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Hodge Theory and Kodaira Dimension

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As explained in S. Iitaka's paper in this volume, "the addition conjecture" by Iitaka is now the central problem in the theory of Kodaira dimension. The theory of variation of Hodge structures by P. Griffiths, W. Schmid, P. Deligne and others provides a powerful tool for the study of this problem. In this paper we shall review such applications of Hodge theory. Section 1 is devoted to explaining the semi-positivity of the direct image sheaf of the relative canonical bundle, which is our fundamental result. In Section 2 we consider the direct image sheaf of higher multiples of the relative canonical bundle in the case where the base space is a curve. In Section 3 we shall discuss some global results by using the period domain. These results are completely covered by the papers [6], [7] and [8], which correspond to Sections 1, 2 and 3, respectively.

The semi-positivity theorems seem to have a close relationship with the existence of the moduli space of algebraic varieties. In this respect, we shall sketch in Section 2 an interesting application of our semi-positivity argument to deformation theory, due to K. Maehara.

All varieties and morphisms in this paper are defined over the complex number field.

For general concepts and results in the classification theory of algebraic varieties, we refer the reader to the papers of Iitaka and Viehweg in this volume, and the lecture notes by Ueno [16], which have become the "bible" in this field or Iitaka's textbook [20].

§ 1. Semi-positivity of the Hodge bundle

What we shall do is the *birational geometry* of algebraic varieties defined over the complex number field: any birationally equivalent varieties or morphisms are considered the same. Thus we usually have only to consider non-singular and projective varieties by Hironaka's resolution theorem. Also, a divisor which appears as the boundary of an open part of a variety will be simplified to a *divisor with normal crossings*, which is

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defined to be a divisor with smooth irreducible components intersecting transversally.

For a non-singular projective variety X, we denote by K_X the *canonical bundle* on X which is defined to be the sheaf of highest differential forms $\Omega_X^{\dim X}$. An *algebraic fiber space* is a morphism $f\colon X\to Y$ of non-singular projective varieties which is surjective and has connected fibers. In this section we shall study the direct image sheaf $f_*K_{X/Y}$, where the *relative canonical bundle* $K_{X/Y}$ is defined to be $K_X\otimes f^*K_Y^{\otimes -1}$. Let Y_0 be the Zariski-open subset of Y over which f is smooth. Put $X_0=f^{-1}(Y_0)$, $f_0=f|_{X_0}$ and $n=\dim X-\dim Y$. *We shall assume that $D=_{\text{def}}Y\setminus Y_0$ is a divisor with normal crossings on Y.* Since the family $f_0\colon X_0\to Y_0$ is locally trivial as a topological fibration, we obtain a local system $H_C=(R^nf_*C_{X_0})_{\text{prim}}$ of primitive cohomology classes with an integral lattice H_Z. Following Griffiths [3], we define the Hodge filtration $\{F_0^p\}_{0\leqslant p\leqslant n}$ on the vector bundle $H_0 =_{\text{def}} H_Z\otimes\mathcal{O}_{Y_0}$, which is a descending filtration by vector subbundles. Cup product followed by the integration along the fibers of f_0 gives a flat bilinear form S on H_0, which is called the *polarization*. The *Weil operator C* acts on H_0 linearly such that it coincides with multiplication by $(\sqrt{-1})^{p-q}$ on each Hodge component $H^{p,q}$. By the Riemann-Hodge bilinear relations, S induces a positive hermitian metric $h(u,v)=_{\text{def}}S(u,C\bar{v})$ on H_0. We note that F_0^n is just the direct image sheaf $f_{0*}K_{X_0/Y_0}$.

