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Standard Monomial Theory and the Work of Demazure

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In collaboration with V. Lakshmibai and C. Musili (cf. [8], [9], [10], [12]), we have given a generalization of the classical Hodge-Young standard monomial theory (cf [4], [5]) of $SL(n)$ to the case of an arbitrary semi-simple linear algebraic group G. The purpose of this generalization is to give an explicit basis for the space $H^0(G/B, L)$ of sections of a line bundle L (associated to a dominant weight) on the flag variety G/B, or more generally for $H^0(X, L)$, where X is a Schubert variety in G/B. Our results provide a complete solution to this problem, when G is classical and only partial answers when G is exceptional. Recall that when the base field is of characteristic zero (Borel-Weil theorem) every irreducible G-module is of the form $H^0(G/B, L)$ (L as above), so that a particular case of this problem is to give an explicit basis for an irreducible G-module. A survey of our results, including its motivation and applications, has been given in [11].

Our first purpose here is to give a proof of the first theorem on standard monomial theory, namely the basis theorem for a fundamental representation (say the field is of characteristic zero), such that its highest weight ω is of *classical type* (see Theorem 3). The proof of this theorem, given here, is not really different from the one given in G/P-IV (cf [9]). However, we have separated out many general considerations with which it is mixed up in [9] and this may be of help in understanding this theorem.

The work of Demazure (cf [2]) is basic to the proof of the main results of standard monomial theory (cf. G/P-IV, [9]), especially his character formula which generalizes the Weyl character formula. Further, standard monomial theory can be considered as a refined version of a conjecture made by Demazure in [2] (see also Remarks 4, 5 and 6). Our second purpose here is to give a fairly self-contained exposition of the results of Demazure, relevant to standard monomial theory. Our exposition of this work of Demazure (see § 2) is basically the same as his; however, it avoids his big inductive machinery (which is perhaps necessary for the desingularisation of Schubert varieties). Consequently, the proofs given here of his vanishing theorems and character formula for line bundles on Schubert

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varieties, appear to be more transparent. We deduce also that *Schubert schemes behave well under base change* (see (ii) of Theorem, also Remark 7). We do not know a simpler proof of this fact. We prove also that Schubert varieties are *normal* (over any field). This proof came as a mild surprise to us (see Remark 8). The *projective normality* of a Schubert variety over a *general field* (say with respect to any ample line bundle) is, however, not known; in fact this can be seen to be equivalent to the conjecture made by Demazure in [2] (see also Remarks 5 and 6)*).

§ 1. Preliminaries

Let G_Z denote a semi-simple, simply-connected, Chevalley group scheme over the ring of integers Z (for many basic facts on Chevalley groups, cf. Steinberg [13]). We fix a maximal torus group scheme T_Z and a Borel subgroup scheme B_Z containing T_Z. We talk of roots, weights etc. with respect to T_Z and B_Z. The Weyl group scheme $N(T_Z)/T_Z$ ($N(T_Z)$ $=$normalizer of T_Z) is a constant group scheme and hence we talk of the Weyl group W of G_Z. If A is any ring, we denote the objects obtained by the base change Spec $A \rightarrow$ Spec Z with the suffix A (unless otherwise stated), e.g., G_A, B_A, T_A etc.

Let U (or to be more precise U_Q) denote the enveloping algebra of Lie G_Q–the Lie algebra of G_Q. Let U_Z (resp. U_Z^+, resp. U_Z^-) denote the canonical Z-form in U i.e. the Z-subalgebra of U spanned by $X_\alpha^n/n!$, α a root (resp. α a positive root, resp. α a negative root), where X_α denotes the usual element in the Chevalley basis of Lie G_Q. We denote by U_α (resp. $U_{\alpha,Z}$) the Q-vector subspace (resp. Z-submodule) of U (resp. U_Z) generated by X_α^n (resp. $X_\alpha^n/n!$). Let $G_{\alpha,Q}$ or simply G_α (resp. $G_{\alpha,Z}$) denote the unipotent subgroup scheme isomorphic to $\boldsymbol{G}_{a,Q}$ (resp. $\boldsymbol{G}_{a,Z}$) of B_Q (resp. B_Z) which corresponds to α. We see that Lie $G_{\alpha,Q} \simeq Q \cdot X_\alpha$.

Let V be a finite dimensional Q-vector space which is also a G_Q-module. Then a *lattice* V_Z in V is said to be an *admissible Z-form* if any of the following three equivalent conditions is satisfied:

(i) V_Z is U_Z-stable

(ii) V_Z is a G_Z-Z-module, i.e., for every commutative ring D (with 1), $V_Z \otimes_Z D$ has a $G_Z(D)$-module structure ($G_Z(D)=$group of D-valued points of G_Z) which is functorial in D.

If V, $D \cdots$ etc. are as above, we observe that for $d \in D$

$$\exp(dX_\alpha) = \sum d^n \cdot X_\alpha^n/n!$$

defines an automorphism of the D-module $V_Z \otimes_Z D$. An important point

*) Much of this work was done during a very pleasant stay in Japan for which the author would like to thank Professor Oda and many other Japanese colleagues.

is that, when we identify $G_{\alpha,Z}(D)$ with D, the action of d on $V_Z \otimes_Z D$ is given by $\exp(dX_\alpha)$. If A is any ring, we set

$$U_A = U_Z \otimes_Z A, \qquad U_{\alpha,A} = U_{\alpha,Z} \otimes_Z A$$

and the above definitions made for $\boldsymbol{Z}$ can be generalized to A.

For a dominant weight λ, let V_λ denote the finite dimensional $\boldsymbol{Q}$-vector space which is the irreducible G_Q-module with highest weight λ. Fix a highest vector $e = e_\lambda$ in V_λ (determined up to a non-zero factor in $\boldsymbol{Q}$). For $\tau \in W$, we write

$$V_{\lambda,Z}(\tau) = U_Z^+ e_\tau, \qquad \text{where } e_\tau = \tau \cdot e$$

(τ can be represented by a $\boldsymbol{Z}$-valued point of G_Z and we see that e_τ is well-determined up to the factor ± 1). We write

$$V_{\lambda,Z}(w_0) = V_{\lambda,Z}, \ w_0 \text{ the element of } W \text{ of maximal length.}$$

One knows that

$$V_{\lambda,Z} \otimes_Z \boldsymbol{Q} = V_\lambda$$

and that $V_{\lambda,Z}$ is a U_Z-stable $\boldsymbol{Z}$-submodule of V_λ or equivalently a G_Z-$\boldsymbol{Z}$-module. We see that

$$V_{\lambda,Z} = U_Z^- e = U_Z^+ e_{w_0}.$$

If A is any ring, we define

$$V_{\lambda,A}(\tau) = V_{\lambda,Z}(\tau) \otimes_Z A$$
$$V_{\lambda,A} = V_{\lambda,Z} \otimes_Z A.$$

We observe that $V_{\lambda,Q} = V_\lambda$. We note also that $V_{\lambda,A}$ is a G_A-A-module. It is not difficult to see that e is a *primitive element* in $V_{\lambda,Z}$, i.e., $\boldsymbol{Z}e$ is a direct summand in $V_{\lambda,Z}$. Consequently we see that every e_τ, $\tau \in W$, is a primitive element in $V_{\lambda,Z}$.

We have now the following:

Lemma 1. *Let $\varphi, \tau \in W$ and*

$$\varphi = s_\alpha \tau, \quad \begin{cases} l(\tau) = l(\varphi) + 1, \ l \text{ being the length function on } W \text{ and } s_\alpha \\ \text{the reflection associated to a simple root } \alpha. \end{cases}$$

Then we have

$$U_{-\alpha,Z} V_{\lambda,Z}(\varphi) = V_{\lambda,Z}(\tau)$$
$$(\text{in particular } V_{\lambda,Z}(\varphi) \subset V_{\lambda,Z}(\tau)).$$

The proof of this lemma is quite straightforward. It is similar to that of Lemma 5.2, [9] and we refer to this.

Let $\tau \in W$. Then if A is any ring, we see that τ determines an A-valued point of the generalised *flag variety* G_A/B_A, which we denote by the same letter τ. Let now k be a field. Then we denote by $X_k(\tau)$ the *Schubert variety* associated to τ, i.e., we define $X_k(\tau)$ to be the closed subvariety of G_k/B_k, which is the Zariski closure of $B_k\tau$ in G_k/B_k, endowed with the canonical reduced structure. Similarly, we define the *Schubert subscheme* $X_{\boldsymbol{Z}}(\tau)$ as the closure of $B_{\boldsymbol{Z}}\tau$ in $G_{\boldsymbol{Z}}/B_{\boldsymbol{Z}}$, endowed with the canonical structure of a closed reduced $\boldsymbol{Z}$-subscheme of $G_{\boldsymbol{Z}}/B_{\boldsymbol{Z}}$. We note that $X_{\boldsymbol{Z}}(\tau)$ is also the *flat* closure of $X_{\boldsymbol{Q}}(\tau)$ in $G_{\boldsymbol{Z}}/B_{\boldsymbol{Z}}$, i.e., the canonical morphism $X_{\boldsymbol{Z}}(\tau) \to \operatorname{Spec} \boldsymbol{Z}$ is $\boldsymbol{Z}$-flat and the generic fibre is $X_{\boldsymbol{Q}}(\tau)$. *However, note that it is not a priori clear that the base change of $X_{\boldsymbol{Z}}(\tau)$ by* Spec $k \to$ Spec $\boldsymbol{Z}$ *coincides with $X_k(\tau)$, where k is an arbitrary field. We shall prove this fact later.* However, we see immediately that

$$X_k(\tau) = (X_{\boldsymbol{Z}}(\tau) \times_{\operatorname{Spec} \boldsymbol{Z}} \operatorname{Spec} k)_{\mathrm{red}}.$$

Let now ω_i, $1 \leqslant i \leqslant l$, denote the fundamental weights, l being the rank of G. Let us now take λ to be of the form

$$\lambda = \sum_{i=1}^{l} a_i \omega_i, \qquad a_i > 0. \tag{1}$$

We now take the *projective space*

$$\boldsymbol{P}(V^*_{\lambda,\boldsymbol{Z}}) = \operatorname{Proj} S(V_{\lambda,\boldsymbol{Z}}).$$

One knows that if D is any ring

$$\boldsymbol{P}(V^*_{\lambda,\boldsymbol{Z}})\ (D) = \text{the set of direct summands of } V_{\lambda,\boldsymbol{Z}} \text{ of rank } 1.$$

If e is the choice of the highest weight vector in V_λ made as above, as we remarked before $\boldsymbol{Z}e$ is a direct summand in $V_{\lambda,\boldsymbol{Z}}$ and we denote by $\bar{e}$ the point of $\boldsymbol{P}(V^*_{\lambda,\boldsymbol{Z}})$. We observe that we have a canonical action of $G_{\boldsymbol{Z}}$ on $\boldsymbol{P}(V^*_{\lambda,\boldsymbol{Z}})$.

Lemma 2. *The isotropy subgroup scheme of $G_{\boldsymbol{Z}}$ at $\bar{e}$ is $B_{\boldsymbol{Z}}$ (note that λ is of the form as in* (1) *above).*

The crucial point in the proof of this lemma is the following lemma due to Deodhar (cf. Lemma 5.8, [9]).

