\sum_{c,d} \chi(c, d)v^{s_0}(|cz+d|^2+|c|^2v^2)^{-s_0}, \qquad (\chi(c, d)=(c/d)_3), \tag{21}$$

where the sum ranges over all pairs $(c, d)$ such that there exists an element $\sigma$ in $\Gamma_3$ of the form $(*, *; c, d)$. Due to the automorphic factor containing the residue symbol, $\theta(u)$ is metaplectic with the degree 3 of the covering: Furthermore, $\theta(u)$ may be regarded as a cubic analogy of the theta function, because the residual form of the Eisenstein series like (21) with quadratic residue symbol essentially coincides with a classical theta function. Whereas [1] gives all coefficients $c_m$ completely, we quote here only $c_m$ for $m\equiv 1 \pmod 3$ as follows:

$$(22)\qquad c_m=\begin{cases} g(m)|m|^{-4/3}, \text{ if } m \text{ is a product of distinct primes,} \\ c_{m'}, \text{ if } m' \ (\equiv 1 \pmod 3) \text{ is a product of distinct primes,} \\ \qquad \text{and } m=m'm_0^3 \text{ with } m_0 \in \mathfrak{o}, \\ 0, \text{ otherwise,} \end{cases}$$

where

$$g(m)=\sum_{d \bmod m} (d/m)_3^{-1} e(2d/m), \qquad (d \in \mathfrak{o}),$$

is a cubic Gauss sum.

Since $(-3)^{3/2}\mathfrak{o}$ is dual to $3\mathfrak{o}$ with respect to $e(2z)$, the operation

$$\theta(u) \longrightarrow \sum_{j \bmod 3} e(-2j)\theta((1,j;\,0,\,1)u), \qquad (j \in (-3)^{3/2}\mathfrak{o}),$$

eliminates all terms of $\theta(u)$ but those with $m\equiv 1 \pmod 3$, (c.f. Section 1). On the other hand, if $\nu$ is a sufficiently high power of 3, and if $\Gamma_\nu$ denotes the principal congruence subgroup mod $\nu$ of $SL(2, \mathfrak{o})$, then $(1,j;\,0,\,1)^{-1}\sigma\cdot(1,j;\,0,\,1)=\sigma' \in \Gamma_3$, and $\chi(\sigma)=\chi(\sigma')$ for any $\sigma \in \Gamma_\nu$ and $j \in (-3)^{3/2}\mathfrak{o}$. This shows that every $\theta((1,j;\,0,\,1)u)$ is a metaplectic automorphic function for the group $\Gamma_\nu$ belonging to the common automorphic factor. Thus,

$$(23)\qquad \theta_1(u)=v^{2/3}\sum_{m\equiv 1 \pmod 3} c_m k(|m|v)e(2mz), \qquad (m \in \mathfrak{o}),$$

with $k(v)=(4\pi v)^{1/3}K_{1/3}(4\pi v)$ is a metaplectic automorphic function with respect to $\Gamma_\nu$.

We now put here

$$(24)\qquad \begin{aligned} &\int_0^\infty \sum_{m\equiv 1 \pmod 3} (v^{2/3}c_m k(|m|v))^n v^{2s-1}dv \\ &\quad = \sum_{m\equiv 1 \pmod 3} c_m^n Nm^{-s-n/3} M(k^n,\, 2s+2n/3)=\Lambda_n(s), \end{aligned}$$

($m \in \mathfrak{o}$, $n=1, 2, 3, \cdots$), and wish to write down more explicit formulas for $\Lambda_n(s)$. If $n=1$, then (22) implies

$$(25)\qquad \Lambda_1(s)=\Big(\sum_{m\equiv 1 \pmod 3} g(m)/|m|\cdot Nm^{-s-1/2}\Big)\zeta_F^*(3s+1)M(k,\, 2s+2/3),$$

($m \in \mathfrak{o}$), with

$$\zeta_F^*(s) = \sum_{m\equiv 1 \pmod 3} Nm^{-s}, \qquad (m \in \mathfrak{o}),$$

i.e., the product of $1-3^{-s}$ and the Dedekind zeta function $\zeta_F(s)$ of $F$, because $g(m)=0$, whenever $m$ has a square factor. If $n=2$, then we have similarly

$$(26) \qquad \Lambda_2(s)(\sum_{m\equiv 1 \pmod 3} g(m)^2/|m|^2 \cdot Nm^{-s-1})\zeta_F^*(3s+2)M(k^2, 2s+4/3),$$

$(m \in \mathfrak{o})$. The case of $n=3$ is somewhat different. First we have again a similar formula

$$(27) \qquad \Lambda_3(s)=(\sum_{m\equiv 1 \pmod 3} g(m)^3/|m|^3 \cdot Nm^{-s-3/2})\zeta_F^*(3s+3)M(k^3, 2s+2),$$