By Schmid [11] we construct the *canonical extension* H of H_0, which is a locally free sheaf on Y defined as follows: Let U be a small open subset of Y which is isomorphic to a polydisc, and let $\{z_1,\cdots,z_d\}$ be a coordinate system on U such that $U\cap D=\{z_1\cdots z_e=0\}$ for some $0\leqslant e\leqslant d$. Let γ_i $(i=1,\cdots,e)$ be the local monodromies of H_Z corresponding to loops around the z_i-axes. It is known that they are all quasi-unipotent. Let $v_1,\cdots,v_r$ be multi-valued flat sections of H_0 on U which make a basis of H_0 at each point. Then the expressions

$$s_j=\exp\left(-\sum_{i=1}^{e}\log\gamma_i\log z_i/2\pi\sqrt{-1}\right)v_j,\qquad j=1,\cdots,r$$

give single-valued holomorphic sections of H_0, where the branches of $\log\gamma_i$ are chosen so that their eigenvalues are in an interval $\sqrt{-1}\,[0,2\pi)$. Let $H|_U$ be a holomorphic vector bundle on U generated by the s_j. Then it can be checked that this construction does not depend on the choice of the z_i and the v_j, and gives a vector bundle H on Y.

Theorem 1. *Let $i\colon Y_0\to Y$ be the inclusion morphism. Then the natural identification $F_0^n\cong f_{0*}K_{X_0/Y_0}$ extends to an isomorphism $H\cap i_*F_0^n\cong f_*K_{X/Y}$* ([7] Lemma 1).

Sketch of proof. The right hand side coincides with the sheaf of square integrable rational differential forms. Thus this is a consequence of the estimation of the growth of the metric h at the boundary given by Schmid [11] p. 253, which says that it is bounded from below and above by a logarithmic function multiplied by a rational power of the coordinate functions whose exponents are determined by the local monodromy.

Corollary 2. *$f_* K_{X/Y}$ is a reflexive sheaf.*

In fact, it is locally free (cf. Viehweg's paper in this volume.)

If all the local monodromies of H_Z at the boundary D are *unipotent*, then each F_0^p $(0 \leqslant p \leqslant n)$ extends to a locally free subbundle F^p of H by Schmid [11]. In general, a line bundle L on a complete variety X is said to be *semi-positive*, if the intersection number $(L.\ C)$ is non-negative for every integral curve C on X. A vector bundle V on X is said to be *semi-positive*, if the tautological line bundle L_V on $\boldsymbol{P}(V)$ is semi-positive.

Theorem 3. *If all the local monodromies are unipotent, then F^n is a semi-positive vector bundle* ([6] Theorem 5).

Sketch of proof. We use the criterion (1) given in the list below. First, we treat the general case: assume that the image $\varphi(C)$ of the curve intersects Y_0. Then the metric h restricted to F_0^n induces a hermitian metric on $Q|_{C_0}$, where $C_0 = \varphi^{-1}(Y_0)$. Griffiths proved in [3] that the curvature of this metric is positive semi-definite. We can calculate the degree of Q as the sum of the integral of the curvature on C_0 plus some boundary contributions at points on $C \setminus C_0$. The former is non-negative by Griffiths' theorem, and the latter is shown to be zero by observing the growth of h at the boundary (in this case it is logarithmic). Next, assume that $\varphi(C)$ is contained in some irreducible component D_1 of D, but not contained in any other irreducible components of D. If U is a small neighborhood of D_1, the (conjugacy class of the) local monodromy γ_1 around D_1 gives a flat ascending filtration $\{W_l\}_{0 \leqslant l \leqslant 2n}$ on $F^n|_{U \setminus D}$, which is called the *weight filtration*, such that γ_1 acts trivially on $\mathrm{Gr}^W(F^n|_{U \setminus D})$. Thus we obtain a flat vector bundle $\mathrm{Gr}^W(F^n|_{D_1^0})$ on $D_1^0 =_{\mathrm{def}} D_1 \setminus (D - D_1)$, and restricting to some adequately defined primitive part, we obtain a variation of Hodge structures again (Schmid [11] Sect. 6). Then the first part of the proof is applied. Finally, if the position of $\varphi(C)$ is more special, then we divide F^n more by taking "Gr" with respect to other local monodromies.

Semi-positive vector bundles in some sense behave like limits of ample vector bundles. We list here some elementary properties of semi-positive vector bundles. These are easily deduced from the corresponding theorems for ample vector bundles by Hartshorne [4].