Lemma 3. *Let β be a positive root such that $X_{-\beta}e \neq 0$. Then $X_{-\beta}e$ is a primitive element in $V_{\lambda,\boldsymbol{Z}}$ (for this lemma we need not suppose that λ satisfies* (1) *above; it could be an arbitrary dominant weight).*

It can be seen that $X_{-\beta}e \neq 0$, where β is any positive root (λ satisfying (1) above). Then using Lemma 3 above, Lemma 2 follows by a fairly easy infinitesimal argument. The arguments are exactly similar to those given on p. 317–320, [9] and for the proof of Lemma 2 and Lemma 3, we refer to the details given there.

Because of Lemma 2, we get a canonical closed immersion

$$j: G_Z/B_Z \longrightarrow \boldsymbol{P}(V^*_{\lambda,Z}).$$

By base change, we obtain for any ring A, a canonical closed immersion

$$j_A: G_A/B_A \longrightarrow \boldsymbol{P}(V^*_{\lambda,A}).$$

Let $\bar{e}_\tau (\tau \in W)$ be the element of $\boldsymbol{P}(V^*_{\lambda,Z})(\boldsymbol{Z})$ corresponding to the direct summand $\boldsymbol{Z}e_\tau$ of $V_{\lambda,Z}$. Recall that τ has been identified as an element of $(G_Z/B_Z)(\boldsymbol{Z})$. Then we see that $j(\tau)=\bar{e}_\tau$. We denote by $L_{\lambda,A}$ the *very ample line bundle* on G_A/B_A, which is the restriction of the tautological ample line bundle on G_A/B_A.

Suppose that λ is a dominant weight which need not satisfy the condition (1) above. By the same arguments as above, we see that the isotropy subgroup scheme of G_Z at $\bar{e} \in \boldsymbol{P}(V^*_{\lambda,Z})$, is a parabolic group scheme P_Z ($P_Z \supset B_Z$). We have now canonical morphisms

$$G_A/B_A \longrightarrow G_A/P_A \longrightarrow \boldsymbol{P}(V^*_{\lambda,A}).$$

The pull-back to G_A/B_A of the tautological ample line bundle on $\boldsymbol{P}(V^*_{\lambda,A})$ is denoted by $L_{\lambda,A}$. The Schubert schemes in G_A/P_A (i.e., the images of the Schubert schemes in G_A/B_A) are parametrized by W/W_P (here P stands for P_A, P_Z). One sees that the line bundle $L_{\lambda,A}$ on G_A/B_A is ample, if and only if λ satisfies (1) above. Associated to λ one has a canonical homomorphism $T_A \to \boldsymbol{G}_{m,A}$ and consequently a canonical homomorphism $B_A \to \boldsymbol{G}_{m,A}$. Then we get a line bundle on G_A/B_A, associated to this homomorphism (in the sense of associated fibre spaces). It can be seen that this line bundle coincides with the $L_{\lambda,A}$ which we just defined.

Let $\tau, \varphi \in W$ be of the form

$$\varphi = s_\alpha \tau;\ l(\tau) = l(\varphi)+1,\ \alpha \text{ simple root.}$$

Then we say that $X_Z(\varphi)$ (resp. $X_k(\varphi)$, k field) is a *Schubert divisor in* $X_Z(\tau)$ (*resp.* $X_k(\tau)$) *moved by a simple root* α. One knows that $X_Z(\tau)$ is stable under $G_{-\alpha,Z}$. Of course every Schubert scheme is stable under $G_{\beta,Z}$ (β any positive root) so that $X_Z(\tau)$ is stable under $G_{-\alpha,Z}$ and $G_{\alpha,Z}$ and hence under the "$SL(2)$" corresponding to α.

Let k be a *field.* Since we have

$$X_Z(\tau)(k) \subset \boldsymbol{P}(V^*_{\lambda,Z})(k) = (V_{\lambda,k} - (0))/k^*$$

we can talk of the k-linear subspace of $V_{\lambda,k}$ generated by $X_Z(\tau)(k)$. We have a canonical linear map

$$V_{\lambda,k}(\tau)=V_{\lambda,Z}(\tau)\otimes_Z k \longrightarrow V_{\lambda,Z}\otimes_Z k=V_{\lambda,k}.$$

We denote by $\operatorname{Im} V_{\lambda,k}(\tau)$ the image of the above map. If k is a field of characteristic zero, note that the above map is *injective* and we can identify $V_{\lambda,k}(\tau)$ as a linear subspace of $V_{\lambda,k}$. We have then:

Lemma 4. *Let k be a field and λ satisfy* (1) *above, i.e., $L_{\lambda,Z}$ is ample on G_Z/B_Z. Then we have*
$\operatorname{Im} V_{\lambda,k}(\tau)=k$-linear subspace of $V_{\lambda,k}$ generated by $X_Z(\tau)$ (k).

Proof. This is done by induction on the length $l(\tau)$ of τ. When $l(\tau)=0$, $V_{\lambda,Z}\approx Ze$ (a direct summand of $V_{\lambda,Z}$) and $X_Z(\tau)\approx \operatorname{Spec} Z$ and we see easily that the lemma follows easily in this case. Let $X_Z(\varphi)$ be a Schubert divisor in $X_Z(\tau)$ moved by a simple root α. We suppose that the lemma is true for $X_Z(\varphi)$.

We see easily that it suffices to prove the lemma when the field k is *algebraically closed.*

Let $q\in V_{\lambda,Z}(\varphi)$ and $\bar{q}$ the canonical image of q in $V_{\lambda,k}(\varphi)\hookrightarrow V_{\lambda,k}$. Now $V_{\lambda,Z}$ is a $G_{-\alpha,Z}$ or equivalently $U_{-\alpha,Z}$-module and $V_{\lambda,k}$ is a $G_{-\alpha,k}$-module or equivalently $U_{-\alpha,k}=U_{-\alpha,Z}\otimes_Z k$-module. The element $t\cdot\bar{q}$, when we identify t with an element of $G_{-\alpha,Z}(k)\approx G_{-\alpha,k}(k)\approx k$, is given as follows (as we remarked above):

(*) $t\cdot\bar{q}=\exp(tX_{-\alpha})\bar{q}=(1+tX_{-\alpha}+\cdots+t^n(X^n_{-\alpha}/n!))\bar{q}$ (we choose n such that $(X^{n+1}_{-\alpha}/n+1)\cdot\bar{q}=0$).

Since *k is algebraically closed,* we can find $t_1,\cdots,t_{n+1}\in k$ such that the Vandermonde determinant

$$\det\Delta=\det\begin{vmatrix}1 & t_1 & \cdots & t_1^n\\ & \cdot & \cdot & \cdot\\ & \cdot & \cdot & \cdot\\ 1 & t_{n+1} & \cdots & t^n_{n+1}\end{vmatrix}\neq 0$$

Set

$$\Delta\cdot\begin{bmatrix}\bar{q}\\ X_{-\alpha}\bar{q}\\ \vdots\\ X^n_{-\alpha}\bar{q}\end{bmatrix}=\begin{bmatrix}\theta_1\\ \theta_2\\ \vdots\\ \theta_n\end{bmatrix}\quad (n\times 1\text{-matrix})$$

so that

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix} = \Delta^{-1} \begin{bmatrix} \bar{q} \\ X_{-\alpha}\bar{q} \\ \vdots \\ X_{-\alpha}^n \bar{q} \end{bmatrix}.$$

This implies that $\forall\ m$

$$(X_{-\alpha}^m/m!)\bar{q} \in k\text{-linear subspace (of } V_{\lambda,k}\text{) spanned by } G_{-\alpha,Z}(k)\bar{q}.$$

(We are in a sense repeating the argument of equivalence between a $U_{-\alpha,Z}$-module structure and that of a $G_{-\alpha,Z}$-Z-module structure). Since $X_Z(\tau)$ is stable under $G_{-\alpha,Z}$, we see, by applying Lemma 1, that

Im $V_{\lambda,k}(\tau) \subset k$-linear subspace of $V_{\lambda,k}$ generated by $X_Z(\tau)(k)$.

It remains to prove the inclusion

$$k\text{-linear subspace of } V_{\lambda,k} \text{ generated by } X_Z(\tau)(k) \subset \text{Im } V_{\lambda,k}(\tau).$$

It is not difficult to see that the image of the map

$$G_{-\alpha,k} \times X_k(\varphi) \longrightarrow X_k(\tau)$$

contains a non-empty open subset of $X_k(\tau)$ (since the image contains $X_k(\varphi)$ and τ etc.). Hence the k-linear subspace of $V_{\lambda,k}$ spanned by $X_k(\tau)(k) = X_Z(\tau)(k)$ is the k-linear subspace of $V_{\lambda,k}$ spanned by $G_{-\alpha,Z}(k) \cdot X_Z(\varphi)(k)$.

We have

$$G_{-\alpha,Z}(k) \cdot X_Z(\varphi)(k) \subset G_{-\alpha,Z}(k)(V_{\lambda,Z}(\varphi) \otimes_Z k).$$

It is obvious that the RHS of the above (see (*)) is contained in

$$(U_{-\alpha,Z} \cdot V_{\lambda \cdot Z}(\varphi)) \otimes_Z k = V_{\lambda,Z}(\tau) \otimes k.$$

This concludes the proof of Lemma 4.

Remark 1. In Lemma 4, *one need not suppose that λ satisfies* (1). One should then work with G_Z/P_Z, where P_Z is the isotropy subgroup scheme of G_Z at $\bar{e}$ and the above proof easily goes through.

Remark 2. We have a canonical isomorphism

$$H^0(\boldsymbol{P}(V_{\lambda,Z}^*), L_{\lambda,Z}) \xrightarrow{\sim} V_{\lambda,Z}^*.$$

Now Lemma 4 says that the smallest projective subspace containing $X_k(\tau)$ (recall $X_k(\tau) = (X_Z(\tau) \times_{\text{Spec } Z} \text{Spec } k)_{\text{red}}$ is $(\text{Im } V_{\lambda,k}(\tau))^*$. Hence the above canonical isomorphism induces a canonical k linear map

$$j_k\colon (V_{\lambda,k}(\tau))^* \longrightarrow H^0(X_k(\tau), L_{\lambda,k})$$

and the image of this linear map can be canonically identified with $(\mathrm{Im}\ V_{\lambda,k}(\tau))^*$. If Char $k=0$, as we have observed before $(V_{\lambda,k}(\tau))^* \xrightarrow{\sim} (\mathrm{Im}\ V_{\lambda,k}(\tau))^*$ and so in this case the map j_k is *injective*. We see that if $V_{\lambda,\boldsymbol{Z}}(\tau)$ is a *direct summand* in $V_{\lambda,\boldsymbol{Z}}$, the canonical map

$$V^*_{\lambda,\boldsymbol{Z}} \longrightarrow H^0(X_{\boldsymbol{Z}}(\tau), L_{\lambda,\boldsymbol{Z}})$$

factors through a canonical $\boldsymbol{Z}$-linear map

$$j_{\boldsymbol{Z}}\colon V^*_{\lambda,\boldsymbol{Z}}(\tau) \longrightarrow H^0(X_{\boldsymbol{Z}}(\tau), L_{\lambda,\boldsymbol{Z}});$$

further with this hypothesis we see that j_k is injective for every field k. Thus if $V_{\lambda,\boldsymbol{Z}}(\tau)$ is a direct summand in $V_{\lambda,\boldsymbol{Z}}$, we conclude that $j_{\boldsymbol{Z}}$ identifies $V^*_{\lambda,\boldsymbol{Z}}(\tau)$ as a direct summand of $H^0(X_{\boldsymbol{Z}}(\tau), L_{\lambda,\boldsymbol{Z}})$; if moreover, say for $k=\boldsymbol{Q}$, j_k is an isomorphism (we shall prove this; see Theorem 1 below) it follows that $V^*_{\lambda,\boldsymbol{Z}}(\tau)$ is canonically isomorphic to $H^0(X_{\boldsymbol{Z}}(\tau), L_{\lambda,\boldsymbol{Z}})$ (since the latter is a free module over $\boldsymbol{Z}$). In particular, taking $X_{\boldsymbol{Z}}(\tau)=G_{\boldsymbol{Z}}/B_{\boldsymbol{Z}}$ (for $\tau=w_0$), in which case $V_{\lambda,\boldsymbol{Z}}(w_0)=V_{\lambda,\boldsymbol{Z}}$, we conclude that *in this case* $j_{\boldsymbol{Z}}$ is an isomorphism (using the well-known fact that $j_{\boldsymbol{Q}}$ is an isomorphism for $\tau=w_0$) and j_k is injective for every field k (if one uses the vanishing theorem, cf. Kempf [7], Haboush [3] or Andersen [1], we see that *in this case* j_k *is an isomorphism for every field* k).