$(m \in \mathfrak{o})$, but, in this case, the congruence relation

$$g(m) \equiv \sum_{d \bmod m} e(2d/m) = \mu(m) \qquad (\mathrm{mod}\ (-3)^{1/2}),$$

with the Möbius function $\mu$ of $F$ implies $g(m)^3 \equiv \mu(m)$ (mod 3) for $m\equiv 1$ (mod 3), and accordingly

$$(28) \qquad g(m)^3/|m|^3 = \bar{\lambda}(m)\mu(m),$$

where $\lambda(m)$ is a Grössencharacter of $F$ mod 3, which is determined by $\lambda(m)=m/|m|$ for $m\equiv 1$ (mod 3), due to the Stickelberger's relation in the general theory of Gauss sums. Therefore, $\Lambda_3(s)$ gains the expression

$$\Lambda_3(s) = L(s+3/2, \bar{\lambda})^{-1}\zeta_F^*(3s+3)M(k^3, 2s+2)$$

containing an $L$-function with a Grössencharakter.

The series in (25) is absolutely convergent for Re $s>1/2$, and since $\Lambda_1(s)$ is immediately connected with an automorphic function, it is easy to discuss the analytic continuation of $\Lambda_1(s)$ by means of routine methods as explained in Section 1. The series in (26) is absolutely convergent for Re $s>0$, and this case can be treated by Rankin's method, as was precisely done by Patterson [2]. In both cases, it is remarkable that the functions are holomorphic at least 1/6 beyond the bound of absolute convergency. If a similar fact would be proved for $\Lambda_3(s)$, for instance by means of an analogy to circle method as explained in Section 2, namely, if $\Lambda_3(s)$ would be holomorphic for Re $s>-2/3$, while the series in (27) is absolutely convergent for Re $s>-1/2$, then a consequence would be that the zeros of $L(s, \bar{\lambda})$ with Re $s>5/6$ should be zeros of $M(k^3, 2s-1)$, because $\zeta^*(3s-3/2)$ has no zeros in the same region.

This kind of imagination can be driven further. Consider the series

(29) $$\sum_{m\equiv 1 \,(\mathrm{mod}\ 3)} (v^{2/3}c_m k(|m|v))^n me(2mz), \qquad (m \in \mathfrak{o}),$$

which is obtained by applying the operator $\partial/\partial z$ to the $n$-fold convolution of $\theta_1(u)$ in (23) as a doubly periodic function of $z$, while the convolution itself appeared in (24). Then, (29) is the convolution of one $\partial/\partial z \cdot \theta_1$ and $(n-1)$-times $\theta_1$ up to a trivial constant. The function $\partial/\partial z \cdot \theta_1$ is not an automorphic function but is a kind of automorphic form, and therefore its behavior near a cusp, as well as that of (29), should not be very hard to determine. On the other hand, the Mellin transform of (29) with $z=0$ leads to

(30) $$\int_0^\infty \sum_{m\equiv 1 \,(\mathrm{mod}\ 3)} (v^{2/3}c_m k(|m|v))^n mv^{2s-1}dv$$
$$= \sum_{m\equiv 1 \,(\mathrm{mod}\ 3)} mc_m^n Nm^{-s-n/3} M(k^n, 2s+2n/3) = \tilde{\Lambda}_n(s),$$

$(m \in \mathfrak{o})$. If in particular $n=3$, the formula (30) compared with (27) shows, together with (28), that

$$\tilde{\Lambda}_3(s) = \zeta_F^*(s+1)^{-1} L(3s+3/2, \lambda) M(k^3, 2s+2).$$

If it were possible, also in this case, to prove that $\tilde{\Lambda}_3(s)$ is holomorphic for $\mathrm{Re}\, s > -1/6$ for instance, then a consequence would be that the zeros of $\zeta_F(s)$ with $\mathrm{Re}\, s > 5/6$, and accordingly the zeros of Riemann's zeta function $\zeta(s)$ in the same region, should be zeros of $M(k^3, 2s)$, because $L(3s+3/2, \lambda)$ has no zeros there.

It must be a phantasy to expect such a partial coincidence between zeros of $\zeta(s)$ and $M(k^3, 2s)$ on the critical line, but whether it may be fact or not would be checked by a computation comparing the zeros of $\zeta(s)$ and the zeros of $M(k^3, 2s) = (4\pi)^{-2s} M(k_{1/3}^3, 2s+1)$.

## References

[1] S. J. Patterson, A cubic analogue of the theta series, J. reine angew. Math., **296** (1977), 125–161.

[2] ——, On Dirichlet series associated with cubic Gauss sums, J. reine angew. Math., **303–304** (1978), 102–125.

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