(1) A vector bundle V on a complete variety X is semi-positive, if

and only if for every non-singular projective curve C, for every morphism φ: $C \to X$ and for every quotient line bundle Q of $\varphi^* V$, we have $\deg_C Q \geqslant 0$.

(2) Every quotient vector bundle of a semi-positive vector bundle is semi-positive. In particular, a vector bundle generated by global sections is semi-positive.

(3) Let f: $X \to Y$ be a morphism of complete varieties and let V be a vector bundle on Y. If V is semi-positive, then $f^* V$ is semi-positive. Conversely, if $f^* V$ is semi-positive and f is surjective, then V is semi-positive.

(4) Let f:$X \to Y$ be a finite and surjective morphism of complete varieties and let V be a vector bundle on X. If $f_* V$ is a vector bundle on Y and is semi-positive, then V is semi-positive on X. (The converse is not necessarily true.)

(5) If a symmetric tensor product $S^m(V)$ is semi-positive for some positive integer m, then V is semi-positive. Let V be a vector bundle on a projective variety X and let L be an ample line bundle. If $S^m(V) \otimes L$ is semi-positive for every positive integer m, then V is semi-positive.

(6) Let V be a vector bundle on a complete variety X. Assume that for every non-singular projective curve C, for every morphism φ: $C \to X$ and for every point p of C, $\varphi^* V \otimes \mathcal{O}_C(p)$ is semi-positive. Then V is semi-positive.

(7) Let V be a semi-positive vector bundle on a projective variety X and let L be an ample line bundle on X. Then $V \otimes L$ is an ample vector bundle. (By Seshadri's criterion.)

(8) If V and W are semi-positive vector bundles, then $V \otimes W$ is also semi-positive. In particular, $V^{\otimes m}$, $S^m(V)$ and $\wedge^m V$ are all semi-positive for any positive integer m.

(9) Let $0 \to V' \to V \to V'' \to 0$ be a short exact sequence of vector bundles. If V' and V'' are both semi-positive, then V is semi-positive.

(10) Let V be a semi-positive vector bundle on a complete variety X of dimension n. Then $c_1(V)^n \geqslant 0$.

History. We review the history of the semi-positivity theorem. After Iitaka posed "Conjecture C_n" for an algebraic fiber space f: $X \to Y$ and proved it in the case $n=2$, i.e., C_2, by using Enriques-Kodaira's classification of surfaces, the first big progress was made by Ueno [17], in which he proved C_2 without using classification. In this case, Ueno proved $\deg(f_* K_{X/Y}) \geqslant 0$, since the fibers of f are curves and their moduli space is the Siegel upper half space, which has plenty of automorphic forms. It is a remarkable thing that almost all the ingredients which are used as tools later were already present in his proof: e.g., moduli space, Grothendieck's duality theorem. He also suggested possible usefulness of the theory of variations of Hodge structures in [18]. Next, Viehweg extended Ueno's proof and obtained $C_{n,1}$ in [19]. His sophisticated proof showed the power of the language of modern algebraic geometry. Many purely algebraic methods were elegantly used, e.g., base change by covering spaces, Grothendieck's duality, moduli space of stable curves, rational singularities, etc. His proof was also based on the existence of automorphic forms on the Siegel upper half space. In the efforts to extend Viehweg's proof to the general case where the moduli space of the general fibers is not well-known, the role of Hodge theory increased. In this respect, Fujita's work [2] was a break-through. In that paper he proved our Theorem 3 in the case where the base space Y is a curve. His proof was based upon elementary calculations of integrals of differential forms, but Hodge theory behind his calculations was obvious. He also recognized the importance of the concept of semi-positivity for the first time. Finally, Theorem 3 was proved in general in the author's paper [6]. When the base space Y is higher dimensional, the boundary behavior of Hodge bundles is more complicated. To study this, Schmid's paper [11] is essential. On the boundary we obtain only abstract variation of Hodge structures, which may have no geometric interpretation, i.e., no differential forms corresponding to them. Hence we need abstract Hodge theory.