§ 2. The work of Demazure

Theorem 1. *Let k be a field of characteristic zero. Then we have the following*:

(i) *There is a canonical isomorphism*

$$H^0(X_k(\tau), L_{\lambda,k}) \xrightarrow{\sim} V_{\lambda,k}(\tau)^* \qquad (\textit{dual of } V_{\lambda,k}(\tau))$$

(ii) $H^i(X_k(\tau), L_{\lambda,k})=0,\ i>0.$

Proof. During the course of this proof *we often drop the subscript k* for the sake of simplicity of notation, e.g., $X_k(\tau)=X(\tau)$ etc.

Let $X(\varphi)$ be a Schubert divisor in $X(\tau)$ moved by a simple root α. Let P_α be the minimal parabolic subgroup of G generated by B and $G_{-\alpha}$ ($G_{\pm\alpha}$ denotes the subgroup of G isomorphic to $\boldsymbol{G}_a$, canonically associated to $\pm\alpha$). Then one knows that (cf. Prop. 1.4, [8]) $X(\tau)$ is stable under P_α. We denote by $SL(2)$ the $SL(2)$ in P_α generated by G_α and $G_{-\alpha}$. Let B_α

denote the Borel subgroup in $SL(2)$ associated to α. Consider now the canonical morphism

$$f_1\colon SL(2)\times X(\tau)\longrightarrow G/B, \qquad (g, x)\longmapsto g\cdot x.$$

Then we see that $\operatorname{Im} f_1 \subset X(\tau)$, so that f_1 gives rise to the map

$$f\colon SL(2)\times X(\tau)\longrightarrow X(\tau), \qquad (g, x)\longmapsto g\cdot x.$$

This map has the property

$$f(gb, b^{-1}x)=f(g, x), \qquad b\in B_\alpha..$$

Hence we get a canonical map

$$\theta\colon SL(2)\times^{B_\alpha} X(\tau)\longrightarrow X(\tau).$$

We have obviously the commutative diagram

$$\begin{tikzcd} SL(2)\times X(\tau) \arrow[r] & X(\tau) \\ SL(2)\times X(\varphi). \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

This yields the following diagram

$$\begin{tikzcd} SL(2)\times^{B_\alpha} X(\tau) \arrow[r, "\theta"] & X(\tau) \\ SL(2)\times^{B_\alpha} X(\varphi) \arrow[u, hook] \arrow[ur, "\Psi"'] & \end{tikzcd}$$

(Ψ denotes the canonical map $SL(2)\times^{B_\alpha} X(\varphi)\to X(\tau)$). Let us identify $\boldsymbol{P}^1$ with $SL(2)/B_\alpha$ and denote by q the canonical map

$$q\colon SL(2)\longrightarrow \boldsymbol{P}^1\simeq SL(2)/B_\alpha$$

so that q is a principal fibre space with structure group B_α. Then $SL(2)\times^{B_\alpha} X(\tau)$ (resp. $SL(2)\times^{B_\alpha} X(\varphi)$) is the *fibre space with fibre* $X(\tau)$ (*resp.* $X(\varphi)$) *associated to* q and we denote the canonical projection maps onto $\boldsymbol{P}^1$ as follows:

$$\pi\colon SL(2)\times^{B_\alpha} X(\tau)=W\longrightarrow \boldsymbol{P}^1$$
$$p\colon SL(2)\times^{B_\alpha} X(\varphi)=Z\longrightarrow \boldsymbol{P}^1.$$

Set $X=X(\tau)$ and $Y=X(\varphi)$. Since $SL(2)$ operates on $X(\tau)=X$ (note that this is not the case for $Y=X(\varphi)$; only B_α operates on Y), it is well-known that the fibration π is *trivial* (in fact, we see that the map δ: $SL(2)\times^{B_\alpha}X\to P^1\times X$ defined by $(g, x)\mapsto(\bar{g}, gx)$, $\bar{g}=gB_\alpha$ admits an inverse, namely $(\bar{h}, x)\to(h, h^{-1}x)$, $\bar{h}=hB_\alpha$ and hence δ is an isomorphism). Identifying W with $P^1\times X$ through δ, we see that the map θ: $W\to X$ can be identified with the canonical projection $P^1\times X\to X$. From this, it follows immediately that if $\mathcal{O}_W$ (resp. $\mathcal{O}_X$) denotes the structure sheaf W(resp. X) we have:

$$(1)\qquad \begin{cases}\theta_*(\mathcal{O}_W)=\mathcal{O}_X\\ R^i\theta_*(\mathcal{O}_X)=0, & i>0.\end{cases}$$

The proof of the theorem is by induction on $\dim X(\tau)$. Hence we can assume that it is true for $X(\varphi)=Y$ (we see also that the theorem is true when $\dim X(\tau)=0$). Thus we suppose that

$$(2)\qquad H^0(X(\varphi), L_\lambda)\xrightarrow{\sim}V_\lambda(\varphi)^* \quad\text{and}\quad H^i(X(\varphi), L_\lambda)=0,\quad i>0.$$

We claim that to prove the theorem it suffices to prove the following:

$$(3)\qquad H^0(Z, M)\xrightarrow{\sim}V_\lambda(\tau)^* \quad\text{and}\quad H^i(Z, M)=0,\quad i>0$$
$$\text{where } M=\psi^*(L_\lambda)$$

This assertion is a consequence of the following ingenious remark of Kempf who used it for a similar purpose (cf. Lemmas 1 and 2, Section 2, [6] or Prop. 2, § 5, [2]).

Lemma 5. *Let W, X be proper schemes over a noetherian ring and L an ample line bundle on X. Suppose that we have a commutative diagram*

$$\begin{tikzcd} W \arrow[r, "\theta"] & X \\ Z \arrow[u, hook, "i"] \arrow[ur, "\psi"'] & \end{tikzcd}\qquad i \text{ a closed immersion}$$

having the following properties:

(a) $\theta_*\mathcal{O}_W=\mathcal{O}_X$

(b) $H^i(W, \theta^*L^n)=0$, $i>0$ *and* $n\gg0$.

(c) *the canonical map*

$$H(W, \theta^*L^n)\longrightarrow H^i(Z, \psi^*L^n)$$

is surjective $\forall i$ *and* $n\gg0$.

Then we have the following:

(i) $\psi_*(\mathcal{O}_Z)=\mathcal{O}_X$,

(ii) $\begin{cases} R^q\psi_*(\mathcal{O}_Z)=0, \ q>0 \\ R^q\theta_*(\mathcal{O}_W)=0, \ q>0 \end{cases}$

(iii) *if F is a vector bundle on X, the canonical maps*

$$H^q(X, F) \longrightarrow H^q(W, \theta^*F); \qquad H^q(X, F) \longrightarrow H^q(Z, \psi^*(F))$$

are isomorphisms for $q \geqslant 0$.

For the proof of Lemma 5, we refer to the papers of Kempf and Demazure quoted above.

Now we shall show that in order to prove the theorem, it suffices to prove (3) above. Let us therefore suppose that (3) is true. *Let us take λ so that the line bundle L_λ on G/B is ample.* We see then that for $L=L_\lambda$, the hypotheses of the above Lemma are satisfied; in fact the hypotheses (a) and (b) are immediate consequences of the fact that θ can be identified with the canonical projection map $X\times \boldsymbol{P}^1 \rightarrow X$; for checking the hypotheses (c) we have only to check it for $i=0$, since (3), in particular, implies that $H^i(Z, \psi^*L_\lambda^n)=0$, $i>0$ and $n \gg 0$. Now consider the canonical maps

$$Z \longrightarrow W \longrightarrow X \longrightarrow G/B \longrightarrow \boldsymbol{P}(V_{n\lambda}^*).$$

The inclusion $X \hookrightarrow \boldsymbol{P}(V_{n\lambda}^*)$ induces a canonical linear map

$$V_{n\lambda}^* \longrightarrow H^0(X(\tau), L_\lambda^n)$$

and we have seen that the image of this map identifies with $V_{n\lambda}^*(\tau)$ (cf. Remark 2). Further, since $\theta\colon Z \rightarrow X$ is dominant, we see also that the canonical linear map

$$H^0(X(\tau), L_\lambda^n) \longrightarrow H^0(Z, \psi^*L_\lambda^n)$$

is injective. These considerations together with the fact that $H^0(Z, \psi^*L_\lambda^n) \simeq (V_{n\lambda}(\tau))^*$ imply that the canonical map $Z \longrightarrow \boldsymbol{P}(V_{n\lambda}^*)$ induces a surjection of $H^0(\boldsymbol{P}(V_{n\lambda}^*))$ onto $H^0(Z, \psi^*L_\lambda^n)$ and since the canonical map $Z \rightarrow \boldsymbol{P}(V_{n\lambda}^*)$ factors through the canonical inclusion $Z \rightarrow W$, we conclude that the canonical map

$$H^0(W, \theta^*L_\lambda^n) \longrightarrow H^0(Z, \psi^*L_\lambda^n)$$

is surjective (in particular for $n \gg 0$). This proves that the hypotheses of (c) is satisfied. Then by the conclusion (iii) of Lemma 5, we see that in order to prove the theorem it suffices to prove (3).

Every fibre of the canonical map $p\colon Z \rightarrow \boldsymbol{P}^1$ is isomorphic to Y and the restriction of M to such a fibre can be identified with $L_\lambda | Y$ (we see easily that the restriction of ψ to the fibres of $Z \rightarrow \boldsymbol{P}^1$ are injective and there is a canonical fibre such that the restriction of ψ to this fibre is an isomorphism onto Y). Because of the inductive hypothesis that $H^i(Y, L_\lambda)=0$, $i>0$ and $H^0(Y, L_\lambda)\simeq V_\lambda(\psi)^*$, we see that

$$H^i(Z, M)\simeq H^i(\boldsymbol{P}^1, p_*(M)), \qquad i>0.$$

Hence to prove (3), we have only to prove that

$$(4)\qquad H^0(\boldsymbol{P}^1, p_*(M))\xrightarrow{\sim} V_\lambda(\tau)^* \quad \text{and} \quad H^i(\boldsymbol{P}^1, p_*(M))=0, \quad i>0.$$

Now it is not difficult to see that the vector bundle $p_*(M)$ on $\boldsymbol{P}^1$ is the vector bundle on $\boldsymbol{P}^1$ associated to the B_α-module $V_\lambda(\varphi)^*$ (since the fibre of the bundle $p_*(M)$ can be identified with $H^0(Y, L_\lambda)\approx V_\lambda(\varphi)^*$, the morphism $Y\to X$ is a B_α-map etc.) Now set

$$E=V_\lambda(\tau), \quad V=V_\lambda(\varphi) \quad \text{and} \quad e=e_\varphi.$$

Recall that we have the following:

$$(5)\quad \begin{cases} \text{(i)} & E \text{ is a } P_\alpha\text{-module.} \\ \text{(ii)} & V=U^+e,\ U_\alpha e=0 \text{ (i.e. } X_\alpha e=0) \text{ and the 1-dimensional linear} \\ & \text{space } ke \text{ is } T\text{-stable. (We see that } V \text{ is a B-module. Note} \\ & \text{that } B \text{ is also the Borel subgroup of } P_\alpha). \\ \text{(iii)} & E=U_{-\alpha}V. \end{cases}$$

Now the crucial result is the following (cf. 2.12, Demazure [2]).

Lemma 6. *Let E be a finite dimensional P_α-module, V a B-submodule and $e\in E$ a weight vector satisfying the conditions* (i), (ii) *and* (iii) *of* (5) *above. Let F be the B-module $(E/V)\otimes\chi$, where χ represents the* 1*-dimensional B_α-module given by the character $t\mapsto t^{-1}$ of its maximal torus $\approx \boldsymbol{G}_m$. Then F is B_α-isomorphic to an $SL(2)$-module.*

For proof we refer to the paper of Demazure quoted above.

Let us denote by $\boldsymbol{E}, \boldsymbol{V}\cdots$ etc. the vector bundles or sheaves on $\boldsymbol{P}^1$ associated to $E, V\cdots$ etc. ($E, V\cdots$ etc. are B_α-modules and the bundles on $\boldsymbol{P}^1$ are those associated to the principal B_α-fibration $SL(2)\to\boldsymbol{P}^1$). We observe that since F is an $SL(2)$-module (not merely a B_α-module), $\boldsymbol{F}$ is a trivial vector bundle on $\boldsymbol{P}^1$. Hence, as an immediate consequence of Lemma 6 we get

Lemma 7. $(\boldsymbol{E}/\boldsymbol{V})^*\simeq\boldsymbol{F}(-1)$, *where $\boldsymbol{F}$ is a trivial vector bundle on $\boldsymbol{P}^1$.*

Now we have the following exact sequence of vector bundles on $\boldsymbol{P}^1$

$$(6)\qquad 0\longrightarrow(\boldsymbol{E}/\boldsymbol{V})^*\longrightarrow\boldsymbol{E}^*\longrightarrow\boldsymbol{V}^*\longrightarrow 0.$$

Since E^* is an $SL(2)$-module, E^* is trivial and $H^0(\boldsymbol{P}^1, \boldsymbol{E}^*) \approx E^*$ and $H^i(\boldsymbol{P}^1, \boldsymbol{E}^*)=0$, $i>0$. By Lemma 7,

$$H^i(\boldsymbol{P}^1, (E/V)^*)=0, \qquad i>0.$$

Hence writing the cohomology exact sequence of (6), we get

$$(7) \qquad H^0(\boldsymbol{P}^1, V^*) \approx E^* \quad \text{and} \quad H^i(\boldsymbol{P}^1, V^*)=0, \quad i>0.$$

We have observed that $V^*=p_*(M)$ and thus we have proved (4), which in turn proves (3). This concludes the proof of Theorem 1.

Corollary (Demazure's Character formula). *As in the theorem, we assume that the ground field k is of characteristic zero. Let $\boldsymbol{Z}[N]$ denote the group ring of the multiplicative group* exp N, *where*

$$\exp N=\{\exp \lambda \mid \lambda \in N\}, \qquad N=\operatorname{Hom}(T, \boldsymbol{G}_m).$$

Let $X_k(\varphi)$ be a Schubert divisor in $X_k(\tau)$ moved by (a simple root) $\alpha(\tau=s_\alpha\varphi)$. Let $L_{\lambda,k}=L_\lambda$ denote the line bundle on G/B, associated to a dominant weight λ as in Theorem 1. Now the characters of the T-modules $H^0(X_k(\varphi), L_\lambda)$ and $H^0(X_k(\tau), L_\lambda)$ are elements of $\boldsymbol{Z}[N]$ and are denoted respectively by $F(\varphi)$ and $F(\tau)$. Let L_α be the linear operator L_α: $\boldsymbol{Z}[N] \to \boldsymbol{Z}[N]$ defined by

$$L_\alpha(\exp \lambda)=\frac{\exp \lambda-\exp s_\alpha(\lambda)}{1-\exp \alpha}, \qquad \lambda \in N.$$

Let M be the operator defined by

$$\begin{aligned} &M_\alpha\colon \boldsymbol{Z}[N] \longrightarrow \boldsymbol{Z}[N], \\ &M_\alpha(\exp \lambda)=(\exp \rho) L_\alpha(\exp(\lambda-\rho)), \\ &(\rho=\textit{half the sum of positive roots}). \end{aligned}$$

Then we have the following iterative character formula:

$$M_\alpha(F(\varphi))=F(\tau) \quad \textit{with} \quad F(w_0)=\exp(-\lambda).$$

(This is equivalent to the formula of Demazure, cf. [2] and [8]).

Proof. Let π be the principal B-fibration

$$\pi\colon P_\alpha \longrightarrow \boldsymbol{P}^1=P_\alpha/B$$

and π_α the restriction of π to the $SL(2)$ in P_α:

$$\pi_\alpha\colon SL(2) \longrightarrow \boldsymbol{P}^1 \qquad (\pi_\alpha\text{-principal } B_\alpha\text{-fibration}).$$

Let W be a finite dimensional B-module and W the associated bundle on $\boldsymbol{P}^1$. Then $H^i(\boldsymbol{P}^1, W)$ is a P_α-module; in particular a T-module and we can talk of its character Char $H^i(\boldsymbol{P}^1, W)$. We write

$$\operatorname{Char} \chi(W) = \operatorname{Char} H^0(\boldsymbol{P}^1, W) - \operatorname{Char} H^1(\boldsymbol{P}^1, W).$$

We observe that if

$$0 \longrightarrow W_1 \longrightarrow W \longrightarrow W_2 \longrightarrow 0$$

is an exact sequence of B-modules, we have

$$\operatorname{Char} \chi(W) = \operatorname{Char} (W_1) - \operatorname{Char} (W_2).$$

Because of Theorem 1 (or rather its proof), we see that

$$\begin{aligned} \operatorname{Char} V_\lambda(\tau) &= \operatorname{Char} H^0(\boldsymbol{P}^1, V_\lambda(\varphi)) \\ &= \operatorname{Char} \chi(V_\lambda(\varphi)), \text{ since } H^1(\boldsymbol{P}^1, V_\lambda(\varphi)) = 0. \end{aligned}$$

Thus it suffices to show that if W is a finite dimensional B-module, then

$$\operatorname{Char} \chi(W) = M_\alpha(\operatorname{Char} W). \tag{1}$$

Since W has a filtration by B-submodules such that its associated graded is a direct sum of 1-dimensional B-modules, by the above additivity property of $W \mapsto \chi(W)$ with respect to exact sequences, we see that it suffices to prove (1) in the case of a 1-dimensional B-module W. *Thus we have only to check that if W is a 1-dimensional B-module (or equivalently a T-module) given by $\mu \in$ Hom $(T, \boldsymbol{G}_m) = N$, then*

$$\chi(W) = M_\alpha(\exp \mu). \tag{2}$$

It is easily checked that (see for example [8])

$$M_\alpha(\exp \mu) = \frac{\exp(\mu - \alpha/2) - \exp s_\alpha(\mu - \alpha/2)}{\exp(-\alpha/2) - \exp \alpha/2}. \tag{3}$$

If μ were only a character of T_α, the above formula (3) is *essentially* the Weyl character formula for $SL(2)$ and (2) follows, interpreted in the sense of characters of T_α-modules (note that the RHS gives the character of the $SL(2)$-irreducible module with lowest weight μ). However, since (2) is an equality of characters of T-modules, a little care has to be taken to deduce (2) from (3). For this purpose, let R be the reductive part of P_α, i.e., $R/T = SL(2)$. We have also $R = P_\alpha/H$, where H is the unipotent radical of P_α. Let then $\mu \in$ Hom $(T, \boldsymbol{G}_m)$. Then the RHS of (3) can be easily seen to be (as a consequence of the usual $SL(2)$ case)

(4) $$\operatorname{Char} H^0(\boldsymbol{P}^1, L_\mu) - \operatorname{Char} H^0(\boldsymbol{P}^1, L_\mu),$$

where L_μ is the line bundle on $\boldsymbol{P}^1$, associated to the principal fibration $R \to \boldsymbol{P}^1$ (where in (4), we take characters of T-modules). Now (2) follows and the proof of Demazure's formula is complete.

Remark 3. Lemmas 6 and 7 have the following $\boldsymbol{Z}$-version, implicit in Demazure [2].

Lemma 6′. *Let E be a $P_{\alpha,\boldsymbol{Z}}$-$\boldsymbol{Z}$-free module of finite rank, V a $B_{\boldsymbol{Z}}$-$\boldsymbol{Z}$-submodule of E which is a direct summand as a $\boldsymbol{Z}$-submodule and $e \in V$ a primitive weight vector* in V (i.e., *$\boldsymbol{Z}\cdot e$ is a $T_{\boldsymbol{Z}}$-$\boldsymbol{Z}$-submodule of V and $\boldsymbol{Z}e$ is a direct summand of V*). *Suppose that the following conditions are satisfied*:

(a) $V = U_{\boldsymbol{Z}}^+ e$, $U_{\alpha,\boldsymbol{Z}} e = 0$

(b) $E = U_{-\alpha,\boldsymbol{Z}} V$.

Let F be the $B_{\alpha,\boldsymbol{Z}}$-$\boldsymbol{Z}$-module $(E/V)\otimes\chi$, where χ represents the $B_{\alpha,\boldsymbol{Z}}$-$\boldsymbol{Z}$-module structure on $\boldsymbol{Z}$, given by the character $t \mapsto t^{-1}$ of its maximal torus $\approx \boldsymbol{G}_{m,\boldsymbol{Z}}$. Then F is $B_{\alpha,\boldsymbol{Z}}$-isomorphic to an $SL_{\boldsymbol{Z}}(2)$-$\boldsymbol{Z}$-module.