Of course, Theorem 3 does not imply C_n: we have only Theorem 4 below. For further development of this method, we refer the reader to the results in Sections 2 and 3. The author does not know whether the method of Hodge theory may admit further exploitation in this field.

Example. We shall see how the proof of Theorem 3 works in special cases.

(1) Positive curvature: Let $Y=\{t \in \boldsymbol{C};\, |t|<1\}$ and let $f\colon X \to Y$ be a projective smooth family of algebraic varieties of relative dimension n. Let Q be a quotient line bundle of $f_*K_{X/Y}$ and let s be a nowhere-vanishing section of Q. The hermitian metric h on $f_*K_{X/Y}$ induces a metric h_Q on Q. Let ω be a representative of s in $f_*K_{X/Y}$ such that $h(\omega(0), \omega(0)) = h_Q(s(0), s(0))$ and fix a diffeomorphism $\psi\colon X \simeq X_0 \times Y$. At a point $x \in X$, a local relative 1-form dz of type $(1, 0)$ near x is transformed by ψ into the 1-form on $X_0 \times Y$ which is congruent to $\alpha dz_0 + t\beta d\bar{z}_0$ modulo t^2, where dz_0 and $d\bar{z}_0$ are 1-forms of types $(1, 0)$ and $(0, 1)$ on X_0, respectively, and α and β are some functions. Therefore, under the identification by ψ

$$\omega \equiv \omega_0 + t\omega_1 \qquad \text{mod } t^2,$$

where $\omega_0 \in H^0(X_0, K_{X_0}) = F^n(H^n(X_0, \boldsymbol{C}))$ and $\omega_1 \in F^{n-1}(H^n(X_0, \boldsymbol{C}))$. Let $\omega_1 = \omega^{n,0} + \omega^{n-1,1}$ be the decomposition according to the types and let ω_2 be a section of $f_*K_{X/Y}$ which gives the value $\omega^{n,0}$ at $0 \in Y$. Put $\omega' = \omega - t\omega_2$ and denote by s' the section of Q induced by ω'. Then $\omega' \equiv \omega_0 + t\omega^{n-1,1}$ mod t^2 under ψ. Since $S(\omega_0, \bar{\omega}^{n-1,1}) = 0$, we have

$$h(\omega'(t), \omega'(t)) = h(\omega_0, \omega_0) - t\bar{t}h(\omega^{n-1,1}, \omega^{n-1,1}) \qquad \text{mod } (t^2, \bar{t}^2),$$

where we note that the minus sign comes from the definition of the operator C. On the other hand, $h_Q(s'(t), s'(t)) \leqslant h(\omega'(t), \omega'(t))$. Hence $h_Q(s'(t), s'(t))$ is a convex function of t at 0. From this follows the semi-positivity of the curvature of Q with respect to h_Q.

The above elementary calculation will be generalized in the proof of Theorem 6.

(2) Boundary behavior: Let $Y = \{t \in \boldsymbol{C}; |t| < 1\}$ and let $f: X \to Y$ be a projective family of algebraic varieties of relative dimension n which is smooth over $Y^* = Y \setminus \{0\}$. We assume moreover that the singular fiber X_0 is a reduced divisor with normal crossings on X, i.e., the fiber space f is semi-stable. By [15] we can calculate explicitly the canonical extension and the weight filtration in this case.

Let $\Omega_X^{n+1}(\log X_0)$ be the sheaf of rational $(n+1)$-forms on X with logarithmic poles along X_0, and let W_r ($r = 0, \cdots, n+1$) be the weight filtration on it. Steenbrink showed that the fiber of the canonical extension F^n at $0 \in Y$ is given by $H^0(X_0, K_{X_0}) = H^0(X_0, W_{n+1}/W_0)$, where we abbreviated $W_r(\Omega_X^{n+1}(\log X_0))$ by W_r. The weight filtration on the cohomology group is induced by that of $\Omega_X^{n+1}(\log X_0)$. We can calculate the associated graded module with respect to this filtration as follows. Let $E_1, \cdots, E_N$ be the irreducible components of X_0, and let

$$E^{(r)} = \coprod_{i_0 < \cdots < i_r} E_{i_0} \cap \cdots \cap E_{i_r} \qquad \text{for } r \geqslant 0.$$