As an immediate consequence of Lemma 6′, we get:

Lemma 7′. (i) *$(E/V)^* \simeq \boldsymbol{F}(-1)$, where $\boldsymbol{F}$ is a trivial vector bundle on $\boldsymbol{P}_{\boldsymbol{Z}}^1$ (where $\boldsymbol{E}\cdots$ etc. denote the vector bundles or sheaves on $\boldsymbol{P}_{\boldsymbol{Z}}^1$ associated to $E\cdots$ etc.).*

(ii) (*an immediate consequence of* (i)) *we have*

$$H^0(\boldsymbol{P}_{\boldsymbol{Z}}^1, \boldsymbol{V}^*) \xrightarrow{\sim} E^*$$

$$H^0(\boldsymbol{P}_k^1, (\boldsymbol{V}\otimes k^*)) \xrightarrow{\sim} (E\otimes k)^*, \qquad \forall \textit{ field } k.$$

Suppose now that for *every* dominant λ in $\operatorname{Hom}(T, \boldsymbol{G}_m)$ and *every* $\tau \in W$, we have

(1) $$V_{\lambda,\boldsymbol{Z}}(\tau) \text{ is a } \textit{direct summand} \text{ in } V_{\lambda,\boldsymbol{Z}}.$$

Then because of Lemmas 6′ and 7′ above, the proof of Theorem 1 given above for $\boldsymbol{Z}$ goes through verbatim over *any base field* and we would obtain the following:

(A) $V_{\lambda,\boldsymbol{Z}}(\tau)^* \rightrightarrows H^0(X_{\boldsymbol{Z}}(\tau), L_{\lambda,\boldsymbol{Z}})$ and
$V_{\lambda,k}(\tau)^* \rightrightarrows H^0(X_k(\tau), L_{\lambda,k})$, $\forall$ field k.

(B) $H^i(X_k(\tau), L_{\lambda,k}) = 0$, $i > 0$, $\forall$ field k and
$H^i(X_{\boldsymbol{Z}}(\tau), L_{\lambda,\boldsymbol{Z}}) = 0$, $i > 0$.

(C) $X_k(\tau) =$ base change of $X_{\boldsymbol{Z}}(\tau)$ by $\operatorname{Spec} k \to \operatorname{Spec} \boldsymbol{Z}$.

The property (1) has been conjectured by Demazure in [2] and has been proved for all classical groups as a consequence of standard monomial theory (cf. [9], [11] and Remarks 5 and 6 below).

A careful analysis of the proof of Theorem 1 above leads to some better assertions than in the foregoing discussion; namely:

Theorem 2. (i) *Let λ be a dominant weight such that the line bundle $L_{\lambda,Z}$ on G_Z/B_Z is ample. Then we have*

$$V_{n\lambda,Z}(\tau) \text{ is a direct summand in } V_{n\lambda,Z} \text{ for } n \gg 0 \text{ and } \forall \tau \in W.$$

(ii) $X_k(\tau)=$ *base change of* $X_Z(\tau)$ *by* Spec $k \to$ Spec Z (*i.e., for this property, we need not suppose Demazure's conjecture to hold as we did in Remark* 3 *above*), *i.e., the fibres of* $X_Z(\tau) \to$ Spec Z *are scheme theoretically Schubert varieties*; *in particular they are reduced.*

(iii) $X_k(\tau)$ *is normal* $\forall$ *field* k. *Consequently* $X_Z(\tau)$ *is also normal.*

(iv) $H^i(X_k(\tau), \mathcal{O}_{X_k(\tau)})=0$, $i>0$, $\forall$ *field* k.

(v) *Suppose that for a given dominant weight* μ $V_{\mu,Z}(\tau)$ *is a direct summand in* $V_{\mu,Z} \forall \tau \in W$. *Then we have*

(A) $V_{\mu,Z}(\tau)^* \rightrightarrows H^0(X_Z(\tau), L_{\mu,Z})$ *and*
$V_{\mu,k}(\tau)^* \rightrightarrows H^0(X_k(\tau), L\mu_{\mu,k})$, $\forall$ *field* k

(B) $H^i(X_Z(\tau), L_{\mu,Z})=0$, $i>0$ *and*
$H^i(X_k(\tau), L_{\mu,k})=0$, $i>0$, $\forall$ *field* k.

Proof. (i) Let k be a field and $Z \to k$ the canonical homomorphism. This homomorphism factors as

$$Z \longrightarrow A \longrightarrow k$$

where A is the local ring at $p \in$ Spec Z and k is an extension of the residue field of A. We can find an integer n_0 such that the canonical map

$$(1) \qquad V^*_{n\lambda,Z} \longrightarrow H^0(X_k(\tau), L^n_{\lambda,k})$$

is surjective for $n \geqslant n_0$ and $\forall \tau \in W$ (in the choice of n_0, we fix the field k).

We shall first prove the slightly weaker assertion than (i), namely that

$$(2) \qquad \begin{cases} \text{the canonical inclusion } V_{n\lambda,A}(\tau) \to V_{n\lambda,A} \text{ is an A-direct summand} \\ \text{for } n \geqslant n_0 \text{ (recall } V_{n\lambda,A}(\tau) = V_{n\lambda,Z}(\tau) \otimes_Z A \text{ etc.).} \end{cases}$$

The proof of (2) is done by induction on $l(\tau)$. We choose a Schubert divisor $X_Z(\varphi)$ in $X_Z(\tau)$ moved by a simple root α. Thus we suppose that (2) is true when τ is replaced by φ. Let us now set

(3) $\qquad X_A(\delta)=$ base change of $X_Z(\delta)$ by Spec $A \to$ Spec $\boldsymbol{Z}$, $\delta \in W$.

Now the hypothesis that $V_{n\lambda,A}(\varphi)$ is a direct summand in $V_{n\lambda,A}$ (for $n \geqslant n_0$) implies (see Remark 2) that the canonical linear map

$$j_Z\colon V^*_{n\lambda,Z}(\varphi) \longrightarrow H^0(X_Z(\varphi), L^n_{\lambda,Z})$$

yields an *isomorphism*

$$(4) \qquad j_A\colon V^*_{n\lambda,A}(\varphi) \xrightarrow{\sim} H^0(X_A(\varphi), L^n_{\lambda,A}), \qquad n \geqslant n_0$$

by the base change Spec $A \to$ Spec $\boldsymbol{Z}$. Further, because of the property (1), one sees easily that j_A induces an isomorphism

$$(5) \qquad j_k\colon V^*_{n\lambda,k}(\varphi) \xrightarrow{\sim} H^0(X_k(\varphi), L^n_{\lambda,k}), \qquad n \geqslant n_0.$$

Let us now see carefully how the above proof of Theorem 1 carries over *when the base is A or k.* By the induction hypothesis $V_{n\lambda,A}(\varphi)$ is a direct summand in $V_{n\lambda,A}$ for $n \geqslant n_0$; it follows a fortiori that

(6) $\qquad V_{n\lambda,A}(\varphi)$ is a direct summand in $V_{n\lambda,A}(\tau)$, $\qquad n \geqslant n_0$.

Now Lemmas 6′ and 7′ obviously generalize over A. Since one has the property (6), it follows that we have a canonical isomorphism

$$(7) \qquad j_A\colon V^*_{n\lambda,A}(\tau) \xrightarrow{\sim} H^0(X_A(\tau), L^n_{\lambda,A}), \qquad n \geqslant n_0$$

and what is important (see (ii) of Lemma 7) is that this induces a canonical isomorphism

$$(8) \qquad j_k\colon V^*_{n\lambda,k}(\tau) \xrightarrow{\sim} H^0(X_k(\tau), L^n_{\lambda,k}), \qquad n \geqslant n_0.$$

We claim that these properties imply that $V_{n\lambda,A}(\tau)$ is a direct summand in $V_{n\lambda,A}$ for $n \geqslant n_0$ (see the discussion in Remark 2); for, as we saw in § 1 and Remark 2, the image of the canonical map

$$(9) \qquad V^*_{n\lambda,k} \longrightarrow H^0(X_k(\tau), L^n_{\lambda,k})$$

can be canonically identified with (Im $V_{n\lambda,k}(\tau))^*$, where Im $V_{n\lambda,k}(\tau)$ is the image under the canonical map $V_{n\lambda,k}(\tau) \to V_{n\lambda,k}$ and because of (8) it follows that (9) factors as

$$\begin{tikzcd} V^*_{n\lambda,k} \arrow[r] \arrow[d, dashed] & H^0(X_k(\tau), L^n_{\lambda,k}). \\ V^*_{n\lambda,k}(\tau) \arrow[ur, hook] & \end{tikzcd}$$

Thus the canonical mapping

$$V_{n\lambda,k}(\tau) \longrightarrow V_{n\lambda,k}$$

is *injective* for $n \geqslant n_0$ so that $V_{n\lambda,A}(\tau)$ is a direct summand in $V_{n\lambda,A}$ $(n \geqslant n_0)$.

This concludes the proof of (2) which is slightly a weaker version of (i). We shall deduce the stronger version after proving (ii).

(ii) Since $X_A(\tau) \to \operatorname{Spec} A$ is *flat*, we see that for $n \gg 0$

$$H^0(X_A(\tau) \times_{\operatorname{Spec} A} \operatorname{Spec} k, L^n_{\lambda,k}) = H^0(X_A(\tau), L^n_{\lambda,A}) \otimes_A k.$$

Thus the properties (7) and (8) above imply that the canonical restriction homomorphism

$$(10) \qquad H^0(X_A(\tau) \times_{\operatorname{Spec} A} \operatorname{Spec} k, L^n_{\lambda,k}) \longrightarrow H^0(X_k(\tau), L^n_{\lambda,k})$$

is an isomorphism for $n \gg 0$. We know that

$$X_k(\tau) = (X_A(\tau) \times_{\operatorname{Spec} A} \operatorname{Spec} k)_{\mathrm{red}}.$$

Now if X is a projective scheme and L is an *ample* line bundle on X such that the canonical homomorphism

$$H^0(X, L^n) \longrightarrow H^0(X_{\mathrm{red}}, L^n)$$

is an isomorphism for $n \gg 0$, we see easily that $X = X_{\mathrm{red}}$. Hence we see that (10) implies that

$$X_k(\tau) = X_A(\tau) \times_{\operatorname{Spec} A} \operatorname{Spec} k.$$

This proves the assertion (ii).

Now we can prove the assertion (i). Since the fibres of $X_{\boldsymbol{Z}}(\tau) \to \operatorname{Spec} \boldsymbol{Z}$ are *reduced*, we see easily that we can find n_0, such that for $n \geqslant n_0$ and *any* field k, the canonical map

$$V^*_{n\lambda,\boldsymbol{Z}} \longrightarrow H^0(X_k(\tau), L^n_{\lambda,k})$$

is surjective $\forall \tau \in W$. Now the proof given above for proving (2) goes through for proving (i). Thus we have proved the assertions (i) and (ii) of Theorem 2.

(iii) Let us look at the proof of Theorem 1 *over the base k*. Because of (i), we see that the canonical morphism (with the notations of Theorem 1, with sometimes a subscript k thrown in)

$$\Psi : Z = SL(2) \times^{B_\alpha} X_k(\varphi) \longrightarrow X_k(\tau)$$

has the property

(11) $$\Psi_*(\mathcal{O}_Z)=\mathcal{O}_{X_k(\tau)}.$$

Now the normality of $X_k(\tau)$ follows by induction on $l(\tau)$. We can suppose that $X_k(\varphi)$ is normal. Then Z is the fibre space with fibre $X_k(\varphi)$ associated to the principal fibration $SL(2)\to \boldsymbol{P}^1$. It follows that Z is normal. Now (11) implies that $X_k(\tau)$ is normal.

(iv) If Ψ is as in (iii) above, we have also

(12) $$R^q\Psi_*(\mathcal{O}_Z)=0, \qquad q>0.$$

Then the assertion that $H^i(X_k(\tau), \mathcal{O}_{X_k(\tau)})=0$, $i>0$ follows by induction on $l(\tau)$ and using (12) above. The proof uses the fibration $Z\to \boldsymbol{P}^1$ with fibre $X_k(\varphi)$ as in (iii) above.

(v) The proof of (v) of Theorem 2 results immediately from the assertion (iii) in the proof of Theorem 1 (using of course (i)).

This completes the proof of Theorem 2.

Remark 4. In the assertion (i) of Theorem 2, one has supposed that λ is such that $L_{\lambda,Z}$ is *ample* on G_Z/B_Z. For *any* dominant weight λ, $L_{\lambda,Z}$ is ample on G_Z/P_Z, where P_Z is the isotropy subgroup scheme of $\bar{e}_\lambda \in \boldsymbol{P}(V^*_{\lambda,Z})$ (corresponding to the highest weight vector $e_\lambda \in V_{\lambda,Z}$). If one works with G_Z/P_Z instead of G_Z/B_Z one sees that the assertion (i) holds for any dominant weight λ. One observes also that *we have an analogue of Theorem 2 for Schubert schemes in G_Z/P_Z, where P_Z is any parabolic subgroup scheme of G_Z.*

Remark 5. The assertion (iv) of Theorem 2 (see also Remark 4) states, in particular, that if λ is a *given* dominant weight such that

(1) $V_{\lambda,Z}(\tau)$ is a direct summand of $V_{\lambda,Z}$ $\forall\tau\in W$, then

(2) $V_{\lambda,Z}(\tau)^*\longrightarrow H^0(X_Z(\tau), L_{\lambda,Z})$ is surjective $\forall\tau\in W$.

Now (2) implies that the canonical map

(3) $V^*_{\lambda,Z}\longrightarrow H^0(X_Z(\tau), L_{\lambda,Z})$ is *surjective* for every $\tau\in W$.

or equivalently (see Remark 1).

(3)′ $H^0(G_Z/B_Z, L_{\lambda,Z})\longrightarrow H^0(X_Z(\tau), L_{\lambda,Z})$ is *surjective* for every $\tau\in W$.

Conversely, we claim that (3) or (3)′ implies (1). For this we observe first that the canonical mapping

$$r\colon H^0(X_Z(\tau), L_{\lambda,Z})\otimes_Z k\longrightarrow H^0(X_k(\tau), L_{\lambda,k})$$

is *injective* $\forall$ field k (we use the fact that $X_k(\tau)$ = base change of $X_Z(\tau)$ by Spec $k \to$ Spec Z, which is the assertion (ii) of Theorem 2). Recall also that the image of the canonical mapping

$$V_{\lambda,k}^* \longrightarrow H^0(X_k(\tau), L_{\lambda,k})$$

can be identified with (Im $V_{\lambda,k}(\tau))^*$ (see Remark 2). Now (3) or (3)′ (together with the injectivity of r above) implies that the dimension of the k-vector space (Im $V_{\lambda,k}(\tau))^*$ or equivalently the dimension of the image of the canonical map

$$V_{\lambda,Z}(\tau)\otimes k = V_{\lambda,k}(\tau) \longrightarrow V_{\lambda,k} = V_{\lambda,Z}\otimes k$$

is independent of k. This implies that $V_{\lambda,Z}(\tau)$ is a direct summand in $V_{\lambda,Z}$.

§ 3. The basis theorem in standard monomial theory for fundamental representations

We shall now suppose that G_Z is *simple*. Let P_Z be a *maximal parabolic* subgroup scheme, *associated to a fundamental weight* ω or the corresponding simple root α (cf. [8]). One way of saying that P_Z is *associated to the fundamental weight* ω is that P_Z is the isotropy subgroup scheme at $\bar{e} \in \boldsymbol{P}(V_{\omega,Z}^*)$, where $\bar{e}$ is associated to the highest weight vector $e \in V_{\omega,Z}$ (to prove the equivalence of this with the usual definition one has to use Lemma 3 due to Deodhar).

We say that P_Z (resp. $p_{R'}$ ω) is of *classical type* if

$$|\langle \omega, \alpha^* \rangle| = |2(\omega, \alpha)/(\alpha, \alpha)| \leqslant 2 \qquad \forall \text{ root } \alpha.$$

If G_Z is a *classical group*, we see that every maximal parabolic subgroup scheme P_Z of G_Z is of classical type (in fact the converse is also true). Further, for an arbitrary G_Z as above, there is always a maximal parabolic subgroup scheme P_Z of classical type and it is not difficult to write the list of all the P_Z of classical type in G_Z (cf. [9]).

For $\tau \in W/W_P$ (W_P = Weyl group of P_Z), let us denote by $[X_k(\tau)]$ (k field), the element of the Chow ring Ch (G_k/P_k) of G_k/P_k, determined by the Schubert variety $X_k(\tau)$ in G_k/P_k. Let H_k denote the unique codimension one Schubert subvariety of G_k/P_k. It can be shown that

$$[X_k(\tau)]\cdot[H_k] = \sum_i d_i[X_k(\tau)], \qquad d_i > 0$$

where $\cdot$ denotes multiplication in Ch (G_k/P_k) and τ_i runs over the set of all $\lambda \in W/W_P$, such that $X_k(\lambda)$ is of codimension one in $X_k(\tau)$. We call d_i the *multiplicity* of $X_k(\tau_i)$ in $X_k(\tau)$. If P_Z is of classical type, we see that

$d_i \leqslant 2$, using a formula of Chevalley (cf. [2]). It is possible that d_i is always $=1$ for a P_Z and the corresponding fundamental weight is characterized by the property of being *minuscule* (cf. [12]).

Note that we have a canonical partial order in W/W_P, namely

$$\tau_1 \geqslant \tau_2 \Leftrightarrow X_k(\tau_1) \supseteq X_k(\tau_2).$$

A pair of elements (τ, φ) in W/W_P is called an *admissible pair* (we suppose that P_Z is of *classical type*), if either $\tau = \varphi$ (in which case, it is called a *trivial admissible pair*) or $\tau \neq \varphi$ and there exist $\{\tau_i\}$, $1 \leqslant i \leqslant s$, $\tau_i \in W/W_P$, such that

(i) $\tau = \tau_1 \geqslant \tau_2 \geqslant \cdots \geqslant \tau_s = \varphi$

(ii) $X(\tau_i)$ is of codimension one in $X(\tau_{i-1})$, $2 \leqslant i \leqslant s$, and the multiplicity of $X(\tau_i)$ *in* $X(\tau_{i-1})$ is exactly 2.

Note that in the minuscule case, every admissible pair is trivial.

Theorem 3. *Suppose that P_Z is of classical type. Then there exist elements $\{P(\tau, \varphi)\}$ of $H^0(G_Z/P_Z, L_{\omega,Z})$, indexed by admissible pairs in W/W_P, such that*:

(i) *$P(\tau, \varphi)$ is a weight vector (under T_Z) of weight*

$$-(\tau(\omega)+\varphi(\omega))/2.$$

(ii) *Let $p(\tau, \varphi)$ be the canonical image of $P(\tau, \varphi)$ in $H^0(G_k/P_k, L_{\omega,k})$ (k being any field). Then the restriction of $p(\tau, \varphi)$ to $X_k(\theta)$ is not identically zero, if and only if $\theta \geqslant \tau$.*

(iii) *$\forall \theta \in W$, the elements $P(\tau, \varphi)$ (resp. $p(\tau, \varphi)$), $\theta \geqslant \tau$, form a basis of $H^0(X_Z(\theta), L_{\omega,Z})$ (resp. $H^0(X_k(\theta), L_{\omega,k})$ (to be precise one has to take the canonical images of $P(\tau, \varphi)$ and $p(\tau, \varphi)$).*

Now we will deduce Theorem 3 as a consequence of the following dual version.

Theorem 3′. *Suppose that P_Z is of classical type. Then there is a basis $\{Q(\tau, \varphi)\}$ of $V_{\omega,Z}$ indexed by (distinct) admissible pairs in W/W_P, such that*

(i) *$Q(\tau, \varphi)$ is a weight vector of weight*

$$(\tau(\omega)+\varphi(\omega))/2.$$

(ii) *If $W(\theta)$ denotes the $\mathbf{Z}$-submodule of $V_{\omega,Z}$ spanned by $Q(\tau, \varphi)$ such that $\theta \geqslant \tau$, then $W(\theta)$ is a basis of $V_{\omega,Z}(\theta)$. In particular, $V_{\omega,Z}(\theta)$ is a direct summand in $V_{\omega,Z}$ for every $\theta \in W/W_P$.*

Now Theorem 3 is an immediate consequence of Theorem 3′. The assertion (iii) of Theorem 3 is a consequence of the direct summand property in the assertion (ii) of Theorem 3 and the assertion (iv) of Theorem 2 (see also Remark 4). We define $\{P(\tau, \varphi)\}$ as the elements of the basis of $V^*_{\omega,Z} \approx H^0(G_Z/P_Z, L_{\omega,Z})$, dual to the basis elements of $\{Q(\tau, \varphi)\}$. We see that the assertions (i) and (ii) of Theorem 3 are respectively equivalent to the assertions (i) and (ii) of Theorem 3′.

The proof of Theorem 3′ is by induction on $l(\tau)$. When $l(\tau)=0$, it is immediate. Let then $X_k(\tau_1)$ be a Schubert divisor in $X_k(\tau)$, moved by a simple root α. We have $\tau=s_\alpha\tau_1$. We now define the following:

(i) α-weight of $\varphi=m(\varphi)$ ($\varphi \in W$ or rather $W/W_P)=-\langle\varphi(\omega), \alpha^*\rangle$ $=-2(\varphi(\omega), \alpha)/(\alpha, \alpha)$

(ii) $m(\varphi)=-\langle\varphi_1(\omega)+\varphi_2(\omega), \alpha^*\rangle/2$, where $\varphi=(\varphi_1, \varphi_2)$ is an admissible pair in W/W_P.

(iii) $I_{\tau_1}=$set of admissible pairs on $X_k(\tau_1)$, i.e., admissible pairs $\varphi=(\varphi_1, \varphi_2)$ such that $\tau_1 \geqslant \varphi_1$.

(iv) $I_{\tau_1}=\{\varphi \in I \mid m(\varphi)=0\}$

$I^-_{\tau_1}=\{\varphi \in I \mid m(\varphi)<0 \text{ and } \tau_1 \geqslant s_\alpha\varphi_1\}$

$I^+_{\tau_1}=\{\varphi \in I \mid m(\varphi)>0\}$

$I^e_{\tau_1}=\{\varphi \in I \mid m(\varphi)<0 \text{ and } \tau_1 \ngeq s_\alpha\varphi_1\}$.

Now Theorem 3′ is an immediate consequence of the following lemma (virtually a reproduction of Lemma 5.5, *G/P*-IV, [9]) which gives the main inductive step in its proof:

Lemma 8. *Suppose that the* **Z**-*module* $V_{\omega,Z}(\tau_1)$ *has a basis* $\{Q_{\lambda,\mu}\}$ *indexed by the (distinct) admissible pairs on* $X(\tau_1)$ (*i.e.*, $\tau_1 \geqslant \lambda$) *having the following properties*;

(i) $Q_{\lambda,\mu}$ *is a weight vector of weight*

$$(\lambda(\omega)+\mu(\omega))/2.$$

(ii) *If* $W(\theta)$ *denotes the* **Z**-*submodule of* $V_{\omega,Z}(\tau_1)$ *spanned by all* $Q_{\lambda,\mu}$ *such that* $\tau_1 \geqslant \theta \geqslant \lambda$, *then*

$$W(\theta)=V_{\omega,Z}(\theta).$$

Let now $(\lambda_1, \mu_1) \in I^e_{\tau_1}$. Then we define

(a) $Q_{\lambda,\mu}=X_{-\alpha}Q_{\lambda_1,\mu_1}$; $(\lambda, \mu)=(s_\alpha\lambda_1, s_\alpha\mu_1)$ if $(\lambda_1, \mu_1)=-1$

(b) *If* $m(\lambda_1, \mu_1)=-2$, we set

$Q_{\lambda,\mu}=(X^2_{-\alpha}/2)Q_{\lambda_1,\mu_1}$; $(\lambda, \mu)=(s_\alpha\lambda_1, s_\alpha\mu_1)$,

$Q_{\lambda,\mu}=X_{-\alpha}Q_{\lambda_1,\mu_1}$, $(\lambda, \mu)=(s_\alpha\lambda_1, \mu_1)$.