The Poincaré residue maps give isomorphisms

$$W_{r+1}/W_r \xrightarrow{\sim} \Omega_{E^{(r)}}^{n-r} = K_{E^{(r)}}.$$

Let $\delta_j: E^{(r)} \longrightarrow E^{(r-1)}$ be the morphisms induced by inclusions

$$E_{i_0} \cap \cdots \cap E_{i_r} \longrightarrow E_{i_0} \cap \cdots \cap E_{i_{j-1}} \cap E_{i_{j+1}} \cap \cdots \cap E_{i_r}$$

for $0 \leqslant j \leqslant r$. Define $d_r: H^{n-r}(E^{(r)}, \boldsymbol{C}) \to H^{n-r+2}(E^{(r-1)}, \boldsymbol{C})$ by $d_r(a) = \sum_{j=0}^{r} (-1)^j \delta_{j*}(a)$, where the δ_{j*} are the Poincaré duals of the pull-back homomorphisms δ_j^*. Then by Corollary (4.20) of [15], we have

$$\mathrm{Gr}_r^W(H^0(X_0, K_{X_0})) = \ker (H^0(E^{(r)}, K_{E^{(r)}}) \xrightarrow{d_r} H^1(E^{(r-1)}, K_{E^{(r-1)}}))$$

for $0 \leqslant r \leqslant n$, where we note that $H^0(E^{(r)}, K_{E^{(r)}}) = F^{n-r}(H^{n-r}(E^{(r)}, \boldsymbol{C}))$. The polarizations on them are given by these of the $H^{n-r}(E^{(r)}, \boldsymbol{C})$. Thus we realized the abstract Hodge structures on the boundary defined in the proof as differential forms on the $E^{(r)}$.

Applying the above Theorems 1 and 3, we can prove the following "addition theorem".

Theorem 4. *Let $f: X \to Y$ be an algebraic fiber space. If $\kappa(X) \geqslant 0$ and $\kappa(Y) = \dim Y$, then $\kappa(X) = \kappa(Y) + \kappa(X_y)$, where X_y denotes the general fiber of f* ([6] Theorem 3).

The proof is rather standard in the classification theory of algebraic varieties. Note that in his paper in this volume, Viehweg uses a similar argument to extend this theorem.

Using Theorem 4, we can prove the following.

Theorem 5. *Let X be a non-singular projective variety such that $\kappa(X) = 0$. Then the Albanese map $\alpha: X \to A(X)$ is an algebraic fiber space. In particular, the irregularity $q(X)$ is bounded from above by* $\dim X$. *Moreover, if $q(X) = \dim X$, then α is a birational morphism. In other words, $\kappa(X) = 0$ and $q(X) = \dim X$ give a characterization of abelian varieties up to birational equivalence* ([6] Theorem 1).

§ 2. Curve-base case

In this section we shall study the direct image sheaf $f_*(K_{X/Y}{}^{\otimes m})$ for a positive integer m in case Y is a curve. The problem is much more difficult, because it admits no direct interpretation from Hodge theory and is not contained in any natural flat bundle. (As explained in T. Oda's paper in this volume, K_X and $\mathcal{O}_X$ carry $\mathcal{D}_X$-module structures but no $K_X^{\otimes m}$ does.) Still we obtain the following theorem.

Theorem 6. *If* $\dim Y = 1$, *then $f_*(K_{X/Y}^{\otimes m})$ is a semi-positive vector bundle* ([7] Theorem 1).

Idea of proof. (This theorem follows also from an algebraic argument explained in Viehweg's paper in this volume.) Fix an m. Let Y_0 be the Zariski-open subset of Y such that f is smooth over Y_0 and the fiber of $f_*(K_{X/Y}^{\otimes m})$ over any point y of Y_0 is naturally isomorphic to $H^0(X_y, K_{X_y}^{\otimes m})$. An element ω of $H^0(X_y, K_{X_y}^{\otimes m})$ is an m-ple differential n-form. We take the m-th root $\sqrt[m]{\omega}$ of ω so that we can apply the Hodge theory of usual differential forms, where $\sqrt[m]{\omega}$ is defined on some m-fold covering of X_y which is similar to the Riemann surface of a branched analytic function.