We call the elements $Q_{\lambda,\mu}$ defined in (a), (b) above as the *new basis elements*. Consider the set of all $Q_{\lambda,\mu}$ above, i.e., the given basis elements of $V_{\omega,Z}(\tau_1)$ $(=W(\tau_1))$ together with the new basis elements. Then we claim that we have the following:

(A) $\{Q_{\lambda,\mu}\}$ give a basis of the $\boldsymbol{Z}$-module $V_{\omega,\boldsymbol{Z}}(\tau)$ indexed by the (distinct) admissible pairs on $X(\tau)$,

(B) $Q_{\lambda,\mu}$ is a weight vector of weight

$$(\lambda(\omega)+\mu(\omega))/2$$

(C) if $W(\theta)$ denotes the $\boldsymbol{Z}$-submodule of $V_{\omega,\boldsymbol{Z}}(\tau)$ spanned by all $Q_{\lambda,\mu}$ such that $\tau \geqslant \theta \geqslant \lambda$, then

$$W(\theta)=V_{\omega,\boldsymbol{Z}}(\theta).$$

(We call these $Q_{\lambda,\mu}$ as the basis elements of $W(\theta)$.)

Proof. Since we will always be working with the dominant weight ω, in the proof let us drop the subscript ω, i.e., we write

$$V_{\boldsymbol{Z}}(\tau) \text{ instead of } V_{\omega,\boldsymbol{Z}}(\tau) \text{ etc.}$$

We also write $I, I^0, \cdots$ for $I_{\tau_1}, I^0_{\tau_1} \cdots$. There are two important facts in the proof of this lemma which we do not prove here but only refer to G/P-IV, [9]. These are the following:

(i) An admissible pair (λ, μ) on $X(\tau)$ which is *not* on $X(\tau_1)$, i.e., $\tau_1 \ngeq \lambda$ is precisely of the form given in (a) and (b) of the statement of the above lemma (cf. (c), Lemma 3.11, G/P-IV, [9]).

(ii) For $k=\boldsymbol{Q}$, we have

$$\operatorname{Char} V_k(\tau)=\sum_{(\lambda,\mu)\in I} \exp\left((\lambda(\omega)+\mu(\omega))/2\right)$$

or equivalently

$$\operatorname{Char} H^0(V_k(\tau), L_k)=\sum_{(\lambda,\mu)\in I} \exp\left(-(\lambda(\omega)+\mu(\omega))/2\right)$$

since one has (cf. Theorem 2)

$$H^0(V_k(\tau), L_k)=V_k(\tau)^*.$$

In particular, we see that

$$\dim V_k(\tau)=\# I.$$

For this we refer to Theorem 4.1, G/P-IV, [9]. Note that as an immediate consequence of (ii), if $v \in V_{\boldsymbol{Z}}(\tau)$ is *any* weight vector, of weight χ

$$(1) \qquad |m(\chi)| \leqslant 2.$$

The above two facts imply that in the statement of Lemma 8, the $\{Q_{\lambda,\mu}\}$ are indexed by the (distinct) admissible pairs on $X_k(\tau)$; further, in order to prove that $\{Q_{\lambda,\mu}\}$ form a basis of $V_{\boldsymbol{Z}}(\tau)$, it suffices to prove that $\{Q_{\lambda,\mu}\}$ *generate* $V_{\boldsymbol{Z}}(\tau)$ as a $\boldsymbol{Z}$-module, for by assertion (ii), the $Q_{\lambda,\mu}$ would be linearly independent over $\boldsymbol{Q}$. Note also that the assertion (B) of Lemma 8 is immediate. Thus we have only to prove the assertion (C) and that $\{Q_{\lambda,\mu}\}$ generate $V_{\boldsymbol{Z}}(\tau)$ as a $\boldsymbol{Z}$-module.

Let V_1 be the $\boldsymbol{Z}$-submodule of $V_{\boldsymbol{Z}}(\tau_1)$ spanned by all $Q_{\lambda,\mu}$ such that $(\lambda, \mu) \in I^e$ and V_2 the submodule spanned by all $Q_{\lambda,\mu}$ such that $(\lambda, \mu) \notin I^e$, i.e., $(\lambda, \mu) \in I^0 \cup I^+ \cup I^-$. We have therefore $V_{\boldsymbol{Z}}(\tau_1) = V_1 + V_2$. We claim that

$$V_{\boldsymbol{Z}}(\tau) = U_{-\alpha,\boldsymbol{Z}} V_1 + V_2. \tag{2}$$

On account of Lemma 2, to prove (2), it suffices to prove

(2)′ (λ, μ) admissible pair on $X(\tau_1)$ and $(\lambda, \mu) \notin I^e$, then $U_{-\alpha,\boldsymbol{Z}} \cdot Q_{\lambda,\mu} \subseteq U_{-\alpha,\boldsymbol{Z}} V_1 + V_2$.

Suppose now that $(\lambda, \mu) \in I^+$, i.e., $m(\lambda, \mu) > 0$. Then we claim that $X_{-\alpha} Q_{\lambda,\mu} = 0$. For otherwise, the weight of $X_{-\alpha} Q_{\lambda,\mu}$ is $(\lambda(\omega) + \mu(\omega))/2 - \alpha$ and we have

$$\langle (\lambda(\omega) + \mu(\omega))/2 - \alpha, \alpha^* \rangle = -m(\lambda, \mu) - 2 \leqslant -3.$$

This easily leads to a contradiction (see (1) above). Thus (2)′ follows when $(\lambda, \mu) \in I^+$.

In the following discussion, we make use of the following simple facts (cf. Lemma 1.2, G/P-IV, [9]):

(i) $m(\varphi) < 0$ iff $X_k(\varphi)$ is a Schubert divisor in $X_k(s_\alpha \varphi)$ moved by α.

(ii) $m(\varphi) > 0$ iff $X_k(s_\alpha \varphi)$ is a Schubert divisor in $X_k(\varphi)$ moved by α or equivalently $X_k(\varphi)$ is stable under the action of the $SL(2)$ corresponding to α.

(iii) $m(\varphi) = 0$ iff $X_k(\varphi) = X_k(s_\alpha \varphi)$.

Suppose now that (λ, μ) is an admissible pair on $X(\tau_1)$ such that $\tau_1 \geqslant s_\alpha \lambda$. (For example $(\lambda, \mu) \in I^-$. To see this use (3) above). Then we have

$$\text{either} \quad \lambda \geqslant s_\alpha \lambda \quad \text{or} \quad \tau_1 \geqslant \theta = s_\alpha \lambda \geqslant \lambda.$$

If $\lambda \geqslant s_\alpha \lambda$, then $X_{\boldsymbol{Z}}(\lambda)$ is stable under the $SL(2)$ corresponding to α and then we have

$$U_{-\alpha,\boldsymbol{Z}} V_{\boldsymbol{Z}}(\lambda) \subseteq V_{\boldsymbol{Z}}(\lambda).$$

If $\tau_1 \geqslant \theta = s_\alpha \lambda > \lambda$, we see again that $X_{\boldsymbol{Z}}(\theta)$ is stable under the $SL(2)$ corresponding to α so that again

$$U_{-\alpha,Z}V_Z(\theta)\subseteq V_Z(\theta).$$

We have of course

$$V_Z(\lambda)\subseteq V_Z(\tau_1) \quad \text{and} \quad V_Z(\theta)\subset V_Z(\tau_1).$$

Thus we get

$$U_{-\alpha,Z}Q_{\lambda,\mu}\subseteq V_Z(\tau_1)=V_1+V_2$$

which proves (2)′ in the case $\tau_1\geqslant s_\alpha\lambda$.

Thus to prove (2)′, it remains only to prove it for the case $(\lambda, \mu)\in I^0$. In this case, set $u=X_\alpha Q_{\lambda,\mu}$. We see that

$$u=X_\alpha Q_{\lambda,\mu}\in V_Z(\tau_1)$$

since $V_Z(\tau_1)$ is stable under B_Z. If now $u=0$, then $Q_{\lambda,\mu}$ is a highest weight vector for the $SL(2)$ corresponding to α. Now one checks that $m(\lambda, \mu)=$ weight with respect to the 1-dimensional torus T_α of this $SL(2)$. Since $m(\lambda, \mu)=0$, it follows by standard $SL(2)$ theory that $X_{-\alpha}Q_{\lambda,\mu}=0$ and then (2)′ is immediate. Thus we have only to consider the case when $u\neq 0$. We now claim that

$$(X^2_{-\alpha}/2)u=X_{-\alpha}Q_{\lambda,\mu}. \tag{4}$$

Let us first show how (4) completes the proof of (2)′. Because of (4), we have only to show that

$$(X^2_{-\alpha}/2)u\in U_{-\alpha,Z}V_1+V_2. \tag{5}$$

From the fact that $m(\lambda, \mu)=0$, we see that the weight of u with respect to T_α is 2 ($T_\alpha=1$ dimensional torus in $SL(2)$). Hence we can write

$$\begin{cases} u=\sum_{(\theta,\sigma)} a_{\theta,\sigma}Q_{\theta,\sigma}, \qquad a_{\theta,\sigma}\in Z \\ (\theta, \sigma) \text{ admissible pair on } X(\tau_1) \text{ with } m(\theta, \sigma)=-2. \end{cases} \tag{6}$$

Thus if (θ, σ) is as in (6), either $(\theta, \sigma)\in I^-$ or I^e. Hence by our discussion above

$$U_{-\alpha,Z}Q_{\theta,\sigma}\subseteq U_{-\alpha,Z}V_1+V_2.$$

This proves (5) and hence (2)′ would follow since $H_\alpha Q_{\lambda,\mu}=0$ (using $m(\lambda, \mu)=0$).

Now to prove (4), we proceed as follows: We have

$$X_{-\alpha}X_{\alpha}Q_{\lambda,\mu}=X_{\alpha}X_{-\alpha}Q_{\lambda,\mu}$$

since $H_{\alpha}Q_{\lambda,\mu}=0$ (using $m(\lambda, \mu)=0$), H_{α} being the usual element $[X_{\alpha}, X_{-\alpha}]$ in $SL(2)$ theory. We have then

$$\begin{aligned}(X^2_{-\alpha}/2)u&=(X_{-\alpha}/2)(X_{-\alpha}X_{\alpha}Q_{\lambda,\mu})\\&=(1/2)X_{-\alpha}(X_{\alpha}X_{-\alpha}Q_{\lambda,\mu})\\&=((1/2)X_{-\alpha}X_{\alpha})(X_{-\alpha}Q_{\lambda,\mu})\\&=(1/2)X_{\alpha}X_{-\alpha}X_{-\alpha}Q_{\lambda,\mu}+X_{-\alpha}Q_{\lambda,\mu},\end{aligned}$$

since we have

$$X_{-\alpha}X_{\alpha}/2=X_{\alpha}X_{-\alpha}/2+H_{\alpha}/2 \quad\text{and}\quad (H_{\alpha}/2)(X_{-\alpha}Q_{\lambda,\mu})=X_{-\alpha}Q_{\lambda,\mu}.$$

We observe now that

$$X^2_{-\alpha}Q_{\lambda,\mu}=0$$

for otherwise, if χ=weight of $X^2_{-\alpha}Q_{\lambda,\mu}$, one has

$$|\langle\chi, \alpha^*\rangle|=4$$

which leads to a contradiction of (1) above. Now (2) follows.

We now claim that

$$(7) \qquad X_{\alpha}Q_{\lambda,\mu}=0 \qquad \text{if } (\lambda, \mu)\in I^e.$$

This is immediate, for if (7) holds

$$\begin{aligned}&\text{weight of } X_{\alpha}Q_{\lambda,\mu}=\alpha+(\lambda(\omega)+\mu(\omega))/2 \qquad \text{and}\\&\langle(\lambda(\omega)+\mu(\omega))/2, \alpha^*\rangle=-m(\lambda, \mu)+2\geqslant 3\end{aligned}$$

which leads to a contradiction of (1). Now from (7), by standard $SL(2)$ theory, it follows that $U_{-\alpha,Z}Q_{\lambda,\mu}$ is spanned as a Z-module by $X_{-\alpha}Q_{\lambda,\mu}$ if $m(\lambda, \mu)=-1$ or by $X_{-\alpha}Q_{\lambda,\mu}$ and $(X^2_{-\alpha}/2)Q_{\lambda,\mu}$ if $m(\lambda, \mu)=-2$. This observation together with (2) above show that $V_Z(\tau)$ is spanned by $Q_{\lambda,\mu}$ as a Z-module.

Thus to conclude the proof of Lemma 8, it remains to prove the assertion (C). Consider first the case when $\tau_1\geqslant\theta$. From the definition of the new basis elements, in this case it is clear that $W(\theta)=V_Z(\theta)$. We have only to consider the case $\tau_1\not\geqslant\theta$. Then we have the simple fact that if $\theta_1=s_{\alpha}\theta$, we have $\tau_1\geqslant\theta_1$ (this simple fact is however the basic idea for the inductive arguments). Hence we have $W(\theta_1)=V_Z(\theta_1)$. Hence we get

$$(8) \qquad U_{-\alpha,Z}W(\theta_1)=V_Z(\theta) \qquad \text{(by Lemma 1)}.$$

Now (8) implies that

$$V_Z(\theta)=U_{-\alpha,Z}V_1(\theta)+W(\theta_1)$$

where $V_1(\theta_1)$ is the $\boldsymbol{Z}$-submodule of $V_Z(\theta)$ generated by Q_{λ_1,μ_1} such that $(\lambda_1, \mu_1) \in I^e_{\theta_1}$ (the proof being similar to that of (A) given above). Suppose now that $(\lambda_1, \mu_1) \in I^e_{\theta_1}$ is such that $(\lambda_1, \mu_1) \in I^e_{\tau_1}$. Then by the way in which the new basis elements (relative to τ_1 and τ) have been defined, we see that $U_{-\alpha,Z}Q_{\lambda_1,\mu_1} \subseteq W(\theta)$. Suppose on the other hand that $(\lambda_1, \mu_1) \in I^e_{\theta_1}$ is such that $(\lambda_1, \mu_1) \notin I^e_{\tau_1}$. This means that $\tau_1 \geqslant \lambda$, where $\lambda = s_\alpha \lambda_1$ and $X_k(\lambda_1)$ is a Schubert divisor in $X_k(\lambda)$ moved by α. By hypothesis, we have $V_Z(\lambda) = W(\lambda)$. Now $X_k(\lambda)$ is stable under the $SL(2)$ corresponding to α, so that we have

$$U_{-\alpha,Z}V_Z(\lambda) \subseteq V_Z(\lambda).$$

Further

$$Q_{\lambda_1,\mu_1} \in V_Z(\lambda_1)=W(\lambda_1) \subseteq V_Z(\lambda)=W(\lambda) \subseteq W(\theta).$$

Hence $U_{-\alpha,Z}Q_{\lambda_1,\mu_1} \subseteq W(\theta)$. Thus we conclude that $U_{-\alpha,Z}V_1(\theta_1) \subseteq W(\theta)$ and since $W(\theta_1) \subseteq W(\theta)$, it follows that $V_Z(\theta) \subseteq W(\theta)$. Thus to show that $V_Z(\theta) = W(\theta)$, it remains to prove that $W(\theta) \subseteq V_Z(\theta)$. Let then $Q_{\lambda,\mu}$ be a basis element of $W(\theta)$. Suppose that $\tau_1 \geqslant \lambda$. Then by hypothesis $Q_{\lambda,\mu} \in V_Z(\lambda)$ (since in this case $V_Z(\lambda) = W(\lambda)$). We have $V_Z(\lambda) \subseteq V_Z(\theta)$ since $\theta \geqslant \lambda$ Thus we have only to consider the case, $\tau_1 \not\geqslant \lambda$. Then if $\lambda_1 = s_\alpha \lambda$, $X_k(\lambda_1)$ is a moving divisor in $X_k(\lambda)$ moved by α and $\theta_1 \geqslant \lambda_1$. Further, by the construction of the new basis elements, we have

$$Q_{\lambda,\mu} \in U_{-\alpha,Z}Q_{\lambda_1,\mu_1}; \ (\lambda_1, \mu_1) \in I^e_{\tau_1}$$

where $Q_{\lambda_1,\mu_1} \in W(\theta_1)$. Now the property (8) implies that $Q_{\lambda,\mu} \in V_Z(\theta)$. Thus we see that $W(\theta) \subseteq V_Z(\theta)$. This concludes the proof of Lemma 8.

This completes the proof of Theorem 3.

For the sake of completeness, we shall now state the main theorem of standard monomial theory on G_Z/P_Z.

We call a *standard monomial of length m* on $X_Z(\tau)$, $\tau \in W/W_P$, (resp. $X_k(\tau)$, k field) an element of $H^0(X_Z(\tau), L^m_Z)$ (resp. $H^0(X_k(\tau), L^m_k)$), represented by

$$P(\tau_1, \varphi_1)P(\tau_2, \varphi_2)\cdots P(\tau_m, \varphi_m), \qquad (\text{resp. } p(\tau_1, \varphi_1)\cdots p(\tau_m, \varphi_m))$$

such that

$$\tau \geqslant \tau_1 \geqslant \varphi_1 \geqslant \tau_2 \geqslant \varphi_2 \cdots \geqslant \tau_m \geqslant \varphi_m.$$

We have then (cf. G/P-IV, [9]):

Theorem 4. *Distinct standard monomials of length m on $X_Z(\tau)$ (resp. $X_k(\tau)$) form a basis of $H^0(X_Z(\tau), L_Z^m)$ (resp. $H^0(X_k(\tau), L_k^m)$).*

Remark 6. Theorem 4 implies in particular that the canonical homomorphism $H^0(G_Z/B_Z, L_Z^m) \to H^0(X_Z(\tau), L_Z^m)$ is surjective. As a consequence, it follows (see Remark 5 and Remark 4) that $V_{m\omega,Z}(\tau)$ is a direct summand of $V_{m\omega,Z}$ for every $m \geqslant 0$.

If G_Z is a *classical group*, one has a suitable generalization of Theorem 4 for G_Z/B_Z and as a consequence one deduces that the canonical homomorphism

$$H^0(G_Z/B_Z, L_{\lambda,Z}) \longrightarrow H^0(X_Z(\tau), L_{\lambda,Z}), \qquad \tau \in W/W_P$$

is surjective for *any* dominant λ. Consequently, one deduces that $V_{\lambda,Z}(\tau)$ is a direct summand in $V_{\lambda,Z}$ for any dominant λ (G_Z being classical).

Remark 7. The assertion (ii) of Theorem 2, namely that

$$(*) \qquad X_k(\tau) = X_Z(\tau) \times_{\operatorname{Spec} Z} \operatorname{Spec} k$$

has been stated and assumed in (G/P-IV, [9]). However, it is not necessary to assume (*) for proving the main theorems on standard monomial theory; in fact (*) follows for all the Schubert varieties for which a standard monomial theory has been proved in [9] (in particular, for all Schubert varieties in the case of a classical group). Let us now indicate a proof of (*) in the case of Schubert varieties in G_k/P_k, where P_k is a maximal parabolic of classical type. One proves Theorem 3′ first and this does not use (*). Since $V_{\omega,Z}(\tau)$ is a direct summand of $V_{\omega,Z}$, the canonical mapping

$$V_{\omega,k}^*(\tau) \longrightarrow H^0(X_k(\tau), L_{\omega,k})$$

is *injective* (as we saw in § 1), where $X_k(\tau)$ is only taken as $(X_Z \times_{\operatorname{Spec} Z} \operatorname{Spec} k)_{\mathrm{red}}$. One defines $P_{\tau,\varphi}$ and $P_{\tau,\varphi}$ as above. We can identify $p_{\tau,\varphi}$ as an element of $H^0(X_k(\tau), L_{\omega,k})$. Now the crucial point is that distinct standard monomials in $p_{\tau,\varphi}$ (say of length m) are linearly independent elements of $H^0(X_k(\tau), L_{\omega,k}^m)$ (see the proof in G/P-IV, [9]). This is a formal step (and does not use (*)). Let $\operatorname{St}(\tau, m)$ denote the Z-submodule of $H^0(X_Z(\tau), L_{\omega,Z}^m)$ generated by standard monomials of length m in $P_{\tau,\varphi}$. Then the canonical mapping

$$(**) \qquad \operatorname{St}(\tau, m) \otimes k \longrightarrow H^0(X_k(\tau), L_{\omega,k}^m)$$

is injective $\forall$ field k. For $k=\boldsymbol{Q}$ this is shown to be an isomorphism. (cf [9]). This implies that the canonical map

$$\mathrm{St}(\tau, m) \longrightarrow H^0(X_Z(\tau), L^m_{\omega,Z})$$

is an isomorphism of $\boldsymbol{Z}$-modules. Now for $m \gg 0$, the canonical map

$$V^*_{m\omega,Z} = H^0(G_Z/P_Z, L^m_{\omega,Z}) \longrightarrow H^0(X_k(\tau), L^m_{\omega,k})$$

is *surjective*. This implies that (**) is an *isomorphism* for $m \gg 0$, i.e., the canonical map

$$H^0(X_Z(\tau), L^m_{\omega,Z}) \otimes k \longrightarrow H^0(X_k(\tau), L^m_{\omega,k})$$

is an isomorphism for $m \gg 0$. This implies (as we saw in the proof of (ii) of Theorem 2) that the canonical map

$$X_Z(\tau) \times_{\mathrm{Spec}\, Z} \mathrm{Spec}\, k \longrightarrow X_k(\tau)$$

is an *isomorphism*, which is the required assertion.

Remark 8. The projective normality of a Schubert variety $X_k(\tau)$ (P_k of classical type, k any field) is a consequence of standard monomial theory (see Remark 4.1, [11]; a simpler deduction has been found by Huneke and Lakshmibai). The normality of such an $X_k(\tau)$ follows as a consequence. We did not suspect that a general proof could be given as in Theorem 2 above.

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