The length $\|\omega\|$ of ω is defined to be $h(\sqrt[m]{\omega}, \sqrt[m]{\omega})^{m/2}$ where h is the metric of the Hodge bundle discussed in Section 1. Using this length, we define a hermitian metric on a line bundle Q which appears as a quotient of a pull back of $f_*(K_{X/Y}^{\otimes m})$. Then an elementary but cumbersome integral calculation leads to the semi-positivity.

By using the above semi-positivity theorem, we can prove the following "addition theorem".

Theorem 7. *Let $f: X \to Y$ be an algebraic fiber space. If* $\dim Y = 1$, *then* $\kappa(X) \geqslant \kappa(Y) + \kappa(X_y)$ ([7] Theorem 2).

Sketch of proof. In case Y is a curve of genus greater than one, the theorem follows easily from Theorem 6. But if Y is an elliptic curve, we need some more argument. Let $f_*(K_{X/Y}^{\otimes m}) = \oplus V_i$ be the decomposition into indecomposable vector bundles. If $\deg V_i > 0$ for some i, then V_i is ample by Hartshorne [5] Theorem 1.3, and we are done. Thus we reduce to the case where $f_*(K_{X/Y}^{\otimes m})$ is a flat vector bundle for every m. Using the results of Atiyah [1] on the structure of such bundles and their behavior under tensor multiplication, we can prove that such an X has a very special property, and using this we get the theorem.

K. Maehara found the following application of Theorem 6, when he studied rational mappings to algebraic varieties of general type.

Theorem 8 (K. Maehara). *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\mu} & X_0 \times T \\ f \downarrow & & \swarrow p \\ Y & & \\ g \downarrow & & \\ T & & \end{array} ,$$

which satisfies the following conditions.

(1) *X, X_0, Y and T are non-singular projective varieties,*

(2) *μ is a birational morphism,*

(3) *f is a generically finite and surjective morphism,*

(4) *g is an algebraic fiber space,*

(5) *p is the projection to the second factor,*

(6) *a general fiber Y_0 of g is a variety of general type,* i.e., $\kappa(Y_0) = \dim Y_0$, *and*

(7) $\dim T = 1$.

Then Y is birationally equivalent to $Y_0 \times T$.

Sketch of proof. For each positive integer m, f induces an injective homomorphism $g_*(K_{Y/T}^{\otimes m}) \to g_* f_*(K_{X/T}^{\otimes m})$, and μ induces an isomorphism $K_{X_0 \times T/T}^{\otimes m} \to \mu_* K_{X/T}^{\otimes m}$. Thus we obtain an injective homomorphism h_m: $g_*(K_{Y/T}^{\otimes m}) \to V_m$, where V_m is a trivial vector bundle over T with a fiber $H^0(X_0, K_{X_0}^{\otimes m})$. Therefore, $g_*(K_{Y/T}^{\otimes m})$ is a semi-negative (i.e., dual of semi-positive) vector bundle. On the other hand, it is semi-positive by Theorem 6. Hence it is flat, and thus it is trivial. Since the h_m are compatible with tensor multiplication, $\mathrm{Proj}\,(\bigoplus_{m \geqslant 0} \mathrm{Im}\,((g_*(K_{Y/T}^{\otimes m_0}))^{\otimes m} \to g_*(K_{Y/T}^{\otimes m m_0}))$ is just a product variety for any m_0. Since Y is birationally equivalent to such a variety for some m_0, we conclude the proof.

§ 3. A global theory

Hodge theory provides us with not only a local theory—the semi-positivity of the Hodge bundle, but also a global theory—the existence of the period domain. We use the notation of Section 1. Let us fix a base point y of Y_0. By Griffiths [3], the variation of Hodge structures on Y_0, induced from the algebraic fiber space f: $X \to Y$, gives a period mapping P: $Y_0 \to \mathscr{D}/\Gamma$, where $\mathscr{D}$ is the classifying space of Hodge structures on $H_{Z,y}$ and Γ is the image of the monodromy representation of $\pi_1(Y_0, y)$ in $\mathrm{Aut}\,(H_{Z,y})$. The period mapping is by definition only an analytic morphism. But by the following theorem, it is essentially algebraic in birational geometry.

Theorem 9. *Let X be an algebraic variety, let Y be an irreducible and reduced complex space and let f: $X \to Y$ be a proper and surjective morphism of complex spaces with connected fibers. Then there exist algebraic varieties X' and Y', a proper birational morphism μ: $X' \to X$, a proper bimeromorphic morphism of complex spaces ν: $Y' \to Y$, and a proper surjective morphism of algebraic varieties f': $X' \to Y'$ such that $f \circ \mu = \nu \circ f'$* ([8] Theorem 11).

By the above theorem we can compactify the period mapping, after the Stein factorization, to make an algebraic fiber space $Y \to Z$. But because of the monodromy, the given variation of Hodge structures over Y_0, which we simply denote by H_0, is not the pull back of some variation of Hodge structures over an open subset Z_0 of Z. However, the monodromy action along the fibers is killed after taking a finite covering. Using the covering technique developed in [6] Section 2 and replacing Y if necessary by a suitable birational model, we obtain the following situation:

$$\begin{array}{ccc} Y' & \xrightarrow[P']{} & Z' \\ {\scriptstyle h}\downarrow & & \\ Y & & \end{array}$$

where Y' and Z' are non-singular projective varieties, h is a finite and surjective morphism, and P' is an algebraic fiber space induced by the period mapping P, such that the pull back h^*H_0 of H_0 by h coincides with the pull back P'^*H_0' of a variation of Hodge structures H_0' over an open subset Z_0' of Z' by P', which means that they are isomorphic as filtered vector bundles with integral lattices. We note that $\dim Z' = \dim(\mathrm{Im}\, P)$.

Then we can extend Kodaira's canonical bundle formula for elliptic surfaces to higher dimensional cases in the following sense. For an elliptic surface $f: X \to Y$ whose fiber contains no exceptional curve of the first kind, Kodaira gave a formula for K_X in terms of K_Y, the J-invariant function, and singular fibers. Since there is no concept of minimal models in higher dimensional cases, we only give a formula for $f_* K_{X/Y}$. (Formulae for $f_*(K_{X/Y}^{\otimes m})$ are not known.) We assume that the canonical bundle of a general fiber of f is trivial. Then $f_* K_{X/Y}$ becomes a line bundle on Y by Corollary 2. Let $H^{\#}$ and H' be the canonical extensions of $h^* H_0$ and H_0' on Y' and Z', respectively. Then $F^n(H^{\#})$ is canonically isomorphic to $P'^*(F^n(H'))$. On the other hand, $h^*(H \cap i_* F_0^n)$ contains $F^n(H^{\#})$, and the difference of these are given by the eigenvalues of local monodromies around the boundary. Thus we can write:

$$h^*(f_* K_{X/Y}) = P'^*(\text{a standard sheaf}) \otimes (\text{monodromy terms}).$$

In the case of elliptic fiber spaces, the first term is given by the J-invariant, and the second term is given by the types of singular fibers at the boundary. We could get a more explicit formulation, if the moduli space of the general fiber is known, e.g., if it is an elliptic curve, a $K3$ surface, or an abelian variety (cf. [8] Theorem 20 and also [10]).

Using the above argument, we can prove the following "addition theorem".

Theorem 10. *Let $f: X \to Y$ be an algebraic fiber space such that its general fiber has a trivial canonical bundle. If $\kappa(Y) \geqslant 0$, then $\kappa(X) \geqslant$* $\max(\kappa(Y), \dim(\mathrm{Im}\, P))$ ([8] Corollary 14).

We can extend the above theorem to the case where the general fiber has a finite covering whose canonical bundle is trivial (see [8]).

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